GRAUERT THEORY AND RECENT COMPLEX GEOMETRY

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Abstract. A short account for some of the papers of Grauert will be given making connections to the talks to be presented in the workshop.

Papers of Grauert

[CCK] *Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik,* Math. Ann. **131** (1956), 38-75.

[App] *Approximationssätzefür holomorphe Funktionen mit Werten in komplexen Räumen,* Math. Ann. **133** (1957), 139-159.

[HFL] *Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen,* Math. Ann. **133** (1957), 450-472.

[AF] *Analytische Faserungen über holomorph-vollständigen Räumen,* Math. Ann. **135** (1958), 263-273.

[LP] *On Levi's problem and the imbedding of real-analytic manifolds,* Ann. of Math. (2) **68** (1958), 460-472.

[ETAG] *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen,* Inst. Hautes ´tudes Sci. Publ. Math. No. 5 (1960), 64 pp.

[ÜM] *Über Modifikationen und exzeptionelle analytische Mengen,* Math. Ann. **146** (1962), 331-368.

[TF] (with Andreotti, A.) *Théorème de finitude pour la cohomologie des espaces complexes,* Bull. Soc. Math. France **90** (1962), 193-259.

[BP] *Bemerkenswerte pseudokonvexe Mannigfaltigkeiten,* Math. Z. **81** (1963), 377-391.

[VAK] (with Riemenschneider, O.) *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen,* Invent. Math. **11** (1970), 263-292.

[KHQ] (with Riemenschneider, O.) *Kählersche Mannigfaltigkeiten mit hyper-q-konvexem Rand,.* Problems in analysis (Lectures Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969), pp. 61-79. Princeton Univ. Press, Princeton, N.J., 1970.

[DIS] *Über die Deformation isolierter Singularitten analytischer Mengen,* Invent. Math. **15** (1972), 171-198.

[QQ] *Theory of q-convexity and q-concavity,* Complex analysis. Several variables, **7**,VII, 259-284, Encyclopaedia Math. Sci., **74**, Springer, Berlin, 1994.

1. [ÜM] in SCV

[ÜM] is a continuation of [LP] where Grauert solved the Levi problem on complex manifolds. In [LP], which generalizes Oka's solution of the Levi problem over $\mathbb{C}^n$, it was proved that every strongly pseudoconvex manifold M has finite-dimensional analytic sheaf cohomology groups in the positive degrees if the coefficients are coherent. As an important consequence of the finite-dimensionality of the first cohomology, it was shown that M is holomorphically convex, which implies further that M can be mapped onto a Stein space by a proper holomorphic map π in such a way that π is one to one outside the set of critical points say E whose image $\pi(E)$ is a finite set. E is called an exceptional set.

Using [LP], properly generalized to analytic spaces with arbitrary singularities, [ÜM] gives a characterization of exceptional sets and studies their geometry. Let us present a brief survey of its content following the review MR137127 given by S.Hitotumatu.

First of all, exceptional sets are characterized as follows.

Given a complex space X, compact analytic subset $A \subset X$ is called *maximal* if $\dim_x A > 0$, $x \in A$, and if every compact analytic subset A' containing no isolated points must be contained in A. Now, a compact analytic set A containing no isolated points is exceptional if and only if there exists a strongly pseudoconvex neighborhood $U = U(A)$ of A relatively compact in X, such that A is the maximal compact analytic set of U (Satz 2.5).

In addition to this famous criterion, exceptional sets are analyzed in several ways.

Generalizing the notion of negative bundle, Grauert introduced a notion of weakly negative vector bundle. A vector bundle V over a complex space X is called *weakly negative* if there exists a strongly pseudoconvex neighborhood $T = T(0)$ of the zero-section 0. A Nakano-negative vector bundle is weakly negative, but not conversely. If a compact complex space X has a weakly negative vector bundle over it, then X is projective algebraic (Satz 3.2). From this result there follows a generalization of Kodaira's theorem that every normal Hodge variety X is projective algebraic. This is true also in the case where X is a normal analytic space.

The complex structure of the neighborhood of an exceptional analytic subset A is discussed, especially when X is a complex manifold and A is an analytic subset of codimension 1. If $\mathfrak{m}$ is a coherent analytic sheaf whose zero-set is A, $A(\mathfrak{m}) = (A, \mathscr{O}_X/\mathfrak{m}|_A)$ has the structure of a complex space with nilpotent elements. Several theorems are discussed when $\mathfrak{m}$ is replaced by a subsheaf $\mathfrak{n}$ with same properties. When $\mathfrak{p}$ is the ideal sheaf of A, the notation $A(\nu)$ is used instead of $A(\mathfrak{p}^\nu)$. Further, $A^* = (A, \mathscr{O}|_A)$ is called the germ of a neighborhood of A. Under the above hypotheses, there exists a ν_0 such that for every $\nu \geq \nu_0$, an isomorphism $\psi : A(\nu) \cong \tilde{A}(\nu)$ is extended to an isomorphism $\phi : A^* \cong \tilde{A}^*$ (Satz 4.6). This means that the complex structure in the neighborhood is computable in this case.

There is given an example of a complex space X which is connected, compact, normal, of dimension 2, has only one singular point and there are two analytically and algebraically independent meromorphic functions over X, but still X is not algebraic (neither in a projective sense nor in Weil's sense).

After [ÜM], Grauert's characterization of exceptional sets has been generalized to the relative cases by Knorr and Schneider in [K-S] and by Nakano and Fujiki in [N, F-N, F]. The deformation theory worked as a motivation of these works. In this context, Grauert proved in [DIS] the existence of a semi-universal deformation of any isolated singularity.

There arose naturally a question on the relation between the deformations of strongly pseudoconvex manifolds and isolated singularities. Kuranishi [K] had an idea of considering the deformation of the boundary of strongly pseudoconvex domains to describe that of isolated singularities. From this viewpoint, some people including Akahori have studied the deformation of compact strictly pseudoconvex CR manifolds as abstract models of the boundary of strongly pseudoconvex domains (cf. [A], [A-G], [A-G-L] and [A-M]).

2. **[TF], [KHQ] and [VAK]**

Grauert's generalization of Kodaira's embedding theorem in [ÜM] is based on the finiteness theorem on strongly pseudoconvex manifolds. It was generalized to finiteness theorems on q-convex spaces and q-concave spaces in [TF] and refined to vanishing theorems on strongly pseudoconvex manifolds in [VAK] and those on hyper q-convex Kähler manifolds in [KHQ].

Although detailed structures of q-convex or q-concave spaces have not been analyzed so much, it is known that they are related to a classical question of algebraic geometry in an interesting way and that the space of $(q-1)$-cycles on q-complete spaces is Stein (cf. [QQ] and [B]). Recently, the q-convexity notion turned out to be effective in the generalization of Hartogs' theorem which characterizes the graph of holomorphic functions in terms of the pseudoconvexity of its complement. Pawlaschyk's thesis [P] (see also [P-S]) and Ohsawa's recent paper [Oh-4] gave completely different proofs of such a generalization. K. Matsumoto has studied the derived sets of q-concave sets and answered some of the questions asked by K. Oka.

[KHQ] was generalized in [Oh-2] and [Dm] to extend the Hodge theory to noncompact manifolds along the line of Akizuki-Nakano's vanishing theorem [A-N] which had come into the picture in the spirit of Lefschetz hyperplane section theorem. Extension of cohomology classes from non-pluriharmonic loci by Tiba may be regarded as a new variant of [KHQ].

[VAK] can be regarded as a prototype of various generalizations of Kodaira's vanishing theorem (cf. [Oh-3]) which are useful in the classification of projective varieties (cf. [E-V]). Based on the vanishing of higher direct images of the canonical sheaves for proper modifications, positivity properties of relative canonical sheaves have been explored for Kähler morphisms. By analyzing canonical sheaves and cotangent sheaves, structure theorems for a distinguished class of varieties have been obtained (cf. [H-P], [G-G-K], [G-K-P]). A characterization of torus quotients will be given in the talk of Greb. Anticanonical divisors also carry important information of projective varieties as one can see from Matsumura's recent work with Ejiri and Iwai [E-I-M]. Hisamoto has introduced a new geometric flow to study certain invariants on Fano manifolds. Iwai will present results on projective manifolds whose tangent bundle contain "positive" subbundles.

3. **Formal principle**

Grauert's result in [ÜM] on the equivalence of neighborhoods of exceptional sets was generalized by Hironaka and Rossi [H-R] so that restrictions on the singularities of X and A as well as the restriction on the codimension of A have been removed. On the other hand, the idea of the problem seems to go back to Poincaré [Pc] as the introdunction of [Kd-S] suggests. Accordingly, the first partial solution to the problem was first given by Nirenberg and Spencer [N-S] in the following form.

From MR0125982 (by S. Hitotumatu): Let V be a compact complex analytic manifold of dimension n and (W, ϕ) be a holomorphic imbedding $\phi : V \to W$, where W is a complex manifold of dimension $n + 1$. We may introduce local coordinates $(z_{i_1}, \cdots, z_{i_n}, w_i)$ in a neighborhood U_i of a point on $\phi(V)$ so that $(z_{i_1}, \cdots, z_{i_n})$ represents tangential coordinates and w_i the normal coordinate. For another imbedding $\phi_0(V)$ into W_0, we define the notion of μ-equivalent imbeddings when there exist biholomorphic maps from a coordinate neighborhood U_i of $\phi(V)$ into a neighborhood corresponding in $\phi_0(V)$ such that the difference of the functions concerning the coordinate transformations vanishes of order $\mu+1$ in the parameter w_i. Now, there has been a conjecture saying that when the dimension n of V exceeds 1 and that (W,ϕ) is an imbedding which induces over V the bundles L, N, where N is a positive bundle (in the sense of Kodaira), there is a number μ_0 such that any imbedding (W_0,ϕ_0) which is μ_0-equivalent to (W,ϕ) is actually equivalent to it[1]. In the present paper, the authors state the following result, and give a brief outline of the proof of it. The imbedding (W,ϕ) is said to be transversely foliated, if the neighborhood U_i of $\phi(V)$ is covered by a family of analytic varieties of dimension 1 which are pairwise disjoint and each of which intersects $\phi(V)$ at only one point. Then, the above conjecture is true, if (W,ϕ) is transversely foliated. Further, they mention some analogues of the above result, and assert that the results also hold when $\dim W - \dim V = q > 1$.

[N-S] was extended by Griffiths [Gf] and later by Commichau and Grauert [FP] on which MR0627752 by K.Wolffhardt says:

Let A be a connected compact complex submanifold of a complex manifold X. The formal principle says: Every isomorphism from the formal neighbourhood $A(\infty)$ of A in X to the formal neighbourhood $B(\infty)$ of a submanifold B of a manifold Y can be continued to an isomorphism from an open neighbourhood of A in X onto an open neighbourhood of B in Y. It is known that the formal principle does not hold in every case [V. I. Arnol'd, Funktsional. Anal. i Prilozhen. 10 (1976), no. 4, 1-12; MR0431285]; however, it does when the normal bundle N of A in X has suitable properties of negativity [H. Grauert, Math. Ann. 146 (1962), 331-368; MR0137127] or positivity [see, e.g., A. Hirschowitz, Ann. of Math. (2) 113 (1981), no. 3, 501-514]. In the present paper the formal principle is proved under the very weak hypothesis of 1-positivity of N.

The method of [Gf] was recently applied by Brinkschulte [Br] to show that there exist no compact C^∞ Levi flat hypersurfaces with positive normal bundle in a connected compact complex manifold of dimension ≥ 3. Some of the Levi flat hypersurfaces in complex surfaces can be described explicitly besides more or less trivial ones given in [BP]. Adachi has been studying these objects since [Ad].

[1]In [N-S] the conjecture is attributed to Kodaira [Kd] where it is not actually stated. It may be the case that the conjecture was one of the expected applications of Kodaira's vanishing theorem.

We note that the formal principle is still unknown to hold even for the case $A \cong \mathbb{P}^1$. In Arnol'd's counterexample, A is an elliptic curve and N is an element of $\mathrm{Pic}_0 A$ such that $\{N^{\otimes m}; m \in \mathbb{N}\}$ satisfies ceratain Diophantine condition. Ueda [U] studied the embeddings of curves with self-intersection 0 and proved in particular that a condition weaker than the 1-positivity of N suffices for the formal principle to hold for the embedding. Koike's talk is about extension of Ueda's theory from the viewpoint of weak positivity notions. On the other hand, Hirschowitz [Hwz] conjectured that the formal principle does hold if A is a member of an analytic family of submanifolds of X whose union contains a neighborhood of A. Recently J.-M.Hwang [Hw] showed that the conjecture is "generically true" by applying a theory of Morimoto [M].

4. Developments from [CCK]

It is said that Grauert once told their students that it would take 25 years for the mathematical community to understand the meaning of his thesis [CCK]. [CCK] deals with conditions for a complete Kähler domains in $\mathbb{C}^n$ to be Stein. It is shown that a Stein manifold M admits a complete Kähler metric of the form $\partial\bar{\partial}\sum|f_j|^2$, where f_j are holomorphic functions on M and the series is uniformly convergent on compact subsets of M. It is also shown that not all complete Kähler domains in $\mathbb{C}^n$ are Stein if $n \geq 2$. As a positive result, it is shown that a complete Kähler domain $D \subset \mathbb{C}^n$ is Stein if ∂D is a real analytic hypersurface. In [Oh-1] the real analyticity condition was replaced by the C^1-smoothness by applying the L^2 method. The idea was to let the induction on the dimension work by extending holomorphic functions on the complex hyperplane section of D to D. By refining this argument an L^2 extension theorem was proved in [Oh-T]. From [CCK] to [Oh-T] it took more than 30 years. Further refinements of [Oh-T] culminated in the solution by [Bł-1] and [G-Z-1,2] of a long-standing conjecture of Suita [Su] on the inequality $\pi K(z,z) \geq c_\beta(z)^2$ between the Bergman kernel K and the logarithmic capacity $c_\beta(z)$ on Riemann surfaces.

On the other hand, Mok and Yau [Mk-Y] showed that a bounded pseudoconvex domain in $\mathbb{C}^n$ admits a complete Kähler-Einstein metric and that a bounded domain in $\mathbb{C}^n$ which admits a complete Hermitian metric satisfying $-c \leq$ Ricci curvature ≤ 0 must be pseudoconvex. The problem of finding a Kähler-Einstein metric on a Kähler manifold is a nonlinear PDE problem of solving a complex Monge-Ampère equation. Andersson, Błocki and Wulcan [A-B-W] recently studied the generalized Monge-Ampère operator acting on plurisubharmonic functions with analytic singularities to extend the important work of Bedford and Taylor [B-T]. Sera will talk about its continuation based on [L-S-W].

5. Before and after [ETAG]

In contrast to [CCK], the importance of [ETAG] was recognized immediately. In [ETAG] it was proved that the direct image sheaves of coherent analytic sheaves are coherent under proper holomorphic maps. It is a natural extension of Kodaira-Spencer's deformation theory. The point is that, for an analytic family of compact complex manifolds M_t with holomorphic vector bundles F_t, with respect to the cohomology groups $H^{p,q}(M_t, F_t)$ more than the upper semi-continuity of $\dim H^{p,q}(M_t, F_t)$ is true. The proof is quite involved and resembles, according to Grauert's commentary in [G], to the argument of the Nash-Moser implicit function theorem. When M_t are the fibers of a projective morphism, the L^2 method is available to say more by much simpler arguments. For instance, it was shown by the L^2 method that plurigenera $\dim H^{0,0}(M_t, K^m_{M_t})$ of M_t are locally constant (cf. [S-1,2], [T], [Pn]). It was surprising that the study of parameter dependence of the Bergman kernel by Maitani and Yamaguchi [M-Y] has led Lempert to give a very short proof of $\pi K \geq c^2_\beta$ (cf. [Bł-2]). Based on a generalization of [M-Y] to higher dimensional families by Berndtsson [Bs], the proof of Lempert was followed by [Bs-L] which gives a simple proof of the optimized L^2 extension theorem obtained in [Bł-1] and [G-Z-1,2]. Hosono [H] has extended the method of [Bs-L] to explore precise L^2 extension theorems from non-reduced subvarieties.

In this range of development, Yoshikawa will talk about the parameter dependence of certain spectra of the Laplacian for degenerating families of compact Riemann surfaces and Tsuji will introduce a new invariant metric associated to L^p holomorphic functions for $p \neq 2$.

6. [App], [HFL], [AF] and Oka manifolds

In complex geometry, topological obstructions come into play in the Runge type approximation problems for holomorphic maps. Grauert has established the following in [App], [HFL] and [AF].

Theorem 1. *For any analytic principal bundle P over a reduced Stein space X, each connected component of the space $C(X,P)$ of continuous sections of P contains one connected component of $\mathcal{O}(X,P) := \{s \in C(X,P); s \text{ is holomorphic }\}$.*

Gromov [Gm] generalized Theorem 1 by introducing the class of elliptic manifolds. Forstnerič [F-1,2] further extended this class and defined the notion of Oka manifolds as follows.

Definition 1. *A complex manifold M is called an Oka manifold if holomorphic maps from compact convex sets in $\mathbb{C}^m$ to M can be approximated uniformly by holomorphic maps from $\mathbb{C}^m$.*

The point is that Theorem 1 holds not only for the principal bundles but also for analytic fiber bundles with Oka fibers. Recently, Kusakabe [Kb] showed that $\mathbb{C}^n \setminus K$ is Oka for any polynomially convex compact set $K \subset \mathbb{C}^n$.

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