Complex Analytic Geometry and Dynamics on Analytic Neighborhood Spaces

Takayuki Koike

University of Tsukuba

Version 1.2
November 4, 2026

Preface

This document is a set of lecture notes prepared for an intensive lecture series scheduled to be held at the University of Tokyo during 26–30 October 2026.

The present note is *not* intended to be read linearly from beginning to end: In particular, the first two chapters are primarily meant to serve as a reference rather than as a self-contained introduction. These chapters collect a number of basic notions, conventions, and background results from various areas of mathematics, such as algebra, topology, functional analysis, and geometry, which will be used throughout the later chapters. The purpose of this collection is *not* to provide an optimal learning path for beginners, but rather to fix notation and terminology, and to make explicit the precise meaning in which certain notions are used in this note. Readers who are already familiar with most of the material in these chapters are encouraged to skip them at first reading, and to consult them only when a specific definition, convention, or background fact needs to be checked later.

Chapter I summarizes basic material drawn mainly from subjects that are commonly included in undergraduate mathematics curricula at many universities in Japan. When the discussion goes beyond this range, additional explanations have been included whenever possible so that the necessary background can still be consulted within the note. The author is grateful to Dr. Maxim Kukol for bringing many valuable references relevant to this chapter to his attention.

Chapter II collects auxiliary results and foundational general theory that will be used repeatedly in the later chapters. Although much of the general material developed there is basic in character, it is not always treated systematically in undergraduate courses. The chapter is intended to assemble, before entering the main discussion of complex analytic geometry, the language, constructions, and general results needed later. The organization and exposition of this chapter, especially of the parts concerning category theory and cohomology, benefited greatly from discussions with Dr. Masataka Iwai, Dr. Yohsuke Matsuzawa, and Dr. Hiroyasu Miyazaki.

Chapter III begins the main discussion of complex analytic geometry and complex dynamics in these notes, developing the classical theory needed in the subsequent chapters. Building on Chapters I and II, it proceeds from local holomorphic functions and analytic algebra to complex manifolds, holomorphic vector bundles, and complex spaces, and then to pluripotential theory, the L^2 method, and Oka–Cartan theory. Currents, divisors, and intersection theory provide the accompanying geometric tools. The final section introduces the complex dynamics used in this note, with an emphasis on local dynamics near fixed

points in one complex variable. Much of the material on several complex variables and complex geometry in this chapter is based on what the author learned as a graduate student under the supervision of Professor Shigeharu Takayama, to whom he is deeply grateful for his guidance. He has also learned much about these subjects from Professor Junjiro Noguchi, Professor Takeo Ohsawa, and Professor Tetsuo Ueda, and is grateful to them for their continuing advice. He is also grateful to Dr. Thomas Pawlaschyk for valuable discussions during a research stay at the University of Wuppertal. These discussions helped the author clarify his understanding of q-convexity, q-completeness, and the bumping arguments presented here. He thanks Dr. Ryo Uchimoto and Dr. Takato Uehara for ongoing discussions that have deepened his understanding of the material in Chapter III, and for recommending textbooks and other references that were helpful in preparing this chapter. The author is indebted to Professor Eric Bedford, Professor Yûsuke Okuyama, and Professor Laurent Stolovitch for what they taught him about complex dynamics, including topics related to the theories of Yoccoz and Pérez-Marco.

Chapter IV brings together normal-form problems for neighborhoods of analytic subsets, as studied by Grauert, Arnol'd, and Ueda, and local dynamical problems near a fixed point in one complex variable. For this purpose, the author introduces in these notes the notions of *analytic neighborhood models and spaces*, and proposes a common formulation of these linearization problems in terms of functors from small categories to the category of analytic neighborhood models. This formulation records both the compatibility of local coordinate changes on overlaps and conjugacy of a holomorphic map and its iterates. Finite jets, formal completions, cohomological estimates, and Ueda theory provide concrete settings in which the relations between these problems can be examined.

Chapter V applies the analytic methods of Chapter III and the framework of Chapter IV to the semi-positivity of holomorphic line bundles and the geometry of neighborhoods and complements. A central theme is how the existence of suitable plurisubharmonic functions and metric weights constrains the complex structure of a neighborhood, and under what conditions semi-positivity or pseudoconvexity forces linearization. These questions connect positivity with Ueda theory and the formal principle for line bundles. The chapter concludes with nine-point blow-ups of the projective plane, where several of these themes come together.

A *Question* records a question of interest to the author whose answer he does not know, although it may already be known to specialists; information or references from readers would be welcome. A *Problem* records a question which, on the basis of his investigations, the author currently believes to be open and to offer scope for new mathematical developments.

The author's research began during his graduate studies, primarily with an interest in metrics with minimal singularities on holomorphic line bundles over projective complex manifolds. What particularly drew him to this subject were phenomena on projective manifolds—objects that are very algebraic in nature—which cannot be fully understood in algebraic terms and in which complex analytic geometry comes to the fore: in other words, situations at the boundary between algebraic and transcendental phenomena in complex geometry. Through discussions with Professors Jean-Pierre Demailly and Shigeharu Takayama, he became interested in criteria for the semi-positivity of nef line bundles, with the anticanonical bundle of nine-point blow-ups of $\mathbb{P}^2$ as a principal example. With advice from both professors and

also from Professor Takeo Ohsawa, he came to Ueda theory and began research in that direction. Later, during his postdoctoral years, he had the opportunity to engage in stimulating discussions with Professor Tetsuo Ueda. When he invited Professor Ueda to give an intensive lecture series at Osaka Metropolitan University (then Osaka City University), Professor Ueda focused on bringing together knowledge from the theory of functions of several complex variables and from the theory of iteration of holomorphic functions and complex dynamics. These lectures were also a major source of inspiration. Through these experiences, the author's interests turned toward Ueda theory itself and, more broadly, toward the connections between complex analytic geometry and complex dynamics. In these notes, he has sought to gather related ideas and results that he has come to know and finds interesting, and to gain an overview of them on a common platform where they can be discussed together; this is why he has chosen to use the term *analytic neighborhood space*. This is still only one possible direction in an attempt at unification at the level of formulation, and he believes that the future aim should be to bring together and understand in a more substantial way these theories, each of which has already developed to a high level. Nevertheless, nothing would make him happier than for this attempt at organizing the material to offer some help toward that goal.

In preparing the mathematical text, the author used ChatGPT (OpenAI) to assist with English expression, LaTeX preparation, and the drafting and review of exposition. The author chose the mathematical content on the basis of his own understanding and the cited literature. In his instructions to ChatGPT, he explicitly described the intended approaches to proofs and arguments.

For established results with several known proofs, he also supplied the relevant context and candidate approaches. He then consulted ChatGPT about possible problems with his plan and about which known approaches, or combinations of them, would best fit that stage of the exposition. The final decisions about the arguments and their presentation remained with the author. The following observations concern this work on the exposition of established results in 2025–2026. AI-assisted checks brought to his attention potentially misleading formulations of assumptions and some hypotheses that had been inadvertently omitted. After examining these suggestions, he revised the relevant passages, improving their precision and mathematical accuracy. In this author-guided setting, he was surprised by how few clearly incorrect mathematical statements he encountered. This is a qualitative account of a particular use, not a systematic assessment of the reliability of AI-generated mathematics. In these discussions of known proof methods, however, ChatGPT often suggested using concepts or results that had not yet been introduced or established at the relevant point in the text. An argument that is mathematically sound in isolation may nevertheless be unsuitable for the intended development of the text. The author therefore needed to examine both the prerequisites and the logical dependencies of such suggestions, not only to preserve the intended pedagogical sequence but also to avoid circular reasoning. AI output was not treated as mathematical justification in its own right, and the author remains responsible for the mathematical content, references, and figures.

The illustrations also draw on the author's research on AI-assisted visualization for geometric intuition in several complex variables. This project explores interactive three-

dimensional and virtual-reality (VR) visualization as tools for examining and communicating geometry. In preparing selected schematic figures, the author used these environments to compare representations of the underlying geometry and to decide which features and viewpoints to retain in a static diagram. These explorations informed the mathematical descriptions and visual specifications supplied to ChatGPT, and the resulting figures were refined through successive instructions and corrections. The diagrams thus provide an educational application of the project's visualization work within graduate-level mathematical exposition.

This visualization project was supported by the MEXT AI for Science Pioneering Research Creation Project (SPReAD: Supporting Pioneering Research through AI for 1,000 Discovery challenges, 26263378), through the research project "AI-assisted visualization for geometric intuition in several complex variables" (June 2026–January 2027). The author was also supported during the preparation of this manuscript by JSPS Fund for the Promotion of Joint International Research (22KK0232), which supported his research visit to the University of Wuppertal during April 2025–March 2026, and by JSPS Grant-in-Aid for Scientific Research (C) (23K03119).

Finally, the author's greatest thanks go to his beloved wife, Tomomi Koike, whom he deeply respects. He began writing these lecture notes in the summer of 2025, while they were staying in Germany. In the unsettled period that followed, with several moves in quick succession, it would have been absolutely impossible for him to continue his mathematical research and to make progress on these notes without her help. More than that, he has always felt that, since long before he began these notes, her unwavering love, understanding, and encouragement, and above all the joyful and fulfilling life they share every day, have formed the foundation of his research achievements, including several of the results presented here. He would like to put into words his gratitude for all of this and his love for her, but "*Hanc marginis exiguitas non caperet*".

Contents

Notation and Conventions **xi**

I Fundamentals for the complex analytic geometry treated in this note **1**
1 Some fundamentals for polynomials . . . 1
2 From Linear Algebra . . . 6
2.1 Matrices and Conjugacy Invariants . . . 6
2.2 Hermitian matrices and Hermitian forms . . . 7
3 From General Topology . . . 8
3.1 Metric Spaces . . . 9
3.2 Properness . . . 14
4 From basic calculus and the theory of ordinary differential equations . . . 16
4.1 From basic calculus . . . 16
4.2 Convexity of real functions . . . 17
4.3 Ordinary differential equations and flows . . . 19
5 From commutative algebra . . . 23
5.1 Some basic notions of the ring theory . . . 23
5.2 Complexes of R-modules and (co)homology . . . 24
5.3 On the duality . . . 29
5.4 Transcendental degree . . . 31
5.5 Completion and graded rings . . . 32
5.6 Hilbert–Serre theorem and Hilbert–Poincaré series . . . 35
5.7 Krull dimension . . . 37
6 From functional analysis . . . 39
6.1 Topological vector spaces and normed vector spaces . . . 39
6.2 Hilbert spaces and operators between them . . . 48
6.3 Locally convex topological vector spaces and their topological duals . . . 55
6.4 Fréchet spaces and strict countable LF-spaces . . . 59
6.5 On extending operators to spaces of Schwartz distributions . . . 65
6.6 On closedness of the image and the duality . . . 69
7 From algebraic topology . . . 75
7.1 Fundamental groups and covering spaces . . . 75
7.2 Singular homology . . . 81
7.3 CW complex . . . 85

7.4 Simplicial complex . . . 89
7.5 Singular cohomology . . . 91
7.6 Fiber bundles . . . 96
7.7 Vector bundles . . . 105
8 From differential geometry . . . 108
8.1 Some basic analytic facts on C^∞ manifolds . . . 110
8.2 Tubular neighborhood theorem . . . 113
8.3 Ehresmann's fibration theorem . . . 116
8.4 Topological aspects of C^∞ manifolds . . . 118
8.5 Whitney approximation theorem and its applications . . . 125
8.6 Spaces of sections of C^∞ vector bundles . . . 126
8.7 Linear partial differential operators and ellipticity . . . 142
8.8 De Rham cohomology . . . 157
8.9 Harmonic forms and the Hodge Laplacian . . . 176
8.10 Foliation . . . 190
9 From complex analysis in one variable . . . 193
Exercises . . . 204

II Preliminaries in some general frameworks **207**
1 A Siegel–type majorant theorem . . . 207
2 On matrices arising from certain graphs . . . 218
3 Categories and sheaves . . . 220
3.1 Categories and related notions . . . 220
3.2 Sheaves on a topological space . . . 260
3.3 Čech cohomology and applications . . . 281
3.4 Ringed and locally ringed spaces . . . 294
3.5 Cohomology over shrinking neighborhoods of a compact subset . . . 314
Exercises . . . 328

III An introduction to complex analytic geometry and complex dynamics **331**
1 Complex analysis on domains in complex Euclidean spaces . . . 331
1.1 Forms on complex domains . . . 331
1.2 Holomorphic functions of several variables . . . 352
1.3 Biholomorphic maps and invariance under holomorphic changes of coordinates . . . 371
1.4 From stalkwise algebra to coherence . . . 376
2 Complex analytic geometry on global objects . . . 396
2.1 Complex manifolds . . . 396
2.2 Holomorphic vector bundles . . . 403
2.3 Hermitian metrics and Chern curvatures . . . 411
2.4 Analytic subsets and complex spaces . . . 415
2.5 Divisors and log resolutions . . . 433
2.6 Pluriharmonic functions . . . 444

2.7 Plurisubharmonic functions and basic pluripotential theory 446
2.8 The $\overline{\partial}$-equation and L^2 methods 458
2.9 Topological invariants of line bundles and divisors 485
2.10 Pseudoconvexity, q-convexity, and Stein spaces 491
2.11 Coherent analytic sheaves and the Oka–Cartan theorem 504
2.12 Riemann surfaces revisited 514
2.13 Strongly 1-convex manifolds and exceptional analytic sets 519
2.14 The Levi problem, Hartogs extension, and Runge-type approximation 528
2.15 Weakly 1-complete manifolds 540
3 Local dynamics in one complex variable 556
3.1 Circle rotations and rational approximation 557
3.2 Circle diffeomorphisms and rotation numbers 559
3.3 Fixed points and normal forms 562
3.4 Siegel compacta and hedgehogs 571
Exercises 585

IV Complex analytic geometry and dynamics on analytic neighborhood spaces 587
1 Analytic neighborhood spaces 587
1.1 The category (ANM) 587
1.2 Sheaves and cohomology on an analytic neighborhood space 593
1.3 The natural topology on cohomology with coherent analytic coefficients 596
2 Finite jets and formal completions 608
2.1 Finite jets 608
2.2 Formal complex spaces and formal jets 609
2.3 Cohomology and restriction to finite and formal jets 618
3 Cohomology of linear bundle neighborhoods 621
3.1 Homogeneous coefficients and their norms 622
3.2 Finite jets, formal cohomology, and the zero class 623
3.3 The best bounded primitive constants 625
3.4 Comparison with estimates for the Dolbeault equation 628
3.5 Positive line bundles and strongly 1-convex neighborhoods 629
4 Comparison problems and finite-order obstructions 630
4.1 Comparison problems and admissible changes 630
4.2 Finite-order discrepancies and obstructions 636
5 Comparison of neighborhoods and Grauert's theorem 639
5.1 Holomorphic tubular neighborhoods and formal comparison 639
5.2 Grauert's tubular neighborhood theorem 642
6 Comparison of invertible sheaves on analytic neighborhood spaces 647
7 Elliptic suspensions and comparison problems 657
8 Ueda theory on analytic neighborhood spaces 661
8.1 Fixing the unitary flat comparison 662
8.2 Defining-function systems and one-step obstructions 663
8.3 Reduced Cartier centers and the Ueda invertible sheaf 664

8.4 Lifting every frame in the reduced Cartier case . . . 667
8.5 A specified divisor with multiplicities . . . 668
8.6 Analytic realization of a transverse comparison . . . 668
8.7 Kœnigs–Siegel linearization as an application . . . 677
8.8 Ueda-type linearization in arbitrary codimension . . . 679
8.9 Ueda-type linearization near singular curves . . . 688
9 From transverse linearization to standard neighborhoods . . . 695

V Line bundle semi-positivity and the geometry and dynamics of analytic neighborhood spaces **701**
1 Global positivity and numerical dimension . . . 701
1.1 Positivity of line bundles and growth of sections . . . 702
1.2 Nefness and numerical dimension . . . 704
1.3 From numerical dimension to semi-positivity . . . 707
2 Semi-positivity of effective line bundles and neighborhood geometry . . . 709
2.1 Effective divisors and plurisubharmonic functions . . . 709
2.2 Semi-positivity on analytic neighborhood spaces . . . 710
2.3 Effective nef divisors of numerical dimension one . . . 711
2.4 Flat neighborhoods and transverse dynamics . . . 716
2.5 Fibrations and Hartogs extension . . . 717
3 Semi-positivity and formal obstructions . . . 719
3.1 A general line bundle and its obstructions . . . 719
3.2 All-order vanishing for divisor bundles . . . 721
3.3 Pseudoconvex neighborhoods and linearization . . . 723
4 Finite Ueda type and pluripotential geometry . . . 725
4.1 Mildest divergence and singular metrics . . . 725
4.2 Ueda's finite-type theorems . . . 728
4.3 Affine line bundles and the critical exponent one . . . 737
4.4 Singular curves and their growth scales . . . 739
4.5 Minimal singular metrics and obstructions to semi-positivity . . . 740
5 Intersection geometry and FPLB on surfaces . . . 741
5.1 Negative largest eigenvalue . . . 742
5.2 Positive largest eigenvalue . . . 743
5.3 Zero largest eigenvalue . . . 747
6 Nine-point blow-ups of the projective plane . . . 749
6.1 The anticanonical bundle . . . 750
6.2 Cubic curves and elliptic pencils . . . 750
6.3 Flatness and comparison of neighborhoods . . . 751
6.4 Cohomology of the complement . . . 752

Appendix **768**
1 A class of double complexes with controlled higher exactness . . . 768
1.1 Set-up and notation . . . 768

1.2 Exactness hypotheses and the statement . . . 770
1.3 The case $m = 0$ . . . 771
1.4 The case $m = 1$ . . . 772
1.5 The case $m = 2$ . . . 774
1.6 Proof of Theorem A.1.1 (for general m) . . . 777
2 Proof of Lemma II.1.4 . . . 782
2.1 Some fundamental lemmata . . . 782
2.2 Proof of Lemma II.1.4 . . . 783
3 Hints and solutions to the exercises . . . 788
3.1 Hints and solutions to the exercises in Chapter I . . . 788
3.2 Hints and solutions to the exercises in Chapter II . . . 871
3.3 Hints and solutions to the exercises in Chapter III . . . 890
3.4 Hints and solutions to the exercises in Chapter IV . . . 907

Revision History **908**

Notation and Conventions

All mathematical arguments in this note are based on the Zermelo–Fraenkel axioms with the axiom of choice (ZFC). In particular, we shall occasionally use the word *collection* in a broader sense than *set* (A collection specified by a property which is not a set is called a *proper class*; we shall not go further into foundational issues than this). Although logical symbols, such as quantifier symbols $\forall$ and $\exists$, will occasionally be used when necessary for clarity, we generally refrain from using them and instead describe statements in plain language, using expressions such as "for all" and "there exists".

The empty set is denoted by $\emptyset$. Let $P(x)$ be a mathematical statement involving a variable x. We denote by

$$\{\, x \mid P(x) \,\}$$

the collection of all x satisfying $P(x)$ (whenever it is clear that it forms a set). If the range of x is restricted to a given set X, then the corresponding subset is denoted by

$$\{\, x \in X \mid P(x) \,\}.$$

For subsets $A, B \subset X$ (where the symbol $\subset$ is *not* used in the strict sense: i.e., $A = X$ or $B = X$ is allowed), the notation

$$A \setminus B$$

stands for the set $\{x \in A \mid x \notin B\}$. For subsets $A, B \subset X$, their intersection and union are denoted by

$$A \cap B \qquad \text{and} \qquad A \cup B,$$

respectively. More generally, for a family of subsets $\{A_\lambda\}_{\lambda \in \Lambda}$ of a given set X, their intersection and union are denoted by

$$\bigcap_{\lambda \in \Lambda} A_\lambda \qquad \text{and} \qquad \bigcup_{\lambda \in \Lambda} A_\lambda,$$

respectively. If $A, B \subset X$ are disjoint subsets (i.e., $A \cap B = \emptyset$), then their union $A \cup B$ is sometimes denoted by

$$A \sqcup B$$

to emphasize that the union is taken under the assumption that the two subsets are disjoint. More generally, for a family of subsets $\{A_\lambda\}_{\lambda \in \Lambda}$ of a given set X which are pairwise disjoint, we may write

$$\coprod_{\lambda \in \Lambda} A_\lambda$$

for their union.

On the other hand, for an abstractly given family of sets $\{A_\lambda\}_{\lambda\in\Lambda}$ (without assuming that they are subsets of a fixed ambient set), the symbol

$$\coprod_{\lambda\in\Lambda} A_\lambda$$

will also be used to denote their disjoint union in the usual sense, which will be realized for example by letting

$$\coprod_{\lambda\in\Lambda} A_\lambda := \{(x,\lambda) \mid \lambda\in\Lambda,\ x\in A_\lambda\},$$

whose elements will often be written simply as $x \in A_\lambda$ (for some $\lambda\in\Lambda$), without explicitly specifying the index λ, when no confusion is likely to arise (For example, if $\Lambda = \{1,2\}$, $A_1 = \{1,2\}$ and $A_2 = \{2,3\}$, then for the set $A_1 \sqcup A_2 := \coprod_{\lambda\in\Lambda} A_\lambda$, which consists of four distinct elements, care is required when writing its elements. On the other hand, if $\Lambda = \{1,2\}$, $A_1 = \{1,2\}$ and $A_2 = \{+,-\}$, then expressions such as "$+ \in A_1 \sqcup A_2$" cause no ambiguity and will be used freely).

For sets X and Y,

$$f\colon X \to Y$$

means that f is a function from X to Y (In this note, we do not make any essential distinction among the words *map*, *mapping*, and *function*; they are used interchangeably. Later, when the language of categories is introduced in Chapter II §3.1, the same notation will also be used for morphisms between objects, although such morphisms need not literally be interpretable as maps between sets. In such cases, the meaning will always be clear from the context). The assignment of elements is written as $x \mapsto f(x)$ for $x \in X$. For a subset $A \subset X$, the restriction of $f\colon X \to Y$ to A is denoted by $f|_A$. The image $f(X)$ of f is usually denoted by $\operatorname{Image} f$. For a subset $B \subset Y$, the preimage $\{x \in X \mid f(x) \in B\}$ is denoted by $f^{-1}(B)$. When the inverse map of f is well defined, it is also denoted by f^{-1}, since no confusion will arise in most situations. Unless otherwise stated, the other set-theoretic notation follows [Ru1, Chapter 1].

Throughout this note, we use the following notation and conventions:

- For a set S, the symbol $\#S$ denotes its cardinality. In particular, if S is a finite set, then $\#S$ denotes the number of elements contained in S. When no confusion is likely to arise, it is also customary to write $\#S = \infty$ to indicate that S is an infinite set.

- $\mathbb{Z}$ denotes the ring of integers, and $\mathbb{Q}$ denotes the field of rational numbers.

- $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. We regard them as metric spaces endowed with the Euclidean metric $(x,y) \mapsto |x-y|$. Unless explicitly stated otherwise, all topological notions concerning $\mathbb{R}$ and $\mathbb{C}$ refer to the topology induced by this metric.

- The constant π denotes the ratio of the circumference of a circle to its diameter, and e denotes Napier's constant. When no confusion is likely to arise, the symbols π and e may also be used to denote other mathematical objects.

- The exponential e^x is also written as $\exp\, x$ or $\exp(x)$.

- For a non-negative integer n, $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the real and complex Euclidean spaces, respectively. Depending on the context, coordinates are written either as $(x^1, x^2, \ldots, x^n)$ or as $(x_1, x_2, \ldots, x_n)$. The former notation is typically used when transformation properties, as in differential geometry, are relevant, whereas the latter may be used when no such consideration is needed.

- For intervals in $\mathbb{R}$, we use the standard notation such as $[a, b)$ for $a, b \in \mathbb{R}$ with $a < b$ (see [Ru1, Chapter 1]). The interval $[a, \infty)$ is often denoted by $\mathbb{R}_{\geq a}$, and (a, ∞) by $\mathbb{R}_{>a}$. Similarly, $\mathbb{Z}_{\geq a}$ and $\mathbb{Z}_{>a}$ denote the intersections of $\mathbb{Z}$ with $\mathbb{R}_{\geq a}$ and $\mathbb{R}_{>a}$, respectively. The notation $\mathbb{R}_{<b}$, etc., is defined analogously.

- For a complex number r, the expressions $r > 0$ and $r \geq 0$ implicitly mean that $r \in \mathbb{R}$ and that the corresponding inequality holds in $\mathbb{R}$.

- For a real number r, the notation $r \gg 1$ means that r is sufficiently large. For a positive real number r, the notation $r \ll 1$ means that r is sufficiently small. In both cases, the precise meaning depends on the context. If the meaning is not clear from the context, the use of these notations will be avoided as much as possible.

- By a subsequence $\{n_k\}_{k \in \mathbb{Z}_{\geq 0}} \subset \mathbb{Z}_{\geq 0}$, we always mean a strictly increasing sequence of integers, i.e., $n_k < n_{k+1}$ for all k.

- For a complex number $z \in \mathbb{C}$, $\operatorname{Re} z$, $\operatorname{Im} z$, and $\overline{z}$ denote the real part, the imaginary part, and the complex conjugate of z, respectively. The imaginary unit is denoted by $\sqrt{-1}$.

- For a subset of an ordered set, the symbols min and max denote the minimum and the maximum, respectively. The symbols inf and sup denote the infimum and the supremum, respectively. For a subset A of an ordered set, if A is empty, then $\inf A$ and $\sup A$ are left undefined in general. However, when it is clear from the context that A is a subset of $\mathbb{R}$, we interpret them as extended real numbers and adopt the conventions

$$\inf \emptyset := +\infty, \qquad \sup \emptyset := -\infty.$$

- $\mathbb{P}^\ell$ denotes the complex projective space of dimension ℓ. For a complex vector space V of (complex) dimension $\ell+1$, the notation $\mathbb{P}(V)$ is understood as the space of hyperplanes in V, whereas the notation $\mathbf{P}(V)$ (or $\mathbf{P}^\ell$) is used to denote the space of complex lines in V itself.

- $\mathbb{C}^*$ denotes the punctured complex plane

$$\mathbb{C} \setminus \{0\} = \{z \in \mathbb{C} \mid z \neq 0\}.$$

- For a positive integer r, $\mathrm{U}(r)$ denotes the unitary group consisting of all $r \times r$ complex matrices M satisfying $M^\dagger M = I$. In particular, $\mathrm{U}(1)$ is naturally identified with the unit circle

$$\{z \in \mathbb{C} \mid |z| = 1\}.$$

- For a positive integer r and a ring R, $\mathrm{GL}_r(R)$ denotes the group of invertible $r \times r$ matrices with entries in R. For a sheaf of rings $\mathcal{R}$, the notation $\mathrm{GL}_r(\mathcal{R})$ denotes the sheaf whose sections on an open set U are $\mathrm{GL}_r(\mathcal{R}(U))$.

- $\mathbb{Z}_+^N$ and $\mathbb{R}_+^N$ denote the sets of all N-dimensional column vectors

$$\mathfrak{m} = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{pmatrix}$$

  such that each m_i is an integer or a real number, respectively, and $m_i > 0$ for all $i = 1, 2, \ldots, N$.

- For a matrix M with complex entries, ${}^tM$ denotes its transpose, $\overline{M}$ denotes the matrix obtained by taking the complex conjugate of each entry, and

$$M^\dagger := \overline{{}^tM}$$

  denotes the adjoint (i.e., the conjugate transpose) of M.

- For a $\mathbb{C}$-valued differentiable function f defined on an open subset of $\mathbb{R}$, and for a holomorphic function g defined on an open subset of $\mathbb{C}$, the symbols f' and g' denote their derivatives with respect to the real and complex variables, respectively.

- The symbols $\simeq$, $\approx$, and $\cong$ are used to indicate homotopy equivalence, homeomorphism, and isomorphism, respectively. Their precise meanings depend on the context in which they appear. The symbol $\cong$ is also used to indicate biholomorphism.

In these notes, the term "compact" is used without assuming the Hausdorff property. The term "connected" always means non-empty and connected. A subset is called *proper* when it is not the whole ambient set; the same usage applies to subspaces. Properness of an inclusion as a topological map is instead stated explicitly, for example by saying "the inclusion is proper" or "properly embedded". Unless explicitly stated otherwise, the term "vector space" always refers to a vector space over the field $\mathbb{C}$. The same convention applies to topological vector spaces, Banach spaces, Hilbert spaces, Fréchet spaces, and related

notions. The word "ring" means a commutative ring with an identity element 1. For the zero ring, which will be simply denoted by 0 by abuse of notation, we agree that $1 = 0$. A ring homomorphism $R \to R'$ is always assumed to map the identity element of R to that of R'. For a topological vector space E, we distinguish between the algebraic dual and the topological dual as follows: The algebraic dual is denoted by E^*, whereas the topological dual is denoted by $E^\vee$. Unless explicitly stated otherwise, varieties and related objects are defined over the field $\mathbb{C}$. Unless otherwise specified, the term "manifold" always means a Hausdorff second-countable manifold without boundary. In particular, in this sense, a compact manifold is the same as a closed manifold. For a variety or a complex manifold, unless otherwise specified, all topological notions such as "open" refer to the topology of the underlying CW complex (in the case of a variety) or of the underlying C^∞ real manifold (in the case of a complex manifold), and not to the Zariski topology. The real-dimension convention in Chapter I and the later convention on complex dimension and pure dimension are distinguished in Remark III.2.41 of Chapter III. The term "analytic" without a modifier means "complex analytic". When real analyticity is intended, we say "real analytic", equivalently "of class C^ω". For functions of complex variables, we usually use the term "holomorphic".

Further basic notations, conventions, and terminology will be introduced in Chapter I.

Chapter I

Fundamentals for the complex analytic geometry treated in this note

In this chapter, we collect several basic notions and conventions that will be used throughout this note. Most of the material in this chapter is drawn from standard topics typically covered in undergraduate courses, possibly with some extensions that can be followed without difficulty when needed. The aim of this chapter is to provide a common background and to fix notation and terminology, so that the reader can consult it as a reference when reading the later chapters. Whenever a basic fact is standard and easily accessible in the literature, we often content ourselves with a reference. On the other hand, when a result, although well known, is difficult to locate in standard sources (for instance, because of differences in terminology across fields or periods), or when a discussion is expected to be helpful for understanding later arguments, we include a proof or at least an outline for the reader's convenience.

1 Some fundamentals for polynomials

In this section we review some basic facts on polynomials $P \in \mathbb{C}[z]$ with complex coefficients. First of all, the *Fundamental Theorem of Algebra* is one of the most important results in complex analysis and algebra:

THEOREM I.1.1 (Fundamental Theorem of Algebra). *Every non-constant polynomial $P(z) \in \mathbb{C}[z]$ with complex coefficients has at least one root in $\mathbb{C}$. In particular, any monic polynomial P of degree $n \geq 1$ can be factorized as*

$$P(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (possibly repeated) complex roots of P.

Theorem I.1.1 admits many proofs in different contexts. Typical examples include algebro-topological arguments based on the winding number, as well as complex-analytic proofs. Refer to §I.9.

Once the existence of roots is known, a natural question is where they are. For example, the next lemma gives a useful estimate of this type.

LEMMA I.1.2 (Cauchy's estimate for the roots of a polynomial). *Let*

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

be a monic polynomial of degree $n \geq 1$, and set $M := \max_{0\leq j\leq n-1} |a_j|$. Then every root λ of P satisfies

$$|\lambda| \leq 1 + M.$$

PROOF. Assume that $|z| > 1+M$. In particular we have $|z| > 1$, and by the definition of M we know that $|a_j| \leq M$ for all $0 \leq j \leq n-1$. Hence each term of the form $a_j z^j$ satisfies

$$|a_j z^j| = |a_j|\,|z|^j \leq M|z|^j.$$

Summing over j from 0 to $n-1$, we obtain

$$\begin{aligned} |a_{n-1}z^{n-1}| + \cdots + |a_1 z| + |a_0| &\leq M\big(|z|^{n-1} + \cdots + |z| + 1\big) \\ &= M\,\frac{|z|^n - 1}{|z| - 1}. \end{aligned}$$

Since $|z| > 1 + M$, we have $|z| - 1 > M$, and therefore

$$M\,\frac{|z|^n - 1}{|z| - 1} < \frac{(|z| - 1)(|z|^n - 1)}{|z| - 1} = |z|^n - 1 < |z|^n.$$

This shows that

$$|a_{n-1}z^{n-1}| + \cdots + |a_1 z| + |a_0| < |z|^n.$$

On the other hand, by the triangle inequality we have

$$|P(z) - z^n| = |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \leq |a_{n-1}z^{n-1}| + \cdots + |a_1 z| + |a_0|.$$

Combining the two estimates, we conclude that

$$|z^n| > |a_{n-1}z^{n-1}| + \cdots + |a_0| \geq |P(z) - z^n|. \tag{1}$$

Hence $|z^n| > |P(z) - z^n|$, which implies $P(z) \neq 0$. Therefore no root of P can satisfy $|z| > 1 + M$, proving the claim. □

The following fact on the continuity of roots for monic polynomial families is also fundamental:

LEMMA I.1.3. *Let Ω be an open subset of $\mathbb{R}^\ell$, and for each $t \in \Omega$ let*

$$P_t(z) = z^n + a_{n-1}(t)z^{n-1} + \cdots + a_0(t)$$

be a monic polynomial of degree n, whose coefficients $a_j : \Omega \to \mathbb{C}$ depend continuously on t. For each $t \in \Omega$, let $R(t) \in \mathrm{Sym}^n(\mathbb{C})$ be the element defined by the roots of P_t (counted with multiplicities), where

$$\mathrm{Sym}^n(\mathbb{C}) := \mathbb{C}^n/S_n$$

is the quotient by the natural action of the symmetric group S_n on $\mathbb{C}^n$ (i.e., the action by permuting the n coordinates. Equivalently, $\mathrm{Sym}^n(\mathbb{C}) := \mathbb{C}^n/\sim$, where $\sim$ is the relation on $\mathbb{C}^n$ defined by $(\lambda_1, \lambda_2, \ldots, \lambda_n) \sim (\mu_1, \mu_2, \ldots, \mu_n)$ if and only if there exists an element $\sigma \in S_n$ such that $\lambda_j = \mu_{\sigma(j)}$ for all $j \in \{1, 2, \ldots, n\}$). Then the map

$$t \longmapsto R(t) \in \mathrm{Sym}^n(\mathbb{C})$$

is continuous, where $\mathrm{Sym}^n(\mathbb{C})$ is endowed with the natural quotient topology of $\mathbb{C}^n$ by the action of the symmetric group.

Refer to Exercise I.1 for the proof of this lemma.

When P has integer coefficients, *Liouville's approximation theorem* should be mentioned as another classical result:

THEOREM I.1.4 (Liouville's approximation theorem). *Let $\alpha \in \mathbb{R}$ be an irrational algebraic number. Suppose that α is a root of a non-zero polynomial*

$$P(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x],$$

with integer coefficients and that P was chosen to have the smallest possible degree among all non-zero polynomials with integer coefficients that vanish at α. Then there exists a constant $C > 0$ such that, for all integers p and q with $q > 0$, we have

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^d}.$$

PROOF. Let p and q be integers with $q > 0$. Since P has integer coefficients and degree d, the value $P(p/q)$ is a rational number whose denominator divides q^d. Thus we can write

$$P\left(\frac{p}{q}\right) = \frac{m}{q^d},$$

where m is an integer.

If $m = 0$, then p/q is a root of P, which contradicts the minimality assumption on the degree of P by virtue of Lemma I.1.5 below. Therefore we have $|m| \geq 1$. It follows that

$$\left|P\left(\frac{p}{q}\right)\right| = \frac{|m|}{q^d} \geq \frac{1}{q^d}.$$

On the other hand, since $P(\alpha) = 0$, we have

$$P(\alpha) - P\left(\frac{p}{q}\right) = -P\left(\frac{p}{q}\right).$$

By the mean value theorem, there exists a real number ξ between α and p/q such that

$$P(\alpha) - P\left(\frac{p}{q}\right) = P'(\xi)\left(\alpha - \frac{p}{q}\right).$$

Taking absolute values and combining the last two formulas, we get

$$|P'(\xi)|\,\left|\alpha - \frac{p}{q}\right| = \left|P\left(\frac{p}{q}\right)\right| \;\geq\; \frac{1}{q^d}.$$

So far, this yields

$$\left|\alpha - \frac{p}{q}\right| \;\geq\; \frac{1}{|P'(\xi)|}\,\frac{1}{q^d}$$

whenever $P'(\xi) \neq 0$. Note that $P'(\alpha) \neq 0$ and hence $P'(\xi) \neq 0$ holds for ξ sufficiently close to α. Indeed, if $P'(\alpha) = 0$, then the derivative $P' \in \mathbb{Z}[x]$ also satisfies $P'(\alpha) = 0$, which contradicts the minimality assumption on P.

Since $P'(\alpha) \neq 0$, there exists $\varepsilon > 0$ such that $P'(x) \neq 0$ for all x with $|x - \alpha| \leq \varepsilon$. Set

$$K := \max\big\{\, |P'(x)| \;\big|\; |x - \alpha| \leq \varepsilon \,\big\} > 0.$$

If $|p/q - \alpha| \leq \varepsilon$, then the intermediate point ξ also satisfies $|\xi - \alpha| \leq \varepsilon$, hence $|P'(\xi)| \leq K$ and

$$\left|\alpha - \frac{p}{q}\right| \;\geq\; \frac{1}{K}\,\frac{1}{q^d}.$$

If $|p/q - \alpha| > \varepsilon$, then

$$\left|\alpha - \frac{p}{q}\right| > \varepsilon \;\geq\; \frac{\varepsilon}{q^d},$$

since $q \geq 1$. Therefore, setting $C := \min\{1/K, \varepsilon\}$, we obtain

$$\left|\alpha - \frac{p}{q}\right| \;\geq\; \frac{C}{q^d}$$

for all integers p and $q > 0$. □

Lemma I.1.5 (A simple consequence of Gauß' lemma). *Let*

$$P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$$

be a non-zero polynomial with integer coefficients. If P has a rational root $r \in \mathbb{Q}$, then there exists a non-zero constant $N \in \mathbb{Z}$ such that

$$N \cdot P(x) = (x - r)\,Q(x)$$

for some polynomial $Q(x) \in \mathbb{Z}[x]$.

PROOF. Since $r \in \mathbb{Q}$, we can write $r = p/q$ with integers p and q such that $q > 0$ and that p and q are coprime. The assumption $P(r) = 0$ implies that $(x - r)$ divides $P(x)$ in $\mathbb{Q}[x]$ by the factor theorem (or equivalently, by considering the polynomial division of $P(x)$ by $(x - r)$). Hence there exists a polynomial $G(x) \in \mathbb{Q}[x]$ such that

$$P(x) = (x - r)\, G(x).$$

Choose a positive integer N such that all coefficients of $G(x)$ are contained in $\frac{1}{N}\mathbb{Z}$, and set $Q(x) := N\, G(x) \in \mathbb{Z}[x]$. Then we have

$$N\, P(x) = (x - r)\, Q(x),$$

which proves the assertion. □

Using the following terminology, Liouville's approximation theorem can be reformulated as follows: every irrational algebraic number is *Diophantine* (Indeed, one may take $\tau = d$ for example, since

$$|q\alpha - p| = q \cdot \left|\alpha - \frac{p}{q}\right| \ \geq \ \frac{C}{q^{d-1}} \geq \frac{C}{q^d}$$

holds for any $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$, where α, d, and C are as in Theorem I.1.4).

DEFINITION I.1.6. We say that a real number α is *Diophantine* if there exist constants $C > 0$ and $\tau > 1$ such that, for all positive integers q,

$$\min_{m \in \mathbb{Z}} |q\alpha - m| \ \geq \ C\, q^{-\tau}$$

holds. □

Note that any Diophantine number is irrational. It is known that the "size" of the set of Diophantine numbers exhibits a rather interesting behavior: from the measure-theoretic point of view, it has the same "size" as $\mathbb{R}$ and can thus be regarded as "large", whereas from the topological point of view it belongs, in a certain sense, to the category of "small" sets. See Exercise I.4 for details.

MINI-EXERCISE I.1.7. Show that there exist uncountably many irrational numbers which are not Diophantine.

In the rest of this section, for a positive integer n, let us finally look at the graded structure of the polynomial ring $A := \mathbb{C}[x_1, x_2, \ldots, x_n]$ (equipped with the standard $\mathbb{Z}_{\geq 0}$-grading $\deg x_i = 1$ for each $i = 1, 2, \ldots, n$). For each integer $m \geq 0$, we denote by A_m the $\mathbb{C}$-vector space of homogeneous polynomials in A of (total) degree m. As will be discussed later in §5.6 in connection with the *Hilbert–Serre theorem*, the *Hilbert–Poincaré series* of A is defined as the formal power series

$$H_A(t) := \sum_{m \geq 0} (\dim A_m)\, t^m.$$

A basis of A_m is given by the monomials

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \qquad (a_1, a_2, \ldots, a_n \in \mathbb{Z}_{\geq 0})$$

satisfying

$$a_1 + a_2 + \cdots + a_n = m.$$

Therefore

$$\dim A_m = \#\big\{(a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \;\big|\; a_1 + a_2 + \cdots + a_n = m\big\} = \binom{m+n-1}{n-1}.$$

Consequently, the Hilbert–Poincaré series of the polynomial ring $A = \mathbb{C}[x_1, x_2, \ldots, x_n]$ is given by

$$H_A(t) = \sum_{m \geq 0} \binom{m+n-1}{n-1} t^m = \frac{1}{(1-t)^n}.$$

Note that the function $m \mapsto \dim A_m$ coincides with $P(m)$ for any $m \in \mathbb{Z}_{\geq 0}$, where $P(X) \in \mathbb{Q}[X]$ is the polynomial of degree $n-1$ defined by

$$P(X) := \binom{X+n-1}{n-1} = \frac{(X+n-1)(X+n-2)\cdots(X+1)}{(n-1)!} = \frac{X^{n-1}}{(n-1)!} + (\text{lower order terms}).$$

This polynomial P is called the *Hilbert polynomial* of $A = \mathbb{C}[x_1, x_2, \ldots, x_n]$.

2 From Linear Algebra

For the basic notions of linear algebra, the reader is referred to standard textbooks such as [Lan2], [HJ]. Throughout this note, unless otherwise stated, the term *vector space* will mean a vector space over $\mathbb{C}$.

2.1 Matrices and Conjugacy Invariants

Let M and M' be square matrices of order n with complex entries. They are said to be *conjugate* (or *similar*) if there exists an invertible matrix P such that $M' = PMP^{-1}$. The notion naturally arises from a change of basis when one considers linear maps that are defined by matrices. Basic invariants under conjugation include the *characteristic polynomial* and, in particular, the *eigenvalues* (and their algebraic multiplicities). These invariants play a fundamental role in the classification of matrices up to conjugation and will be used repeatedly in what follows.

The reader should take care not to confuse this notion with that of congruence: two matrices M and M' are said to be *congruent* if $M' = PMP^{\dagger}$ for some invertible P. The notion of congruence naturally arises from a change of basis in the context of Hermitian forms.

2.2 Hermitian matrices and Hermitian forms

The reader is assumed to be familiar with *Hermitian matrices* and the notions of *positive (semi-)definiteness* (refer to [HJ, Chapter 4] and [Lan2, Chapter XV] for example). In particular, the following fact is fundamental: For a positive integer n and a square matrix H of order n, H is Hermitian if and only if there is a unitary matrix U of order n and a real diagonal Λ of order n such that

$$H = U\Lambda U^{\dagger} \tag{2}$$

(see [HJ, Theorem 2.5.6, Theorem 4.1.5] for example). In particular, any eigenvalue of an Hermitian matrix is real. For two Hermitian matrices A and B of the same size, we write $A \geq B$ (resp. $A > B$) if $A - B$ is positive semi-definite (resp. positive definite).

Mini-Exercise I.2.1. Let n be an integer with $n \geq 2$, ℓ an integer with $1 \leq \ell < n$, and $A = (a_{jk})_{j,k=1}^{n}$ be an Hermitian matrix of order n. Assume that A is positive semi-definite and that $a_{jj} = 0$ for each integer j with $j > \ell$. Show that $a_{jk} = 0$ for any integers j and k with $\max\{j, k\} > \ell$.

For a positive definite (Hermitian) matrix H of order n, we denote by $\sqrt{H}$ the positive definite matrix A of order n such that $A^2 = H$.

Mini-Exercise I.2.2. Show the well-definedness of $\sqrt{H}$ for a positive definite (Hermitian) matrix H.

Let V be a (complex) vector space. A map

$$h : V \times V \to \mathbb{C}$$

is called an *Hermitian form* if it is linear in the first variable, conjugate-linear (anti-linear) in the second, and satisfies

$$h(x, y) = \overline{h(y, x)} \qquad \text{for all } x, y \in V.$$

An Hermitian form is *positive definite* (resp. *positive semi-definite*) if, for all $v \in V$, $h(v, v) \geq 0$ and $h(v, v) = 0$ implies $v = 0$ (resp. for all $v \in V$, $h(v, v) \geq 0$). Note that

$$|x|_h := \sqrt{h(x, x)}$$

is well-defined as a non-negative real number for any $x \in V$ and for any positive semi-definite Hermitian form, since $h(x, x) \geq 0$. When h is positive definite, $| \bullet |_h$ defines a norm on V. A complex vector space V equipped with a positive definite Hermitian form h is called an *Hermitian space*. Note that a positive-definite Hermitian form is sometimes also called an *(Hermitian) inner product*. The reader is encouraged to recall the *polarization identity* and the *Schwarz inequality* associated with these notions (See [Lan2, Chapter XV §5] for example).

REMARK I.2.3. An Hermitian form (or inner product) $h(\bullet, \bullet)$ is often written using angle brackets, for instance as $\langle \bullet, \bullet \rangle_h$ (Depending on the context or the field, other notations such as $(\bullet, \bullet)_h$ are also frequently used). When the Hermitian form (or inner product) is fixed, or clear from the context, it is common to omit the subscript and simply write $\langle \bullet, \bullet \rangle$. In some situations, one also writes $\langle \bullet, \bullet \rangle_V$ to emphasize the underlying vector space. □

An Hermitian form h is said to be *non-degenerate* when the following holds: $v = 0$ if and only if $h(v, w) = 0$ for all $w \in V$ (or equivalently, $w = 0$ if and only if $h(v, w) = 0$ for all $v \in V$).

In what follows, let us consider the case where V is finite dimensional. Fix a basis $\{e_1, \ldots, e_n\}$ of V. The matrix $H = (h(e_i, e_j))_{1 \leq i,j \leq n}$ is then an Hermitian matrix, and

$$h\left(\sum_{j=1}^{n} x_j e_j,\ \sum_{j=1}^{n} y_j e_j\right) = {}^t x\, H\, \overline{y}, \quad \text{for all } x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n.$$

By *Sylvester's law of inertia* (see [HJ, Theorem 4.5.8] for example), the triple (p, q, r), where p (resp. q, r) denotes the number of positive (resp. negative, zero) eigenvalues of any Hermitian matrix representing h, is independent of the choice of basis. This triple is called the *signature* (or *inertia*) of h. The form h is positive definite if and only if $q = r = 0$, and positive semi-definite if and only if $q = 0$. Moreover, h is non-degenerate if and only if $r = 0$. In this case, it is customary to write the signature simply as (p, q).

We now describe the following elementary but useful lemma for the case where the Hermitian form has signature $(1, n-1)$.

LEMMA I.2.4. *Let V be a vector space of dimension n and h be a non-degenerate Hermitian form on V of signature $(1, n-1)$. Let $A, D \in V$ be elements such that $h(A, A) > 0$ and $h(A, D) = 0$. Then $h(D, D) \leq 0$, and the equality holds if and only if $D = 0$.*

PROOF. Denote by $A^\perp$ the linear subspace $\{v \in V \mid h(A, v) = 0\}$. Then we have the direct sum decomposition

$$V = \mathbb{C}A \oplus A^\perp.$$

One has

$$h(aA + v, bA + w) = a\bar{b} \cdot h(A, A) + h(v, w)$$

for any $a, b \in \mathbb{C}$ and $v, w \in A^\perp$. Therefore, by considering a basis $e_1, e_2, \ldots, e_n$ of V with $e_1 = A$ and $e_2, e_3, \ldots, e_n \in A^\perp$, it follows that the restriction of h to $A^\perp$ is *negative definite* (i.e., the restriction of $-h$ to $A^\perp$ is positive definite), from which the assertion follows. □

3 From General Topology

Unless otherwise stated, any *connected set* is assumed to be non-empty. The term *Hausdorff* is used in the standard sense of the T_2 separation axiom, and the term *compact* is

used in the sense of *quasi-compactness* (i.e., no Hausdorff condition is included in this terminology): a topological space is said to be compact if every open covering admits a finite subcovering. In the same spirit, a topological space is said to be *locally compact* if every point admits a compact neighborhood, where compactness is understood in the above sense. By a *neighborhood* of a subset $A \subset X$ of a topological space X, we mean any subset of X containing an open subset that contains A. When a neighborhood is required to be open, this will always be stated explicitly by saying *open neighborhood*. For the other terms and fundamentals, refer to some standard textbooks such as [Ke] and [Mu2] (See also [Lee-Jo2, Appendix A]).

For a subset A of a topological space X, we denote by $\operatorname{Int} A$ its interior, by $\overline{A}$ its closure, and by ∂A its boundary. Note that in some contexts, for instance when $X \subset \mathbb{C}$, the notation $\overline{A}$ may also be used to denote the set of complex conjugates $\{\overline{x} \mid x \in A\}$. In this note, however, unless explicitly stated otherwise, $\overline{A}$ always denotes the closure of A. Similarly, the symbol ∂ is used in various other contexts, such as boundary operators in homological algebra or the complex exterior derivative of type $(1,0)$. Whenever a possible ambiguity may arise, we shall distinguish these meanings explicitly to avoid confusion. We remark that some texts denote the collection of all the open subsets of a topological space X by the symbol $\mathcal{O}_X$. In this note, however, the notation $\mathcal{O}_X$ is reserved for the structure sheaf of X, and we avoid using this symbol to denote the topology itself.

Mini-Exercise I.3.1. Let X be a compact topological space and Y a Hausdorff space. Show that any continuous map $f\colon X \to Y$ is a closed map.

Mini-Exercise I.3.2. Let X be a topological space and K be a compact topological space. Show that the *first projection* (the projection onto the first factor) $\pi_X\colon X \times K \to X$ is a closed map.

Mini-Exercise I.3.3. Let n, N be positive integers, $\Omega \subset \mathbb{R}^N$ an open set, and $x_0 \in \Omega$. For $x \in \Omega$, let $A(x)$ and $B(x)$ be Hermitian $n \times n$ matrices whose entries depend continuously on x.
(i) Assume that $A(x_0) > 0$. Show that there exist a neighborhood U of x_0 and a constant $\varepsilon > 0$ such that $A(x) > c\,B(x)$ for all $x \in U$ and $c \in (-\varepsilon, \varepsilon)$.
(ii) Assume that $A(x_0) > B(x_0)$. Show that there exists a neighborhood U of x_0 such that $A(x) > B(x)$ for all $x \in U$.

3.1 Metric Spaces

A *metric space* is a pair (X, d) consisting of a set X and a distance function $d\colon X \times X \to [0, \infty)$ satisfying the usual axioms: $d(x,y) = 0$ if and only if $x = y$ $(x, y \in X)$, symmetry, and the triangle inequality. The topology induced by d is generated by the open balls $B(x,r) := \{y \in X \mid d(x,y) < r\}$ for $r > 0$.

Remark I.3.4. In this note, the term *metric* will be used primarily in the sense of an inner product on a vector space or a vector bundle (such as a Riemannian or Hermitian metric). To avoid possible confusion, we therefore often refer to a function $d\colon X \times X \to [0, \infty)$ defining a

metric space simply as a *distance function*. Nevertheless, the standard term "metric space" will be retained, as it seems the most widely accepted terminology, and no ambiguity is expected to arise from the context. □

Recall that a metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point of X. We also recall the following fundamental facts concerning completeness (see [Ke, Chapter 6 22, 32] and [Mu2, Chapter 7] for example).

- For a complete metric space (X, d), a subset $F \subset X$ is closed if and only if it is complete with respect to the induced distance function.
- A subset K of a metric space is compact if and only if it is complete and *totally bounded*.
- In general, closed and bounded subsets of a complete metric space need *not* be compact.

Mini-Exercise I.3.5.
(i) Recall the definitions of *bounded* and *totally bounded* subsets of a metric space. Show that every totally bounded subset is bounded. Then give an example of a bounded subset which is not totally bounded.
(ii) Give a concrete example showing that a closed and bounded subset of a complete metric space need not be compact.
(iii) Recall the *Heine–Borel theorem* for finite-dimensional Euclidean spaces. Compare it with your example in (ii) and explain why the conclusion of the Heine–Borel type theorem fails in this example.

Mini-Exercise I.3.6. Let (X, d) be a metric space. We say that X is *sequentially compact* if every sequence in X admits a convergent subsequence.
(i) Show that if X is compact, then X is sequentially compact.
(ii) Show that if X is sequentially compact, then X is totally bounded.
(iii) Deduce that X is compact if and only if X is sequentially compact.

Mini-Exercise I.3.7. Let (X, d) be a metric space. For a subset $A \subset X$ and $\varepsilon > 0$, set $U(A, \varepsilon) := \{x \in X \mid d(x, A) < \varepsilon\}$. Let $\mathcal{H}_X$ denote the set of all non-empty compact subsets of X. For $A, B \in \mathcal{H}_X$, define

$$d_H(A, B) := \inf\big\{\varepsilon > 0 \;\big|\; A \subset U(B, \varepsilon) \text{ and } B \subset U(A, \varepsilon)\big\}.$$

The function d_H is called the *Hausdorff distance* on $\mathcal{H}_X$.
(i) For any $A, B \in \mathcal{H}_X$, show

$$d_H(A, B) = \max\Big\{\sup_{a \in A} d(a, B),\ \sup_{b \in B} d(b, A)\Big\}.$$

(ii) Show that d_H defines a metric on $\mathcal{H}_X$.
(iii) Let $\{A_\nu\}_{\nu \in \mathbb{Z}_{\geq 0}} \subset \mathcal{H}_X$ be a sequence converging to $A \in \mathcal{H}_X$ with respect to d_H. Assume that each A_ν is connected. Show that A is connected.
(iv) Assume that (X, d) is complete. Show that $(\mathcal{H}_X, d_H)$ is complete.
(v) Assume that (X, d) is totally bounded. Show that $(\mathcal{H}_X, d_H)$ is totally bounded.
(vi) Assume that (X, d) is compact. Show that $(\mathcal{H}_X, d_H)$ is compact.

One of the most important theorems concerning complete metric spaces is the following:

THEOREM I.3.8 (Banach fixed point theorem). *Let (X, d) be a non-empty complete metric space and $T\colon X \to X$ be a contraction, i.e., there exists $r \in (0,1)$ such that $d(T(x), T(y)) \leq r \cdot d(x,y)$ for all $x, y \in X$. Then T has a unique fixed point $x^* \in X$, and for every $x_0 \in X$ the sequence $x_{n+1} := T(x_n)$ converges to x^*.*

MINI-EXERCISE I.3.9. Show Theorem I.3.8.

We recall the *Baire category theorem* in the setting of metric spaces: if (X, d) is a complete metric space, then the intersection of any countable family of dense open subsets of X is again dense in X (see [Ke, Chapter 6 34], [Mu2, Theorem 48.2] or [Ru1, 5.6] for example). A subset of a topological space is called

- *meagre* if it can be written as a countable union of nowhere dense subsets, and
- *comeagre* if its complement is meagre.

With this terminology, the Baire category theorem implies the following: in a non-empty complete metric space, the whole space is *not* meagre.

We finally recall the basic separation axioms, which naturally appear in classical characterizations of metrizability. Let X be a topological space.

- X is said to satisfy the separation axiom T_1 if every singleton subset $\{x\} \subset X$ is closed.
- X is said to satisfy T_2 if it is Hausdorff.
- X is said to satisfy T_3 if it is T_1 and for any point $x \in X$ and any closed subset $F \subset X$ with $x \notin F$, there exist disjoint open neighborhoods of x and F.
- X is said to satisfy T_4 if it is Hausdorff and for any two disjoint closed subsets $A, B \subset X$, there exist disjoint open neighborhoods of A and B.

We remark that the precise formulation of the separation axioms, in particular T_3, may vary slightly in the literature. Throughout this note, we adopt the above conventions. We also remark that the separation axioms T_3 and T_4 are often described using the terms *regular* and *normal*, respectively, in the literature. However, as these terms are also used with different meanings in other areas, notably in algebraic geometry, we refrain from using these terms in this note, to avoid possible confusion.

MINI-EXERCISE I.3.10. Let X be a topological space satisfying the separation axiom T_3.
(i) Let $x \in X$, and let $U \subset X$ be an open neighborhood of x. Show that there exists an open neighborhood V of x such that $\overline{V} \subset U$.
(ii) Let $Z \subset X$ be compact and $\{U_j\}_{j \in J}$ be a finite family of open subsets of X such that

$$Z \subset \bigcup_{j \in J} U_j.$$

For every $j \in J$, let $W_j \subset U_j$ be an open subset satisfying $W_j \cap Z = U_j \cap Z$. Show that there exist open subsets

$$V_j \subset W_j \qquad (j \in J)$$

such that $Z \subset \bigcup_{j\in J} V_j$ and $\overline{V_j} \subset W_j \subset U_j$ for each $j \in J$.
(*iii*) Under the assumptions of (*ii*), suppose that, for every pair $(j,k) \in J^2$, an open subset $W_{jk} \subset U_j \cap U_k$ is given and satisfies

$$W_{jk} \cap Z = U_j \cap U_k \cap Z.$$

Apply (*ii*) to the open subsets $W_j := W_{jj} \subset U_j$ for each $j \in J$, and choose open subsets V_j as in (*ii*). For a pair (j,k) with $j \neq k$, set

$$T_{jk} := \overline{V_j \cap V_k \setminus W_{jk}}.$$

Show that $T_{jk} \cap Z = \emptyset$, and that replacing V_j with $V_j \setminus T_{jk}$ does not remove any point of Z and gives $(V_j \setminus T_{jk}) \cap V_k \subset W_{jk}$.
(*iv*) Show that there exist open subsets $U'_j \subset U_j$ for each $j \in J$ such that $Z \subset \bigcup_{j\in J} U'_j$, $\overline{U'_j} \subset U_j$, and $U'_j \cap U'_k \subset W_{jk}$ for every pair $(j,k) \in J^2$.

It is a classical fact that every metrizable space satisfies T_4 (Indeed, if X is metrizable, for a distance function d inducing its topology and for any two non-empty disjoint closed subsets $A, B \subset X$, the function

$$x \longmapsto \frac{d(x,A)}{d(x,A)+d(x,B)} \tag{3}$$

is well-defined and continuous, and separates A and B. Refer to the following Mini-Exercise I.3.11 (*iii*) for the details. If $A = \emptyset$, use the constant function 1; if $B = \emptyset$, use the constant function 0). Conversely, it is known that a second-countable T_4 space is metrizable (*Urysohn's metrization theorem*, see [Ke, Chapter 4 16] or [Mu2, Theorem 34.1]).

Mini-Exercise I.3.11. Let (X,d) be a metric space, and Y and Z be topological spaces. Assume that Z is non-empty and compact.
(*i*) Let $f\colon Y \times Z \to \mathbb{R}$ be an upper semi-continuous (resp. lower semi-continuous) function. Show that the function

$$y \longmapsto \max_{z\in Z} f(y,z) \quad (\text{resp. } y \longmapsto \min_{z\in Z} f(y,z))$$

is (well-defined and) upper semicontinuous (resp. lower semicontinuous) on Y.
(*ii*) For a non-empty subset $A \subset X$, show that the map $x \mapsto d(x,A) := \inf_{a\in A} d(x,a)$ is continuous. Moreover, if A is closed, show that $d(x,A) = 0$ holds if and only if $x \in A$.
(*iii*) For non-empty disjoint closed subsets $A, B \subset X$, show that the function (3) is well-defined and continuous on X.

Lemma I.3.12. *Let X be a paracompact Hausdorff topological space. Then X satisfies the separation axiom T_4.*

PROOF. First we show that X satisfies T_3. Fix $x \in X$ and a closed subset $F \subset X$ with $x \notin F$. For each $y \in F$, since X is Hausdorff, we may choose disjoint open sets $V_y \ni x$ and $W_y \ni y$. In particular, $x \notin \overline{W_y}$. Then $\{X \setminus F\} \cup \{W_y\}_{y \in F}$ is an open covering of X. By paracompactness, it admits a locally finite open refinement $\{U_i\}_{i \in I}$. Let $I_F \subset I$ be the set of indices i such that $U_i \cap F \neq \emptyset$. Then each U_i for $i \in I_F$ is contained in some W_y, hence $x \notin \overline{U_i}$. Put

$$U := \bigcup_{i \in I_F} U_i.$$

Then U is an open neighborhood of F. By the locally finite nature of the family $\{U_i\}_{i \in I}$, it follows from Lemma I.3.13 below that

$$\overline{U} = \bigcup_{i \in I_F} \overline{U_i}.$$

In particular, $x \notin \overline{U}$. Therefore

$$V := X \setminus \overline{U}$$

is an open neighborhood of x disjoint from U.

Now, to show that X satisfies T_4, let $A, B \subset X$ be disjoint closed subsets. We construct disjoint open neighborhoods of A and B by applying the same type of argument as above. As X satisfies T_3, for each point $a \in A$, one can take an open neighborhood O_a of a such that $\overline{O_a} \cap B = \emptyset$. Then $\{X \setminus A\} \cup \{O_a\}_{a \in A}$ is an open covering of X. By paracompactness, it admits a locally finite open refinement $\{U_i\}_{i \in I}$.

Let $I_A \subset I$ be the set of indices i with $U_i \cap A \neq \emptyset$. Then each U_i for $i \in I_A$ is contained in some O_a, and hence $\overline{U_i} \cap B = \emptyset$. Put

$$U := \bigcup_{i \in I_A} U_i.$$

Then U is an open neighborhood of A, and again by the locally finite nature of the family $\{U_i\}_{i \in I}$, from Lemma I.3.13 we have

$$\overline{U} = \bigcup_{i \in I_A} \overline{U_i} \subset X \setminus B.$$

Finally, set

$$V := X \setminus \overline{U}.$$

Then V is an open neighborhood of B, and $U \cap V = \emptyset$. Thus X satisfies T_4. □

LEMMA I.3.13. *Let Y be a topological space and $\{U_i\}_{i \in I}$ a locally finite family of subsets of Y. Then*

$$\overline{\bigcup_{i \in I} U_i} = \bigcup_{i \in I} \overline{U_i}.$$

PROOF. The inclusion "$\supset$" is obvious. For the converse, let $y \notin \bigcup_{i\in I} \overline{U_i}$. For each i, choose an open neighborhood V_i of y with $V_i \cap U_i = \emptyset$. Since the family $\{U_i\}_{i\in I}$ is locally finite, there exists an open neighborhood V of y meeting only finitely many U_i's, say $U_{i_1}, \ldots, U_{i_m}$. Then

$$V' := V \cap V_{i_1} \cap \cdots \cap V_{i_m}$$

is an open neighborhood of y disjoint from $\bigcup_{i\in I} U_i$. Hence $y \notin \overline{\bigcup_{i\in I} U_i}$. □

MINI-EXERCISE I.3.14. Let X be a paracompact Hausdorff topological space.
(i) Let $A, B \subset X$ be disjoint closed subsets. Show that there exists a continuous function $f\colon X \to [0,1]$ such that $f|_A = 0$ and $f|_B = 1$.
(ii) Show that, for any locally finite open covering $\{U_i\}_{i\in I}$ of X, there exists a family of continuous functions $\{\varphi_i\}_{i\in I}$ on X such that $0 \le \varphi_i \le 1$, $\operatorname{Supp}\varphi_i := \overline{\{x \in X \mid \varphi_i(x) \neq 0\}} \subset U_i$, and $\sum_{i\in I} \varphi_i = 1$ (a *partition of unity subordinate to* $\{U_i\}_{i\in I}$).

3.2 Properness

A continuous map $f\colon X \to Y$ between topological spaces is said to be *proper* if the preimage $f^{-1}(K)$ of every compact subset K of Y is compact in X. Note that, in this note, we shall also consider projective varieties over $\mathbb{C}$ and suitable morphisms between them, for which properness may be defined in a different algebro-geometric manner. In this setting, however, these notions of properness are known to be compatible with the above topological definition. This compatibility depends on the precise setting under consideration, but can, for instance, be explained by combining [Se2, Proposition 6] with the following:

LEMMA I.3.15. *Let $f\colon X \to Y$ be a continuous map between Hausdorff spaces. Assume that f is closed. Then f is proper if and only if the fiber $f^{-1}(y)$ is compact for any $y \in Y$.*

PROOF. Assume first that f is proper. Since Y is Hausdorff, the singleton $\{y\}$ is compact for every $y \in Y$. Hence $f^{-1}(y) = f^{-1}(\{y\})$ is compact in X.

Conversely, assume that the fiber $f^{-1}(y)$ is compact for every $y \in Y$, and that f is closed. Let $K \subset Y$ be a compact subset. Fix an open cover $\{U_j\}_{j\in J}$ of $f^{-1}(K)$.

To show the compactness of $f^{-1}(K)$, it suffices to show the following: for every $y \in K$, there exists an open neighborhood W_y of y such that $f^{-1}(W_y)$ is covered by finitely many members of $\{U_j\}_{j\in J}$. Indeed, once this is established, the compactness of K yields finitely many points $y_1, \ldots, y_N \in K$ such that

$$K \subset W_{y_1} \cup \cdots \cup W_{y_N},$$

and hence

$$f^{-1}(K) \subset f^{-1}(W_{y_1}) \cup \cdots \cup f^{-1}(W_{y_N}),$$

which is covered by finitely many members of $\{U_j\}_{j\in J}$.

We now fix $y \in K$ and show the existence of such a neighborhood W_y. Since $f^{-1}(y)$ is compact, there exist $j_1, \ldots, j_m \in J$ such that

$$f^{-1}(y) \subset U_{j_1} \cup \cdots \cup U_{j_m}.$$

Set

$$V_y := U_{j_1} \cup \cdots \cup U_{j_m},$$

which is an open subset of X containing $f^{-1}(y)$. Let us show the existence of an open neighborhood $W_y \subset Y$ of y such that

$$f^{-1}(W_y) \subset V_y. \tag{4}$$

Set $F_y := X \setminus V_y$, which is closed in X. Then

$$F_y \cap f^{-1}(y) = \emptyset,$$

and hence $y \notin f(F_y)$. Since f is closed, $f(F_y)$ is closed in Y. Therefore there exists an open neighborhood W_y of y in Y such that

$$W_y \cap f(F_y) = \emptyset.$$

This implies

$$f^{-1}(W_y) \cap F_y = \emptyset,$$

which is equivalent to (4). □

Mini-Exercise I.3.16. Let X and Y be Hausdorff, locally compact topological spaces.
(*i*) Show that any continuous proper map $f\colon X \to Y$ is a closed map.
(*ii*) Deduce that a continuous proper bijection $f\colon X \to Y$ is a homeomorphism.

The following lemma is elementary but fundamental, and will be useful when studying neighborhoods of compact subsets in locally compact spaces.

Lemma I.3.17. *Let Y be a locally compact Hausdorff topological space, (Z, d) be a metric space, and $f\colon Y \to Z$ be a continuous open map such that a point $z_0 \in Z$ is contained in the image $f(Y)$ and its preimage $f^{-1}(z_0)$ is compact in Y. Then there exists an open neighborhood $V \subset Y$ of $f^{-1}(z_0)$ and a positive number δ such that the following holds: For $\Delta := \{z \in Z \mid d(z_0, z) < \delta\}$, $f(V) = \Delta$ and $f|_V : V \to \Delta$ is proper.*

Proof. The proof is essentially the same as that of [KUeh2, Lemma 2.2]. From the assumption, there exists an open neighborhood U of $f^{-1}(z_0)$ such that its closure $\overline{U}$ is compact and thus so is its boundary $\partial U = \overline{U} \setminus U$. Since f is continuous, the image $f(\partial U)$ is compact in Z with $z_0 \notin f(\partial U)$.

Since $f(\partial U)$ is compact in the metric space Z and does not contain z_0, and since $f(U)$ is open, we may choose $\delta > 0$ so that $\Delta := \{z \in Z \mid d(z_0, z) < \delta\}$ is contained in $f(U) \setminus f(\partial U)$.

Put $V := U \cap f^{-1}(\Delta)$. Then $f|_V : V \to \Delta$ is surjective by construction. Moreover, for any compact subset $K \subset \Delta$, we have

$$(f|_V)^{-1}(K) = V \cap f^{-1}(K) = U \cap f^{-1}(K) = \overline{U} \cap f^{-1}(K),$$

where the third equality follows from $\Delta \cap f(\partial U) = \emptyset$. Since $\overline{U}$ is compact and $f^{-1}(K)$ is closed in Y, the set $(f|_V)^{-1}(K)$ is compact. Hence $f|_V$ is proper. □

4 From basic calculus and the theory of ordinary differential equations

4.1 From basic calculus

For basic notation and standard facts in calculus, the reader is referred to classical textbooks such as [Sp] (also refer to [Lee-Jo2, Appendix C]).

MINI-EXERCISE I.4.1. Define a function $h\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$h(x,y) := \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Show that, for every fixed $y \in \mathbb{R}$, the function

$$\mathbb{R} \ni x \longmapsto h(x,y)$$

is of class C^∞, and that, for every fixed $x \in \mathbb{R}$, the function

$$\mathbb{R} \ni y \longmapsto h(x,y)$$

is of class C^∞. Also show that h is not continuous at $(0,0)$.

In this note, for positive integers n and m, an open set $U \subset \mathbb{R}^n$, and a differentiable map $f\colon U \to \mathbb{R}^m$, we denote by Df the *Jacobian matrix* of f: i.e.,

$$(Df)(x) := \begin{pmatrix} \dfrac{\partial f_1}{\partial x^1}(x) & \dfrac{\partial f_1}{\partial x^2}(x) & \cdots & \dfrac{\partial f_1}{\partial x^n}(x) \\ \dfrac{\partial f_2}{\partial x^1}(x) & \dfrac{\partial f_2}{\partial x^2}(x) & \cdots & \dfrac{\partial f_2}{\partial x^n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x^1}(x) & \dfrac{\partial f_m}{\partial x^2}(x) & \cdots & \dfrac{\partial f_m}{\partial x^n}(x) \end{pmatrix},$$

where $x = (x^1, x^2, \ldots, x^n)$ denotes the standard coordinate system on $\mathbb{R}^n$ and f_j's are the functions defined by

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \in \mathbb{R}^m.$$

In the special case where $n = m = 1$, the Jacobian matrix Df is a 1×1 matrix whose unique entry is the derivative f' (and it is customary to identify Df with f'). Moreover, in this note, when we say that a function f is *of class* C^k for $k \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, we mean the following: f admits continuous partial derivatives of all orders $\leq k$ if $k < \infty$; f is of class C^ℓ for any $\ell \in \mathbb{Z}_{\geq 0}$ if $k = \infty$; and f is real analytic if $k = \omega$.

The following elementary fact is well known, but since it frequently appears in applications in complex analytic geometry, we record it here for later use.

LEMMA I.4.2. *Let $\varepsilon > 0$ be fixed. For a real number a, the integral*

$$\int_0^\varepsilon t^a \, dt$$

converges (i.e., has a finite value) if and only if $a > -1$.

MINI-EXERCISE I.4.3. Show Lemma I.4.2.

MINI-EXERCISE I.4.4. Let Ω be a neighborhood of the origin in the complex plane $\mathbb{C}$ with the standard coordinate w, and $\Phi\colon \Omega \to \mathbb{R}_{\geq 0}$ be a function. Assume that, for a positive integer n, Φ satisfies

$$\Phi(w) = \sum_{p=0}^{n} c_p \cdot w^p \overline{w}^{n-p} + O(|w|^{n+1})$$

as $|w| \to 0$, where c_p's are complex constants.
(i) When n is odd, show that $c_p = 0$ holds for any $p \in \{0, 1, 2, \ldots, n\}$.
(ii) When n is even, show that the constant $c_{n/2}$ is a non-negative real number, and that any of the other constants are zero if $c_{n/2} = 0$.

Related fundamental results such as the *inverse function theorem* and the *implicit function theorem* are assumed to be well known to the reader.

MINI-EXERCISE I.4.5. Let $U \subset \mathbb{R}^2$ be an open subset, $F\colon U \to \mathbb{R}$ be a C^1 function, and $(a, b) \in U$ satisfy

$$F(a, b) = 0, \qquad \frac{\partial F}{\partial y}(a, b) \neq 0.$$

Applying Banach fixed point theorem (Theorem I.3.8), show that there exist neighborhoods $I \subset \mathbb{R}$ of a, $J \subset \mathbb{R}$ of b, and a continuous function $f\colon I \to J$ such that $F(x, y) = 0$ holds on $(x, y) \in I \times J$ if and only if $y = f(x)$.

4.2 Convexity of real functions

Let n be a positive integer, $\Omega \subset \mathbb{R}^n$ be a non-empty convex subset, and $f\colon \Omega \longrightarrow \mathbb{R}$ be a function. We say that f is *convex* (on Ω) if for all $x, y \in \Omega$ and all $t \in [0, 1]$, one has

$$f\big((1-t)x + ty\big) \;\leq\; (1-t)f(x) + tf(y).$$

We say that f is *strictly convex at a point* $x \in \Omega$ if, for every C^2 function φ defined in a neighborhood of x, there exist a neighborhood U of x in Ω and a constant $\varepsilon > 0$ such that $f - \varepsilon\varphi$ is convex on U. We say that f is *locally strictly convex* if it is strictly convex at every point of Ω.

REMARK I.4.6. There are several notions that may reasonably be regarded as expressing a "strict" form of convexity, and the terminology seems not to be entirely uniform in the literature. The choice of definition often depends on the field and the intended applications.

For instance, one may consider the property that the graph of a convex function admits a supporting hyperplane which touches the graph at exactly one point. From this point of view, the function $\mathbb{R} \ni x \mapsto x^4 \in \mathbb{R}$ can be regarded as "strictly convex". However, in our sense, this function is *not* strictly convex at 0, as follows immediately from Proposition I.4.7 (*ii*) below.

The definition adopted here is chosen so as to be compatible with the terminology and the basic philosophy of (pluri)subharmonicity in complex analysis, which will play a central role later in this note. □

It is known that the convexity of f can be characterized in terms of the *Hessian matrix*

$$\mathrm{Hess}_x(f) := \left(\frac{\partial^2 f}{\partial x^j \partial x^k}(x) \right)_{1 \le j,k \le n}$$

of f as follows when f is of class C^2, where $(x^1, x^2, \ldots, x^n)$ denotes the standard coordinates of $\mathbb{R}^n$ and $x \in \Omega$.

PROPOSITION I.4.7. *Assume that $\Omega \subset \mathbb{R}^n$ is a non-empty convex open subset and that $f \colon \Omega \to \mathbb{R}$ is of class C^2. Then the following holds.*
(*i*) *The function f is convex if and only if $\mathrm{Hess}_x(f)$ is positive semi-definite for every $x \in \Omega$.*
(*ii*) *For $x \in \Omega$, the function f is strictly convex at x if and only if $\mathrm{Hess}_x(f)$ is positive definite.*

PROOF. For (*i*), see [Roc, Part 1, § 4]. We prove (*ii*). Assume that f is strictly convex at a point $x \in \Omega$. By definition, for the function $\varphi(y) := \frac{1}{2}\|y - x\|^2$, where $\| \bullet \|$ denotes the Euclidean (L^2-)norm on $\mathbb{R}^n$, there exist a neighborhood U of x and a constant $\varepsilon > 0$ such that $f - \varepsilon\varphi$ is convex on U. By (*i*), this implies that

$$\mathrm{Hess}_x(f) - \varepsilon\, \mathbf{I} = \mathrm{Hess}_x(f) - \varepsilon\, \mathrm{Hess}_x(\varphi) = \mathrm{Hess}_x(f - \varepsilon\varphi)$$

is positive semi-definite, where $\mathbf{I}$ denotes the identity matrix of order n. Therefore we have that $\mathrm{Hess}_x(f) \ge \varepsilon\, \mathbf{I}$, and hence $\mathrm{Hess}_x(f)$ is positive definite. Conversely, assume that $\mathrm{Hess}_x(f)$ is positive definite. Then, for a C^2 function φ defined in a neighborhood of x, it follows by applying Mini-Exercise I.3.3 (*i*) (to $A(y) := \mathrm{Hess}_y(f)$ and $B(y) := \mathrm{Hess}_y(\varphi)$) that

$$\mathrm{Hess}_y(f) > \varepsilon\, \mathrm{Hess}_y(\varphi) \qquad (\text{for all } y \in U)$$

holds for a neighborhood U of x and a constant $\varepsilon > 0$. Therefore, by (*i*), $f - \varepsilon\varphi$ is convex on U. □

The following examples and facts are fundamental, and will be used repeatedly in connection with plurisubharmonic functions discussed later.

EXAMPLE I.4.8 ((=[KUed2, Example 2.1])). Let χ be either of the functions defined by $t \mapsto \log(1 + e^t)$ on $\mathbb{R}$, $t \mapsto t + e^{-at}$ on $\{t \in \mathbb{R} \mid t > (\log a)/a\}$ $(a > 0)$, or $t - t^b$ on $\{t \in \mathbb{R} \mid t > b^{1/(1-b)}\}$ $(0 < b < 1)$. Then χ is an increasing and strictly convex function such that $\chi(t)/t \to 1$ as $t \to \infty$. □

EXAMPLE I.4.9. One can modify the function $(t^1, t^2) \mapsto \max\{t^1, t^2\}$ to construct an example of a C^∞ function of two real variables which is non-decreasing in each variable and convex as follows (*a regularized maximum function*, see [De4, §5.E] for the details. Here we describe this example with two variables, which is just for simplicity). Take a positive constant ε and a C^∞ non-negative function $\theta\colon \mathbb{R} \to \mathbb{R}$ with support in the interval $[-1, 1]$ such that $\int_{\mathbb{R}} \theta(t)dt = 1$ and $\int_{\mathbb{R}} t\theta(t)dt = 0$ hold (Indeed, one can easily construct such a function from the standard *cutoff functions*; see, for example, [Lee-Jo2, Chapter 2]). Define a function $\mathrm{RegMax} = \mathrm{RegMax}_{\varepsilon,\theta}\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$\mathrm{RegMax}(s^1, s^2) := \varepsilon^{-2} \int_{\mathbb{R}^2} \max\{s^1 + t^1, s^2 + t^2\} \cdot \theta(t^1/\varepsilon) \cdot \theta(t^2/\varepsilon)\, dt^1 dt^2.$$

One can show the following by simple computations: RegMax is non-decreasing in each variable and convex, and $\mathrm{RegMax}(s^1, s^2) = \max\{s^1, s^2\}$ holds if $|s^1 - s^2| > 2\varepsilon$. Note that $\mathrm{RegMax}(s^1 + a, s^2 + a) = \mathrm{RegMax}(s^1, s^2) + a$ holds. □

MINI-EXERCISE I.4.10. Let n be a positive integer and $(x^1, x^2, \ldots, x^n)$ the standard coordinates on $\mathbb{R}^n$. Define functions $f, g\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(x^1, x^2, \ldots, x^n) := \log \sum_{j=1}^{n} e^{x^j}, \qquad g(x^1, x^2, \ldots, x^n) := \log \left(1 + \sum_{j=1}^{n} e^{x^j}\right).$$

Show that f is convex, while g is locally strictly convex.

MINI-EXERCISE I.4.11. Show the following:
(i) For any sequence $\{y_n\}_{n\in\mathbb{Z}_{>0}} \subset \mathbb{R}$, there exists a C^∞ function $\theta\colon \mathbb{R}_{>0} \to \mathbb{R}$ such that $y_n < \theta(n)$ for every $n \in \mathbb{Z}_{>0}$, and $\theta'(x) > 0$ and $\theta''(x) > 0$ for every $x \in \mathbb{R}_{>0}$.
(ii) For any continuous function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$, there exists a C^∞ function $\chi\colon \mathbb{R}_{>0} \to \mathbb{R}$ such that $f(x) < \chi(x)$, $\chi'(x) > 0$, and $\chi''(x) > 0$ for every $x \in \mathbb{R}_{>0}$.

4.3 Ordinary differential equations and flows

As a preparation from very basic ordinary differential equation theory, let us first recall the following classical result.

THEOREM I.4.12 (Cauchy–Lipschitz theorem / Picard–Lindelöf theorem). *Fix a positive integer n. Let $\alpha \in \mathbb{R}$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n$, and $a, b > 0$ be positive real numbers. Consider the subset*

$$D := \{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid |x - \alpha| \leq a,\ \|y - \beta\| \leq b\},$$

where for $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ we set

$$\|y - \beta\| := \max\{|y_1 - \beta_1|,\ |y_2 - \beta_2|,\ \ldots,\ |y_n - \beta_n|\}.$$

Let $f\colon D \to \mathbb{R}^n$ be a continuous function which is Lipschitz continuous with respect to the second variable in the following sense: there exists a constant $L > 0$ such that for all $x \in \mathbb{R}$ and all $y, z \in \mathbb{R}^n$,

$$(x, y), (x, z) \in D \quad \textit{implies} \quad \|f(x, y) - f(x, z)\| \leq L \cdot \|y - z\|,$$

where the norm $\|\bullet\|$ is the same as above, namely the maximum of the absolute values of the components. Assume moreover that f is bounded on D, i.e., there exists a constant $M > 0$ such that

$$\|f(x, y)\| \leq M \qquad \textit{for all } (x, y) \in D.$$

Define

$$I := \left\{ x \in \mathbb{R} \;\middle|\; |x - \alpha| \leq \min\left\{ a,\ \frac{b}{M},\ \frac{1}{2L} \right\} \right\}.$$

Then there exists a unique function $y\colon \operatorname{Int} I \to \mathbb{R}^n$ such that y is differentiable, $(x, y(x)) \in D$ for all $x \in \operatorname{Int} I$, and

$$\begin{cases} y'(x) = f(x, y(x)) & \textit{for all } x \in \operatorname{Int} I, \\ y(\alpha) = \beta. \end{cases}$$

Here we write $y = (y_1, y_2, \ldots, y_n)$, and y' denotes the vector-valued function $(y_1', y_2', \ldots, y_n')$.

Theorem I.4.12 can be proved by applying the Banach fixed point theorem (Theorem I.3.8) to the following metric space (X, d).

We define X by

$$X := \left\{ \varphi\colon I \to \mathbb{R}^n \;\middle|\; \max_{x \in I} \|\varphi(x) - \beta\| \leq b, \text{ and } \varphi \text{ is } M\text{-Lipschitz continuous} \right\},$$

where, by saying that $\varphi\colon I \to \mathbb{R}^n$ is M-Lipschitz continuous, we mean that

$$\|\varphi(x_0) - \varphi(x_1)\| \leq M \cdot |x_0 - x_1| \qquad \text{for all } x_0, x_1 \in I.$$

Note that for every $x \in I$ and every $\varphi \in X$, we have $(x, \varphi(x)) \in D$ by the definition of X, and hence the expression "$f(x, \varphi(x))$" is well-defined.

For $\varphi, \psi \in X$, we define

$$d(\varphi, \psi) := \max_{x \in I} \|\varphi(x) - \psi(x)\|.$$

The maximum on the right-hand side indeed exists, since the function $x \mapsto \|\varphi(x) - \psi(x)\|$ is continuous on the compact interval I.

Mini-Exercise I.4.13. Show that (X, d) is a complete metric space.

For $\varphi \in X$, we define a function $T(\varphi)\colon I \to \mathbb{R}^n$ by

$$T(\varphi)(x) := \beta + \int_\alpha^x f(t, \varphi(t))\, dt,$$

namely, denoting by $f_j\colon D \to \mathbb{R}$ the j-th component of $f\colon D \to \mathbb{R}^n$, we set

$$T(\varphi)(x) := \left(\beta_1 + \int_\alpha^x f_1(t, \varphi(t))\, dt,\ \ldots,\ \beta_n + \int_\alpha^x f_n(t, \varphi(t))\, dt \right).$$

MINI-EXERCISE I.4.14. Show that T defines a contraction map on X.

As X is non-empty (since it contains the constant function β), the Banach fixed point theorem (Theorem I.3.8) implies that T has a unique fixed point $y \in X$, which satisfies

$$y(x) = \beta + \int_\alpha^x f(t, y(t))\, dt \qquad \text{for all } x \in I$$

by definition of T. This implies the assertion of Theorem I.4.12.

MINI-EXERCISE I.4.15. Let $r > 0$ and $a \in (0, 1]$ be constants, and $F\colon [0, r) \to \mathbb{R}$ be a continuous function. Set

$$G(x) := \frac{1}{x} + x^{a-1} F(x) \qquad (x \in (0, r)).$$

Show that there exists a unique continuous function $H\colon [0, r) \to \mathbb{R}$ such that $H(0) = 0$, H is of class C^2 on $(0, r)$, and the function $\chi(x) := \log x + H(x)$ satisfies

$$\chi''(x) + G(x) \cdot \chi'(x) = 0$$

for all $x \in (0, r)$.

For further standard results on ordinary differential equations, including regularity properties of solutions and classical methods of solving initial value problems, we refer the reader to [Lee-Jo2, Appendix D] for example.

Let n be a positive integer, and $x = (x^1, x^2, \ldots, x^n)$ denote the standard coordinates on the Euclidean space $\mathbb{R}^n$. Throughout this subsection, for a positive number $\varepsilon > 0$ we set

$$I_\varepsilon := \{t \in \mathbb{R} \mid |t| < \varepsilon\}.$$

Let $U \subset \mathbb{R}^n$ be open, let $\varepsilon > 0$ be a constant, and let $\varphi\colon I_\varepsilon \times U \to \mathbb{R}^n$ be a differentiable map. For $t_0 \in I_\varepsilon$ and $x \in U$, we define $(\dot\varphi)_{(t_0,x)}$ to be the tangent vector at the point $\varphi(t_0, x)$ to the curve $t \mapsto \varphi(t, x)$. Writing φ_j for the j-th component of φ, this means that

$$(\dot\varphi)_{(t_0,x)} := \sum_{j=1}^n \frac{\partial \varphi_j}{\partial t}(t_0, x)\ \left.\frac{\partial}{\partial x^j}\right|_{\varphi(t_0,x)}.$$

Under the standard identification of tangent vectors with directional derivatives (see, for example, [Lee-Jo2, Chapter 3]), this definition can be reformulated as follows: for any real-valued differentiable function ρ defined in a neighborhood of $\varphi(t_0, x)$,

$$(\dot\varphi)_{(t_0,x)}(\rho) := \left.\frac{\partial}{\partial t}\right|_{t=t_0} \rho(\varphi(t, x)).$$

DEFINITION I.4.16. Let v be a vector field on an open set $U \subset \mathbb{R}^n$. For a positive number $\varepsilon > 0$ and an open subset $V \subset U$, a differentiable map

$$\varphi\colon I_\varepsilon \times V \to U$$

is called a *(local) flow of the vector field* v if $\varphi(0, -)\colon V \to U$ coincides with the inclusion map $V \hookrightarrow U$, and if for all $(t, x) \in I_\varepsilon \times V$ one has $(\dot\varphi)_{(t,x)} = v_{\varphi(t,x)}$. □

THEOREM I.4.17. *Let $U \subset \mathbb{R}^n$ be an open set and v be a C^1 vector field on U. For any point $x_0 \in U$, there exist a positive number $\varepsilon > 0$, an open neighborhood $V \subset U$ of x_0, and a differentiable map*

$$\varphi\colon I_\varepsilon \times V \to U$$

such that φ is a flow of v. Moreover, such a map φ is uniquely determined in a neighborhood of $(0, x_0)$ inside $I_\varepsilon \times V$. If the vector field v is of class C^k for a positive integer k, then φ is of class C^k as a map on $I_\varepsilon \times V$.

MINI-EXERCISE I.4.18. For U, v, and x_0 as in Theorem I.4.17, by applying Theorem I.4.12, show the existence of an open neighborhood $V \subset U$ of x_0 and a map $\varphi\colon I_\varepsilon \times V \to U$ which is differentiable in the variable $t \in I_\varepsilon$ such that φ is a flow of v. Show also the uniqueness of such a map φ in a neighborhood of $(0, x_0)$.

Note that the assertion of Theorem I.4.17 concerning the regularity of φ follows from standard results on the regularity of solutions of ordinary differential equations, such as [Lee-Jo2, Theorem D.5], or from a more careful direct application of Theorem I.4.12.

In applications, we will mainly consider flows of C^∞ vector fields. It is therefore useful to record explicitly the smoothness of the flow and its basic property as a local one-parameter group of transformations.

COROLLARY I.4.19. *Let $U \subset \mathbb{R}^n$ be an open set, v a C^∞ vector field on U, and $x_0 \in U$ be a point. Then there exist a positive number $\varepsilon > 0$, an open neighborhood $W \subset U$ of x_0, and a C^∞ map*

$$\varphi\colon I_\varepsilon \times W \to U$$

such that φ is a flow of v. Moreover, such a map φ is uniquely determined in a neighborhood of $(0, x_0)$ inside $I_\varepsilon \times W$. Furthermore, for all $s, t \in I_\varepsilon$ and all $x \in W$ such that $s + t \in I_\varepsilon$ and $\varphi(s, x) \in W$, one has

$$\varphi(t, \varphi(s, x)) = \varphi(s + t, x).$$

PROOF. The existence of a flow φ and its local uniqueness follow from Theorem I.4.17. Assume that $s \in I_\varepsilon$ and $x \in W$ satisfy $\varphi(s, x) \in W$, and set $x_1 := \varphi(s, x)$. Consider the maps

$$c\colon t \longmapsto \varphi(t + s, x), \qquad \gamma\colon t \longmapsto \varphi(t, \varphi(s, x)) = \varphi(t, x_1),$$

both defined on the interval $\{t \in I_\varepsilon \mid t + s \in I_\varepsilon\}$, which contains 0. Both curves take values in U, satisfy $c(0) = \gamma(0) = x_1$, and their tangent vectors are given by

$$\dot{c}_{t_0} = (\dot{\varphi})_{(t_0+s,x)} = v_{c(t_0)}, \qquad \dot{\gamma}_{t_0} = (\dot{\varphi})_{(t_0,x_1)} = v_{\gamma(t_0)}.$$

By the local uniqueness assertion of Theorem I.4.12, the two curves agree on this whole interval (Their coincidence set is non-empty, closed by continuity, and open by local uniqueness). This proves the desired identity. □

MINI-EXERCISE I.4.20. Determine explicitly the flow of the vector field

$$v := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

on $\mathbb{R}^2$, where (x, y) are the standard coordinates of $\mathbb{R}^2$.

5 From commutative algebra

5.1 Some basic notions of the ring theory

Throughout this note the word "ring" shall mean a commutative ring with an identity element 1 (for the zero ring, which will be simply denoted by 0 by abuse of notation, we agree that $1 = 0$). A ring homomorphism $R \to R'$ is supposed to map the identity element of R to that of R'. For the basics of the commutative ring theory, we refer to [AM], [Lan2], [Mat] for example.

Mini-Exercise I.5.1. State the definitions of a *prime ideal*, a *maximal ideal*, and the *radical* $\sqrt{I}$ of an ideal I in a ring R. Explain these notions through a few concrete examples.

Recall that a ring R is said to be *reduced* if its *nilradical* $\sqrt{\{0\}}$ is equal to $\{0\}$. In connection with the notion of radical, note that *Hilbert's Nullstellensatz* is a fundamental and important fact, see [AM, p. 69], [Lan2, Chapter IX §1], [Mat, Chapter 5], and [De3, Chapter II (1.24)] for example.

Mini-Exercise I.5.2. Recall the definitions of the following notions for a ring: *Noetherian*, *integral domain* (or *entire*), *PID*(=*principal ideal domain*), and *UFD* (=*unique factorization domain*, or *factorial (entire) ring*/*unique factorization ring*). Explain these notions through a few concrete examples.

Mini-Exercise I.5.3. Let R be a ring which is an integral domain. Recall the definition of the *localization* $S^{-1}R$ of R at a multiplicative subset $S \subset R$, and recall also the definition of the *fraction field* $\mathrm{Frac}(R)$ of R. Assume now that $S \subset R \setminus \{0\}$ so that $S^{-1}R$ is again an integral domain. Describe the relation between $\mathrm{Frac}(R)$ and $\mathrm{Frac}(S^{-1}R)$.

The reader is encouraged at this point to review the structure theorem for finitely generated abelian groups. See, for example, [Lan2, Chapter I §8, Chapter III §7]. Let A be a finitely generated abelian group. An element $a \in A$ is said to be *torsion* if there exists a positive integer n such that

$$na := \underbrace{a + a + \cdots + a}_{n \text{ times}} = 0.$$

The set of all torsion elements of A forms a subgroup of A, called the *torsion subgroup* of A, and is denoted by A_{tor}. By the structure theorem for finitely generated abelian groups, the quotient

$$A_{\mathrm{free}} := A/A_{\mathrm{tor}}$$

is isomorphic to

$$\mathbb{Z}^{\oplus r} := \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r \text{ times}}$$

for some non-negative integer r. This integer r is called the *rank* of A and denoted by $\mathrm{rank}\, A$.

The reader is also encouraged to review the definitions and basic properties of *Noetherian rings* and *Noetherian modules*; see, for example, [AM, Chapters 6 and 7].

MINI-EXERCISE I.5.4. Let R be a ring, M be a finitely generated R-module, and $N \subset M$ be an R-submodule.
(i) Show that N is also finitely generated as an R-module when R is Noetherian.
(ii) Give an example of R, M, and N such that N is not finitely generated as an R-module.

To prepare for the ring-theoretic arguments in Chapter III, the reader is also encouraged to learn the following standard facts at this point: a Noetherian ring has finitely many minimal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$, and its nilradical is their intersection; see [Mat, §6] and [St, Tag 00FR, Tag 00E0]. Here a *minimal prime ideal* means a prime ideal minimal with respect to inclusion.

5.2 Complexes of R-modules and (co)homology

In this subsection, we fix a ring R. For the details of the results in this subsection, refer to [Os, Chapter 3] and [Weib, Chapter 1, 3].

Chain complexes and homology

A *(chain) complex of R-modules* is a sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

of R-modules together with R-module homomorphisms d_n such that

$$d_n \circ d_{n+1} = 0 \qquad \text{for all integers } n.$$

The maps d_n are called the *differentials* of the complex. We denote such a complex by $C_\bullet = (C_n, d_n)$.

Let $C_\bullet = (C_n, d_n)$ and $D_\bullet = (D_n, d'_n)$ be complexes of R-modules. A *morphism of complexes*

$$f_\bullet \colon C_\bullet \longrightarrow D_\bullet$$

is a family of R-module homomorphisms

$$f_n \colon C_n \longrightarrow D_n \qquad (n \in \mathbb{Z})$$

such that

$$d'_n \circ f_n = f_{n-1} \circ d_n \qquad \text{for all } n.$$

Let $C_\bullet = (C_n, d_n)$ be a complex of R-modules. For each integer n, we set

$$Z_n(C_\bullet) := \operatorname{Ker} d_n, \qquad B_n(C_\bullet) := \operatorname{Image} d_{n+1}.$$

Since $d_n \circ d_{n+1} = 0$, we have

$$B_n(C_\bullet) \subset Z_n(C_\bullet).$$

The *n-th homology module* of the complex $C_\bullet$ is defined by

$$H_n(C_\bullet) := Z_n(C_\bullet)/B_n(C_\bullet).$$

A morphism of complexes $f_\bullet \colon C_\bullet \to D_\bullet$ induces R-module homomorphisms

$$H_n(C_\bullet) \longrightarrow H_n(D_\bullet) \qquad (n \in \mathbb{Z}).$$

Cochain complexes and cohomology

Similarly, a *cochain complex of R-modules* $C^\bullet$ is a sequence

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots$$

of R-modules and R-module homomorphisms satisfying

$$d^n \circ d^{n-1} = 0 \qquad \text{for all } n.$$

For such a complex, we set

$$Z^n(C^\bullet) := \operatorname{Ker} d^n, \qquad B^n(C^\bullet) := \operatorname{Image} d^{n-1}, \qquad H^n(C^\bullet) := Z^n(C^\bullet)/B^n(C^\bullet),$$

and call $H^n(C^\bullet)$ the *n-th cohomology module*. A morphism of cochain complexes $f^\bullet\colon C^\bullet \to D^\bullet$, which is defined in the same manner as that of chain complexes, induces R-module homomorphisms

$$H^n(C^\bullet) \longrightarrow H^n(D^\bullet) \qquad (n \in \mathbb{Z}).$$

The reader may have noticed that the above descriptions of chain complexes and cochain complexes are essentially the same, the only difference being the direction of indices and the notation. Nevertheless, we have chosen to present both notions explicitly, for the convenience of the reader in later applications. Indeed, depending on the situation, one of the two viewpoints is often more natural than the other. Although there are many reasons for this, we deliberately refrain from discussing them here, and instead only remark at the present stage the following basic guideline, which will be sufficient for our purposes: In many situations, both chain and cochain complexes vanish in sufficiently low degrees. After shifting the indices, one can then work with index sets of the form $\mathbb{Z}_{\geq 0}$ in both cases.

At this point, the reader may find it helpful to recall the notion of *homotopy* between (co)chain complexes and its basic properties, in particular the fact that homotopic morphisms induce the same maps on (co)homology.

Short exact sequences of (co)chain complexes and long exact sequences

A sequence of complexes of R-modules

$$0 \longrightarrow A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \longrightarrow 0$$

is called a *short exact sequence* if, for each integer n,

$$0 \longrightarrow A_n \xrightarrow{\iota_n} B_n \xrightarrow{\pi_n} C_n \longrightarrow 0$$

is a short exact sequence of R-modules. It is a basic and fundamental fact that such a short exact sequence of complexes induces a *long exact sequence* of homology modules

$$\cdots \longrightarrow H_{n+1}(C_\bullet) \longrightarrow H_n(A_\bullet) \longrightarrow H_n(B_\bullet) \longrightarrow H_n(C_\bullet) \longrightarrow H_{n-1}(A_\bullet) \longrightarrow \cdots .$$

Similarly, from a short exact sequence of cochain complexes of R-modules

$$0 \longrightarrow A^\bullet \xrightarrow{\iota^\bullet} B^\bullet \xrightarrow{\pi^\bullet} C^\bullet \longrightarrow 0,$$

whose definition is entirely analogous to that of chain complexes, one obtains a long exact sequence of cohomology modules

$$\cdots \longrightarrow H^{n-1}(C^\bullet) \longrightarrow H^n(A^\bullet) \longrightarrow H^n(B^\bullet) \longrightarrow H^n(C^\bullet) \longrightarrow H^{n+1}(A^\bullet) \longrightarrow \cdots .$$

We refer the reader to [Weib, §1.3] for detailed constructions, in particular of the *connecting homomorphisms*, and for proofs of the exactness. The reader is also encouraged to review basic tools such as the *five lemma* and the *snake lemma* at this moment. It is also recommended to check the discussions and concrete examples in some textbook on algebraic topology, such as [Dol, Chapter VI] and [Lee-Jo1, Chapter 13].

Definition of Tor and its basic properties

Let M and N be R-modules. We first choose a *projective resolution* of M, namely, an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

in which each P_i is a projective R-module. Such a resolution is known to exist for any R-module, since every R-module is a quotient of a free (hence projective) R-module.

By applying the tensor product $- \otimes_R N$ termwise, we obtain a chain complex

$$\cdots \longrightarrow P_2 \otimes_R N \longrightarrow P_1 \otimes_R N \longrightarrow P_0 \otimes_R N \longrightarrow 0.$$

For each integer $n \geq 0$, we define

$$\mathrm{Tor}_n^R(M, N) := H_n(P_\bullet \otimes_R N),$$

and call it the *n-th Tor module* of M and N. It is known that $\mathrm{Tor}_\bullet^R(M, N)$ is well-defined up to an isomorphism, that is, it does not depend on the choice of the projective resolution $P_\bullet$ of M.

From the viewpoint of applications, the following property is of particular importance. Given a short exact sequence of R-module homomorphisms

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

and an R-module K, one obtains a naturally induced long exact sequence

$$\ldots \longrightarrow \mathrm{Tor}_2^R(L, K) \longrightarrow \mathrm{Tor}_2^R(M, K) \longrightarrow \mathrm{Tor}_2^R(N, K) \qquad .$$

$$\longrightarrow \mathrm{Tor}_1^R(L, K) \longrightarrow \mathrm{Tor}_1^R(M, K) \longrightarrow \mathrm{Tor}_1^R(N, K)$$

$$\longrightarrow L \otimes_R K \longrightarrow M \otimes_R K \longrightarrow N \otimes_R K \longrightarrow 0$$

Mini-Exercise I.5.5. Let R be a ring, A an R-module, and $r \in R$ an element which is not a zero-divisor. Show that $\mathrm{Tor}_1^R(R/rR, A) \cong \{a \in A \mid ra = 0\}$.

Definition of Ext **and its basic properties**

Let M and N be R-modules. We again start by choosing a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of M. Applying $\mathrm{Hom}_R(-, N)$ termwise, we obtain a cochain complex

$$\mathrm{Hom}_R(P_0, N) \longrightarrow \mathrm{Hom}_R(P_1, N) \longrightarrow \mathrm{Hom}_R(P_2, N) \longrightarrow \cdots .$$

For each integer $n \geq 0$, we set

$$\mathrm{Ext}^n_R(M, N) := H^n\big(\mathrm{Hom}_R(P_\bullet, N)\big).$$

Alternatively, one may define $\mathrm{Ext}^\bullet_R(M, N)$ by using an *injective resolution* of N, namely, an exact sequence

$$0 \longrightarrow N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

in which each I^j is an injective R-module. Such a resolution is also known to exist for any R-module, although its construction is generally less explicit than in the projective case (See [Weib, §2.3] for the details). By applying $\mathrm{Hom}_R(M, -)$ termwise, we obtain a cochain complex

$$\mathrm{Hom}_R(M, I^0) \longrightarrow \mathrm{Hom}_R(M, I^1) \longrightarrow \mathrm{Hom}_R(M, I^2) \longrightarrow \cdots ,$$

and we define

$$\mathrm{Ext}^n_R(M, N) := H^n\big(\mathrm{Hom}_R(M, I^\bullet)\big).$$

It is known that the two constructions yield canonically isomorphic R-modules ([Weib, Theorem 2.7.6]), and that $\mathrm{Ext}^n_R(M, N)$ is well-defined up to an isomorphism, independently of the choice of the projective or injective resolution used in its definition.

From the viewpoint of applications, the most important feature of Ext is the existence of the following long exact sequences: For a short exact sequence of R-module homomorphisms

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

and an R-module K, long exact sequences

$$0 \longrightarrow \mathrm{Hom}_R(K, L) \longrightarrow \mathrm{Hom}_R(K, M) \longrightarrow \mathrm{Hom}_R(K, N)$$

$$\longrightarrow \mathrm{Ext}^1_R(K, L) \longrightarrow \mathrm{Ext}^1_R(K, M) \longrightarrow \mathrm{Ext}^1_R(K, N)$$

$$\longrightarrow \mathrm{Ext}^2_R(K, L) \longrightarrow \mathrm{Ext}^2_R(K, M) \longrightarrow \mathrm{Ext}^2_R(K, N) \longrightarrow \ldots$$

and

$$0 \longrightarrow \mathrm{Hom}_R(N,K) \longrightarrow \mathrm{Hom}_R(M,K) \longrightarrow \mathrm{Hom}_R(L,K)$$

$$\longrightarrow \mathrm{Ext}^1_R(N,K) \longrightarrow \mathrm{Ext}^1_R(M,K) \longrightarrow \mathrm{Ext}^1_R(L,K)$$

$$\longrightarrow \mathrm{Ext}^2_R(N,K) \longrightarrow \mathrm{Ext}^2_R(M,K) \longrightarrow \mathrm{Ext}^2_R(L,K) \longrightarrow \dots$$

are induced.

Mini-Exercise I.5.6. For a finitely generated abelian group A, show that $\mathrm{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z}) \cong A_{\mathrm{tor}}$.

Let R be a ring, and let L and N be R-modules. By an *extension of N by L*, we mean a short exact sequence of R-module homomorphisms

$$0 \longrightarrow L \longrightarrow E \longrightarrow N \longrightarrow 0.$$

Two extensions

$$0 \longrightarrow L \longrightarrow E \longrightarrow N \longrightarrow 0, \qquad 0 \longrightarrow L \longrightarrow E' \longrightarrow N \longrightarrow 0$$

are said to be *equivalent* if there exists an R-module isomorphism $E \xrightarrow{\cong} E'$ making the natural diagram commute, that is, inducing the identity on both L and N.

An extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow N \longrightarrow 0$$

is said to be *split* if it is equivalent to the short exact sequence

$$0 \longrightarrow L \longrightarrow L \oplus N \longrightarrow N \longrightarrow 0,$$

or equivalently, if there exists an R-module homomorphism $N \longrightarrow E$ whose composition with $E \longrightarrow N$ is the identity on N.

It is a fundamental fact that the set of equivalence classes of extensions of N by L is naturally in bijection with $\mathrm{Ext}^1_R(N,L)$. Under this correspondence, the zero element of $\mathrm{Ext}^1_R(N,L)$ corresponds to the split extension (see [Weib, Theorem 3.4.3]). This correspondence can be described as follows. Given an extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow N \longrightarrow 0,$$

consider the long exact sequence induced by applying $\mathrm{Hom}_R(-,L)$. The associated connecting homomorphism

$$\mathrm{Hom}_R(L,L) \longrightarrow \mathrm{Ext}^1_R(N,L)$$

appears in this sequence, and the image of the identity map $\mathrm{id}_L \in \mathrm{Hom}_R(L,L)$ under this homomorphism is precisely the class corresponding to the given extension (the *extension class*).

5.3 On the duality

In this subsection, we show the following:

LEMMA I.5.7. *Let R be a ring, M be an R-module, and*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a sequence of R-modules (i.e., A, B, and C are R-modules and both $f\colon A \to B$ and $g\colon B \to C$ are R-module homomorphisms) such that $g \circ f = 0$. Set $\mathfrak{A}_M := \mathrm{Hom}_R(A, M)$, $\mathfrak{B}_M := \mathrm{Hom}_R(B, M)$, and $\mathfrak{C}_M := \mathrm{Hom}_R(C, M)$, and denote by

$$\mathfrak{A}_M \xleftarrow{\mathfrak{f}_M} \mathfrak{B}_M \xleftarrow{\mathfrak{g}_M} \mathfrak{C}_M$$

the dual sequence, where $\mathfrak{f}_M$ and $\mathfrak{g}_M$ are defined by $\mathfrak{f}_M(\varphi) := \varphi \circ f$ and $\mathfrak{g}_M(\psi) := \psi \circ g$. Define

$$H := \frac{\mathrm{Ker}\, g}{\mathrm{Image}\, f}, \qquad \mathfrak{H}_M := \frac{\mathrm{Ker}\, \mathfrak{f}_M}{\mathrm{Image}\, \mathfrak{g}_M}.$$

Then there exists a natural homomorphism

$$\iota_M\colon \mathfrak{H}_M \longrightarrow \mathrm{Hom}_R(H, M).$$

Moreover the following holds:
(i) There is a natural isomorphism

$$\mathrm{Ker}\, \iota_M \;\cong\; \mathrm{Ker}\big(\mathrm{Ext}^1_R(\mathrm{Coker}\, g, M) \longrightarrow \mathrm{Ext}^1_R(C, M)\big)\,.$$

(ii) There is a natural injection

$$\mathrm{Coker}\, \iota_M \;\hookrightarrow\; \mathrm{Ker}\big(\mathrm{Ext}^1_R(\mathrm{Image}\, g, M) \longrightarrow \mathrm{Ext}^1_R(B, M)\big)\,.$$

PROOF. First we construct ι_M and check that it is well-defined. Since $g \circ f = 0$, we have $\mathrm{Image}\, f \subset \mathrm{Ker}\, g$. Let $\varphi \in \mathrm{Ker}\, \mathfrak{f}_M \subset \mathfrak{B}_M$. The condition $\varphi \in \mathrm{Ker}\, \mathfrak{f}_M$ means $\varphi \circ f = 0$, hence φ vanishes on $\mathrm{Image}\, f$. Therefore the restriction $\varphi|_{\mathrm{Ker}\, g}$ also vanishes on $\mathrm{Image}\, f$, and by the universal property of the quotient it induces a unique homomorphism

$$\widetilde{\varphi} \in \mathrm{Hom}_R\left(\frac{\mathrm{Ker}\, g}{\mathrm{Image}\, f},\, M\right) = \mathrm{Hom}_R(H, M).$$

We define $\iota_M([\varphi]) := \widetilde{\varphi}$. If φ is replaced by $\varphi + \mathfrak{g}_M(\psi) = \varphi + \psi \circ g$ for some $\psi \in \mathfrak{C}_M$, then $(\psi \circ g)|_{\mathrm{Ker}\, g} = 0$, hence the induced map $\widetilde{\varphi}$ does not change. Thus ι_M is well-defined.

To prove (i), consider the short exact sequence

$$0 \longrightarrow \mathrm{Image}\, g \longrightarrow C \longrightarrow \mathrm{Coker}\, g \longrightarrow 0.$$

Put $Q := \mathrm{Coker}\, g$. Applying $\mathrm{Hom}_R(-, M)$, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(Q, M) \longrightarrow \mathfrak{C}_M \longrightarrow \mathrm{Hom}_R(\mathrm{Image}\, g, M) \xrightarrow{\delta} \mathrm{Ext}^1_R(Q, M) \longrightarrow \mathrm{Ext}^1_R(C, M),$$

where δ is the connecting homomorphism.

On the other hand, an element $[\varphi] \in \mathfrak{H}_M$ lies in $\operatorname{Ker} \iota_M$ if and only if it admits a representative $\varphi \in \operatorname{Ker} \mathfrak{f}_M$ such that $\varphi|_{\operatorname{Ker} g} = 0$. This is equivalent to saying that φ factors through the quotient $B/\operatorname{Ker} g \cong \operatorname{Image} g$, hence determines an element of $\operatorname{Hom}_R(\operatorname{Image} g, M)$. Moreover, changing φ by an element of $\operatorname{Image} \mathfrak{g}_M$ exactly corresponds to changing the induced homomorphism on $\operatorname{Image} g$ by restriction of an element of $\mathfrak{C}_M$. Consequently, we have a natural identification

$$\begin{aligned} \operatorname{Ker} \iota_M &\cong \frac{\operatorname{Hom}_R(\operatorname{Image} g, M)}{\operatorname{Image}\big(\mathfrak{C}_M \to \operatorname{Hom}_R(\operatorname{Image} g, M)\big)} \\ &= \operatorname{Image} \delta \;=\; \operatorname{Ker}\big(\operatorname{Ext}^1_R(Q, M) \to \operatorname{Ext}^1_R(C, M)\big)\,, \end{aligned}$$

which proves (i).

Next we prove (ii). Consider the short exact sequence

$$0 \longrightarrow \operatorname{Ker} g \longrightarrow B \longrightarrow \operatorname{Image} g \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(-, M)$, we obtain the long exact sequence

$$\operatorname{Hom}_R(\operatorname{Image} g, M) \longrightarrow \mathfrak{B}_M \longrightarrow \operatorname{Hom}_R(\operatorname{Ker} g, M) \xrightarrow{\partial} \operatorname{Ext}^1_R(\operatorname{Image} g, M) \longrightarrow \operatorname{Ext}^1_R(B, M).$$

By definition, $\operatorname{Hom}_R(H, M)$ may be identified with the submodule of $\operatorname{Hom}_R(\operatorname{Ker} g, M)$ consisting of those homomorphisms that vanish on $\operatorname{Image} f$. Under this identification, the image of ι_M consists exactly of those elements of $\operatorname{Hom}_R(H, M)$ that arise as restrictions of elements of $\mathfrak{B}_M$. Therefore $\operatorname{Coker} \iota_M$ embeds into the quotient

$$\frac{\operatorname{Hom}_R(\operatorname{Ker} g, M)}{\operatorname{Image}\big(\mathfrak{B}_M \to \operatorname{Hom}_R(\operatorname{Ker} g, M)\big)}.$$

By exactness of the above long exact sequence, this quotient is canonically identified with $\operatorname{Image} \partial$, which equals

$$\operatorname{Ker}\big(\operatorname{Ext}^1_R(\operatorname{Image} g, M) \longrightarrow \operatorname{Ext}^1_R(B, M)\big)\,.$$

This proves (ii). □

For a ring R and an R-module M, we denote by M^* the dual $\operatorname{Hom}_R(M, R)$. For an R-module homomorphism $f\colon M_0 \to M_1$ between R-modules, we denote by $\operatorname{Ker} f$ and $\operatorname{Coker} f$ the kernel and the cokernel of f, respectively, and by $f^*\colon M_1^* \to M_0^*$ the R-module homomorphism defined by $f^*(\varphi) = \varphi \circ f$. Then, as a corollary, we have the following:

Corollary I.5.8. *Let R be a ring and*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a sequence of R-modules such that $g \circ f = 0$. Set $\mathfrak{A} := A^*$, $\mathfrak{B} := B^*$, and $\mathfrak{C} := C^*$, and denote by

$$\mathfrak{A} \xleftarrow{\mathfrak{f}} \mathfrak{B} \xleftarrow{\mathfrak{g}} \mathfrak{C}$$

the dual sequence, where $\mathfrak{f} = f^*$ and $\mathfrak{g} = g^*$. Define

$$H := \frac{\operatorname{Ker} g}{\operatorname{Image} f}, \qquad \mathfrak{H} := \frac{\operatorname{Ker} \mathfrak{f}}{\operatorname{Image} \mathfrak{g}}.$$

Then there exists a natural homomorphism

$$\iota \colon \mathfrak{H} \longrightarrow H^*.$$

Moreover the following holds:
(i) There is a natural isomorphism

$$\operatorname{Ker} \iota \;\cong\; \operatorname{Ker}\left(\operatorname{Ext}^1_R(\operatorname{Coker} g, R) \longrightarrow \operatorname{Ext}^1_R(C, R)\right).$$

(ii) There is a natural injection

$$\operatorname{Coker} \iota \;\hookrightarrow\; \operatorname{Ker}\left(\operatorname{Ext}^1_R(\operatorname{Image} g, R) \longrightarrow \operatorname{Ext}^1_R(B, R)\right).$$

Proof. Apply Lemma I.5.7 for $M = R$. □

Corollary I.5.9. *Let*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a sequence of vector spaces (i.e., A, B, and C are vector spaces and both f and g are linear) such that $g \circ f = 0$. Define

$$H := \frac{\operatorname{Ker} g}{\operatorname{Image} f}, \qquad \mathfrak{H} := \frac{\operatorname{Ker} f^*}{\operatorname{Image} g^*}.$$

Then the natural homomorphism

$$\iota : \mathfrak{H} \longrightarrow H^*$$

constructed in Corollary I.5.8 is an isomorphism.

Proof. As every (complex) vector space is a free $\mathbb{C}$-module, one has $\operatorname{Ext}^1_{\mathbb{C}}(V, \mathbb{C}) = 0$ for any vector space V. Applying Corollary I.5.8, it follows that both $\operatorname{Ker} \iota$ and $\operatorname{Coker} \iota$ vanish, from which the assertion follows. □

5.4 Transcendental degree

Let K be a field extension of $\mathbb{C}$. A subset $\{x_1, \ldots, x_r\} \subset K$ is said to be *algebraically independent over* $\mathbb{C}$ if there is no non-zero polynomial $P \in \mathbb{C}[X_1, X_2, \ldots, X_r]$ such that

$$P(x_1, x_2, \ldots, x_r) = 0.$$

Equivalently, no algebraic relation with coefficients in $\mathbb{C}$ holds among $x_1, x_2, \ldots, x_r$.

The *transcendental degree* (or *transcendence degree*) of K over $\mathbb{C}$ is defined as the maximal cardinality of an algebraically independent subset of K (see [Lan2, Chapter VIII, Theorem 1.1]), and is denoted by

$$\mathrm{tr.deg}_{\mathbb{C}}(K).$$

As a basic example,

$$\mathrm{tr.deg}_{\mathbb{C}}(\mathbb{C}(x_1, x_2 \ldots, x_n)) = n$$

holds for indeterminates $x_1, x_2 \ldots, x_n$, where $\mathbb{C}(x_1, x_2 \ldots, x_n)$ denotes the field of rational functions in n variables $x_1, x_2 \ldots, x_n$.

For a finitely generated $\mathbb{C}$-algebra R which is an integral domain, the number $\mathrm{tr.deg}_{\mathbb{C}}(\mathrm{Frac}(R))$ is a fundamental invariant of R. Related to this, let us recall the following:

THEOREM I.5.10 (Noether's normalization theorem, see [AM, p. 69] and [Lan2, Chapter VIII, Theorem 2.1] for example). *Let R be a finitely generated $\mathbb{C}$-algebra which is an integral domain with $\mathrm{tr.deg}_{\mathbb{C}}(\mathrm{Frac}(R)) = d$ ($d \in \mathbb{Z}_{>0}$). Choose generators $x_1, x_2, \ldots, x_n$ of R as a $\mathbb{C}$-algebra. Then there exist linear forms*

$$y_i = \sum_{j=1}^{n} c_{ij} x_j \qquad (c_{ij} \in \mathbb{C})$$

such that R is integral over the polynomial ring $\mathbb{C}[y_1, y_2, \ldots, y_d]$.

5.5 Completion and graded rings

Let R be a ring, M an R-module, and I be an ideal of R. Then, for the *I-adic completions*

$$\widehat{R} := \varprojlim_n R/I^n \quad \text{and} \quad \widehat{M} := \varprojlim_n M/I^n M,$$

$\widehat{R}$ naturally has a ring structure such that the natural map $R \to \widehat{R}$ is a ring homomorphism, and $\widehat{M}$ can be naturally regarded as an $\widehat{R}$-module.

MINI-EXERCISE I.5.11. Determine the completion $\widehat{R}$ for $R = \mathbb{C}[z]$ and $I = (z) := z\mathbb{C}[z]$.

To investigate the completions, it is important to study *graded rings* and *graded modules*: A graded ring is a ring R together with a family $\{R_n\}_{n=0}^{\infty}$ of subgroups of the additive group R, such that $R = \bigoplus_{n=0}^{\infty} R_n$ and that $R_n R_m \subset R_{n+m}$ for all $n, m \in \mathbb{Z}_{\geq 0}$. For a graded ring R, a graded R-module is an R-module M together with a family $\{M_n\}_{n=0}^{\infty}$ of subgroups of M such that $M = \bigoplus_{n=0}^{\infty} M_n$ and that $R_n M_m \subset M_{n+m}$ for all $n, m \in \mathbb{Z}_{\geq 0}$. For graded R-modules $M = \bigoplus_{n=0}^{\infty} M_n$ and $N = \bigoplus_{n=0}^{\infty} N_n$, a *homomorphism of graded R-modules* is an R-module homomorphism $f\colon M \to N$ such that $f(M_n) \subset N_n$ for all $n \in \mathbb{Z}_{\geq 0}$. For the details, refer to [AM, §10], [Lan2, Chapter X §5] and [Mat, §10].

EXAMPLE I.5.12. Let R be a ring, M an R-module, and I be an ideal of R. Then

$$S_I(R) := \bigoplus_{n=0}^{\infty} I^n$$

is a graded ring (the *first associated graded ring* or the *Rees ring*) and

$$S_I(M) := \bigoplus_{n=0}^{\infty} I^n M$$

is a graded $S_I(R)$-module. When R is Noetherian, $S_I(R)$ is generated as an R-algebra by $x_1t, x_2t, \ldots, x_nt$, where t is an indeterminate and $x_1, x_2, \ldots, x_n$ are generators of I (Note that I is finitely generated, since R is Noetherian), and hence $S_I(R)$ is also Noetherian by Hilbert's basis theorem. When R is Noetherian and M is finitely generated as an R-module, then it is known that $S_I(M)$ is a finitely generated $S_I(R)$-module (See [AM, Lemma 10.8] or [Lan2, Lemma 5.3]). □

EXAMPLE I.5.13. Let R be a ring, M an R-module, and I be an ideal of R. Then

$$\mathrm{gr}_I(R) := \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$$

is a graded ring (the *second associated graded ring*) and

$$\mathrm{gr}_I(M) := \bigoplus_{n=0}^{\infty} I^n M/I^{n+1}M$$

is a graded $\mathrm{gr}_I(R)$-module. When R is Noetherian, $\mathrm{gr}_I(R)$ is generated as an R/I-algebra by the images of $x_1, \ldots, x_n$ in I/I^2, where $x_1, x_2, \ldots, x_n$ are generators of I, hence $\mathrm{gr}_I(R)$ is Noetherian by Hilbert's basis theorem. When R is Noetherian and M is finitely generated as an R-module, then it is known that $\mathrm{gr}_I(M)$ is a finitely generated $\mathrm{gr}_I(R)$-module (See [AM, Lemma 10.22] or [Lan2, Lemma 5.8]). □

One of the main reasons for introducing the first associated graded ring (Example I.5.12) is that it provides a convenient framework to derive several fundamental results concerning I-adic filtrations and completions. In particular, it allows us to relate the finite generation properties of graded modules to stability phenomena such as the *Artin–Rees theorem*, which in turn lead to the *Krull intersection theorem*. These results play a crucial role in understanding the behavior of I-adic completions of rings and modules.

PROPOSITION I.5.14 (Artin–Rees theorem for the I-adic filtration). *Let R be a Noetherian ring, M a finitely generated R-module, $I \subset R$ an ideal, and $N \subset M$ a submodule. Then there exists an integer $k \geq 0$ such that*

$$I^n M \cap N = I^{n-k}\big(I^k M \cap N\big) \quad \textit{for all } n \geq k.$$

PROOF. Consider the graded $S_I(R)$-module

$$\mathcal{N} := \bigoplus_{n=0}^{\infty} (I^n M \cap N),$$

which is a graded submodule of $S_I(M) = \bigoplus_{n\geq 0} I^n M$. Since R is Noetherian and M is finitely generated, the graded ring $S_I(R)$ is Noetherian, and $S_I(M)$ is a finitely generated $S_I(R)$-module. Hence $\mathcal{N}$ is also a finitely generated $S_I(R)$-module. Moreover, choosing homogeneous generators, we may assume that $\mathcal{N}$ is generated by finitely many homogeneous elements $x_1, \ldots, x_t \in \mathcal{N}$ with $\deg(x_j) = d_j$ $(j \in \{1, 2, \ldots, t\})$.

Let $k := \max\{d_1, \ldots, d_t\}$. Fix $n \geq k$ and take any element $y \in \mathcal{N}_n = I^n M \cap N$. Since $x_1, \ldots, x_t$ generate $\mathcal{N}$ as an $S_I(R)$-module, we can write

$$y = \sum_{j=1}^{t} a_j x_j \quad \text{with} \quad a_j \in S_I(R)_{n-d_j} = I^{n-d_j}.$$

As $d_j \leq k$, we have $I^{n-d_j} \subset I^{n-k}$, hence

$$y \in \sum_{j=1}^{t} I^{n-d_j} x_j \subset I^{n-k} \sum_{j=1}^{t} S_I(R)_{k-d_j} x_j \subset I^{n-k} \mathcal{N}_k.$$

Therefore $\mathcal{N}_n \subset I^{n-k}\mathcal{N}_k$; i.e.,

$$I^n M \cap N \subset I^{n-k}(I^k M \cap N).$$

The reverse inclusion is clear, since, if $u \in I^k M \cap N$, then $I^{n-k} u \subset I^n M$ and also $I^{n-k} u \subset N$, so $I^{n-k} u \subset I^n M \cap N$. □

THEOREM I.5.15 (Krull intersection theorem). *Let R be a Noetherian ring, $I \subset R$ an ideal contained in the Jacobson radical of R (i.e., I is contained in every maximal ideal of R), and M a finitely generated R-module. Then*

$$\bigcap_{n=0}^{\infty} I^n M = 0.$$

PROOF. Set

$$K := \bigcap_{n=0}^{\infty} I^n M.$$

Then $K \subset M$ is a finitely generated R-submodule (As M is finitely generated, so is K, since R is Noetherian). Applying Proposition I.5.14 to $N = K$, we obtain an integer $k \geq 0$ such that

$$I^n M \cap K = I^{n-k}(I^k M \cap K) \quad \text{for all } n \geq k.$$

Since $K \subset I^n M$ for every n, the left-hand side equals K. In particular, for $n = k+1$ we have

$$K = I(I^k M \cap K) \subset IK \subset K,$$

hence $K = IK$. Since I is contained in the Jacobson radical of R, Nakayama–Azumaya–Krull lemma (Nakayama's lemma, refer to [Mat, Chapter 1]) implies that $K = 0$. □

By combining the above results, it is known that one can further derive several fundamental properties of I-adic completions. More precisely, let R be a Noetherian ring, $I \subset R$ an ideal, and let L, M, and N be finitely generated R-modules. Denote by $\widehat{R}$, $\widehat{L}$, $\widehat{M}$, and $\widehat{N}$ their I-adic completions. Then the following facts are well known:

- If a short exact sequence of R-modules

  $$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

  is given, then the induced sequence of $\widehat{R}$-modules

  $$0 \longrightarrow \widehat{L} \longrightarrow \widehat{M} \longrightarrow \widehat{N} \longrightarrow 0$$

  is also exact ([AM, Proposition 10.12]).

- The natural morphism

  $$\widehat{R} \otimes_R M \longrightarrow \widehat{M}$$

  is an isomorphism ([AM, Proposition 10.13]).

- If I is contained in the Jacobson radical of R, then the I-adic topology on M is Hausdorff ([AM, Corollary 10.19]).

From some arguments on the second associated graded ring (Example I.5.13), it is known that one can deduce the following important fact:

- For a Noetherian ring R and its ideal $I \subset R$, the I-adic completion $\widehat{R}$ of R is Noetherian ([AM, Theorem 10.26]).

5.6 Hilbert–Serre theorem and Hilbert–Poincaré series

In this subsection, we briefly review the Hilbert–Serre theorem and its consequences for graded algebras. For simplicity and in view of later applications, we restrict ourselves to graded Noetherian $\mathbb{C}$-algebras, and consider formal power series whose coefficients are given by the complex dimensions of homogeneous components.

Let

$$A = \bigoplus_{n=0}^{\infty} A_n$$

be a graded Noetherian $\mathbb{C}$-algebra such that $A_0 = \mathbb{C}$. Note that

$$A^+ := \bigoplus_{n=1}^{\infty} A_n$$

is an ideal of A. Since A is Noetherian, A^+ is generated (as an A-module) by finitely many elements, say $x_1, x_2, \ldots, x_m \in A^+$. Replacing each x_j by its homogeneous components, we may assume that each x_j is homogeneous of some positive degree d_j. Then, by an inductive argument on $n \in \mathbb{Z}_{\geq 0}$, one can show the following fact: For each $n \in \mathbb{Z}_{\geq 0}$, the homogeneous component A_n is a finite-dimensional vector space spanned by

$$\left\{ \prod_{j=1}^{m} x_j^{a_j} \,\middle|\, a_1, \ldots, a_m \in \mathbb{Z}_{\geq 0},\ \sum_{j=1}^{m} a_j d_j = n \right\}.$$

We define the *Hilbert–Poincaré series* of A by

$$H_A(t) := \sum_{n=0}^{\infty} (\dim A_n)\, t^n \ \in\ \mathbb{Z}[[t]].$$

THEOREM I.5.16 (Hilbert–Serre). *Let $A = \bigoplus_{n\geq 0} A_n$, $x_1, x_2, \ldots, x_m$ and $d_1, d_2, \ldots, d_m$ be as above. Then there exists a polynomial $f(t) \in \mathbb{Z}[t]$ such that*

$$H_A(t) = \frac{f(t)}{\prod_{j=1}^{m}(1 - t^{d_j})}$$

holds. In particular,

$$H_A(t) = \frac{f(t)}{(1-t)^m}$$

holds if $d_1 = d_2 = \cdots = d_m = 1$.

PROOF. Refer to the proofs of [AM, Theorem 11.1] and [Lan2, Chapter X Theorem 6.1]. □

COROLLARY I.5.17. *Let $A = \bigoplus_{n\geq 0} A_n$, $x_1, x_2, \ldots, x_m$ and $d_1, d_2, \ldots, d_m$ be as above. Assume that $d_1 = d_2 = \cdots = d_m = 1$. Then there uniquely exists a polynomial $P(n) \in \mathbb{Q}[n]$, either zero or of degree at most $\max\{0, m-1\}$, such that*

$$\dim A_n = P(n)$$

holds for all sufficiently large $n \in \mathbb{Z}_{\geq 0}$.

PROOF. If $m = 0$, then $A = \mathbb{C}$ and the assertion holds with $P = 0$. We may therefore assume that $m \geq 1$. By Theorem I.5.16, under the assumption $d_1 = \cdots = d_m = 1$, the Hilbert–Poincaré series of A can be written in the form

$$H_A(t) = \frac{f(t)}{(1-t)^m}$$

for some polynomial $f(t) \in \mathbb{Z}[t]$. Using the binomial expansion

$$\frac{1}{(1-t)^m} = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} t^n,$$

we see that the coefficient of t^n in $H_A(t)$ is given, for all sufficiently large n, by a polynomial function of n which is either zero or of degree at most $\max\{0, m-1\}$. The uniqueness of such a polynomial is clear, since a polynomial of degree at most $\max\{0, m-1\}$ is uniquely determined by its values at infinitely many integers. □

The polynomial P appearing in Corollary I.5.17 is called the *Hilbert polynomial* of the graded algebra A.

5.7 Krull dimension

For a ring R and its prime ideal $I \subset R$, the *height* of I, denoted by $\mathrm{ht}(I)$, is defined as the supremum of the lengths n of all strictly increasing chains of prime ideals

$$I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = I$$

(note that, if R is an integral domain, one may assume $I_0 = 0$, since 0 is a prime ideal in this case). The *Krull dimension* of R, denoted by $\dim R$, is defined as the supremum of the heights of all prime ideals of R (note that R itself is *not* a prime ideal of R by definition). By definition, the height of a prime ideal $I \subset R$ coincides with the Krull dimension $\dim R_I$ of the localization R_I of R at I.

Now let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and assume that the residue field $R/\mathfrak{m}$ is isomorphic to $\mathbb{C}$. Consider the second associated graded ring

$$\mathrm{gr}_{\mathfrak{m}}(R) := \bigoplus_{n=0}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

As mentioned in Example I.5.13, since R is Noetherian, the graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is also Noetherian. We define its Hilbert–Poincaré series by

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) := \sum_{n=0}^{\infty} \left(\dim_{\mathbb{C}} \mathfrak{m}^n/\mathfrak{m}^{n+1}\right) t^n \ \in \ \mathbb{Z}[[t]].$$

By Theorem I.5.16, this formal power series can be written in the form

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = \frac{f(t)}{(1-t)^m}$$

for some polynomial $f(t) \in \mathbb{Z}[t]$. Let $d(R)$ denote the order of the pole of this function at $t = 1$. Then it is known that

$$d(R) = \dim R \tag{5}$$

which is usually referred to as the *dimension theorem* (see [AM, Theorem 11.14] and [Mat, Chapter 5, Theorem 17]). In particular, this shows that $\dim R < \infty$ under the present assumptions. We emphasize that the hypothesis that the residue field $R/\mathfrak{m}$ is isomorphic to $\mathbb{C}$ is imposed only for the sake of simplicity of exposition: In fact, by a similar argument, one can show more generally that the Krull dimension of any Noetherian local ring is finite (see [AM, Theorem 11.14] and [Mat, Chapter 5, Theorem 17] also for this fact).

EXAMPLE I.5.18. Let n be a positive integer, and consider the ring

$$R := \mathbb{C}[[x_1, x_2, \ldots, x_n]]$$

of formal power series in n variables over $\mathbb{C}$. This ring is obtained as the J-adic completion of the polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ with respect to the ideal

$$J := \{f \in \mathbb{C}[x_1, x_2, \ldots, x_n] \mid f(0, 0, \ldots, 0) = 0\} = (x_1, x_2, \ldots, x_n),$$

and hence R is a Noetherian ring. Let $\mathfrak{m} \subset R$ be the ideal consisting of all formal power series with vanishing constant term. Then the ring homomorphism $R \to \mathbb{C}$ which assigns to each formal power series its constant term induces an isomorphism $R/\mathfrak{m} \cong \mathbb{C}$. Moreover, it is easy to see that a formal power series with nonzero constant term is a unit in R. Therefore R is a Noetherian local ring with maximal ideal $\mathfrak{m}$.

In this case, for each $m \in \mathbb{Z}_{\geq 0}$, the quotient $\mathfrak{m}^m/\mathfrak{m}^{m+1}$ can be naturally identified with the $\mathbb{C}$-vector space having as a basis the monomials

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \qquad (a_1, a_2, \ldots, a_n \in \mathbb{Z}_{\geq 0})$$

satisfying

$$a_1 + a_2 + \cdots + a_n = m.$$

Consequently, we have

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = \sum_{m \geq 0} \binom{m+n-1}{n-1} t^m = \frac{1}{(1-t)^n}.$$

In particular, we obtain $d(R) = n$, and hence $\dim R = n$ by the dimension theorem.

We also note that R admits a chain of prime ideals

$$0 \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, x_2, \ldots, x_n) = \mathfrak{m}$$

of length n in this example. □

We conclude this subsection by recalling a well-known characterization of prime ideals of codimension one in Noetherian local rings.

THEOREM I.5.19. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ such that the residue field $R/\mathfrak{m}$ is isomorphic to $\mathbb{C}$, and let $I \subset R$ be a prime ideal. Then the implication $(i) \implies (ii)$ holds for the following assertions. If, moreover, R is a UFD, then (i) and (ii) are equivalent.*
(i) There exists an element $f \in \mathfrak{m}$ which is not a zero-divisor (i.e., $fg \neq 0$ holds for any $g \in R \setminus \{0\}$) such that $I = f \cdot R$.
(ii) $\dim R/I = \dim R - 1$.

PROOF. The implication $(i) \implies (ii)$ follows from Krull's principal ideal theorem; see [AM, Corollaries 11.17 and 11.18].

Assume conversely that R is a UFD and that (ii) holds. Choose a chain of prime ideals

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_r = I$$

whose length is $r \in \mathbb{Z}_{\geq 0}$. Next, choose a chain of prime ideals in R/I,

$$0 = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_s,$$

whose length s is equal to $\dim R/I$ (which is finite, since R/I is also a Noetherian local ring). Let $\widetilde{J}_\nu \subset R$ be the preimage of J_ν under the quotient map $R \to R/I$. Then each $\widetilde{J}_\nu$ is a prime ideal of R containing I, and we obtain a chain

$$I_0 \subsetneq \cdots \subsetneq I_r = I = \widetilde{J}_0 \subsetneq \widetilde{J}_1 \subsetneq \cdots \subsetneq \widetilde{J}_s.$$

Hence, by definition of the Krull dimension of R, we have $\dim R \geq r + s = r + \dim R/I$. Therefore we have that $r \leq \dim R - \dim R/I = 1$. Since the above argument applies to any such chain, we obtain that $\mathrm{ht}(I) \leq 1$. Thus $\mathrm{ht}(I) = 1$, since otherwise $I = 0$ and hence $\dim R/I = \dim R$, contrary to (ii).

Since R is a UFD, every prime ideal of height one is principal: Indeed, this can be easily shown by considering the factorization of a non-zero element of the ideal. Hence there exists an element $f \in \mathfrak{m}$ such that $I = f \cdot R$ (since $I \subset \mathfrak{m}$). As R is an integral domain, the element f is not a zero-divisor. This proves (i) and completes the proof. □

6 From functional analysis

In this section, we collect and summarize several topics from functional analysis that will be used later. We assume that the reader is already familiar with the basic language of *measure spaces* (including the notion of *Lebesgue measure* and the associated *Lebesgue integral*), the definition of L^p-spaces for $p \in \mathbb{R}_{>1}$, and the fundamental properties of *Banach spaces* arising from them. For background material, the reader may consult standard textbooks such as [Ru1].

Before proceeding, however, we make one terminological convention explicit. Although this is not standard in the functional analysis literature, we shall use the following terminology throughout this note: If a measure satisfies the usual non-negativity condition, we shall refer to it as *semi-positive*, rather than *positive*. For example, the measure that assigns the value 0 to every measurable set is, in our terminology, a semi-positive measure, but not a positive one.

6.1 Topological vector spaces and normed vector spaces

A *topological vector space* (abbreviated as *TVS*) V is a vector space (over $\mathbb{C}$) endowed with a topology such that the vector addition $V \times V \to V$ and the scalar multiplication $\mathbb{C} \times V \to V$

are continuous maps, where the domains are endowed with the product topologies. Such a topology is called a *TVS topology* on V. In the literature, it is sometimes assumed in addition that a TVS is Hausdorff. In this note, however, we do *not* impose the Hausdorff condition for TVSs unless explicitly stated otherwise.

We fix the following notation. For an element $x \in V$, subsets $A, B \subset V$ of a TVS V, and a complex number $r \in \mathbb{C}$, we set

$$x \pm A := \{x \pm a \mid a \in A\}, \quad A \pm B := \{a \pm b \mid a \in A,\ b \in B\}, \quad rA := \{ra \mid a \in A\}.$$

A subset $A \subset V$ of a TVS V is said to be *balanced* if $\lambda A \subset A$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, and is said to be *absorbing* if for every $x \in V$ there exists a real number $t > 0$ such that $x \in tA$.

Mini-Exercise I.6.1. Let V be a TVS and $A, B \subset V$ be subsets.
(i) Show that, for each $b \in V$, the translation map $T_b : V \to V$ defined by $T_b(x) := x + b$ is a homeomorphism.
(ii) Show that every neighborhood of $0 \in V$ is absorbing.
(iii) Show that, for every neighborhood U of $0 \in V$, there exists a neighborhood W of $0 \in V$ such that $W + W \subset U$.
(iv) If A is open, show that $A + B$ is open.
(v) Show that, for every neighborhood U of $0 \in V$, there exists a neighborhood W of $0 \in V$ such that $W \subset U$ and W is balanced and absorbing.
(vi) If A is closed and B is compact, show that $A + B$ is closed.
(vii) If A is a vector subspace of V, and if A is endowed with the subspace topology induced from V while V/A is endowed with the quotient topology with respect to the canonical surjection

$$q : V \longrightarrow V/A,$$

show that both A and V/A are TVSs.

In what follows in this note, unless otherwise stated, for a TVS V and a vector subspace $A \subset V$, we always consider A as a TVS with the subspace topology induced from V, and the quotient V/A as a TVS with the quotient topology.

Lemma I.6.2. *Let V be a TVS and $A \subset V$ be a subset. Then the following holds:*
(i) *If $\mathcal{B}$ is a neighborhood basis of $0 \in V$, then*

$$\overline{A} = \bigcap_{U \in \mathcal{B}} (A + U).$$

(ii) *The space V is Hausdorff if and only if the singleton $\{0\}$ is closed.*
(iii) *When A is a vector subspace of V, V/A is Hausdorff iff A is closed.*

Proof. (i) Let $x \in V$. Then $x \in \overline{A}$ holds if and only if every open neighborhood W of x satisfies $W \cap A \neq \emptyset$. Since $\mathcal{B}$ is a neighborhood basis of 0, the family $\{x - U \mid U \in \mathcal{B}\}$ is a neighborhood basis of x. Hence $x \in \overline{A}$ is equivalent to $(x - U) \cap A \neq \emptyset$ for all $U \in \mathcal{B}$. This is equivalent to $x \in A + U$ for all $U \in \mathcal{B}$, because

$$(x - U) \cap A \neq \emptyset \quad \Longleftrightarrow \quad \exists\, a \in A,\ \exists\, u \in U \text{ such that } a = x - u \quad \Longleftrightarrow \quad x \in A + U.$$

Thus $\overline{A} = \bigcap_{U \in \mathcal{B}} (A + U)$.

(ii) Assume that V is Hausdorff. Then every singleton is closed, in particular $\{0\}$ is closed. Conversely, assume that $\{0\}$ is closed. By (i) applied to $A = \{0\}$, we have

$$\{0\} = \overline{\{0\}} = \bigcap_{U \in \mathcal{B}} U,$$

where $\mathcal{B}$ is a neighborhood basis of $0 \in V$ consisting of open sets. Therefore, for any x and y with $x \neq y$, one can find $U \in \mathcal{B}$ such that $x - y \notin U$. By continuity of subtraction at $(0, 0)$, choose an open neighborhood W of 0 such that $W - W \subset U$. The open neighborhoods $x + W$ and $y + W$ are disjoint: otherwise, $x + w_1 = y + w_2$ for some $w_1, w_2 \in W$ would give $x - y = w_2 - w_1 \in W - W \subset U$, a contradiction. Thus V is Hausdorff.

(iii) Let $q : V \to V/A$ be the canonical surjection (and endow V/A with the quotient topology). By definition of the quotient topology, a subset $C \subset V/A$ is closed if and only if $q^{-1}(C)$ is closed in V. In particular, $\{0\} \subset V/A$ is closed if and only if A is closed in V. Therefore the assertion follows from (ii) applied to the TVS V/A. □

As an immediate consequence of Lemma I.6.2, for any TVS V, the quotient space $V/\overline{\{0\}}$ is always a Hausdorff TVS. We denote this quotient by $V_{\text{Hausdorff}}$ and call it the *Hausdorffization* of V.

Let V be a Hausdorff TVS. A family $\mathcal{F}$ of subsets of V is said to be a *filter base* if it is non-empty, $\emptyset \notin \mathcal{F}$, and, for any $A, B \in \mathcal{F}$, there exists $C \subset A \cap B$ such that $C \in \mathcal{F}$. A filter base $\mathcal{F}$ is said to be *Cauchy* if the following holds: For every open neighborhood U of $0 \in V$, there exists $A \in \mathcal{F}$ such that $A - A \subset U$. The TVS V is said to be *complete* if, for every Cauchy filter base $\mathcal{F}$ of V, there exists a point $x \in V$ such that $\mathcal{F} \to x$ holds: i.e., For any neighborhood U of $x \in V$, there exists an element $A \in \mathcal{F}$ such that $A \subset U$.

Mini-Exercise I.6.3. Assume that there exists a translation-invariant distance function d (i.e., d is a distance function on V such that $d(x + a, y + a) = d(x, y)$ for all $x, y, a \in V$) such that the topology on V which is induced by d coincides with the TVS topology of V. Show that V is complete if and only if the metric space (V, d) is complete.

As a basic and most familiar class of examples of TVSs, let us briefly recall *normed vector spaces* (or *normed spaces* for short). Let V be a complex vector space and $\| \bullet \|$ be a *norm* on V, that is, a map $\| \bullet \| : V \to \mathbb{R}_{\geq 0}$ satisfying $\|x\| = 0$ if and only if $x = 0$ $(x \in V)$, $\|\lambda x\| = |\lambda| \, \|x\|$ for $\lambda \in \mathbb{C}$ and $x \in V$, and the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$ $(x, y \in V)$ holds. Then the distance function

$$d(x, y) := \|x - y\| \qquad (x, y \in V)$$

induces a topology on V, which makes V into a Hausdorff TVS.

Recall that a normed vector space V is said to be *complete* if it is complete as a metric space with respect to the metric induced by the norm, and that a complete normed vector space is called a *Banach space*. Also recall the fundamental facts for normed vector spaces explained in [Ru1, Chapter 5] such as the following:

- For a linear map $f\colon X \to Y$ between normed vector spaces, f is continuous if and only if f is *bounded*: i.e., its *operator norm*

$$\|f\|_{\mathrm{op}} := \sup\{\|f(x)\|_Y \mid x \in X,\ \|x\|_X \leq 1\}$$

satisfies $\|f\|_{\mathrm{op}} < \infty$, where $\| \bullet \|_X$ and $\| \bullet \|_Y$ are the norms of X and Y, respectively ([Ru1, 5.4]).

- Any surjective continuous linear map between Banach spaces is an open mapping (the *open mapping theorem*, [Ru1, 5.9]).

- For a normed vector space V, its vector subspace $W \subset V$, and a continuous linear function $f\colon W \to \mathbb{C}$, there exists a continuous linear function $F\colon V \to \mathbb{C}$ such that $F|_W = f$ and $\|F\|_{\mathrm{op}} = \|f\|_{\mathrm{op}}$ (the *Hahn–Banach Theorem*, [Ru1, 5.16]).

Concerning finite-dimensional Hausdorff TVSs, the following is important:

THEOREM I.6.4. *Let V be a finite-dimensional Hausdorff TVS and W be a TVS. Then, for $n := \dim V$, the following holds:*
(i) The space V is TVS isomorphic *to $\mathbb{C}^n$ endowed with the standard (Euclidean) topology. More precisely, for any basis $(e_1, e_2, \ldots, e_n)$ of V, the map $f : \mathbb{C}^n \to V$ defined by*

$$f(z_1, z_2, \ldots, z_n) := \sum_{j=1}^{n} z_j e_j$$

*is a linear bijective homeomorphism (*TVS isomorphism*).*
(ii) Any linear map from V to W is continuous.
(iii) If W is Hausdorff and locally compact, then $\dim W < \infty$.

PROOF. For the assertions *(i)* and *(ii)*, refer to [SW, 3.2 and 3.4] or [Tr, Theorem 9.1]. For *(iii)*, refer to [SW, 3.6] or [Tr, Theorem 9.2]. For the reader's convenience, we here briefly sketch the idea of the proof of *(iii)* following [Tr, Theorem 9.2].

As W is locally compact, from Mini-Exercise I.6.1 *(v)* it follows that there exists a compact neighborhood K of $0 \in W$ that is balanced (and absorbing). As K is compact and $\frac{1}{2}K$ is also a neighborhood of 0, there exist finitely many points $x_1, \ldots, x_r \in W$ such that

$$K \subset (x_1 + \tfrac{1}{2}K) \ \cup \ \cdots \ \cup \ (x_r + \tfrac{1}{2}K).$$

Let M be the linear subspace spanned by $x_1, x_2, \ldots, x_r$. Then M is finite-dimensional; hence M is closed in W (since W is Hausdorff, see Mini-Exercise I.6.6 *(i)* below). Let $\phi\colon W \to W/M$ be the quotient map. As M is closed, the quotient W/M is Hausdorff (Lemma I.6.2 *(iii)*). Moreover, from $K \subset M + \frac{1}{2}K$ we obtain

$$\phi(K) \subset \tfrac{1}{2}\phi(K),$$

and hence, by iteration, one has

$$\phi(2^n K) \subset \phi(K)$$

for any $n \in \mathbb{Z}_{\geq 0}$.

On the other hand, since K is balanced and absorbing, we have

$$W = \bigcup_{n=0}^{\infty} 2^n K.$$

Therefore

$$\phi(W) = \bigcup_{n=0}^{\infty} \phi(2^n K) \subset \phi(K),$$

hence W/M is compact. Since W/M is a Hausdorff TVS, compactness forces $W/M = \{0\}$ (see Mini-Exercise I.6.6 (*ii*) below), from which the assertion follows. □

Remark I.6.5. Assertion (*iii*) of Theorem I.6.4 is often attributed to F. Riesz in the literature. On the other hand, some sources indicate that a proof goes back at least to A. Weil. The author does not attempt here to settle the question of attribution. It should also be noted that the Hausdorff assumption in I.6.4 (*iii*) cannot be dropped. Indeed, for any vector space V, one can make V into a TVS by endowing it with the indiscrete topology, which is compact (hence locally compact). □

Mini-Exercise I.6.6. Let V be a Hausdorff TVS and $A \subset V$ be a vector subspace.
(*i*) If A is a finite-dimensional vector subspace of V, show that A is closed in V.
(*ii*) If V is compact, show that $\dim V = 0$.
(*iii*) If A is of codimension one, show that A is either closed or dense in V.
(*iv*) Give an example of a Hausdorff TVS V and a linear map $f\colon V \to \mathbb{C}$ such that f is not continuous.
(*v*) Give an example of a Hausdorff TVS V and a vector subspace $A \subset V$ of codimension one which is not closed.

Remark I.6.7. Given two TVSs A and B, one can form their direct sum $A \oplus B$ as a TVS. By definition, this is the direct sum of A and B as vector spaces, endowed with the product topology of $A \times B$.

On the other hand, given a TVS V and linear subspaces $A, B \subset V$, the statement "$V = A \oplus B$" may be used in two different senses, which should be carefully distinguished. Sometimes one merely means this algebraically, namely that

$$V = A + B \qquad \text{and} \qquad A \cap B = \{0\}.$$

In this case no topological compatibility is required.

In other situations, one means that V is the direct sum of A and B as a TVS, where A and B are endowed with the subspace topologies induced from V. This is equivalent to requiring that the canonical projections

$$\mathrm{Pr}_A\colon V \longrightarrow A, \qquad \mathrm{Pr}_B\colon V \longrightarrow B$$

associated with the algebraic decomposition be continuous. Indeed, if V is TVS isomorphic to $A \oplus B$, then these projections are clearly continuous. Conversely, if both Pr_A and Pr_B are continuous, then the map

$$V \longrightarrow A \oplus B, \qquad v \longmapsto (\mathrm{Pr}_A(v), \mathrm{Pr}_B(v))$$

is a continuous linear inverse to the addition map $A \oplus B \to V$, which is continuous by definition of a TVS.

It is therefore important not to confuse these two notions. For example, if V is Hausdorff and $A \subset V$ is a linear subspace which is not closed, then although an algebraic complement B of A always exists, such a complement can never be a direct summand in the sense of TVSs. Indeed, in this Hausdorff setting, in a topological direct sum decomposition $V = A \oplus B$, both A and B must be closed subsets of V, since they are kernels of the continuous projections Pr_B and Pr_A, respectively.

In this note, we shall therefore distinguish these two situations explicitly. When no topological compatibility is intended, we will say that $V = A \oplus B$ *algebraically*. When the above continuity conditions are satisfied and V is isomorphic to the TVS direct sum $A \oplus B$, we will say that $V = A \oplus B$ *as TVSs*.

It is worth noting that there exists a Banach space V together with a *closed* vector subspace $A \subset V$ such that, for any vector subspace $B \subset V$, the decomposition $V = A \oplus B$ does *not* hold as TVSs: See, for instance, [Wh] for details. □

Mini-Exercise I.6.8. Let V be a Hausdorff TVS and $A \subset V$ be a finite-dimensional vector subspace. Assume that the restriction map $V^\vee \to A^\vee$ is surjective. Show that there exists a closed vector subspace $B \subset V$ such that $V = A \oplus B$ as TVSs.

Mini-Exercise I.6.6 (*iv*) shows that, even for a Hausdorff TVS V, there exist linear maps $f\colon V \to \mathbb{C}$ which are not continuous. This observation makes it necessary to distinguish between *all* linear functionals and those which are compatible with the given topology of V (In this note, the term *functional* is used to mean a complex-valued linear map. Continuity is not assumed unless explicitly stated. In situations where confusion may arise, we shall avoid the use of this term).

Recall that the *algebraic dual* of a complex vector space V is defined as

$$V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}),$$

the vector space of all linear maps from V to $\mathbb{C}$, without any continuity requirement.

Definition I.6.9. Let V be a TVS. The *topological dual* of V is defined to be

$$V^\vee := \{\, \ell \in V^* \mid \ell \text{ is continuous as a map } V \to \mathbb{C} \,\}.$$

□

Although we have defined $V^\vee$ merely as a vector space in the above, in practice it is important to treat $V^\vee$ also as a TVS by endowing it with a TVS topology. It is also important

to note that there is in general no single "best" choice of such a topology: one often needs to choose different topologies on $V^\vee$ depending on the situation. We postpone the discussion of such topologies for general TVSs to a subsequent section §6.3. Here, at this stage, let us only briefly recall the most basic ones in the normed setting.

Assume that V is a normed vector space. Then $V^\vee$ carries at least the following two standard TVS topologies:

- The topology induced by the operator norm $\| \bullet \|_{\mathrm{op}}$ on $V^\vee$. This is called the *norm topology*. It is also often called the *strong topology* on $V^\vee$ in the normed setting.

- The topology of pointwise convergence on V. Equivalently, view $V^\vee$ as a subspace of the product space $\mathbb{C}^V := \prod_{x \in V} \mathbb{C}$ via the evaluation map $\ell \mapsto (\ell(x))_{x \in V}$, and endow $V^\vee$ with the subspace topology. This topology is called the *weak-$*$ topology* on $V^\vee$.

Unless otherwise stated, when V is a normed vector space, we regard $V^\vee$ as a normed vector space with the norm topology (When we use the weak-$*$ topology, we will always state it explicitly).

A fundamental fact about the weak-$*$ topology is the following (the *Banach–Alaoglu theorem*): for a normed vector space V, the closed unit ball

$$\{\ell \in V^\vee \mid \|\ell\|_{\mathrm{op}} \leq 1\}$$

is compact in $V^\vee$ with respect to the weak-$*$ topology. This can be proved quickly from *Tychonoff's theorem* (see [Ke, Chapter 5, 13] for example) by viewing the unit ball as a closed subset of the product space

$$\prod_{x \in V} \{z \in \mathbb{C} \mid |z| \leq \|x\|_V\} \ (\subset \mathbb{C}^V),$$

where $\| \bullet \|_V$ is the norm of V. In connection with this theorem, readers who are not yet accustomed to comparing different topologies on the same vector space may find it instructive to clarify the following point: for a normed vector space V, (under which topology) is the above closed unit ball compact? Indeed, with respect to the weak-$*$ topology, the Banach–Alaoglu theorem asserts that it is always compact. On the other hand, with respect to the norm topology, as seen in Theorem I.6.4 (*iii*), the compactness of the closed unit ball is equivalent to the finite-dimensionality of V.

Let $T \colon X \to Y$ be a continuous linear map between TVSs. By composing continuous linear functionals on Y with T, one obtains a linear map

$$T^\vee \colon Y^\vee \longrightarrow X^\vee, \qquad T^\vee(\varphi) := \varphi \circ T \qquad (\varphi \in Y^\vee),$$

which is called the *adjoint* (or *dual*) of T. In the normed setting, the adjoint map is automatically continuous; more precisely, it is bounded with respect to the operator norms (see Mini-Exercise I.6.10 (*i*) below).

MINI-EXERCISE I.6.10. Let X and Y be normed vector spaces and $T\colon X \to Y$ be a continuous linear map.
(i) Show that $\|T^\vee\|_{\mathrm{op}} \leq \|T\|_{\mathrm{op}}$ holds.
(ii) Let $J_X\colon X \to X^{\vee\vee} := (X^\vee)^\vee$ be the *canonical embedding*: i.e., $(J_X(x))(\varphi) := \varphi(x)$ for any $x \in X$ and $\varphi \in X^\vee$. Show that J_X is an isometric embedding.
(iii) Show that

$$\overline{\operatorname{Image} T} = \big\{\, y \in Y \;\big|\; \forall\, \varphi \in \operatorname{Ker} T^\vee,\ \varphi(y) = 0 \big\}.$$

(iv) Show that

$$\operatorname{Ker} T^\vee = \{\varphi \in Y^\vee \mid \varphi|_{\operatorname{Image} T} \equiv 0\}.$$

(v) If X is Banach, show that $X^\vee$ is also Banach.

We now recall a famous criterion for the closedness of images, which can also be checked on the adjoint side.

THEOREM I.6.11 (The closed range theorem for continuous linear maps between Banach spaces). *Let X and Y be Banach spaces and $T\colon X \to Y$ a continuous linear map. Then the following are equivalent:*
(i) $\operatorname{Image} T$ *is closed in* Y.
(ii) $\operatorname{Image} T^\vee$ *is closed in* $X^\vee$ *with respect to the weak-$*$ topology.*
(iii) $\operatorname{Image} T^\vee$ *is closed in* $X^\vee$ *(with respect to the norm topology).*
Moreover, if these conditions are satisfied, then

$$\operatorname{Image} T = (\operatorname{Ker} T^\vee)^\perp := \{\, y \in Y \mid \varphi(y) = 0 \text{ for all } \varphi \in \operatorname{Ker} T^\vee \,\},$$

and

$$\operatorname{Image} T^\vee = (\operatorname{Ker} T)^\perp := \{\, \psi \in X^\vee \mid \psi(x) = 0 \text{ for all } x \in \operatorname{Ker} T \,\}.$$

PROOF. For the equivalence of (i), (ii) and (iii), refer to [Ru2, 4.14] (see also Remark I.6.54 below). Assume that these equivalent conditions are satisfied. Since $\operatorname{Image} T$ is closed in Y, applying Mini-Exercise I.6.10 (iii) to the map $T\colon X \to Y$, we obtain $\operatorname{Image} T = (\operatorname{Ker} T^\vee)^\perp$.

In what follows we prove the identity $\operatorname{Image} T^\vee = (\operatorname{Ker} T)^\perp$. The inclusion $\operatorname{Image} T^\vee \subset (\operatorname{Ker} T)^\perp$ is immediate: Indeed, for $\varphi \in Y^\vee$ and $x \in \operatorname{Ker} T$, one has $(T^\vee(\varphi))(x) = \varphi(T(x)) = \varphi(0) = 0$. Conversely, by assuming (i) and taking $\ell \in (\operatorname{Ker} T)^\perp$, let us show that $\ell \in \operatorname{Image} T^\vee$. Consider the factorization

$$X \xrightarrow{q} X/\operatorname{Ker} T \xrightarrow{\widetilde{T}} \operatorname{Image} T \hookrightarrow Y,$$

where q is the quotient map and $\widetilde{T}(q(x)) := T(x)$. Since X is Banach and $\operatorname{Ker} T$ is closed, the quotient $X/\operatorname{Ker} T$ is Banach. From the assumption (i), we have that $\operatorname{Image} T$ is also Banach. Hence, as the map $\widetilde{T}$ is a continuous linear bijection between Banach spaces, it follows from the open mapping theorem that $\widetilde{T}^{-1}$ is continuous.

Define $\widetilde{\ell} \in (X/\operatorname{Ker} T)^\vee$ by

$$\widetilde{\ell}(q(x)) := \ell(x) \qquad (x \in X).$$

This is well-defined because ℓ vanishes on $\operatorname{Ker} T$. Now define a linear functional ψ on $\operatorname{Image} T$ by

$$\psi(T(x)) := \ell(x) \qquad (x \in X).$$

Equivalently, $\psi = \widetilde{\ell} \circ \widetilde{T}^{-1}$. In particular, ψ is continuous on $\operatorname{Image} T$ because both $\widetilde{\ell}$ and $\widetilde{T}^{-1}$ are continuous. By the Hahn–Banach theorem, ψ extends to some $\varphi \in Y^\vee$. Then for every $x \in X$ we have

$$(T^\vee(\varphi))(x) = \varphi(T(x)) = \psi(T(x)) = \ell(x),$$

hence $\ell = T^\vee(\varphi) \in \operatorname{Image} T^\vee$, which proves the assertion. □

Let X and Y be Banach spaces. A continuous linear map $T\colon X \to Y$ is said to be *compact* if there exists a neighborhood U of $0 \in X$ such that the closure $\overline{T(U)}$ is compact in Y. Equivalently, T is compact if for every bounded sequence $\{x_j\}_{j\in\mathbb{Z}_{\geq 0}}$ in X, the sequence $\{T(x_j)\}$ admits a convergent subsequence in Y.

THEOREM I.6.12 (Schauder's theorem on compact operators). *Let X and Y be Banach spaces and $T\colon X \to Y$ a continuous linear map. Then T is compact if and only if $T^\vee$ is compact.*

PROOF. Assume first that T is compact. Let $\{\varphi_j\}_{j\in\mathbb{Z}_{\geq 0}}$ be a bounded sequence in $Y^\vee$. We show that $\{T^\vee(\varphi_j)\}$ admits a convergent subsequence in $X^\vee$. Let $B_X := \{x \in X \mid \|x\|_X \leq 1\}$ be the closed unit ball, where $\| \bullet \|_X$ denotes the norm of X. Compactness of T implies that $K := \overline{T(B_X)}$ is compact in Y. For each j, consider the continuous function $f_j\colon K \to \mathbb{C}$ defined by $f_j(y) := \varphi_j(y)$. Since $\sup_j \|\varphi_j\|_{\mathrm{op}} < \infty$, the family $\{f_j\}$ is uniformly bounded. Moreover it is equicontinuous, because for $y, y' \in K$ one has

$$|f_j(y) - f_j(y')| = |\varphi_j(y - y')| \leq \|\varphi_j\|_{\mathrm{op}}\, \|y - y'\|_Y,$$

where $\| \bullet \|_Y$ is the norm of Y, and the right-hand side is bounded by a constant multiple of $\|y - y'\|_Y$ independent of j. Hence by the Arzelà–Ascoli theorem, there exists a subsequence $\{f_{j_k}\}$ converging uniformly on K. As the uniform convergence on $T(B_X) \subset K$ means

$$\sup_{\|x\|_X \leq 1} \left|\varphi_{j_k}(T(x)) - \varphi_{j_m}(T(x))\right| \longrightarrow 0 \qquad (k, m \to \infty),$$

$\{T^\vee(\varphi_{j_k})\}$ is a Cauchy sequence in $X^\vee$ for the operator norm. Since $X^\vee$ is complete (see Mini-Exercise I.6.10 (v)), this subsequence converges. Thus $T^\vee$ is compact.

Conversely, assume that $T^\vee$ is compact. Let $J_Y\colon Y \to Y^{\vee\vee}$ be the canonical embedding. It is an isometric linear embedding, hence continuous. Consider the second adjoint $T^{\vee\vee}\colon X^{\vee\vee} \to Y^{\vee\vee}$. Since $T^\vee$ is compact, the first part applied to $T^\vee$ shows that $T^{\vee\vee}$ is compact (Note that both $X^\vee$ and $Y^\vee$ are Banach, refer to Mini-Exercise I.6.10 (v)). Therefore $J_Y \circ T = T^{\vee\vee} \circ J_X\colon X \to Y^{\vee\vee}$ is compact.

We claim that compactness of $J_Y \circ T$ implies compactness of T. Indeed, let $\{x_j\}$ be a bounded sequence in X. Then $\{J_Y(T(x_j))\}$ has a convergent subsequence in $Y^{\vee\vee}$. Since Y is complete and J_Y is an isometric embedding (see Mini-Exercise I.6.10 (ii)), the image $J_Y(Y)$ is a closed subspace of $Y^{\vee\vee}$; hence the limit of this subsequence lies in $J_Y(Y)$. Therefore, convergence of $\{J_Y(y_j)\}$ in $Y^{\vee\vee}$ is equivalent to convergence of $\{y_j\}$ in Y for any sequence $\{y_j\} \subset Y$. Hence $\{T(x_j)\}$ admits a convergent subsequence in Y. □

6.2 Hilbert spaces and operators between them

We now turn to Hilbert spaces, which form one of the most important classes of Banach spaces in analysis. A *Hilbert space* is an Hermitian space whose associated norm makes it into a complete normed vector space, or equivalently, a Banach space whose norm comes from an Hermitian inner product. The reader may consult standard textbooks such as [Ru1, Chapter 4] and [Tr, Chapter 12] for basic notions and results on Hilbert spaces. Among them, we list below several fundamental ones for a Hilbert space H endowed with an inner product $\langle \bullet, \bullet \rangle_H$:

- For any vector subspace $M \subset H$, its *orthogonal complement*

$$M^\perp := \{x \in H \mid \langle x, m \rangle_H = 0 \text{ for all } m \in M\}$$

  is a closed vector subspace of H ([Ru1, 4.9]).

- If $M \subset H$ is a closed vector subspace, then every element $x \in H$ admits a unique decomposition

$$x = m + n, \qquad m \in M,\ n \in M^\perp.$$

  Equivalently, one has an orthogonal direct sum decomposition $H = M \oplus M^\perp$ as Hilbert spaces (i.e., H is isometrically TVS isomorphic to the Hilbert direct sum $M \oplus M^\perp$, see Remark I.6.13 below). The map $P_M \colon H \to M$ sending x to its M-component m is a bounded linear map satisfying $\|P_M\|_{\mathrm{op}} \leq 1$; it is called the *orthogonal projection* onto M ([Ru1, 4.11], [Tr, p. 121]).

- The map

$$H \longrightarrow H^\vee, \qquad x \longmapsto \langle \bullet, x \rangle_H$$

  is a conjugate-linear isometric bijection (the *Riesz representation theorem*, [Ru1, 4.12] [Tr, Theorem 12.2]).

Remark I.6.13. Here let us briefly recall the meaning of direct sums in the Hilbert space setting. Let H_1 and H_2 be Hilbert spaces. Their *Hilbert direct sum* $H_1 \oplus H_2$ is defined as the vector space direct sum equipped with the inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle_{H_1 \oplus H_2} := \langle x_1, y_1 \rangle_{H_1} + \langle x_2, y_2 \rangle_{H_2},$$

where $\langle \bullet, \bullet \rangle_{H_j}$ denotes the inner product of H_j $(j = 1, 2)$. With respect to this inner product, $H_1 \oplus H_2$ is again a Hilbert space. □

Remark I.6.14. Since a Hilbert space H endowed with an inner product $\langle \bullet, \bullet \rangle_H$ is, in particular, a Banach space, one may be tempted to interpret, in view of the notation introduced earlier in this section, the symbol "$M^\perp$" for a vector subspace $M \subset H$ as

$$\{\varphi \in H^\vee \mid \varphi|_M \equiv 0\}.$$

At first sight, this may seem inconsistent with the notation of the orthogonal complement given above. However, this ambiguity is harmless in practice. Indeed, by the Riesz representation theorem, the space above is canonically identified with

$$\{\langle \bullet, x\rangle_H \in H^\vee \mid x \in M^\perp\},$$

where "$x \in M^\perp$" is in the sense of orthogonality. Thus, via the Riesz representation, the two notions of $M^\perp$ are canonically identified. □

Since we will deal with linear maps whose domains are not necessarily the whole space, it is convenient to introduce the following terminology.

Definition I.6.15. Let X and Y be Hilbert spaces. An *operator* T from X to Y is a linear map

$$T\colon \mathrm{Dom}\, T \longrightarrow Y$$

defined on a vector subspace $\mathrm{Dom}\, T \subset X$, which is not assumed to be closed nor dense unless explicitly stated. We define the *range* and the *kernel* of T by

$$\mathrm{Ran}\, T := T(\mathrm{Dom}\, T) \subset Y, \qquad \mathrm{Ker}\, T := \{x \in \mathrm{Dom}\, T \mid T(x) = 0\}.$$

The *graph* of T is the subset

$$\mathrm{Graph}(T) := \{(x, T(x)) \in X \oplus Y \mid x \in \mathrm{Dom}\, T\},$$

where $X \oplus Y$ is endowed with the Hilbert direct sum structure. □

Let X and Y be Hilbert spaces. When one says that T is an operator from X to Y, it is common in the literature to write $T\colon X \to Y$. However, according to the above definition, T is in general *not* a map defined on the whole space X, but rather a linear map whose domain $\mathrm{Dom}\, T$ is a (possibly proper) vector subspace of X. To avoid confusion on this point, we shall distinguish operators from ordinary maps by notation. Throughout this note, the statement that "T is an operator from X to Y" will be written as

$$T\colon X \xrightarrow{\mathrm{op}} Y.$$

Recall that an operator $T\colon X \xrightarrow{\mathrm{op}} Y$ is said to be *bounded* if there exists a constant $C \geq 0$ such that

$$\|T(x)\|_Y \leq C\, \|x\|_X \qquad (\text{for all } x \in \mathrm{Dom}\, T),$$

where $\| \bullet \|_X$ and $\| \bullet \|_Y$ denote the norms of X and Y, respectively. In this case T extends uniquely to a continuous linear map $\widehat{T}\colon \overline{\mathrm{Dom}\, T} \to Y$. If T is *densely defined* (i.e., $\mathrm{Dom}\, T$ is dense in X), this extension further extends uniquely to a continuous linear map $\widehat{T}\colon X \to Y$. An operator $T\colon X \xrightarrow{\mathrm{op}} Y$ between Hilbert spaces is said to be *closed* if $\mathrm{Graph}(T)$ is closed in $X \oplus Y$. Equivalently, T is closed if and only if, whenever a sequence $\{x_n\}_{n \in \mathbb{Z}_{\geq 0}} \subset \mathrm{Dom}\, T$ satisfies $x_n \to x$ in X and $T(x_n) \to y$ in Y as $n \to \infty$ for some $x \in X$ and $y \in Y$, one has $x \in \mathrm{Dom}\, T$ and $T(x) = y$. An operator $T\colon X \xrightarrow{\mathrm{op}} Y$ between Hilbert spaces is said to be *closable* if the closure of its graph in $X \oplus Y$ is again the graph of an operator $\widehat{T}\colon X \xrightarrow{\mathrm{op}} Y$, which can be easily checked to be unique (such $\widehat{T}$ is called the *closure* of T).

Mini-Exercise I.6.16. Let $T\colon X \xrightarrow{\mathrm{op}} Y$ be an operator between Hilbert spaces.
(i) Show that T is closable if and only if the following implication holds for any sequence $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of $\operatorname{Dom} T$ and $y \in Y$:

$$x_n \to 0 \text{ in } X \text{ and } T(x_n) \to y \text{ in } Y \text{ as } n \to \infty \quad \Longrightarrow \quad y = 0.$$

(ii) If T is densely defined and bounded, show that T is automatically closable, and its closure coincides with the unique continuous extension $\widehat{T}\colon X \to Y$.
(iii) Give an example of an operator $T\colon X \xrightarrow{\mathrm{op}} Y$ between Hilbert spaces such that T is densely defined, closable, and not extendable to a continuous operator defined on the whole space X.

Using the Riesz representation theorem, one can define the adjoint of a densely defined operator in the sense of von Neumann as follows.

Definition I.6.17. Let X and Y be Hilbert spaces and $T\colon X \xrightarrow{\mathrm{op}} Y$ be a densely defined operator. The *von Neumann adjoint* of T is the operator

$$T^\dagger\colon Y \xrightarrow{\mathrm{op}} X$$

defined as follows:

- The domain $\operatorname{Dom} T^\dagger$ consists of all elements $y \in Y$ such that the map

  $$\operatorname{Dom} T \longrightarrow \mathbb{C}, \qquad x \longmapsto \langle T(x), y\rangle_Y$$

  is continuous;

- For each $y \in \operatorname{Dom} T^\dagger$, the element $T^\dagger(y) \in X$ is defined by the identity

  $$\langle T(x), y\rangle_Y = \langle x, T^\dagger(y)\rangle_X \qquad (\text{for all } x \in \operatorname{Dom} T),$$

  whose existence and uniqueness follow from the Riesz representation theorem.

□

Note that, when $T\colon X \to Y$ is a bounded operator defined on the whole space X, $\operatorname{Dom} T^\dagger = Y$, and the above definition coincides with the usual adjoint.

Theorem I.6.18 (Von Neumann). *Let $T\colon X \xrightarrow{\mathrm{op}} Y$ be a densely defined closed operator between Hilbert spaces. Then its adjoint $T^\dagger$ is also densely defined and closed. Moreover, $(T^\dagger)^\dagger = T$ and*

$$(\operatorname{Ran} T)^\perp = \operatorname{Ker} T^\dagger, \qquad (\operatorname{Ker} T)^\perp = \overline{\operatorname{Ran} T^\dagger}$$

hold.

PROOF. In the following proof, we identify $Y \oplus X$ with $X \oplus Y$ by the unitary map $(y, x) \mapsto (x, y)$. First we claim that

$$\mathrm{Graph}(T^\dagger) = \mathrm{Graph}(-T)^\perp \subset X \oplus Y. \tag{6}$$

Indeed, let $(u, v) \in X \oplus Y$. Then $(u, v) \in \mathrm{Graph}(-T)^\perp$ means that

$$0 = \langle (u, v), (x, -T(x)) \rangle_{X \oplus Y} = \langle u, x \rangle_X - \langle v, T(x) \rangle_Y \qquad (\text{for all } x \in \mathrm{Dom}\, T).$$

Equivalently,

$$\langle T(x), v \rangle_Y = \langle x, u \rangle_X \qquad (\text{for all } x \in \mathrm{Dom}\, T).$$

By definition of $T^\dagger$, this is precisely the condition that $v \in \mathrm{Dom}\, T^\dagger$ and $u = T^\dagger(v)$, i.e., $(u, v) \in \mathrm{Graph}(T^\dagger)$. This proves (6).

Since T is closed, $\mathrm{Graph}(-T)$ is closed in $X \oplus Y$, hence $\mathrm{Graph}(T^\dagger) = \mathrm{Graph}(-T)^\perp$ is also closed. Therefore $T^\dagger$ is closed.

By (6), we obtain the orthogonal decomposition

$$X \oplus Y = \mathrm{Graph}(-T) \oplus \mathrm{Graph}(T^\dagger), \tag{7}$$

that is, every $(u, v) \in X \oplus Y$ can be written uniquely as

$$(u, v) = (x, -T(x)) + (T^\dagger(y), y), \qquad x \in \mathrm{Dom}\, T,\ y \in \mathrm{Dom}\, T^\dagger.$$

Apply this to $u = 0$. Then there exist $x \in \mathrm{Dom}\, T$ and $y \in \mathrm{Dom}\, T^\dagger$ such that

$$x + T^\dagger(y) = 0, \qquad v = y - T(x).$$

Using $x = -T^\dagger(y)$ in the second identity, we get

$$v = y + T(T^\dagger(y)).$$

Now let $v \in (\mathrm{Dom}\, T^\dagger)^\perp \subset Y$. Taking the inner product with $y \in \mathrm{Dom}\, T^\dagger$, we find

$$\langle v, y \rangle_Y = \langle y, y \rangle_Y + \langle T(T^\dagger(y)), y \rangle_Y = \|y\|_Y^2 + \langle T^\dagger(y), T^\dagger(y) \rangle_X = \|y\|_Y^2 + \|T^\dagger(y)\|_X^2.$$

Since $v \perp \mathrm{Dom}\, T^\dagger$, the left-hand side is 0, hence $y = 0$ and then $v = 0$. Thus $(\mathrm{Dom}\, T^\dagger)^\perp = \{0\}$, i.e., $\mathrm{Dom}\, T^\dagger$ is dense in Y.

Next let us show $(T^\dagger)^\dagger = T$. From (6) we get

$$\mathrm{Graph}(T^\dagger)^\perp = \mathrm{Graph}(-T)^{\perp\perp} = \mathrm{Graph}(-T),$$

because $\mathrm{Graph}(-T)$ is closed. Applying the same identity (6) with $T^\dagger$ in place of T, we have

$$\mathrm{Graph}\big((T^\dagger)^\dagger\big) = \mathrm{Graph}(-T^\dagger)^\perp.$$

On the other hand,

$$\begin{aligned}\mathrm{Graph}(-T^\dagger)^\perp &= \{(x,y)\in X\oplus Y \mid (x,-y)\in \mathrm{Graph}(T^\dagger)^\perp\}\\ &= \{(x,y)\in X\oplus Y \mid (x,-y)\in \mathrm{Graph}(-T)\}\\ &= \mathrm{Graph}(T).\end{aligned}$$

Hence $\mathrm{Graph}((T^\dagger)^\dagger)=\mathrm{Graph}(T)$, i.e., $(T^\dagger)^\dagger = T$.

Let $y\in Y$. Then $y\in \operatorname{Ker} T^\dagger$ if and only if $y\in \operatorname{Dom} T^\dagger$ and $T^\dagger(y)=0$, namely

$$\langle T(x), y\rangle_Y = \langle x, T^\dagger(y)\rangle_X = 0 \qquad (\text{for all } x\in \operatorname{Dom} T).$$

This is equivalent to $y\perp \operatorname{Ran} T$, hence $\operatorname{Ker} T^\dagger = (\operatorname{Ran} T)^\perp$.

Finally, applying the already proved identity to $T^\dagger$ (which is densely defined and closed), we obtain

$$\operatorname{Ker}\big((T^\dagger)^\dagger\big) = (\operatorname{Ran} T^\dagger)^\perp.$$

Since $(T^\dagger)^\dagger = T$, this gives

$$\operatorname{Ker} T = (\operatorname{Ran} T^\dagger)^\perp.$$

Taking orthogonal complements in X and using $(M^\perp)^\perp = \overline{M}$ for a subspace M of a Hilbert space, we conclude $(\operatorname{Ker} T)^\perp = \overline{\operatorname{Ran} T^\dagger}$. □

The theory of operators on Hilbert spaces due to Kodaira and Hörmander plays a fundamental role in modern complex analytic geometry. In particular, the systematic use of elliptic operators of Laplace–Beltrami type and their L^2-estimates has led to a highly refined and powerful framework, which underlies many central results in the subject. Because of its importance, this theory has been extensively studied, developed, and presented in a number of comprehensive and largely self-contained references such as [De4] and [Oh5]. In view of this, we do not attempt to give a systematic treatment of the theory of unbounded operators on Hilbert spaces in this note. Instead, we here restrict ourselves to showing only the following:

Theorem I.6.19 ([De4, Chapter VIII (1.2)], [Oh5, Theorem 2.2]). *Let H_1, H_2, H_3 be Hilbert spaces, and $T\colon H_1 \overset{\mathrm{op}}{\longrightarrow} H_2$ and $S\colon H_2 \overset{\mathrm{op}}{\longrightarrow} H_3$ be densely defined closed operators satisfying $\operatorname{Ran} T \subset \operatorname{Ker} S$. Assume that there exists a constant $C>0$ such that*

$$\|u\|_{H_2}^2 \le C\big(\|T^\dagger u\|_{H_1}^2 + \|Su\|_{H_3}^2\big)$$

for all $u\in \operatorname{Dom} T^\dagger \cap \operatorname{Dom} S \cap (\operatorname{Ker} S \cap \operatorname{Ker} T^\dagger)^\perp$. Then one has the orthogonal decomposition

$$H_2 = \operatorname{Ran} T \oplus \operatorname{Ran} S^\dagger \oplus (\operatorname{Ker} S \cap \operatorname{Ker} T^\dagger).$$

Proof. Put

$$\mathcal{H} := \operatorname{Ker} S \cap \operatorname{Ker} T^\dagger \subset H_2.$$

Since $\operatorname{Ran} T \subset \operatorname{Ker} S$, one has $S \circ T = 0$ on $\operatorname{Dom} T$. As is well known (a Kodaira-type orthogonal decomposition; c.f., [Oh5, Theorem 2.1], see also Mini-Exercise I.6.20 below), this implies the orthogonal decomposition

$$H_2 = \overline{\operatorname{Ran} T} \oplus \overline{\operatorname{Ran} S^\dagger} \oplus \mathcal{H}. \tag{8}$$

Thus it suffices to show that the closures in (8) can be removed. More precisely, it is enough to prove

$$\overline{\operatorname{Ran} T} \cap \mathcal{H}^\perp = \operatorname{Ran} T \qquad \text{and} \qquad \overline{\operatorname{Ran} S^\dagger} \cap \mathcal{H}^\perp = \operatorname{Ran} S^\dagger. \tag{9}$$

By symmetry, we shall only prove the first equality in (9); the second one can be shown in the same way.

To show this, we shall use the following well-known criterion due to Hörmander (c.f., [Oh5, Lemma 2.1], see also Mini-Exercise I.6.20 below): for a densely defined closed operator $T\colon H_1 \overset{\text{op}}{\to} H_2$ and $v \in H_2$,

$$v \in \operatorname{Ran} T \iff \exists\, C_v \geq 0 \text{ such that } |\langle u, v\rangle_{H_2}| \leq C_v \, \|T^\dagger(u)\|_{H_1} \quad (\forall\, u \in \operatorname{Dom} T^\dagger). \tag{10}$$

Take $v \in \operatorname{Ker} S \cap \mathcal{H}^\perp$ and $u \in \operatorname{Dom} T^\dagger$. Decompose $u = u_0 + u_1$ with $u_0 \in \operatorname{Ker} S$ and $u_1 \in (\operatorname{Ker} S)^\perp$. Then $\langle u, v\rangle_{H_2} = \langle u_0, v\rangle_{H_2}$. Next decompose $u_0 = u_0' + h$ with $u_0' \in \operatorname{Ker} S \cap \mathcal{H}^\perp$ and $h \in \mathcal{H}$. Since $v \in \mathcal{H}^\perp$, we have $\langle u, v\rangle_{H_2} = \langle u_0', v\rangle_{H_2}$. For u_0' we may apply the assumed estimate (note that $S(u_0') = 0$), hence

$$\|u_0'\|_{H_2}^2 \leq C \, \|T^\dagger(u_0')\|_{H_1}^2 = C \, \|T^\dagger(u_0)\|_{H_1}^2 = C \, \|T^\dagger(u)\|_{H_1}^2,$$

where we used $h \in \mathcal{H} \subset \operatorname{Ker} T^\dagger$ and $u_1 \in (\operatorname{Ker} S)^\perp = \overline{\operatorname{Ran} S^\dagger} \subset \operatorname{Ker} T^\dagger$ (since $T^\dagger \circ S^\dagger = 0$). Therefore

$$|\langle u, v\rangle_{H_2}| = |\langle u_0', v\rangle_{H_2}| \leq \|u_0'\|_{H_2} \, \|v\|_{H_2} \leq \sqrt{C} \, \|T^\dagger(u)\|_{H_1} \, \|v\|_{H_2}.$$

By (10), this proves

$$\operatorname{Ker} S \cap \mathcal{H}^\perp \subset \operatorname{Ran} T.$$

Since $\operatorname{Ran} T \subset \operatorname{Ker} S$ and $\operatorname{Ker} S$ is closed, we have $\overline{\operatorname{Ran} T} \subset \operatorname{Ker} S$. Hence $\overline{\operatorname{Ran} T} \cap \mathcal{H}^\perp \subset \operatorname{Ker} S \cap \mathcal{H}^\perp \subset \operatorname{Ran} T$. On the other hand, $\operatorname{Ran} T \subset \mathcal{H}^\perp$ holds since $\mathcal{H} \subset \operatorname{Ker} T^\dagger$. Therefore $\operatorname{Ran} T \subset \overline{\operatorname{Ran} T} \cap \mathcal{H}^\perp$, and the first equality in (9) follows. □

Mini-Exercise I.6.20. Under the configuration in the proof of Theorem I.6.19 above, show the following:
(*i*) Show that $\mathcal{H} = (\operatorname{Ran} T + \operatorname{Ran} S^\dagger)^\perp$.
(*ii*) Deduce (8) from (*i*).
(*iii*) For a densely defined closed operator $T\colon H_1 \overset{\text{op}}{\to} H_2$ and $v \in H_2$, assume that there exists $C_v \geq 0$ such that the condition in (10) holds. Show that the linear map $\ell\colon \operatorname{Ran} T^\dagger \to \mathbb{C}$

$$\ell(T^\dagger(u)) := \langle u, v\rangle_{H_2} \qquad (\forall\, u \in \operatorname{Dom} T^\dagger)$$

is well-defined and continuous.
(*iv*) Show (10).

In many analytic situations, Hilbert spaces naturally arise as L^2-spaces associated with measure spaces. Since *separability* (i.e., the existence of a countable dense subset) will be required when orthonormal bases and series expansions are used, we record the following standard sufficient condition.

LEMMA I.6.21. *Let (X, Σ, μ) be a σ-finite measure space. Assume that Σ is countably generated modulo μ-null sets: there is a countable family $\mathcal{G} \subset \Sigma$ such that every set in Σ agrees up to a μ-null set with a member of the σ-algebra generated by $\mathcal{G}$. Then the Hilbert space $L^2(X, \mu)$ is separable.*

Here (X, Σ, μ) denotes a measure space, where Σ is a σ-algebra of subsets of X and μ is a measure on Σ. In situations where the underlying σ-algebra is clear from the context, we will often write (X, μ) instead of (X, Σ, μ). The space $L^2(X, \mu)$ denotes the Hilbert space of (equivalence classes of) complex-valued measurable functions f on X such that

$$\int_X |f|^2 \, d\mu < \infty,$$

endowed with the inner product

$$\langle f, g \rangle := \int_X f \, \overline{g} \, d\mu.$$

In particular, when X is a Euclidean space (or a smooth manifold), μ will often be the Lebesgue measure (or a measure induced from it), and the reader is encouraged to briefly review the basic properties of Lebesgue measure if necessary.

MINI-EXERCISE I.6.22. Show Lemma I.6.21.

Assume now that H is separable. Then there exists a sequence $\{e_j\}_{j=1}^N$ in H ($N = \dim H$ if H is finite-dimensional and $N = \infty$ if $\dim H = \infty$) which forms a *complete orthonormal system* (or *CONS* for short), that is, $\{e_j\}_{j=1}^N \subset H$ such that

$$\langle e_j, e_k \rangle_H = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

and that the vector subspace spanned by $\{e_j\}_{j=1}^N$ is dense in H. See for instance [Ru1, Chapter 4] for the further details.

REMARK I.6.23. Such a system is often called a *complete orthonormal basis* (*CONB* for short) in the literature. However, since this terminology may suggest an algebraic basis, which is called a *Hamel basis* in this area, we will avoid this term, at least when the space is infinite-dimensional. □

6.3 Locally convex topological vector spaces and their topological duals

Up to this point we have mainly worked with TVSs arising from norms such as Banach and Hilbert spaces. However, in differential and complex geometry one often encounters naturally defined TVSs which are not normable; to treat them systematically we now move to topologies generated by families of seminorms.

A *seminorm* on a vector space V is a function $p\colon V \to \mathbb{R}_{\geq 0}$ satisfying

$$p(x+y) \leq p(x) + p(y) \quad \text{and} \quad p(\lambda x) = |\lambda|\, p(x)$$

for all $x, y \in V$ and $\lambda \in \mathbb{C}$.

Mini-Exercise I.6.24. Let V be a vector space, p a seminorm, and $r > 0$ be a constant. Show that the sublevel set $B_r(p) := \{x \in V \mid p(x) < r\}$ is *convex* (i.e., $tx + (1-t)y \in B_r(p)$ whenever $x, y \in B_r(p)$ and $t \in [0,1]$), balanced, and absorbing.

In contrast to a norm, a seminorm is allowed to have a nontrivial kernel $\{x \in V \mid p(x) = 0\}$. Therefore, to distinguish points, it is usual to consider a family of seminorms. A *locally convex topological vector space* (*LCTVS* for short) is a TVS V whose topology is generated by a family of seminorms: i.e., for each point $x_0 \in V$, the sets

$$\bigcap_{j=1}^{n} \big\{\, x \in V \mid p_j(x - x_0) < \varepsilon_j \,\big\},$$

where $p_1, p_2, \ldots, p_n$ are finitely many members of the defining family of seminorms and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$, form a neighborhood base at each point x_0 (It is straightforward to check that the topology defined in this way indeed makes V into a TVS).

Mini-Exercise I.6.25. Let V be an LCTVS whose topology is generated by a non-empty family of continuous seminorms $\{p_i\}_{i \in I}$. Let $f\colon V \to \mathbb{C}$ be a linear map. Show that the following are equivalent.
(*i*) f is continuous.
(*ii*) There exists a continuous seminorm p on V such that

$$|f(v)| \leq p(v) \qquad (\forall v \in V).$$

(*iii*) There exist a non-empty finite subset $\{i_1, \ldots, i_k\} \subset I$ and a constant $C > 0$ such that

$$|f(v)| \leq C \max_{1 \leq j \leq k} p_{i_j}(v) \qquad (\forall v \in V).$$

The assumption of local convexity is crucial for the richness of the topological dual: Compare the following variant of the Hahn–Banach theorem and Mini-Exercise I.6.27 below.

Theorem I.6.26 (The Hahn–Banach theorem for LCTVSs). *For an LCTVS V, and a vector subspace $W \subset V$ endowed with the subspace topology, the restriction map $V^\vee \to W^\vee$ is surjective.*

PROOF. Take $f \in W^\vee$. Let $\{p_i\}_{i\in I}$ be a non-empty family of continuous seminorms on V which generates the topology of V. Then the subspace topology on W is generated by the family $\{p_i|_W\}_{i\in I}$. Since f is continuous on W, Mini-Exercise I.6.25 applied to W yields a non-empty finite subset $\{i_1, \dots, i_k\} \subset I$ and a constant $C > 0$ such that

$$|f(w)| \le C \max_{1\le j\le k} p_{i_j}(w) \qquad (\forall w \in W).$$

Set

$$p(v) := C \max_{1\le j\le k} p_{i_j}(v) \qquad (v \in V).$$

Then p is a continuous seminorm on V and $|f(w)| \le p(w)$ holds for all $w \in W$. By the Hahn–Banach theorem (in the seminorm form, see [SW, Chapter II 3.2] or [Tr, Theorem 18.1]), there exists a linear map $F\colon V \to \mathbb{C}$ such that $F|_W = f$ and

$$|F(v)| \le p(v) \qquad (\forall v \in V).$$

Since p is continuous, the inequality implies that F is continuous (again by Mini-Exercise I.6.25). □

MINI-EXERCISE I.6.27. Let $p \in \mathbb{R}_{>0}$. Denote by $L^p([0,1])$ the vector space of (equivalence classes of, refer to Remark I.6.28 below) measurable functions $f\colon [0,1] \to \mathbb{C}$ on the unit interval $[0,1]$ (endowed with the Lebesgue measure) such that

$$\int_0^1 |f(x)|^p \, dx < \infty.$$

Define a function $\| \bullet \|_p\colon L^p([0,1]) \to \mathbb{R}_{\ge 0}$ by letting

$$\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} \qquad (f \in L^p([0,1])).$$

Define a translation-invariant distance function d_p on $L^p([0,1])$ by

$$d_p(f,g) := \begin{cases} \|f-g\|_p & \text{if } p \ge 1, \\ \|f-g\|_p^p & \text{if } 0 < p < 1, \end{cases}$$

and endow $L^p([0,1])$ with the TVS topology induced by d_p.
(*i*) Show that $d_p(f,g)$ is a distance function.
(*ii*) Show that $\| \bullet \|_p$ is a norm on $L^p([0,1])$ when $p \ge 1$.
(*iii*) Show that $\| \bullet \|_p$ is *not* a norm on $L^p([0,1])$ when $p < 1$.
(*iv*) Show $(L^p([0,1]))^\vee = 0$ when $p < 1$.

REMARK I.6.28. In the definition of the L^p space in Mini-Exercise I.6.27, the phrase "(equivalence classes of) measurable functions" means that two measurable functions $f, g\colon [0,1] \to \mathbb{C}$ are identified whenever they agree almost everywhere: i.e., whenever the set $\{x \in [0,1] \mid f(x) \ne g(x)\}$ is of measure zero. This identification is standard when dealing with function spaces such as L^p spaces. Unless confusion may arise, in this note we will not explicitly mention "equivalence classes" in what follows. □

Using the notion of LCTVSs, one can study the following question in a systematic way: given a TVS V, which topology should be imposed on $V^\vee$? As already mentioned in the previous subsection, it is rather rare that there exists a single "best" topology answering this question in all situations. Instead, usually it is important to work with several different topologies on $V^\vee$, depending on the context and the purpose of the discussion. Moreover, in many situations one is naturally led to reconsider the topology on V itself in the light of the topology imposed on $V^\vee$. With these considerations in mind, and in order to treat V and $V^\vee$ on an equal footing, the following notion is convenient.

Definition I.6.29. A *dual pair* $\langle E, F\rangle$ consists of two vector spaces E and F together with a bilinear mapping

$$\langle \bullet, \bullet\rangle \colon E \times F \longrightarrow \mathbb{C}.$$

A dual pair $\langle E, F\rangle$ is said to be *separating* if both maps

$$E \ni x \longmapsto \langle x, \bullet\rangle \in F^* \quad \text{and} \quad F \ni y \longmapsto \langle \bullet, y\rangle \in E^*$$

are injective. □

For example, given a topological vector space V, the pairs $\langle V, V^\vee\rangle$ and $\langle V^\vee, V\rangle$ become dual pairs by defining the bilinear mappings

$$\langle x, \varphi\rangle := \varphi(x) \qquad (x \in V,\ \varphi \in V^\vee),$$

and

$$\langle \varphi, x\rangle := \varphi(x) \qquad (\varphi \in V^\vee,\ x \in V),$$

respectively.

Mini-Exercise I.6.30. Let V be a Hausdorff LCTVS. Show that the dual pair $\langle V, V^\vee\rangle$ is separating.

Given a dual pair $\langle E, F\rangle$, one of the most natural ways to endow E with a topology is to require that, for each $y \in F$, the linear functional

$$\langle \bullet, y\rangle \colon E \longrightarrow \mathbb{C}$$

be continuous, and to impose the weakest topology with this property (that is, the *initial topology* with respect to this family of linear maps). This topology is denoted by $\sigma(E, F)$ and is called the *weak topology* on E. By definition, the weak topology $\sigma(E, F)$ is the weakest TVS topology on E for which all seminorms

$$p_y(x) := |\langle x, y\rangle| \qquad (y \in F)$$

are continuous. Equivalently, $(E, \sigma(E, F))$ is precisely the LCTVS whose topology is generated by the family of seminorms $\{p_y\}_{y\in F}$.

Mini-Exercise I.6.31. Let V be a normed vector space. Show that, for the dual pair $\langle V^\vee, V\rangle$, the topology $\sigma(V^\vee, V)$ on $V^\vee$ coincides with the weak-$*$ topology.

Mini-Exercise I.6.32. If a dual pair $\langle E, F\rangle$ is separating, show that $(E, \sigma(E, F))$ is Hausdorff.

As a natural extension of this construction, given a dual pair $\langle E, F\rangle$, it is natural to ask the following question: Given a family $\mathcal{A}$ of non-empty subsets of F, when does the assignment

$$p_A(x) := \sup_{y\in A} |\langle x, y\rangle|$$

define a well-defined seminorm on E? For families $\mathcal{A}$ for which this is the case, the collection of seminorms $\{p_A\}_{A\in\mathcal{A}}$ defines a locally convex topology on E. Topologies obtained in this way are closely related to the theory of *polar topologies*. Note that $\mathcal{A} := \{A \subset F \mid A \text{ is a non-empty finite set}\}$ yields the same topology as the weak topology $\sigma(E, F)$.

In order to answer this question in a systematic way, we first introduce the notion of boundedness for TVSs.

Definition I.6.33. Let V be a topological vector space. A subset $B \subset V$ is said to be *TVS bounded* if, for every neighborhood U of $0 \in V$, there exists a real number $t > 0$ such that $B \subset tU$. □

Remark I.6.34. In the literature, a subset satisfying the above condition is usually called simply *bounded*. In this note, however, in order to avoid confusion with other notions of boundedness (for instance, boundedness in metric spaces), we will use the term "TVS bounded". □

Remark I.6.35. For a Hausdorff LCTVS V, it is known that a subset $B \subset V$ is TVS bounded in V if and only if it is TVS bounded in $(V, \sigma(V, V^\vee))$ (*Mackey's theorem*, [V, Theorem 6.1]). □

Mini-Exercise I.6.36. Let V be an LCTVS whose topology is generated by a family of continuous seminorms $\{p_i\}_{i\in I}$. Show that a subset $B \subset V$ is TVS bounded if and only if

$$\sup_{x\in B} p_i(x) < \infty \qquad \text{for every } i \in I.$$

We can now answer the question above as follows: Let $\langle E, F\rangle$ be a dual pair and $A \subset F$ be a non-empty subset. Then

$$p_A(x) := \sup_{y\in A} |\langle x, y\rangle| \qquad (x \in E)$$

is a well-defined seminorm on E if A is TVS bounded in $(F, \sigma(F, E))$. Indeed, since the weak topology $\sigma(F, E)$ is generated by the seminorms

$$q_x(y) := |\langle x, y\rangle| \qquad (x \in E),$$

TVS boundedness of A in $(F, \sigma(F,E))$ implies that

$$\sup_{y\in A} |\langle x, y\rangle| < \infty \qquad (\forall x \in E)$$

(refer to Mini-Exercise I.6.36), which exactly means that $p_A(x)$ is finite for all $x \in E$. The seminorm axioms follow immediately from the definition.

We are now led to the following definition.

Definition I.6.37. Let $\langle E, F\rangle$ be a dual pair. The *strong topology* on E, denoted by $\beta(E,F)$, is the TVS topology defined by the family of seminorms

$$p_A(x) := \sup_{y\in A} |\langle x, y\rangle|,$$

where A ranges over all non-empty TVS bounded subsets of $(F, \sigma(F,E))$. □

By construction, the strong topology $\beta(E,F)$ is finer than the weak topology $\sigma(E,F)$.

Mini-Exercise I.6.38. Let V be a normed vector space. Show that the strong topology $\beta(V^\vee, V)$ on $V^\vee$ coincides with the topology induced by the operator norm.

In this subsection, we have only discussed two extreme choices of families $\mathcal{A}$: non-empty finite subsets of F, which give rise to the weak topology, and non-empty TVS bounded subsets of $(F, \sigma(F,E))$, which define the strong topology. More generally, one can consider various intermediate families $\mathcal{A}$, leading to a rich class of locally convex topologies on E, commonly referred to as *polar topologies* (see [SW, Chapter IV], [Tr, §19], and [V, Chapter 3] for further details). In this note, however, we shall not pursue this direction further.

Unless otherwise stated, throughout the remainder of this note we shall always endow the dual space $V^\vee$ of a TVS V with the strong topology $\beta(V^\vee, V)$. This convention is adopted for convenience and does not preclude the use of other topologies on $V^\vee$ when appropriate; such choices will always be explicitly indicated.

Mini-Exercise I.6.39. Let V and W be LCTVSs and $f\colon V \to W$ be a continuous linear map.
(i) Show that, for any TVS bounded subset $B \subset V$, $f(B)$ is TVS bounded in W.
(ii) Show that the adjoint $f^\vee\colon W^\vee \to V^\vee$ is continuous.

Remark I.6.40. On the dual space $V^\vee$ of an LCTVS V, there are two natural weak-type topologies: The topology $\sigma(V^\vee, V)$ is called the *weak-$*$ topology*, while $\sigma(V^\vee, V^{\vee\vee})$ is called the *weak topology*. □

6.4 Fréchet spaces and strict countable LF-spaces

In the previous subsection we introduced LCTVSs which are not necessarily normed because function spaces naturally arising on geometric objects treated in this note often

belong to this class. However, in fact, almost all such examples appearing later in this note fall into one of the following two categories (or topological duals of them): Fréchet spaces or (strict countable) LF-spaces. In this subsection, we review their definitions and basic properties.

It should be noted that, depending on the literature, the Hausdorff condition is sometimes included in the definition of these spaces and sometimes not. In this note, however, we shall always include the Hausdorff condition in the definition of both Fréchet spaces and (strict countable) LF-spaces (or adopt definitions for which the Hausdorff property holds automatically).

A *Fréchet space* is a Hausdorff LCTVS F whose topology is defined by a countable family of seminorms $\{p_j\}_{j\in\mathbb{Z}_{\geq 0}}$, and which is complete. Equivalently, F is complete with respect to the translation-invariant distance function defined by

$$d(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{p_j(x-y)}{1+p_j(x-y)}$$

(See Mini-Exercise I.6.3). For example, every Banach space is a Fréchet space.

Example I.6.41. Let n be a positive integer, $\Omega \subset \mathbb{R}^n$ a non-empty open set, and $\{K_\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ be a sequence of non-empty compact subsets of Ω such that

$$K_\nu \subset \operatorname{Int} K_{\nu+1} \text{ (for any } \nu \in \mathbb{Z}_{\geq 0}), \qquad \Omega = \bigcup_{\nu=0}^{\infty} K_\nu.$$

Denote by $C^\infty(\Omega)$ the vector space of complex-valued C^∞ functions on Ω. For each $j, \nu \in \mathbb{Z}_{\geq 0}$, define a seminorm $\rho_{\nu,j}$ on $C^\infty(\Omega)$ by

$$\rho_{\nu,j}(f) := \sup_{x\in K_\nu} \max \left\{ \left| \frac{\partial^{a_1+\cdots+a_n} f}{(\partial x_1)^{a_1}\cdots(\partial x_n)^{a_n}}(x) \right| \;\middle|\; (a_1,\ldots,a_n) \in (\mathbb{Z}_{\geq 0})^n,\ a_1+\cdots+a_n \leq j \right\}$$

for $f \in C^\infty(\Omega)$. The family $\{\rho_{\nu,j}\}_{(\nu,j)\in(\mathbb{Z}_{\geq 0})^2}$ defines a Fréchet space structure on $C^\infty(\Omega)$. Note that every closed and TVS bounded subset of $C^\infty(\Omega)$ is compact (the *Heine–Borel property* or the *Montel property*, See [Ru2, 1.46] for the details). From this property and Theorem I.6.4 (*iii*), it follows that $C^\infty(\Omega)$ is not a Banach space. For $\Omega = \emptyset$, we regard $C^\infty(\Omega) = \{0\}$ with its unique TVS topology. □

Mini-Exercise I.6.42. Let F be a Fréchet space whose topology is defined by a family of seminorms $\{p_j\}_{j\in\mathbb{Z}_{\geq 0}}$. Let d be the translation-invariant distance function defined above.
(*i*) Show that the topology of F coincides with the topology induced by the distance function d.
(*ii*) Assume that $F \neq \{0\}$. Give an example of a subset $A \subset F$ which is bounded in the metric space (F,d) but is not TVS bounded.
(*iii*) For a closed vector subspace $A \subset F$, show that both A and the quotient F/A are Fréchet spaces.
(*iv*) Show that countable products of Fréchet spaces, endowed with the product topology, are Fréchet spaces.

Let $\{F^\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ be a family of Fréchet spaces, and assume that for each $\nu \in \mathbb{Z}_{\geq 0}$ a continuous linear map

$$\pi^\nu \colon F^{\nu+1} \longrightarrow F^\nu$$

is given. In this note, we call the family $\{\pi^\nu \colon F^{\nu+1} \to F^\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ a *projective sequence of Fréchet spaces*. Note that, by composing the maps π^ν, such a family defines a projective system indexed by the countable totally ordered set $\mathbb{Z}_{\geq 0}$. The projective limit of this system, in the sense of vector spaces, is realized by

$$\varprojlim_\nu F^\nu := \Big\{(x_\nu)_\nu \in \prod_{\nu=0}^\infty F^\nu \;\Big|\; \pi^\nu(x_{\nu+1}) = x_\nu \text{ for all } \nu \in \mathbb{Z}_{\geq 0}\Big\}.$$

Endowing $\prod_{\nu=0}^\infty F^\nu$ with the product topology, it follows from Mini-Exercise I.6.42 (*iv*) that this product is a Fréchet space. By definition, $\varprojlim_\nu F^\nu$ is a closed vector subspace of this product. Therefore, by Mini-Exercise I.6.42 (*iii*), $\varprojlim_\nu F^\nu$ carries a natural Fréchet space structure.

Mini-Exercise I.6.43. Let $\{\pi^\nu \colon F^{\nu+1} \to F^\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ be a projective sequence of Fréchet spaces. Show that the topology on $\varprojlim_\nu F^\nu$ defined above is the weakest topology for which all the canonical maps

$$f_\nu \colon \varprojlim_\nu F^\nu \longrightarrow F^\nu$$

are continuous (the *projective topology*).

Similarly, in this note, by an *inductive sequence of Fréchet spaces* $\{\iota_\nu \colon F_\nu \to F_{\nu+1}\}_{\nu\in\mathbb{Z}_{\geq 0}}$ we mean a family of Fréchet spaces $\{F_\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ together with continuous linear maps $\iota_\nu \colon F_\nu \to F_{\nu+1}$ for all $\nu \in \mathbb{Z}_{\geq 0}$. For the inductive system indexed by the countable totally ordered set $\mathbb{Z}_{\geq 0}$ that is defined by composing the maps ι_ν's, the inductive limit as a vector space is realized as the quotient

$$\varinjlim_\nu F_\nu := \Big(\bigoplus_{\nu=0}^\infty F_\nu\Big)\Big/ R(\{\iota_\nu\}_\nu),$$

for a suitable subspace $R(\{\iota_\nu\}_\nu) \subset \bigoplus_{\nu=0}^\infty F_\nu$.

In contrast to the case of projective limits, it is known that more care is required in choosing a topology on $\varinjlim_\nu F_\nu$. The most naive approach is to endow $\varinjlim_\nu F_\nu$ with the strongest topology for which all canonical maps $f_\nu \colon F_\nu \to \varinjlim_\nu F_\nu$ are continuous (the final topology). However, with this choice, it is not clear in general whether $\varinjlim_\nu F_\nu$ becomes a (locally convex) topological vector space (At least this is not obvious, although the author does not know any concrete counterexamples).

In order to make $\varinjlim_\nu F_\nu$ an LCTVS, usually it is endowed with the strongest topology among all the TVS topologies of $\varinjlim_\nu F_\nu$ which makes it an LCTVS such that all the maps $f_\nu \colon F_\nu \to \varinjlim_\nu F_\nu$ are continuous. This topology is called the *locally convex final topology*, whose existence and well-definedness are well-known: see [SW, Chapter II, §6] and [V, Theorem 10.1 (a)] for example. With this topology, the following universal property holds: for any LCTVS V and any linear map $g \colon \varinjlim_\nu F_\nu \to V$, the map g is continuous if and only if all

compositions $g \circ f_\nu \colon F_\nu \longrightarrow V$ are continuous for every $\nu \in \mathbb{Z}_{\geq 0}$ ([SW, Chapter II, 6.1], [V, Theorem 10.1 (b)]). Note that the set of all the *absolutely convex* (i.e., convex and balanced) absorbing subsets $U \subset \varinjlim_\nu F_\nu$ such that $f_\nu^{-1}(U)$ is a neighborhood of 0 in F_ν for any $\nu \in \mathbb{Z}_{\geq 0}$ is a neighborhood base of 0 in this topology ([SW, Chapter II, §6] and [V, Theorem 10.5]).

In this note, we say that an inductive sequence of Fréchet spaces $\{\iota_\nu \colon F_\nu \to F_{\nu+1}\}_{\nu \in \mathbb{Z}_{\geq 0}}$ is *strict* if each $\iota_\nu(F_\nu)$ is closed in $F_{\nu+1}$ and ι_ν induces a TVS isomorphism between F_ν and $\iota_\nu(F_\nu)$, where the latter is endowed with the topology induced from $F_{\nu+1}$. A locally convex topological vector space F is called a *strict countable LF-space* if there exists a strict inductive sequence of Fréchet spaces $\{\iota_\nu \colon F_\nu \to F_{\nu+1}\}_{\nu \in \mathbb{Z}_{\geq 0}}$ such that F is TVS isomorphic to $\varinjlim_\nu F_\nu$ endowed with the locally convex final topology. In this situation, we shall refer to such a system $\{\iota_\nu \colon F_\nu \to F_{\nu+1}\}_{\nu \in \mathbb{Z}_{\geq 0}}$ as a *strict inductive sequence of Fréchet spaces adapted to* F.

According to results of Dieudonné–Schwartz [DS], under this notion of strictness, the following properties are known for a strict countable LF-space F together with an adapted strict inductive sequence $\{\iota_\nu \colon F_\nu \to F_{\nu+1}\}_{\nu \in \mathbb{Z}_{\geq 0}}$ and the canonical maps $f_\nu \colon F_\nu \to F$ (see also [SW, Chapter II, 6.4, 6.5, 6.6] and [V, Theorem 10.8]):

- F is Hausdorff and complete.
- Each f_ν is a TVS isomorphism from F_ν onto its image $f_\nu(F_\nu)$.
- A subset $B \subset F$ is TVS bounded if and only if there exists $\nu \in \mathbb{Z}_{\geq 0}$ such that $B \subset f_\nu(F_\nu)$ and $f_\nu^{-1}(B)$ is TVS bounded in F_ν.

Example I.6.44. Let n be a positive integer. For a compact subset $K \subset \mathbb{R}^n$, let C^∞_K denote the vector space of complex-valued C^∞ functions f on $\mathbb{R}^n$ whose support $\operatorname{Supp} f := \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}$ is contained in K. When $K \neq \emptyset$, for each $j \in \mathbb{Z}_{\geq 0}$, define a seminorm $\rho_{K,j}$ on C^∞_K by

$$\rho_{K,j}(f) := \sup_{x \in K} \max \left\{ \left| \frac{\partial^{a_1+\cdots+a_n} f}{(\partial x_1)^{a_1} \cdots (\partial x_n)^{a_n}}(x) \right| \;\middle|\; (a_1, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n,\ a_1 + \cdots + a_n \leq j \right\}$$

for $f \in C^\infty_K$. The family $\{\rho_{K,j}\}_{j \in \mathbb{Z}_{\geq 0}}$ defines a Fréchet space structure on C^∞_K (See [Ru2, 1.46] for example). For $K = \emptyset$, we endow $C^\infty_K = \{0\}$ with its unique TVS topology.

Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\{K_\nu\}_{\nu \in \mathbb{Z}_{\geq 0}}$ be a sequence of compact subsets of Ω such that

$$K_\nu \subset \operatorname{Int} K_{\nu+1} \text{ (for any } \nu \in \mathbb{Z}_{\geq 0}), \qquad \Omega = \bigcup_{\nu=0}^{\infty} K_\nu.$$

For each $\nu \in \mathbb{Z}_{\geq 0}$, define $\iota_\nu \colon C^\infty_{K_\nu} \to C^\infty_{K_{\nu+1}}$ to be the inclusion map. Then $\{\iota_\nu \colon C^\infty_{K_\nu} \to C^\infty_{K_{\nu+1}}\}_{\nu \in \mathbb{Z}_{\geq 0}}$ is a strict inductive sequence of Fréchet spaces. The resulting strict countable LF-space is denoted by $C^\infty_c(\Omega)$ (also refer to [Ru2, §§ 6.2–6.5] for further details). □

Mini-Exercise I.6.45. Let n be a positive integer and $\Omega \subset \mathbb{R}^n$ be a non-empty open set. Show that the strict countable LF-space $C^\infty_c(\Omega)$ in Example I.6.44 is not a Fréchet space.

For Fréchet spaces and strict countable LF-spaces, the following is important.

THEOREM I.6.46 (Open mapping theorem for Fréchet spaces and strict countable LF-spaces). *Let E and F be TVSs, and $f\colon E \to F$ be a continuous surjective linear map. Assume that both E and F are Fréchet spaces, or that both E and F are strict countable LF-spaces. Then f is an open map.*

PROOF. When both E and F are Fréchet spaces, the proof is almost the same as that of the open mapping theorem for Banach spaces: Indeed, one can run the argument based on the Baire category theorem (since Fréchet spaces can be regarded as complete metric spaces) just by replacing norm balls with suitable neighborhoods defined by seminorms. See [Tr, Theorem 17.1] for the details.

When both E and F are strict countable LF-spaces, the theorem is due to Dieudonné–Schwartz [DS]. Let $\{\iota_\nu\colon E_\nu \to E_{\nu+1}\}_{\nu\in\mathbb{Z}_{\geq 0}}$ and $\{\jmath_\nu\colon F_\nu \to F_{\nu+1}\}_{\nu\in\mathbb{Z}_{\geq 0}}$ be strict inductive sequences of Fréchet spaces adapted to E and F, respectively. Denote by $i_\nu\colon E_\nu \to E$ and $j_\nu\colon F_\nu \to F$ the canonical maps. Take a neighborhood U of 0 in E. It suffices to show that $f(U)$ is a neighborhood of 0 in F. As is mentioned above, the set of all the absolutely convex absorbing subsets $W \subset \varinjlim_\nu F_\nu$ such that $W \cap j_\nu(F_\nu)$ is a neighborhood of 0 in $j_\nu(F_\nu)$ for any $\nu \in \mathbb{Z}_{\geq 0}$ is a neighborhood base of 0 in this topology ([SW, Chapter II, §6] and [V, Theorem 10.5], recall that $j_\nu\colon F_\nu \to j_\nu(F_\nu)$ is a TVS isomorphism). Therefore all we have to do is to show that $f(U) \cap j_\nu(F_\nu)$ is a neighborhood of 0 in $j_\nu(F_\nu)$ for any $\nu \in \mathbb{Z}_{\geq 0}$, because by shrinking if necessary, we may assume that U is absolutely convex (since E is LCTVS, here we also used the equation $F = \bigcup_\nu j_\nu(F_\nu)$ and Mini-Exercise I.6.1 (*ii*) to say that $f(U)$ is absorbing if $f(U) \cap j_\nu(F_\nu)$ is a neighborhood of 0 in $j_\nu(F_\nu)$ for any $\nu \in \mathbb{Z}_{\geq 0}$).

Let ν be a non-negative integer. Consider the continuous linear map

$$f_{\mu,\nu} := f|_{i_\mu(E_\mu)\cap f^{-1}(j_\nu(F_\nu))}\colon i_\mu(E_\mu) \cap f^{-1}(j_\nu(F_\nu)) \longrightarrow j_\nu(F_\nu).$$

Note that the subspace $i_\mu(E_\mu) \cap f^{-1}(j_\nu(F_\nu))$ is a closed vector subspace of $i_\mu(E_\mu)$, hence it is Fréchet (see Mini-Exercise I.6.42 (*iii*)). As f is surjective,

$$j_\nu(F_\nu) = \bigcup_{\mu=0}^{\infty} (f(i_\mu(E_\mu)) \cap j_\nu(F_\nu)) = \bigcup_{\mu=0}^{\infty} \operatorname{Image} f_{\mu,\nu} \tag{11}$$

holds. One can take a $\mu \in \mathbb{Z}_{\geq 0}$ such that $f_{\mu,\nu}$ is surjective, since otherwise it follows from (11) and the following Theorem I.6.47 that the Fréchet space F_ν can be decomposed into a countable union of meagre subsets, which contradicts the Baire category theorem. By the open mapping theorem for Fréchet spaces, the map $f_{\mu,\nu}$ is open. In particular, the image

$$f_{\mu,\nu}\big((U \cap i_\mu(E_\mu)) \cap f^{-1}(j_\nu(F_\nu))\big) = f(U \cap i_\mu(E_\mu)) \cap j_\nu(F_\nu)\ (\subset f(U) \cap j_\nu(F_\nu))$$

is a neighborhood of 0 in $j_\nu(F_\nu)$, which completes the proof. □

THEOREM I.6.47 (=[Bana, p. 38 Théorème 3]). *Let E and F be Fréchet spaces and $f\colon E \to F$ be a continuous linear map. Then* Image f *is either meagre in F or coincides with F.*

PROOF. Put $H := \operatorname{Image} f \subset F$. We fix translation-invariant distance functions d_E on E and d_F on F which induce their TVS topologies, and write

$$B_E(r) := \{x \in E \mid d_E(x,0) < r\}, \qquad B_F(r) := \{y \in F \mid d_F(y,0) < r\} \quad (r > 0).$$

Assume that H is not meagre in F. First we show the following: For every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$B_F(\eta) \subset \overline{f\big(B_E(\varepsilon)\big)}, \tag{12}$$

where the closure is taken in F.

To show this, fix $\varepsilon > 0$ and for each $n \in \mathbb{Z}_{\geq 1}$ set

$$G_n := n\, B_E(\varepsilon/2), \qquad H_n := f(G_n) = n\, f\big(B_E(\varepsilon/2)\big) \subset H.$$

Since $E = \bigcup_{n\geq 1} G_n$, we have $H = \bigcup_{n=1}^{\infty} H_n$. If every H_n were meagre in F, then H would be meagre as a countable union of meagre sets, contradicting our assumption. Hence there exists n_0 such that H_{n_0} is not meagre in F.

Now, since H_{n_0} is not meagre, its closure $\overline{H_{n_0}}$ has an interior point. Therefore we can find $y_1 \in F$ and $\eta_1 > 0$ such that

$$y_1 + B_F(\eta_1) \subset \overline{H_{n_0}},$$

from which we obtain

$$\frac{y_1}{n_0} + B_F(\eta_1/n_0) \subset \overline{f\big(B_E(\varepsilon/2)\big)}.$$

Taking differences of two points in the left-hand side, we get

$$B_F(\eta_1/n_0) \subset \overline{f\big(B_E(\varepsilon/2)\big)} - \overline{f\big(B_E(\varepsilon/2)\big)} \subset \overline{f\big(B_E(\varepsilon)\big)},$$

because $B_E(\varepsilon/2) - B_E(\varepsilon/2) \subset B_E(\varepsilon)$ and f is linear. Thus (12) holds with $\eta := \eta_1/n_0$.

Now let us show $H = F$ by assuming H is not meagre. Choose $\varepsilon_1 > 0$ and let $\eta_1 > 0$ be a constant such that (12) holds for $\varepsilon = \varepsilon_1$. Inductively, set $\varepsilon_{k+1} := \varepsilon_k/2$ and choose $\eta_{k+1} > 0$ corresponding to $\varepsilon = \varepsilon_{k+1}$ in (12). By replacing η_{k+1} with $\min\{\eta_{k+1}, \eta_k/2\}$, we may assume

$$\eta_{k+1} \leq \eta_k/2 \qquad (\text{forall } k \in \mathbb{Z}_{\geq 1}). \tag{13}$$

Take $y \in F$. Choose $m \in \mathbb{Z}_{\geq 1}$ so large that $y/m \in B_F(\eta_1)$ (which is possible since scalar multiplication by $1/m$ tends to 0 in a TVS). By (12) for $k = 1$, we can choose $x_1 \in B_E(\varepsilon_1)$ such that

$$d_F\big(f(x_1),\, y/m\big) < \eta_2.$$

Set $r_2 := y/m - f(x_1)$, so that $r_2 \in B_F(\eta_2)$. Again by (12) for $k = 2$, we can choose $x_2 \in B_E(\varepsilon_2)$ such that

$$d_F\big(f(x_2),\, r_2\big) < \eta_3.$$

Set $r_3 := r_2 - f(x_2) \in B_F(\eta_3)$. Continuing inductively, we construct sequences $(x_k)_{k\geq 1}$ in E and $(r_k)_{k\geq 2}$ in F such that

$$x_k \in B_E(\varepsilon_k), \qquad r_{k+1} = r_k - f(x_k) \in B_F(\eta_{k+1}) \quad (\text{forall } k \in \mathbb{Z}_{\geq 2}).$$

Since $\varepsilon_{k+1} = \varepsilon_k/2$, the series $\{\sum_{k=1}^{N} x_k\}_N$ is Cauchy in E, hence convergent because E is complete. Let $x := \sum_{k=1}^{\infty} x_k \in E$. On the other hand, by (13) we have $\eta_k \to 0$, hence $r_k \to 0$ in F. From the telescoping identity

$$\frac{y}{m} - \sum_{k=1}^{N} f(x_k) = r_{N+1},$$

letting $N \to \infty$ and using continuity of f, we obtain

$$\frac{y}{m} = f(x).$$

Therefore $y = f(mx) \in H$, which shows $H = F$. □

Remark I.6.48 (A Fréchet closed-range theorem). Besides the open mapping theorem, several other Banach-space results have natural Fréchet analogues. We record one that will be useful later, without giving a general account. The closed-range theorem (Theorem I.6.11) has the following form: if E, F are Fréchet spaces and $T\colon E \to F$ is a continuous linear map, then

$$\operatorname{Image} T \text{ is closed in } F \quad \Longleftrightarrow \quad \operatorname{Image} T^{\vee} \text{ is weak-}* \text{ closed in } E^{\vee},$$

where $T^{\vee}\colon F^{\vee} \to E^{\vee}$ is the transpose and weak-$*$ means $\sigma(E^{\vee}, E)$. In this case the induced map $E/\operatorname{Ker} T \to \operatorname{Image} T$ is a TVS isomorphism, with the relative topology on the image. The displayed equivalence is precisely the equivalence of (c) and (e) in [DS, §12, Théorème 7, p. 92].

For strict countable LF-spaces, applying the open mapping theorem also requires checking its hypotheses on the image with its relative topology. Closedness of the image alone does not, by definition, supply it with a strict countable LF-space structure. Thus the Fréchet argument should not be transferred to that setting without checking this additional point (See also [LL, Problem 1.7, p. 306]). □

6.5 On extending operators to spaces of Schwartz distributions

Let n be a positive integer and let $\Omega \subset \mathbb{R}^n$ be an open set. Let $C^{\infty}(\Omega)$ and $C_c^{\infty}(\Omega)$ be the LCTVSs defined in Examples I.6.41 and I.6.44. Elements of $(C_c^{\infty}(\Omega))^{\vee}$ are called *Schwartz distributions*.

Let $\langle C^{\infty}(\Omega), C_c^{\infty}(\Omega)\rangle$ be the separating dual pair defined by

$$\langle f, \varphi\rangle := \int_{\Omega} f \cdot \varphi \, d\lambda \qquad \big(f \in C^{\infty}(\Omega),\ \varphi \in C_c^{\infty}(\Omega)\big),$$

where $d\lambda$ denotes the Lebesgue measure. Via the linear map

$$I\colon C^{\infty}(\Omega) \longrightarrow (C_c^{\infty}(\Omega))^{\vee}, \qquad I(f) := \langle f, \bullet\rangle,$$

we may regard $C^{\infty}(\Omega)$ as a vector subspace of $(C_c^{\infty}(\Omega))^{\vee}$. In this sense, Schwartz distributions are regarded as a generalized notion of functions on Ω. Linear operators from $C^{\infty}(\Omega)$ to itself

(such as differential operators or multiplication by a fixed smooth function) often admit a natural extension to linear operators on $(C_c^\infty(\Omega))^\vee$. The same phenomenon will appear later for currents on manifolds. In order to describe this extension procedure in a systematic way, we now consider the following abstract setting.

Let A^+ and A^- be Fréchet spaces, and let $T\colon A^+ \to A^-$ be a continuous linear map. For each $\bullet \in \{+,-\}$, let $\{\iota_\nu^\bullet\colon A_\nu^\bullet \to A_{\nu+1}^\bullet\}_{\nu\in\mathbb{Z}_{\geq 0}}$ be a strict inductive sequence of Fréchet spaces, and set

$$A_c^\bullet := \varinjlim_\nu A_\nu^\bullet,$$

endowed with the locally convex final topology. Thus $A_c^\bullet$ is a strict countable LF-space. We identify each $A_\nu^\bullet$ with its canonical image in $A_c^\bullet$.

Assume moreover that for each $\bullet \in \{+,-\}$ there exists a bilinear map

$$\langle -,-\rangle_\bullet\colon A^\bullet \times A_c^\bullet \longrightarrow \mathbb{C}$$

whose restriction to $A^\bullet \times A_\nu^\bullet$ is continuous for every $\nu \in \mathbb{Z}_{\geq 0}$. For each $f \in A^\bullet$, the linear functional $\langle f, \bullet\rangle_\bullet$ is continuous on every stage and hence on $A_c^\bullet$ by the universal property of the locally convex final topology. Assume that the resulting linear map

$$I^\bullet\colon A^\bullet \longrightarrow (A_c^\bullet)^\vee, \qquad I^\bullet(f) := \langle f, \bullet\rangle_\bullet,$$

is injective.

Assume further that there exists a continuous linear map

$$T^*_{\mathrm{formal}}\colon A_c^- \longrightarrow A_c^+$$

such that

$$\langle T(f), \varphi\rangle_- = \langle f, T^*_{\mathrm{formal}}(\varphi)\rangle_+ \qquad (\forall f \in A^+,\ \forall \varphi \in A_c^-).$$

We call T^*_{formal} the *formal adjoint* of T with respect to the above pairings.

Proposition I.6.49. *Under the above assumptions, define a linear map*

$$\widetilde{T}\colon (A_c^+)^\vee \longrightarrow (A_c^-)^\vee$$

by $\widetilde{T} := (T^*_{\mathrm{formal}})^\vee$*: i.e.,*

$$(\widetilde{T}(u))(\varphi) := u\big(T^*_{\mathrm{formal}}(\varphi)\big) \qquad \big(u \in (A_c^+)^\vee,\ \varphi \in A_c^-\big).$$

Then the following statements hold.
(*i*) *For each* $\bullet \in \{+,-\}$*, the map* $I^\bullet\colon A^\bullet \to (A_c^\bullet)^\vee$ *is an injective continuous linear map.*
(*ii*) *The map* $\widetilde{T}$ *is a well-defined continuous linear map.*
(*iii*) *The diagram*

$$\begin{array}{ccc} A^+ & \xrightarrow{T} & A^- \\ \downarrow{\scriptstyle I^+} & & \downarrow{\scriptstyle I^-} \\ (A_c^+)^\vee & \xrightarrow{\widetilde{T}} & (A_c^-)^\vee \end{array}$$

is commutative.

PROOF. We first prove (i). Fix $\bullet \in \{+,-\}$ and write $A := A^\bullet$, $A_c := A_c^\bullet$, $\langle -,-\rangle := \langle -,-\rangle_\bullet$, and $I := I^\bullet$. As $(A_c)^\vee$ is endowed with the strong topology $\beta\big((A_c)^\vee, A_c\big)$, its topology is generated by the seminorms

$$p_B(u) := \sup_{\varphi\in B} |u(\varphi)|,$$

where B runs over all non-empty TVS bounded subsets of A_c. It therefore suffices to show that for every non-empty TVS bounded subset $B \subset A_c$, the seminorm

$$q_B(f) := p_B(I(f)) = \sup_{\varphi\in B} \big|\langle f,\varphi\rangle\big| \qquad (f \in A)$$

is continuous on A. Let $B \subset A_c$ be non-empty and TVS bounded. By the bounded-set property of strict countable LF-spaces recalled in §6.4, there exists $\nu \in \mathbb{Z}_{\geq 0}$ such that $B \subset A_\nu^\bullet$ and B is TVS bounded in $A_\nu^\bullet$. Let $\varepsilon > 0$ be a positive number. By continuity of the pairing on $A \times A_\nu^\bullet$ at $(0,0)$, there exist neighborhoods $U \subset A$ and $V \subset A_\nu^\bullet$ of 0 such that $|\langle u, v\rangle| < \varepsilon$ for all $u \in U$ and $v \in V$. Since B is TVS bounded in $A_\nu^\bullet$, there exists $t > 0$ such that $B \subset tV$. Then for every $u \in U$ one has

$$q_B(u) = \sup_{\varphi\in B} |\langle u,\varphi\rangle| \leq \sup_{\psi\in V} |\langle u, t\psi\rangle| = t \sup_{\psi\in V} |\langle u,\psi\rangle| \leq t\varepsilon.$$

This shows that q_B is continuous at 0, hence continuous on A. Thus $I = I^\bullet$ is an injective continuous linear map (since the injectivity is assumed in the above setting). (ii) follows from Mini-Exercise I.6.39 (ii). Finally, (iii) follows since one has

$$\big(\widetilde{T}(I^+(f))\big)(\varphi) = I^+(f)\big(T^*_{\text{formal}}(\varphi)\big) = \langle f, T^*_{\text{formal}}(\varphi)\rangle_+ = \langle T(f),\varphi\rangle_- = I^-(T(f))(\varphi)$$

for any $f \in A^+$ and $\varphi \in A_c^-$. □

Let us briefly summarize how Proposition I.6.49 will be used in practice. In applications, one first fixes a linear map $T\colon A^+ \to A^-$ between Fréchet spaces together with sequences of Fréchet spaces and continuous linear maps $\{\iota_\nu^\bullet\colon A_\nu^\bullet \to A_{\nu+1}^\bullet\}_{\nu\in\mathbb{Z}_{\geq 0}}$. For the vector spaces $A_c^\bullet = \varinjlim_\nu A_\nu^\bullet$, $A^\bullet \times A_c^\bullet$ often have a naturally defined bilinear pairing $\langle -,-\rangle_\bullet$. Then we check the following:

- Each $\{\iota_\nu^\bullet\colon A_\nu^\bullet \to A_{\nu+1}^\bullet\}_\nu$ is a strict inductive sequence;
- $I^\bullet\colon A^\bullet \to (A_c^\bullet)^\vee$ is injective;
- $\langle -,-\rangle_\bullet$ is continuous on $A^\bullet \times A_\nu^\bullet$ for every $\nu \in \mathbb{Z}_{\geq 0}$;
- T is continuous;
- T admits a formal adjoint $T^*_{\text{formal}}\colon A_c^- \longrightarrow A_c^+$;
- T^*_{formal} is continuous.

Once one verifies them, Proposition I.6.49 can be applied to obtain a continuous extension $\widetilde{T}\colon (A_c^+)^\vee \longrightarrow (A_c^-)^\vee$ between the strong duals of strict countable LF-spaces.

Note that, in concrete situations, especially when working "in the sense of distributions", it is common to suppress the maps $I^\bullet$ from the notation and to regard elements of $A^\bullet$ as continuous linear functionals on $A_c^\bullet$ via the above pairings. This identification is justified by Proposition I.6.49 (i), which ensures that $I^\bullet$ is a continuous injective linear map.

Example I.6.50. In the notation of Examples I.6.41 and I.6.44, let

$$A^\pm := C^\infty(\mathbb{R}), \qquad A_c^\pm := C_c^\infty(\mathbb{R}), \qquad A_\nu^\pm := C^\infty_{[-\nu-1,\nu+1]}$$

for $\nu \in \mathbb{Z}_{\geq 0}$. Let $\langle -,-\rangle_\pm := \langle -,-\rangle$ be the pairing defined by

$$\langle f, \varphi\rangle := \int_{\mathbb{R}} f(x)\,\varphi(x)\,dx \qquad \big(f \in C^\infty(\mathbb{R}),\ \varphi \in C_c^\infty(\mathbb{R})\big).$$

Then each $A_\nu^\pm$ is clearly a closed subspace of $C^\infty(\mathbb{R})$. Consider the linear map

$$T\colon C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}), \qquad T(f) := f'.$$

By applying Proposition I.6.49 (see Mini-Exercise I.6.51), one has the extension

$$\widetilde{T}\colon (C_c^\infty(\mathbb{R}))^\vee \longrightarrow (C_c^\infty(\mathbb{R}))^\vee,$$

which is called the *derivative in the sense of distributions*.

Note that any locally integrable function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ defines a distribution by

$$\varphi \longmapsto \int_{\mathbb{R}} f(x)\,\varphi(x)\,dx,$$

which will often be identified with f itself by abuse of notation (which can be regarded as a natural extension of our abuse of notation concerning $I^\bullet$ for smooth functions). In this sense, the derivative of any locally integrable function can be defined in the sense of distributions. □

Mini-Exercise I.6.51. In this mini-exercise, we work in the setting of Example I.6.50.
(i) Show that all the assumptions of Proposition I.6.49 are satisfied.
(ii) Define the *Heaviside function* $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(x) := \begin{cases} 0 & (x \leq 0), \\ 1 & (0 < x). \end{cases}$$

Compute the derivative of h in the sense of distributions.

6.6 On closedness of the image and the duality

To obtain TVS analogues of Corollary I.5.9, it is classical that closedness of images naturally enters the picture. As a TVS variant tailored to the Fréchet/strict countable LF-space setting, we prove the following:

LEMMA I.6.52. *Let A, B, C be LCTVSs, and let $f\colon A \to B$ and $g\colon B \to C$ be continuous linear maps such that $g \circ f = 0$. Put*

$$\mathfrak{A} := A^\vee, \qquad \mathfrak{B} := B^\vee, \qquad \mathfrak{C} := C^\vee,$$

and denote by

$$\mathfrak{f} := f^\vee\colon \mathfrak{B} \to \mathfrak{A}, \qquad \mathfrak{g} := g^\vee\colon \mathfrak{C} \to \mathfrak{B}$$

the transpose maps. Define

$$H := \frac{\operatorname{Ker} g}{\operatorname{Image} f}, \qquad \mathfrak{H} := \frac{\operatorname{Ker} \mathfrak{f}}{\operatorname{Image} \mathfrak{g}}.$$

We endow $\operatorname{Ker} g$ *and* $\operatorname{Image} f$ *with the relative topologies induced from B, and H with the quotient topology; thus H is a TVS, not necessarily Hausdorff. All duals carry the strong topology, and $\mathfrak{H}$ carries the quotient of the subspace topology on* $\operatorname{Ker}\mathfrak{f} \subset B^\vee$. *Assume that A, B, and C are Fréchet spaces, or that A, B and C are strict countable LF-spaces. Then there exists a natural linear map*

$$\iota\colon \mathfrak{H} \longrightarrow H^\vee,$$

and the following holds.
(i) ι is surjective.
(ii) Assume that $\operatorname{Image} g$ *is a closed subspace of C. When A, B, and C are strict countable LF-spaces, additionally assume that* $\operatorname{Image} g$ *(endowed with the relative topology) is also a strict countable LF-space. Then ι is a bijection.*
(iii) Suppose that every TVS bounded subset of $H_{\text{Hausdorff}}$ is contained in the image of some TVS bounded subset of $\operatorname{Ker} g$ *under the quotient map* $\operatorname{Ker} g \to H_{\text{Hausdorff}}$, *where the Hausdorffization $H_{\text{Hausdorff}}$ of H is naturally identified with* $\operatorname{Ker} g/\overline{\operatorname{Image} f}$ *with the quotient topology and the closure is taken in* $\operatorname{Ker} g$. *Then ι is continuous for the strong dual topology on $H^\vee$.*

PROOF. The construction of the linear map ι is formally identical to the purely algebraic construction given in Lemma I.5.7. For completeness, we briefly recall it and then explain the additional topological aspects.

Since $g \circ f = 0$, one has $\operatorname{Image} f \subset \operatorname{Ker} g$. Let $[\Phi] \in \mathfrak{H} = \operatorname{Ker}\mathfrak{f}/\operatorname{Image}\mathfrak{g}$, and choose a representative $\Phi \in \operatorname{Ker}\mathfrak{f} \subset B^\vee$. The condition $\Phi \in \operatorname{Ker}\mathfrak{f}$ means $\Phi \circ f = 0$, hence Φ vanishes on $\operatorname{Image} f$. Therefore the restriction $\Phi|_{\operatorname{Ker} g}$ also vanishes on $\operatorname{Image} f$, and by the universal property of the quotient it induces a unique linear functional

$$\widetilde{\Phi} \in H^\vee = \left(\frac{\operatorname{Ker} g}{\operatorname{Image} f}\right)^\vee.$$

We define $\iota([\Phi]) := \widetilde{\Phi}$. If Φ is replaced with $\Phi + \psi \circ g$ for some $\psi \in C^\vee$, then $(\psi \circ g)|_{\operatorname{Ker} g} = 0$, so the induced functional $\widetilde{\Phi}$ does not change. Thus ι is well-defined (and linear by construction).

Let $\lambda \in H^\vee$ be arbitrary. Composing with the quotient map $\operatorname{Ker} g \twoheadrightarrow H$, we regard λ as an element of $(\operatorname{Ker} g)^\vee$ that vanishes on $\operatorname{Image} f$. By the Hahn–Banach theorem for LCTVSs (Theorem I.6.26), λ extends to a continuous linear functional $\Phi \in B^\vee$. Because $\operatorname{Image} f \subset \operatorname{Ker} g$ and λ vanishes on $\operatorname{Image} f$, the extension Φ also satisfies $\Phi \circ f = 0$, hence $\Phi \in \operatorname{Ker} \mathfrak{f}$. By construction one has $\iota([\Phi]) = \lambda$, which shows that ι is surjective. This proves (i).

Assume that $\operatorname{Image} g$ is a closed subspace of C. When A, B, and C are Fréchet spaces, $\operatorname{Image} g$ is again a Fréchet space. When A, B, and C are strict countable LF-spaces, we assume in addition that $\operatorname{Image} g$ is also a strict countable LF-space. In either case, by applying the open mapping theorem (Theorem I.6.46), we have that the induced surjection

$$B \longrightarrow \operatorname{Image} g$$

is open. Therefore the induced map

$$\overline{g} \colon B/\operatorname{Ker} g \longrightarrow \operatorname{Image} g, \qquad [b] \longmapsto g(b)$$

is a TVS isomorphism.

To show (ii), it suffices to show the injectivity of ι, since the surjectivity has already been established. Take $[\Phi] \in \mathfrak{H}$ and assume that $\iota([\Phi]) = 0$. Choose a representative $\Phi \in \operatorname{Ker} \mathfrak{f} \subset B^\vee$. The condition $\iota([\Phi]) = 0$ means that the induced functional $\widetilde{\Phi} \in H^\vee$ is zero, hence Φ vanishes on $\operatorname{Ker} g$. Therefore Φ factors through the quotient $B/\operatorname{Ker} g$, from which we have that there exists a unique linear functional

$$\phi \in (B/\operatorname{Ker} g)^\vee$$

such that $\Phi = \phi \circ q$, where $q \colon B \to B/\operatorname{Ker} g$ is the quotient map. Transport ϕ via the TVS isomorphism $\overline{g}$ and define $\ell \in (\operatorname{Image} g)^\vee$ by letting

$$\ell(y) := \phi\big(\overline{g}^{-1}(y)\big) \quad (y \in \operatorname{Image} g).$$

Then ℓ is continuous and satisfies $\Phi = \ell \circ g$. As $\operatorname{Image} g$ is a vector subspace of the LCTVS C, the Hahn–Banach theorem for LCTVSs (Theorem I.6.26) implies that ℓ extends to some $\psi \in C^\vee$. For this ψ, one has $\psi \circ g = \ell \circ g = \Phi$. Thus we obtain $\Phi \in \operatorname{Image} \mathfrak{g}$, which completes the proof of (ii).

To prove (iii), let $S \subset H$ be non-empty and TVS bounded in $(H, \sigma(H, H^\vee))$. Its image in $H_{\text{Hausdorff}}$ is weakly bounded and hence TVS bounded by Mackey's theorem recalled above, so choose a TVS bounded set $D \subset \operatorname{Ker} g$ whose image in $H_{\text{Hausdorff}}$ contains it. For each $\Phi \in \operatorname{Ker} \mathfrak{f}$, the functional $\iota([\Phi])$ factors through $H_{\text{Hausdorff}}$, so the choice of D gives

$$\sup_{s \in S} |\iota([\Phi])(s)| \leq \sup_{z \in D} |\Phi(z)|.$$

The right-hand side is a continuous seminorm on $B^\vee$ with its strong topology, because D is also TVS bounded in B. Thus the map from $\operatorname{Ker} \mathfrak{f}$ to $H^\vee$ is continuous for every defining strong-dual seminorm. Passing to the quotient by $\operatorname{Image} \mathfrak{g}$ now proves the continuity of ι. $\square$

REMARK I.6.53. In Lemma I.6.52, the algebraic surjectivity in (i) uses only local convexity and the Hahn–Banach theorem; it does not require the Fréchet/LF hypotheses. Those hypotheses enter (ii) through the open mapping theorem. Continuity for the strong dual topology is a separate issue: TVS bounded subsets of a quotient need not have TVS bounded lifts merely by the definition of the quotient topology. Assertion (iii) supplies a sufficient condition. It holds, for example, if $H_{\mathrm{Hausdorff}}$ is finite-dimensional: lift a basis to $\operatorname{Ker} g$ to obtain a linear section of the quotient map. This section is continuous by Theorem I.6.4 (ii) and therefore maps TVS bounded sets to TVS bounded sets. It also holds when $\operatorname{Image} f = 0$. The stronger topological duality results for Fréchet–Schwartz complexes used in Chapter III are given in [Cas, §3, Theorems 4–5]; the general Hahn–Banach argument alone is not being used to assert those results. □

REMARK I.6.54. We keep the same setting and assumptions as in Lemma I.6.52, and consider the special case where $A = 0$, so that $f = 0$. Then we have

$$H = \operatorname{Ker} g, \qquad \mathfrak{H} = \frac{B^\vee}{\operatorname{Image} \mathfrak{g}}.$$

By Lemma I.6.52, under the assumption stated in the assertion (ii) on A, B, C and $\operatorname{Image} g$, the natural map $\iota\colon B^\vee/\operatorname{Image}\mathfrak{g} \to H^\vee$ is injective. It is also continuous by (iii), whose condition on TVS bounded subsets is automatic here because $H = H_{\mathrm{Hausdorff}} = \operatorname{Ker} g$. Since $H^\vee$ is Hausdorff, its zero element $\{0\}$ is closed. Therefore it follows that $\operatorname{Image}\mathfrak{g}$ is a closed subspace of $B^\vee$ if the assumption as in Lemma I.6.52 (ii) holds. □

For a sufficient condition for the image of a continuous linear map between TVSs to be closed, Serre's theorem [Se1, Lemme 2] is fundamental. Here let us see the proof of this theorem, and its variant for strict countable LF-spaces.

LEMMA I.6.55 ([Se1, Lemme 2] and its strict countable LF-space version). *Let E and F be TVSs and $f\colon E \to F$ be a continuous linear map. Assume that both E and F are Fréchet spaces, or that both E and F are strict countable LF-spaces. If*

$$\dim \operatorname{Coker} f < \infty,$$

then $\operatorname{Image} f$ *is closed in F, and* $\operatorname{Coker} f$ *is Hausdorff.*

PROOF. By the assumption, there exists a finite-dimensional vector subspace $V \subset F$ such that

$$F = \operatorname{Image} f \oplus V$$

holds algebraically. Since F is Hausdorff and $\dim V < \infty$, the subspace V is closed in F (see Mini-Exercise I.6.6 (i)).

Consider the continuous linear map

$$g\colon E \oplus V \longrightarrow F, \qquad g(x, v) := f(x) + v.$$

By construction, g is surjective. When E is Fréchet, it is clear that $E \oplus V$ is also Fréchet. When E is a strict countable LF-space that comes from a strict inductive sequence $\{\iota_\nu\colon E_\nu \to$

$E_{\nu+1}\}_{\nu\in\mathbb{Z}_{\geq 0}}$ of Fréchet spaces, it follows by considering $\{\iota_\nu \oplus \mathrm{id}_V \colon E_\nu \oplus V \to E_{\nu+1} \oplus V\}_{\nu\in\mathbb{Z}_{\geq 0}}$ that $E\oplus V$ is again a strict countable LF-space, where id_V is the identity map on V. Therefore, by the open mapping theorem for Fréchet spaces and strict countable LF-spaces (Theorem I.6.46), g is an open map.

We claim that

$$\mathrm{Ker}\, g = (\mathrm{Ker}\, f) \oplus \{0\}$$

as vector subspaces of $E\oplus V$. Indeed, the inclusion $\supset$ is clear. Conversely, if $g(x,v) = 0$, then $f(x) = -v \in \mathrm{Image}\, f \cap V$. Since $F = \mathrm{Image}\, f \oplus V$ algebraically, we have $\mathrm{Image}\, f \cap V = \{0\}$, hence $v = 0$ and $f(x) = 0$, i.e., $x \in \mathrm{Ker}\, f$.

Denote by

$$\overline{g} \colon (E \oplus V)/\mathrm{Ker}\, g \longrightarrow F$$

the induced map, which is clearly a continuous linear bijection. Moreover, since g is open, the quotient map $E\oplus V \to (E\oplus V)/\mathrm{Ker}\, g$ is open, hence $\overline{g}$ is a TVS isomorphism. Under this TVS isomorphism, the subspace $\mathrm{Image}\, f \subset F$ corresponds to the image of $E \oplus \{0\} \subset E \oplus V$ in the quotient $(E \oplus V)/\mathrm{Ker}\, g$. Since

$$(E \oplus \{0\}) \cap \mathrm{Ker}\, g = (\mathrm{Ker}\, f) \oplus \{0\},$$

this image can be identified with $E/\mathrm{Ker}\, f$ via

$$E/\mathrm{Ker}\, f \;\cong\; (E \oplus \{0\})/((\mathrm{Ker}\, f) \oplus \{0\}).$$

On the other hand, the natural inclusion $E \oplus \{0\} \hookrightarrow E \oplus V$ is a topological embedding with closed image (it is the kernel of the continuous projection $E \oplus V \to V$). Therefore $\mathrm{Image}\, f$ is closed in F.

Finally, since F is Hausdorff and $\mathrm{Image}\, f$ is closed, it follows from Lemma I.6.2 (*iii*) that the quotient $\mathrm{Coker}\, f = F/\mathrm{Image}\, f$ is Hausdorff. □

Next we show the following theorem of L. Schwartz. The proof below is in accordance with the argument in [De4, Chapter IX §1.B] (Though one can find another proof of this theorem in [GuR, Appendix B], the author could not verify (*ii*) in the proof of [GuR, Appendix B 12] there).

THEOREM I.6.56 (Schwartz). *Let X and Y be Fréchet spaces, $T \colon X \to Y$ a surjective continuous linear map, and $K \colon X \to Y$ a compact linear map. Then the quotient space $Y/\mathrm{Image}(T+K)$ is finite-dimensional and Hausdorff. In particular,* $\mathrm{Image}(T+K)$ *is closed in Y.*

PROOF. Since K is compact, there exists an absolutely convex open neighborhood U of 0 in X such that the closure

$$\overline{K(U)} \subset Y$$

is compact. Since T is surjective and continuous, T is an open map by the open mapping theorem for Fréchet spaces (Theorem I.6.46). Hence

$$V := T(U)$$

is a neighborhood of 0 in Y.

Since $\overline{K(U)}$ is compact and $\frac{1}{2}V$ is an open neighborhood of 0 in Y, the family of open sets $\{b+\frac{1}{2}V\}_{b\in Y}$ covers $\overline{K(U)}$. Hence there exist $b_1,\dots,b_N\in\overline{K(U)}$ such that

$$\overline{K(U)}\subset\bigcup_{j=1}^{N}\left(b_j+\tfrac{1}{2}V\right). \tag{14}$$

Let S be the finite-dimensional vector subspace of Y spanned by $b_1,b_2,\dots,b_N$. Note that S is closed by Mini-Exercise I.6.6 (i). Denote by

$$q\colon Y\to Y/S$$

the quotient map and set

$$f:=T+K,\qquad \widetilde{T}:=q\circ T,\qquad \widetilde{K}:=q\circ K,\qquad \widetilde{f}:=q\circ f=\widetilde{T}+\widetilde{K}.$$

Put $\widetilde{V}:=\widetilde{T}(U)=q(V)$. Then $\widetilde{T}$ is surjective (because T is surjective), hence $\widetilde{V}$ is a neighborhood of 0 in Y/S. Moreover, by (14),

$$\overline{\widetilde{K}(U)}=\overline{q(K(U))}\subset q\Big(\overline{K(U)}\Big)\subset\bigcup_{j=1}^{N}q\big(b_j+\tfrac{1}{2}V\big)=\bigcup_{j=1}^{N}\Big(q(b_j)+\tfrac{1}{2}\widetilde{V}\Big)=\tfrac{1}{2}\widetilde{V},$$

because $q(b_j)=0$ for $b_j\in S$. Thus we have

$$\overline{\widetilde{K}(U)}\subset\tfrac{1}{2}\widetilde{V}. \tag{15}$$

Let $\{p_\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ be a system of seminorms that defines the TVS topology of X. By inductively replacing p_ν with $2\cdot\max\{p_0,p_1,\dots,p_\nu\}$, we may assume $2p_\nu\leq p_{\nu+1}$ for all $\nu\in\mathbb{Z}_{\geq 0}$. Set $U_\nu:=\{x\in X\mid p_\nu(x)<1\}$. Then $\{U_\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ is a fundamental system of absolutely convex neighborhoods of 0 in X such that

$$U_{\nu+1}\subset\tfrac{1}{2}U_\nu\qquad(\text{for all }\nu\in\mathbb{Z}_{\geq 0}). \tag{16}$$

Fix $\nu\in\mathbb{Z}_{\geq 0}$. For each $n\in\mathbb{Z}_{\geq 0}$, the set

$$\widetilde{T}\big(2^nU_\nu\cap\tfrac{1}{2}U\big)$$

is an open subset of $\widetilde{T}(\frac{1}{2}U)=\frac{1}{2}\widetilde{V}$, because $\widetilde{T}$ is open. Moreover,

$$\bigcup_{n\geq 0}\widetilde{T}\big(2^nU_\nu\cap\tfrac{1}{2}U\big)=\widetilde{T}(\tfrac{1}{2}U)=\tfrac{1}{2}\widetilde{V},$$

since $\bigcup_{n\geq 0}2^nU_\nu=X$. By (15), the compact set $\overline{\widetilde{K}(U)}\subset\frac{1}{2}\widetilde{V}$ is covered by this family of open sets, hence there exists $N(\nu)\in\mathbb{Z}_{\geq 0}$ such that

$$\overline{\widetilde{K}(U)}\subset\widetilde{T}\big(2^{N(\nu)}U_\nu\cap\tfrac{1}{2}U\big). \tag{17}$$

Fix such $N(\nu)$'s inductively so that

$$1 < N(0) < N(1) < \cdots$$

holds.

Now let us show that $\widetilde{f}\colon X \to Y/S$ is surjective. Since $\widetilde{V}$ is a neighborhood of 0 in the TVS Y/S, it is absorbing (Mini-Exercise I.6.1 (ii)). Hence it suffices to show that every $y_0 \in \widetilde{V}$ lies in Image $\widetilde{f}$.

Fix $y_0 \in \widetilde{V}$. Choose $x_0 \in U$ with $\widetilde{T}(x_0) = y_0$. Set

$$y_1 := y_0 - \widetilde{f}(x_0) = -\widetilde{K}(x_0) \in \widetilde{K}(U) \subset \frac{1}{2}\widetilde{V},$$

where the last inclusion follows from (15). Therefore one can take $x_1 \in \frac{1}{2}U$ such that $\widetilde{T}(x_1) = y_1$.

Inductively, assume that for some $n \geq 0$ we have constructed $x_0, x_1, \ldots, x_n \in U$ and $y_0, y_1, \ldots, y_n \in \widetilde{V}$ with the following properties for any $m \in \{0, 1, \ldots, n\}$ and $\nu \in \mathbb{Z}_{\geq 0}$:

- $x_m \in 2^{-m}U$;
- If $N(\nu) < m \leq N(\nu+1)$, $x_m \in 2^{N(\nu)+1-m}U_\nu \cap 2^{-m}U$;
- $\widetilde{T}(x_m) = y_m$ $(\in 2^{-m}\widetilde{V})$;
- If $m > 0$, $y_m = y_{m-1} - \widetilde{f}(x_{m-1})$ $(= -\widetilde{K}(x_{m-1}))$.

Then $x_n \in 2^{-n}U$ implies $\widetilde{K}(x_n) \in 2^{-n}\widetilde{K}(U)$, hence

$$y_{n+1} := -\widetilde{K}(x_n) \in 2^{-n}\widetilde{K}(U) \subset 2^{-n-1}\widetilde{V}$$

again by (15). Choose $x_{n+1} \in 2^{-(n+1)}U$ such that $\widetilde{T}(x_{n+1}) = y_{n+1}$; this is possible because $y_{n+1} \in 2^{-(n+1)}\widetilde{V} = \widetilde{T}(2^{-(n+1)}U)$. If $n+1 > N(0)$, let ν be the unique non-negative integer such that $N(\nu) < n+1 \leq N(\nu+1)$. In this case, since $y_{n+1} \in 2^{-n}\widetilde{K}(U)$, by (17), we can choose x_{n+1} so that

$$x_{n+1} \in 2^{N(\nu)-n}U_\nu \cap 2^{-n-1}U \quad \text{and} \quad \widetilde{T}(x_{n+1}) = y_{n+1}.$$

Therefore we obtain sequences $\{x_m\}_{m\in\mathbb{Z}_{\geq 0}}$ and $\{y_m\}_{m\in\mathbb{Z}_{\geq 0}}$ that satisfy the properties above.

As

$$\begin{aligned} x_{N(\nu)+1} + \cdots + x_{N(\nu+1)} &\in \big(2^{N(\nu)+1-(N(\nu)+1)} + \cdots + 2^{N(\nu)+1-N(\nu+1)}\big)U_\nu \\ &\subset (1 + 2^{-1} + 2^{-2} + \cdots)U_\nu \subset 2U_\nu, \end{aligned}$$

from (16) it follows that the series $\sum_{n\geq 0} x_n$ is Cauchy in X. As X is complete, it converges to some $x \in X$. As $\overline{\widetilde{K}(U)}$ is compact, $\{y_m\}_{m\in\mathbb{Z}_{\geq 0}}$ admits a convergent subsequence, whose limit has to be $0 \in Y/S$ since $y_{m+1} \in 2^{-m}\overline{\widetilde{K}(U)}$ for any m. Therefore, from the equation

$$y_{n+1} = y_0 - \widetilde{f}(x_0 + \cdots + x_n) \qquad (\forall n \geq 0)$$

we have that $\widetilde{f}(x) = y_0$. Thus $y_0 \in \operatorname{Image} \widetilde{f}$, and hence $\widetilde{f}$ is surjective.

The surjectivity of $\widetilde{f} = q \circ f$ means $Y = \operatorname{Image} f + S$. Hence $\dim\bigl(Y/\operatorname{Image} f\bigr) \leq \dim S < \infty$. Thus the assertion follows from Lemma I.6.55. □

Mini-Exercise I.6.57. Let $f_+ \colon C_+^\flat \to C_+^\natural$, $g_+ \colon C_+^\natural \to C_+^\sharp$, $f_- \colon C_-^\flat \to C_-^\natural$, and $g_- \colon C_-^\natural \to C_-^\sharp$ be continuous linear maps between Fréchet spaces such that $g_\pm \circ f_\pm = 0$. Let $r^\bullet \colon C_+^\bullet \to C_-^\bullet$ ($\bullet \in \{\flat, \natural, \sharp\}$) be continuous linear maps which make the diagram

$$\begin{array}{ccccc} C_+^\flat & \xrightarrow{f_+} & C_+^\natural & \xrightarrow{g_+} & C_+^\sharp \\ \downarrow{\scriptstyle r^\flat} & & \downarrow{\scriptstyle r^\natural} & & \downarrow{\scriptstyle r^\sharp} \\ C_-^\flat & \xrightarrow{f_-} & C_-^\natural & \xrightarrow{g_-} & C_-^\sharp \end{array}$$

commutative. Assume that the induced map

$$\frac{\operatorname{Ker} g_+}{\operatorname{Image} f_+} \to \frac{\operatorname{Ker} g_-}{\operatorname{Image} f_-}$$

is surjective, and that $r^\natural$ is compact.
(*i*) Show that the map $T \colon \operatorname{Ker} g_+ \oplus C_-^\flat \to \operatorname{Ker} g_-$ defined by

$$T(x, y) := -r^\natural(x) + f_-(y)$$

is surjective.
(*ii*) Show that $\operatorname{Ker} g_-/\operatorname{Image} f_-$ is a finite-dimensional Hausdorff TVS.

7 From algebraic topology

The basic terminology and standard facts from algebraic topology briefly reviewed in this section can be found, for example, in [Dol], [Ha1], and [Lee-Jo1]. The reader is assumed to be familiar with some fundamental notions such as *homotopy*, *contractible spaces*, *deformation retracts*, the *fundamental group*, *simply connected spaces*, and *path connectedness* (In this note, we adopt the convention that a path-connected space is always assumed to be non-empty), together with the basic results related to them. Unless otherwise stated, for a topological space X and a subset $A \subset X$, we always regard A as a topological space endowed with the relative topology.

7.1 Fundamental groups and covering spaces

Let X be a topological space. We begin by fixing the notation concerning paths in X that will be used throughout this note.

A *path* in X is a continuous map $\gamma \colon I \longrightarrow X$ from a bounded closed interval $I = [a, b] \subset \mathbb{R}$ of positive length, where $a, b \in \mathbb{R}$ and $a < b$. The point $\gamma(a)$ is called the *initial point* of

γ, and $\gamma(b)$ is called the *terminal point* of γ. A path whose initial point and terminal point coincide is called a *loop*. In what follows, when we say that "$\gamma\colon I \to X$ is a path in X", it is understood that I is a bounded closed interval of positive length in $\mathbb{R}$. In most topological arguments, the explicit choice of the interval I is inessential, since any two bounded closed intervals of positive length are homeomorphic to each other via affine transformations. For this reason, whenever it is harmless or conceptually convenient, we shall fix the domain of a path to be $I = [0, 1]$.

Two given paths γ_1 and γ_2 in X are said to be *homotopic* if, after identifying their domains with a common interval $I \subset \mathbb{R}$ as above, there exists a homotopy $H\colon I \times [0, 1] \longrightarrow X$ connecting them (i.e., $H(-, 0) = \gamma_1$ and $H(-, 1) = \gamma_2$). Assume that $I = [a, b]$, where $a, b \in \mathbb{R}$ and $a < b$. If $\gamma_1(a) = \gamma_2(a)$ and there exists such a homotopy H with $H(a, -)$ being a constant map, then γ_1 and γ_2 are said to be *homotopic relative to the initial point*. Similarly, if $\gamma_1(b) = \gamma_2(b)$ and $H(b, -)$ is constant, we say that they are *homotopic relative to the terminal point*. If both conditions are satisfied, namely if $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$ and there exists a homotopy H such that both $H(a, -)$ and $H(b, -)$ are constant maps, then γ_1 and γ_2 are said to be *homotopic relative to their endpoints*.

It is important to consider an operation that combines two given paths to form a new one. However, the appropriate notation for this operation depends on the context and on the equivalence relation under which paths are identified. In this subsection, which concerns the fundamental group, it is natural to consider such an operation only when the terminal point of one path coincides with the initial point of the other: For paths γ_1 and γ_2 in X such that the terminal point of γ_1 coincides with the initial point of γ_2, after identifying the domain of γ_1 with $[0, 1]$ and that of γ_2 with $[1, 2]$, we define a path

$$[0, 2] \ni t \longmapsto \begin{cases} \gamma_1(t) & \text{if } t \in [0, 1] \\ \gamma_2(t) & \text{if } t \in [1, 2] \end{cases},$$

which will be denoted by $\gamma_1 *_{\mathrm{path}} \gamma_2$ in this note (Although this operation is often denoted simply by $*$ in the literature, we avoid this notation since the symbol $*$ will be used for several other purposes in this note. Other conventions, such as multiplicative notation or composition-like notation, also appear in the literature, but we do not adopt them in order to avoid possible confusion. Depending on the context, we may implicitly compose the paths with suitable orientation-preserving affine homeomorphisms $I \to [0, 1]$, $I \to [1, 2]$, $I \to [0, 2]$, or their inverses so as to regard $\gamma_1 *_{\mathrm{path}} \gamma_2$ as being defined or interpreted on a fixed interval I). Note that this operation is not strictly associative at the level of paths, due to the choice of parametrizations, although it becomes associative after passing to appropriate homotopy classes. Even when γ_1 and γ_2 are loops with the same initial and terminal points, the paths $\gamma_1 *_{\mathrm{path}} \gamma_2$ and $\gamma_2 *_{\mathrm{path}} \gamma_1$ should in general be distinguished (see Example I.7.7 for example). In this sense, the operation $*_{\mathrm{path}}$ should be regarded as non-commutative in general.

By contrast, in the context of homology, which will be discussed in the next subsection, paths are viewed as "1-chains". In that setting, the initial and terminal points of paths play no role in defining their "addition", and the order of addition is irrelevant (For example, when X is a Euclidean space or a smooth manifold, this can be understood by thinking of

smooth paths as integration paths for 1-forms). In this homological context, the sum of two paths will be simply written as $\gamma_1 + \gamma_2$ $(= \gamma_2 + \gamma_1)$, or rather identified with its appropriate equivalence class.

From the same point of view, for a path $\gamma\colon [0,1] \to X$, the path

$$[0,1] \ni t \longmapsto \gamma(1-t) \in X,$$

which is often denoted by (or identified with) $-\gamma$ in the homological context, will be denoted by $\mathrm{Inv}_{\mathrm{path}}(\gamma)$ in this subsection.

It is well known that for a non-empty topological space X and a point $x \in X$, the set of homotopy classes of loops $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1) = x$, taken relative to their endpoints, admits a natural group structure (The unit element is given by the constant map valued at x, the product is induced by the operation $*_{\mathrm{path}}$, and the inverse is induced by $\mathrm{Inv}_{\mathrm{path}}$). This group is called the *fundamental group* of X (based at x) and is denoted by $\pi_1(X,x)$. When the choice of the base point $x \in X$ is irrelevant in the given context (for instance, when X is path-connected; see Mini-Exercise I.7.1 (*iii*) below), we simply write $\pi_1(X,*)$ (We deliberately avoid the notation $\pi_1(X)$, since in some contexts it is used to denote the group isomorphism class of the fundamental group, and not the fundamental group itself). For a loop γ of X based at $x \in X$ (i.e., whose initial/terminal point is x), we denote by $[\gamma]$ the class of $\pi_1(X,x)$ represented by γ. For a continuous map $f\colon X \to Y$ between topological spaces, we denote the induced group homomorphism by $f_*\colon \pi_1(X,*) \longrightarrow \pi_1(Y,*)$: Here the symbol $*$ is used in the sense as above, namely to indicate that the choice of the base point is not made explicit. If one wishes to specify the base point, then for $x \in X$ the induced homomorphism is written as $f_*\colon \pi_1(X,x) \longrightarrow \pi_1(Y,f(x))$. Moreover, if two continuous maps $f, g\colon X \to Y$ are homotopic through a homotopy that keeps a chosen base point fixed, then the induced homomorphisms on fundamental groups coincide: i.e., $f_* = g_*$ in this case. In particular, it follows that every contractible topological space is simply connected.

Mini-Exercise I.7.1. Let X be a topological space.
(*i*) Show that if X is path-connected, then X is connected.
(*ii*) Show that if X is locally path-connected and connected, then X is path-connected.
(*iii*) Show that if X is path-connected, then $\pi_1(X,*)$ does not depend on the choice of the base point, up to group isomorphism.
(*iv*) Give an example of a locally path-connected and path-connected space X such that $\pi_1(X,*)$ is non-commutative.

Let X be a locally path-connected topological space. A map $p\colon Z \to X$ from a topological space Z is called a *covering map* if p is continuous and surjective, and if every point $x \in X$ is *evenly covered by* p; that is, there exists an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of (open and connected) subsets of Z, and such that the restriction of p to each connected component of $p^{-1}(U)$ is a homeomorphism onto U. In this case, Z (or, more precisely, the pair (Z,p)) is called a *covering space* of X. In this note, we will also use the shorter expression "$p\colon Z \to X$ is a covering of X."

Note that in some references it is additionally assumed that Z is connected, which we do *not* impose in this note (When Z is connected, we explicitly say that Z is a *connected covering space* of X, or that "$p\colon Z \to X$ is a connected covering of X").

EXAMPLE I.7.2. Let X be a connected and locally path-connected topological space. Consider the map

$$p\colon X \sqcup X \longrightarrow X$$

whose restriction to each copy of X is the identity map. Then p is a covering map. However, since $X \sqcup X$ is not connected, this covering space would not be regarded as a covering space in some references that require the total space to be connected. □

EXAMPLE I.7.3. Let $X := \mathrm{U}(1) \subset \mathbb{C}$ be the unit circle. The map

$$\mathbb{R} \ni x \longmapsto e^{2\pi\sqrt{-1}\,x} \in X$$

is a covering map. On the other hand, the restriction of this map to $(0, 2)$ is continuous, surjective, and *locally homeomorphic* (that is, for each $x \in (0, 2)$, its restriction to a sufficiently small open neighborhood of x is a homeomorphism onto its image), but it is not a covering map. □

Note also that in the context of complex analytic geometry, the notion of a *ramified covering space* (or *branched covering space*) appears. Such a space is not a covering space in the sense defined here in general. In that context, a covering space in the present sense is often called an *unramified covering space*.

The following theorem is one of the most fundamental and important results relating fundamental groups and covering spaces.

THEOREM I.7.4 ([Lee-Jo1, Theorem 11.12 and Theorem 11.18]). *Let X and Y be connected locally path-connected topological spaces, $p\colon Z \to X$ a connected covering of X, $f\colon Y \to X$ a continuous map, and let $x_0 \in X$, $y_0 \in f^{-1}(x_0)$, and $z_0 \in p^{-1}(x_0)$ be points. Then the following are equivalent:*
(i) There exists a continuous map $F\colon Y \to Z$ such that $F(y_0) = z_0$ and $p \circ F = f$.
(ii) $\mathrm{Image}\big(f_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)\big) \subset \mathrm{Image}\big(p_*\colon \pi_1(Z, z_0) \to \pi_1(X, x_0)\big)$.
Moreover, if these conditions are satisfied, the map F in (i) is unique (and is called a lift *of f).*

PROOF. If a map F as in (i) exists, then $f_* = (p \circ F)_* = p_* \circ F_*$ and hence (ii) follows immediately. Therefore it suffices to construct F uniquely under the assumption (ii). Here we only sketch the idea of the construction and refer to [Lee-Jo1, Theorem 11.12–11.18] for the details. For a point $y \in Y$, choose a path $\gamma\colon I \to Y$ with initial point y_0 and terminal point y. Consider the path $f \circ \gamma\colon I \to X$. One attempts to define $F(y)$ as the terminal point of a lift of this path starting at z_0. Since X is locally path-connected and $x_0 = f(y_0)$ is evenly covered by p, it is easy to check that $F(y)$ is uniquely determined in this way when y is sufficiently close to y_0. In general, one subdivides the interval I into sufficiently small subintervals and applies the same argument step by step. Condition (ii) is used to verify

that the resulting point $F(y)$ does not depend on the choice of the path γ (Note that p_* is injective for a connected covering $p\colon Z \to X$, which is also shown by a similar argument by lifting the homotopy map, see [Ha1, Proposition 1.31]). □

As an application of Theorem I.7.4, here let us determine the fundamental group of the unit circle $\mathrm{U}(1) \subset \mathbb{C}$. Let $p\colon \mathbb{R} \longrightarrow \mathrm{U}(1)$ be as in Example I.7.3. Let $\gamma\colon [0,1] \to \mathrm{U}(1)$ be a loop based at $1 \in \mathrm{U}(1)$. Denote by $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}$ the lift of γ with $\widetilde{\gamma}(0) = 0$, whose existence and uniqueness is ensured by Theorem I.7.4. Since $\gamma(1) = \gamma(0) = 1$, we have

$$e^{2\pi\sqrt{-1}\,\widetilde{\gamma}(1)} = p(\widetilde{\gamma}(1)) = \gamma(1) = 1,$$

hence $\widetilde{\gamma}(1) \in \mathbb{Z}$. We define the *winding number* of γ by

$$\mathrm{wind}(\gamma) := \widetilde{\gamma}(1) \in \mathbb{Z}.$$

One checks that $\mathrm{wind}(\gamma)$ depends only on the endpoint-fixed homotopy class $[\gamma] \in \pi_1(\mathrm{U}(1), 1)$: Indeed, for loops $\gamma_1, \gamma_2\colon [0,1] \to \mathrm{U}(1)$ based at 1 such that $[\gamma_1] = [\gamma_2]$, their homotopy $H\colon [0,1] \times [0,1] \to \mathrm{U}(1)$ also admits a lift again by Theorem I.7.4, from which we have that $\mathrm{wind}(\gamma_1) = \mathrm{wind}(\gamma_2)$. The induced map

$$\mathrm{wind}\colon \pi_1(\mathrm{U}(1), 1) \longrightarrow \mathbb{Z}$$

is a group isomorphism (see Mini-Exercise I.7.5 below). In particular, $\pi_1(\mathrm{U}(1), 1) \cong \mathbb{Z}$.

Mini-Exercise I.7.5. Show that the map $\mathrm{wind}\colon \pi_1(\mathrm{U}(1), 1) \to \mathbb{Z}$ is a group isomorphism.

Let X be a connected locally path-connected topological space and $p\colon Z \to X$ a connected covering of X. As mentioned in the proof of Theorem I.7.4, the induced homomorphism $p_*\colon \pi_1(Z, *) \to \pi_1(X, *)$ is injective. Hence we may (and often will) regard $\pi_1(Z, *)$ as a subgroup of $\pi_1(X, *)$ via p_*.

Example I.7.6. For an integer $n \in \mathbb{Z}_{\geq 1}$, consider the map

$$q_n\colon \mathrm{U}(1) \longrightarrow \mathrm{U}(1), \qquad q_n(z) := z^n.$$

Then q_n is a connected covering of degree n. Identifying $\pi_1(\mathrm{U}(1), 1)$ with $\mathbb{Z}$ via the winding number, the induced homomorphism $(q_n)_*\colon \pi_1(\mathrm{U}(1), 1) \to \pi_1(\mathrm{U}(1), 1)$ corresponds to multiplication by n on $\mathbb{Z}$. In particular, $\mathrm{Image}\,(q_n)_* = n\mathbb{Z} \subset \mathbb{Z}$. □

Let X be a connected locally path-connected topological space. A covering $p\colon Z \to X$ of X is said to be a *universal covering* if Z is connected and simply connected. Every connected, locally path-connected, and locally simply connected topological space X has a universal covering: Indeed, one can construct such a space by considering

$$\widetilde{X} := \{\ell\colon [0,1] \to X \mid \ell \text{ is continuous and } \ell(0) = x_0\}/\sim,$$

where $x_0 \in X$ is a fixed base point and "$\sim$" denotes homotopy equivalence relative to their endpoints, together with the map $\pi\colon \widetilde{X} \to X$ defined by letting $\pi([\ell]) := \ell(1)$. See (the proof

of) [Lee-Jo1, Theorem 11.43] for the details. Note that, by construction, one can naturally identify the fiber $\pi^{-1}(x_0)$ with the fundamental group $\pi_1(X, x_0)$. Note also that, for any loop γ of X based at x_0, the map

$$\widetilde{X} \ni [\ell] \longmapsto [\gamma *_{\mathrm{path}} \ell] \in \widetilde{X}$$

defines a natural action of the group $\pi_1(X, x_0)$ on $\widetilde{X}$.

Let $p\colon Z \to X$ and $p'\colon Z' \to X$ be connected coverings, and let $x_0 \in X$, $z_0 \in p^{-1}(x_0)$, and $z_0' \in (p')^{-1}(x_0)$ be points. Assume that $p'_*\big(\pi_1(Z', z_0')\big) \subset p_*\big(\pi_1(Z, z_0)\big)$ holds as subgroups of $\pi_1(X, x_0)$. Then, by applying Theorem I.7.4, one uniquely obtains a continuous map $\varphi\colon Z' \to Z$ such that $p \circ \varphi = p'$ and $\varphi(z_0') = z_0$. It can be checked that such a map φ is also a covering map; see [Lee-Jo1, Proposition 11.36].

In particular, for a connected, locally path-connected, and locally simply connected topological space X, a connected covering $p\colon Z \to X$, and the universal covering $\pi\colon \widetilde{X} \to X$ constructed above, one has the following:

- There exists a covering $\widetilde{p}\colon \widetilde{X} \to Z$ such that $p \circ \widetilde{p} = \pi$. If Z is also a universal covering, then $\widetilde{p}$ is a homeomorphism (consider the case where $(Z', p') = (\widetilde{X}, \pi)$).

- The group

  $$\mathrm{Aut}(Z/X) := \{\varphi\colon Z \to Z \mid \varphi \text{ is a homeomorphism and } p \circ \varphi = p\}$$

  of *deck transformations* of $p\colon Z \to X$ acts transitively on the fiber $p^{-1}(x_0)$ if and only if $p_*(\pi_1(Z, z_0))$ is a normal subgroup of $\pi_1(X, x_0)$ (consider the case where $(Z', p') = (Z, p)$; see (the proof of) [Ha1, Proposition 1.39 (a)], [Lee-Jo1, Theorems 11.40, 12.4, Corollary 12.5] for details). The covering $p\colon Z \to X$ is said to be *normal* or *Galois* if these conditions are satisfied.

- The group $\mathrm{Aut}(\widetilde{X}/X)$ can be naturally identified with $\pi_1(X, x_0)$ (consider the case where $(Z', p') = (Z, p) = (\widetilde{X}, \pi)$).

Based on the facts stated above, one may identify the universal covering space of X with the covering $\pi\colon \widetilde{X} \to X$ constructed above.

At this point, the reader is encouraged to review the correspondence between (normal) subgroups of $\mathrm{Aut}(\widetilde{X}/X)$ and (normal) covering spaces of X, known as the *Galois correspondence for coverings*. See [Lee-Jo1, Chapter 12] and [Ha1, § 1.3] for details.

Example I.7.7. Let $n \in \mathbb{Z}_{\geq 1}$. Consider the map

$$p\colon \mathbb{R}^n \longrightarrow \mathrm{U}(1)^n, \qquad p(x_1, x_2, \ldots, x_n) := \big(e^{2\pi\sqrt{-1}x_1}, e^{2\pi\sqrt{-1}x_2}, \ldots, e^{2\pi\sqrt{-1}x_n}\big).$$

This map is a covering map. Since $\mathbb{R}^n$ is contractible, it is simply connected. Therefore $p\colon \mathbb{R}^n \to \mathrm{U}(1)^n$ is the universal covering of $\mathrm{U}(1)^n$. As p induces a homeomorphism $\mathbb{R}^n/\mathbb{Z}^n \to \mathrm{U}(1)^n$ and the deck transformations of p are given by translations

$$(x_1, x_2, \ldots, x_n) \longmapsto (x_1 + m_1, x_2 + m_2, \ldots, x_n + m_n), \qquad (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n,$$

we have that

$$\mathrm{Aut}(\mathbb{R}^n/\mathrm{U}(1)^n) \cong \pi_1(\mathrm{U}(1)^n, *) \cong \mathbb{Z}^n.$$

Note that one may take $D = [0,1)^n$ as a *fundamental domain* for the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ (i.e., a subset $D \subset \mathbb{R}^n$ such that every orbit meets D in exactly one point, except possibly on the boundary). In particular, in the case $n = 2$, a fundamental domain can be chosen to be a square. From this point of view, in what follows we sometimes regard a "doubly periodic function" on $\mathbb{R}^2$ (or a function on $[0,1] \times [0,1]$ satisfying periodic boundary conditions) as a function on $\mathrm{U}(1) \times \mathrm{U}(1)$. Note also that one can apply the *Seifert–Van Kampen theorem* to determine the fundamental groups of all compact orientable surfaces, by realizing them as a connected sum of finitely many copies of $\mathrm{U}(1) \times \mathrm{U}(1)$: They are known to be generated by $2g$ generators $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ with a single defining relation

$$\prod_{j=1}^{g} \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} = 1$$

for each integer $g \in \mathbb{Z}_{\geq 0}$. Refer to §8.4 or [Lee-Jo1, Chapter 10] for the details. □

7.2 Singular homology

The reader is assumed to be familiar with the definition of *singular homology* for a topological space X. For the sake of fixing notation, we briefly recall the basic terminology.

For each $q \in \mathbb{Z}_{\geq 0}$, the free abelian group generated by the set of all continuous maps $\sigma\colon (\mathrm{Simplex})_q \to X$ is denoted by $C_q^{\mathrm{sing}}(X)$, and its elements are called *singular q-chains* of X (The generating maps σ are called *singular q-simplices* of X), where

$$(\mathrm{Simplex})_q := \{(x_0, x_1, \dots, x_q) \in (\mathbb{R}_{\geq 0})^{q+1} \mid x_0 + x_1 + \cdots + x_q = 1\}$$

is the *standard q-simplex*. As in the case of paths discussed in the previous subsection, the specific choice of the model for the standard simplex is not essential: any space obtained from $(\mathrm{Simplex})_q$ by an affine homeomorphism may be used equivalently as the domain of a singular simplex. Accordingly, the expression on the right-hand side above should be regarded not as a rigid definition, but rather as a convenient representative of this equivalence class. Indeed, in what follows, we often identify $(\mathrm{Simplex})_q$ with a face of $(\mathrm{Simplex})_{q+1}$ via a suitable homeomorphism. When it is necessary to specify which face is chosen and how the identification is made, we use the following notation: For $n, m \in \mathbb{Z}_{\geq 0}$, and affinely independent points $v_0, v_1, \dots, v_m$ in $\{(x_0, x_1, \dots, x_n) \in (\mathbb{R}_{\geq 0})^{n+1} \mid x_0 + x_1 + \cdots + x_n = 1\}$, we write

$$\mathrm{Conv}(v_0, v_1, \dots, v_m) := \left\{ \sum_{j=0}^{m} t_j v_j \;\middle|\; (t_0, t_1, \dots, t_m) \in (\mathbb{R}_{\geq 0})^{m+1},\ \sum_{j=0}^{m} t_j = 1 \right\},$$

and

$$\mathrm{Conv}^{\circ}(v_0, v_1, \dots, v_m) := \left\{ \sum_{j=0}^{m} t_j v_j \;\middle|\; (t_0, t_1, \dots, t_m) \in (\mathbb{R}_{>0})^{m+1},\ \sum_{j=0}^{m} t_j = 1 \right\}.$$

Here an ordering $(v_0, v_1, \dots, v_m)$ is important, since it is interpreted geometrically as the choice of an *orientation* of the corresponding singular simplex.

REMARK I.7.8. The standard n-simplex is often denoted by Δ^n in the literature. However, in this note, the symbol Δ^n is reserved for the polydisc

$$\{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n \mid |z_j| < 1 \text{ for all } j = 1, 2, \dots, n\}.$$

For this reason, we avoid using the notation Δ^n for the standard n-simplex. □

For $q \in \mathbb{Z}_{>0}$, the boundary homomorphism

$$\partial_q \colon C_q^{\mathrm{sing}}(X) \longrightarrow C_{q-1}^{\mathrm{sing}}(X)$$

is defined on a singular q-simplex $\sigma \colon (\mathrm{Simplex})_q = \mathrm{Conv}(v_0, v_1, \dots, v_q) \to X$ by

$$\partial_q(\sigma) := \sum_{j=0}^{q} (-1)^j \, \sigma \circ (\iota_j \colon (\mathrm{Simplex})_{q-1} \to \mathrm{Conv}(v_0, \dots, \widehat{v_j}, \dots, v_q)),$$

where $(v_0, \dots, \widehat{v_j}, \dots, v_q)$ denotes the ordered q-tuple obtained from $(v_0, v_1, \dots, v_q)$ by omitting v_j, and $\iota_j \colon (\mathrm{Simplex})_{q-1} \to \mathrm{Conv}(v_0, \dots, \widehat{v_j}, \dots, v_q)$ is the affine homeomorphism that preserves the displayed vertex order and is given by

$$\iota_j(x_0, x_1, \dots, x_{q-1}) := \sum_{k=0}^{j-1} x_k v_k + \sum_{k=j}^{q-1} x_k v_{k+1}$$

(As mentioned above, we may use any affinely homeomorphic model of the standard simplex, provided that the vertex orderings and face maps are transported consistently under the affine identifications).

When no confusion is likely to arise, we often suppress the degree and simply write ∂ for the boundary homomorphism ∂_q.

EXAMPLE I.7.9. A point $x \in X$ can be naturally identified with the singular 0-simplex $\sigma_x \colon (\mathrm{Simplex})_0 \to X$ defined by $\mathrm{Image}\, \sigma_x = \{x\}$. A path $\gamma \colon [0, 1] \to X$ of X can be regarded as a singular 1-simplex. With this identification, the boundary of γ is given by

$$\partial\gamma = -\gamma(0) + \gamma(1),$$

which should be compared with the usual interpretation of the boundary of an oriented interval. In particular, we have $\partial\gamma = 0 \in C_0^{\mathrm{sing}}(X)$ if γ is a loop. □

By definition, one can easily show that $\partial_{q-1} \circ \partial_q = 0$ for any $q \in \mathbb{Z}_{\geq 1}$ (see [Lee-Jo1, Lemma 13.1]). Therefore

$$\cdots \longrightarrow C_{n+1}^{\mathrm{sing}}(X) \xrightarrow{\partial_{n+1}} C_n^{\mathrm{sing}}(X) \xrightarrow{\partial_n} C_{n-1}^{\mathrm{sing}}(X) \longrightarrow \cdots \longrightarrow C_1^{\mathrm{sing}}(X) \xrightarrow{\partial_1} C_0^{\mathrm{sing}}(X) \to 0$$

is a chain complex of $\mathbb{Z}$-modules. The homology group

$$H_q(X) := H_q\big(C_\bullet^{\mathrm{sing}}(X)\big) = \begin{cases} \operatorname{Ker}\partial_q/\operatorname{Image}\partial_{q+1} & \text{if } q > 0 \\ C_0^{\mathrm{sing}}(X)/\operatorname{Image}\partial_1 & \text{if } q = 0 \end{cases}$$

of the chain complex $C_\bullet^{\mathrm{sing}}(X)$ is called the *q-th singular homology group* of X. An element of $\operatorname{Ker}\partial_q$ is called a *q-cycle*, and an element of $\operatorname{Image}\partial_{q+1}$ is called a *q-boundary*. See [Dol, Chapter III], [Ha1, § 2.1], and [Lee-Jo1, Chapter 13] for further details on singular homology groups. The following properties are fundamental:

- For a continuous map $f\colon X \to Y$ between topological spaces, the morphism of complexes $C_\bullet^{\mathrm{sing}}(X) \to C_\bullet^{\mathrm{sing}}(Y)$ can be naturally defined (by considering the composition of each $\sigma\colon (\mathrm{Simplex})_\bullet \to X$ and f), from which the group homomorphisms

  $$f_*\colon H_\bullet(X) \longrightarrow H_\bullet(Y)$$

  are induced (see [Dol, Chapter VI 7.2]).

- If two continuous maps $f, g\colon X \to Y$ are homotopic, then the induced homomorphisms on homology coincide; that is, $f_* = g_*$ (see [Dol, Chapter VI 7.4], [Lee-Jo1, Corollary 13.8]).

- For a topological space X and its open subsets $U, V \subset X$ with $X = U \cup V$, there exists a long exact sequence

  $$\begin{aligned} &\cdots \longrightarrow H_2(U \cap V) \longrightarrow H_2(U) \oplus H_2(V) \longrightarrow H_2(X) \qquad , \\ &\longrightarrow H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(X) \\ &\longrightarrow H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(X) \longrightarrow 0 \end{aligned}$$

  which is called the *Mayer–Vietoris sequence* (see [Dol, Chapter VI 7.6], [Lee-Jo1, Theorem 13.16]). In particular, if $U \cap V = \emptyset$, then $H_\bullet(X) \cong H_\bullet(U) \oplus H_\bullet(V)$.

Remark I.7.10. In the above setting, when one actually performs computations, it is usually harmless to think of the Mayer–Vietoris sequence as being induced from the following sequence of chain complexes of $\mathbb{Z}$-modules, *as if* it were a short exact sequence:

$$0 \longrightarrow C_\bullet^{\mathrm{sing}}(U \cap V) \xrightarrow{\iota_\bullet} C_\bullet^{\mathrm{sing}}(U) \oplus C_\bullet^{\mathrm{sing}}(V) \xrightarrow{\pi_\bullet} C_\bullet^{\mathrm{sing}}(X),$$

where $\iota_\bullet(\sigma) := (\sigma, \sigma)$ and $\pi_\bullet(\xi, \eta) := \xi - \eta$. However, the map $\pi_\bullet$ is not surjective in general, and hence this sequence is not a short exact sequence at the level of chain complexes.

For example, let $X = \mathbb{R}$, $U = (-\infty, 3)$, and $V = (1, \infty)$. The singular 1-simplex $[0, 1] \to X$, $t \mapsto 4t$, does not lie in the image of π_1. This issue is resolved by the standard subdivision

technique: although a given simplex may not lie in the image of $\pi_\bullet$, it can be subdivided into finitely many smaller simplices that do. At the level of homology, this subtlety causes no problem and leads to the long exact Mayer–Vietoris sequence. In the above example, this corresponds to subdividing the interval $[0,1]$ into $[0,1/2]$ and $[1/2,1]$, so that the resulting simplices can be regarded as coming from U and V. Refer to [Ha1, §2.2] and [Lee-Jo1, Chapter 13] for the precise arguments.

As the purpose of this section is merely to review some basic facts from algebraic topology, we do not pursue a more abstract treatment of the phenomenon, which can be regarded as a kind of "essential surjectivity" (which is used only in an informal sense, and does *not* refer to any established notion). Nevertheless, it may be helpful for the reader to keep in mind that such techniques as subdivision/refinement (and considering the inductive systems and their inductive limits) are useful for revealing such "essential surjectivity": In particular, this perspective may serve as a mild conceptual preparation for later topics in this note, such as the notion of surjectivity for morphisms of sheaves and the definition of Čech cohomology. □

Example I.7.11. Let X be a non-empty path-connected topological space. As explained in Example I.7.9, each point $x \in X$ can be regarded as an element of $C_0^{\mathrm{sing}}(X)$. For any two points $x, y \in X$, by assumption, there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. By Example I.7.9, we have

$$\partial\gamma = -\{x\} + \{y\} \in \operatorname{Image}\partial_1.$$

Hence any two generators $\{x\}, \{y\} \in C_0^{\mathrm{sing}}(X)$ become equal in the quotient by $\operatorname{Image}\partial_1$. It follows that

$$H_0(X) = \frac{\bigoplus_{x\in X}\mathbb{Z}\cdot\{x\}}{\operatorname{Image}\partial_1} \cong \mathbb{Z}.$$

□

Example I.7.12. Let X be a non-empty topological space. As observed in Example I.7.9, any loop in X can be regarded as a singular 1-simplex. Let $\gamma\colon [0,1] \to X$ be a loop based at $x_0 \in X$ such that $[\gamma] = 1 \in \pi_1(X, x_0)$. By definition, there exists a homotopy relative to the endpoints $H\colon [0,1]\times[0,1] \to X$ such that

$$H(-,0) = \gamma, \qquad H(-,1) \equiv x_0, \qquad H(0,-) \equiv x_0, \qquad H(1,-) \equiv x_0.$$

Set

$$S := \{(s,t) \in (\mathbb{R}_{\geq 0})^2 \mid s + t \leq 1\} \subset \mathbb{R}^2.$$

Let

$$\alpha\colon (\mathrm{Simplex})_2 \longrightarrow S, \qquad \alpha(x_0,x_1,x_2) := (x_1,x_2),$$

which is an affine homeomorphism. Define a continuous map $K\colon S \to X$ by

$$K(s,t) := \begin{cases} H\big(s/(1-t),t\big) & (t<1),\\ x_0 & ((s,t)=(0,1)). \end{cases}$$

The continuity at $(0,1)$ follows from the continuity of H and its constant value on the compact top edge $[0,1]\times\{1\}$. Define a singular 2-simplex $\sigma\colon (\mathrm{Simplex})_2 \to X$ by

$$\sigma := K \circ \alpha.$$

For the ordered face maps $\iota_0, \iota_1, \iota_2$ fixed above, one has

$$\sigma\circ\iota_0 = \ell_{x_0}, \qquad \sigma\circ\iota_1 = \ell_{x_0}, \qquad \sigma\circ\iota_2 = \gamma,$$

where ℓ_{x_0} denotes the constant singular 1-simplex with value x_0. Hence

$$\partial\sigma = (\sigma\circ\iota_0) - (\sigma\circ\iota_1) + (\sigma\circ\iota_2) = \ell_{x_0} - \ell_{x_0} + \gamma = \gamma$$

in $C_1^{\mathrm{sing}}(X)$, and γ is a 1-boundary. Therefore we have a natural map $\pi_1(X,x_0)\to H_1(X)$, which can be checked to be a group homomorphism. If X is path-connected, it is known that this homomorphism is surjective, and its kernel coincides with the commutator subgroup of $\pi_1(X,x_0)$ ([Ha1, Theorem 2.A.1]). In particular, for a path-connected space X, the first singular homology group $H_1(X)$ is naturally isomorphic to the abelianization $\pi_1(X,x_0)^{\mathrm{ab}}$ of the fundamental group. □

Mini-Exercise I.7.13. Let X be a non-empty contractible topological space. Determine the singular homology groups $H_q(X)$ for all $q\in\mathbb{Z}_{\geq 0}$.

7.3 CW complex

Let X be a non-empty Hausdorff topological space. Following [Lee-Jo1, Chapter 5], we use the following notation in this note.

- For $n\in\mathbb{Z}_{\geq 0}$, a subset $B\subset X$ is called an *open n-cell* (resp. a *closed n-cell*) if B is homeomorphic to $\mathbb{B}_{\mathbb{R}^n} := \{(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n \mid x_1^2+x_2^2+\cdots+x_n^2<1\}$ (resp. to $\overline{\mathbb{B}_{\mathbb{R}^n}}$).
- A family $\mathfrak{E} = \{e_\lambda\}_{\lambda\in\Lambda}$ of subsets of X is said to be a *cell decomposition* of X if there exists a map
$$\Lambda \ni \lambda \mapsto n_\lambda \in \mathbb{Z}_{\geq 0}$$
such that $\mathfrak{E}$ and the *n-skeleton*
$$X^n := \bigcup\{e_\lambda \mid \lambda\in\Lambda,\ n_\lambda\leq n\} \quad (n\in\mathbb{Z}_{\geq 0})$$
satisfy the following conditions:
  - $X = \bigcup\{e_\lambda \mid \lambda\in\Lambda\}$ holds.
  - For any $\lambda,\mu\in\Lambda$, we have $e_\lambda\cap e_\mu = \emptyset$ if $\lambda\neq\mu$.
  - X^0 is a discrete subset of X.
  - For each $\lambda\in\Lambda$, e_λ is a point of X if $n_\lambda = 0$, and is an open n_λ-cell if $n_\lambda > 0$.

  - For each $\lambda \in \Lambda$ with $n_\lambda > 0$, there exists a continuous map $\Phi_\lambda \colon \overline{\mathbb{B}_{\mathbb{R}^{n_\lambda}}} \to X$ (called a *characteristic map*) such that the restriction $\Phi_\lambda|_{\mathbb{B}_{\mathbb{R}^{n_\lambda}}}$ is a homeomorphism from $\mathbb{B}_{\mathbb{R}^{n_\lambda}}$ onto e_λ, and that $\Phi_\lambda(\partial\mathbb{B}_{\mathbb{R}^{n_\lambda}}) \subset X^{n_\lambda - 1}$.

- A *cell complex* $(X, \mathfrak{E})$ is a pair consisting of X and a cell decomposition $\mathfrak{E}$.

- A subset $A \subset X$ of a cell complex $(X, \mathfrak{E} = \{e_\lambda\}_{\lambda\in\Lambda})$ is said to be a *subcomplex* if it is closed and $A = \bigcup\{e_\lambda \mid \lambda \in \Lambda'\}$ holds for some $\Lambda' \subset \Lambda$.

- A *CW complex* is a cell complex $(X, \mathfrak{E} = \{e_\lambda\}_{\lambda\in\Lambda})$ which satisfies the following two conditions:

  (C) For each $\lambda \in \Lambda$, there exists a finite subset $\Lambda_\lambda \subset \Lambda$ such that

  $$\overline{e_\lambda} \subset \bigcup\{e_\ell \mid \ell \in \Lambda_\lambda\}$$

  (*closure finiteness condition*).

  (W) A subset $U \subset X$ is open (resp. closed) if and only if $U \cap \overline{e_\lambda}$ is open (resp. closed) in $\overline{e_\lambda}$ for each $\lambda \in \Lambda$ (*weak topology condition*).

- A CW complex $(X, \mathfrak{E})$ is said to be *finite-dimensional* if its n-skeleton coincides with X for some $n \in \mathbb{Z}_{\geq 0}$. In this case, the minimum of such n is called the *dimension* of X.

Note that, based on the following proposition, the domain of a characteristic map is often taken to be another suitable compact convex subset D with $\operatorname{Int} D \neq \emptyset$: Typical examples include the standard simplices.

Proposition I.7.14. *Let $n \in \mathbb{Z}_{\geq 0}$ and let $D \subset \mathbb{R}^n$ be a subset such that $\operatorname{Int} D \neq \emptyset$. Assume that D is compact and convex. Then D is a closed n-cell and $\operatorname{Int} D$ is an open n-cell.*

Proof. The case $n = 0$ is immediate, so assume $n \geq 1$. By considering the translation, we may assume $0 \in \operatorname{Int} D$. Take a positive number ε such that the closed ball centered at 0 with radius ε is included in $\operatorname{Int} D$. By scaling, we may assume that $\varepsilon = 1$: i.e., $\overline{\mathbb{B}_{\mathbb{R}^n}} \subset \operatorname{Int} D$. Then, by a usual argument by using the compactness and the convexity of D, we have the following: For any $x \in \partial\mathbb{B}_{\mathbb{R}^n}$, there uniquely exists $r(x) > 0$ such that $r(x) \cdot x \in \partial D$. The map $\ell \cdot x \mapsto \ell r(x) \cdot x$ ($\ell \in [0, 1]$, $x \in \partial\mathbb{B}_{\mathbb{R}^n}$) gives a homeomorphism between $\overline{\mathbb{B}_{\mathbb{R}^n}}$ and D, whose restriction to $\mathbb{B}_{\mathbb{R}^n}$ is a homeomorphism onto $\operatorname{Int} D$. See [Lee-Jo1, Proposition 5.1] for the details. □

In what follows, if there is no risk of confusion (or if the choice of the cell decomposition $\mathfrak{E}$ is not important in the given context and it suffices to endow X with some CW complex structure), we shall simply refer to a CW complex $(X, \mathfrak{E})$ as a "CW complex X".

For CW complexes, the following facts are basic.

- A cell complex $(X, \mathfrak{E} = \{e_\lambda\}_{\lambda\in\Lambda})$ is a CW complex if it is *locally finite*, that is, for any $x \in X$, there exists a neighborhood $U \subset X$ of x such that the set
$$\{\lambda \in \Lambda \mid e_\lambda \cap U \neq \emptyset\}$$
is finite ([Lee-Jo1, Proposition 5.4]). In particular, if $(X, \mathfrak{E})$ is a *finite cell complex* (i.e., if $\mathfrak{E}$ is a finite set), then $(X, \mathfrak{E})$ is a CW complex.
- CW complexes are Hausdorff and paracompact ([Ha1, Proposition A.3], [Ha2, Proposition 1.20]). Hence they satisfy the separation axiom T_4 by Lemma I.3.12.
- CW complexes are locally contractible ([Ha1, Proposition A.4]).
- A CW complex is connected if and only if it is path-connected ([Lee-Jo1, Theorem 5.11]).
- A CW complex is compact if and only if it is a finite cell complex ([Dol, Chapter V, 2.6], [Lee-Jo1, Corollary 5.6]).
- Any subcomplex A of a CW complex X has an open neighborhood $N \subset X$ that deformation retracts onto A ([Ha1, Proposition A.5]): i.e., there exists a continuous map $r\colon N \to A$ such that $r|_A$ is the identity id_A on A and the inclusion $A \hookrightarrow N$ is a homotopy inverse of r (equivalently, $r \circ (A \hookrightarrow N) = \mathrm{id}_A$ and $(A \hookrightarrow N) \circ r \simeq \mathrm{id}_N$ through a homotopy fixed on A). In particular, if A is a compact subcomplex, then A admits a fundamental system of neighborhoods each of which is homotopy equivalent to A.

For a given CW complex, there exists a very effective method for computing its singular homology groups (*cellular homology*, see [Dol, Chapter V], [Ha1, §2.2]). As the singular homology groups of most concrete examples appearing in this note can in fact be computed also by other methods (and as we do not introduce the notion of *homology groups of pairs* in this note), we refrain from giving a detailed account here. However let us mention that, from this, one obtains in particular the following important facts for a *finite* CW complex $(X, \mathfrak{E})$:

- Each singular homology group $H_q(X)$ is finitely generated;
- For $q \in \mathbb{Z}_{\geq 0}$, if the cell decomposition $\mathfrak{E}$ contains no q-cell, then $H_q(X) = 0$;

(see [Dol, Chapter V, §4], [Ha1, Theorem 2.35 and the subsequent applications $(i), (ii)$]).

Let X be an n-dimensional finite CW complex. Then it follows from the fact above that $H_q(X) = 0$ for all $q \in \mathbb{Z}_{>n}$. The (finite) number
$$b_q(X) := \operatorname{rank} H_q(X)$$
is called the *Betti number* of X. Furthermore, the integer
$$\chi_{\mathrm{top}}(X) := \sum_{q=0}^{n} (-1)^q \, b_q(X)$$
is called the *Euler–Poincaré characteristic* of X.

Example I.7.15. In this note, by a *graph* we mean a CW complex of dimension one (Of course, this notion is different from the graph of a function or an operator, but no confusion should arise in this note). For a graph G, the 0-cells are called *vertices*, and the 1-cells are called *edges*.

In what follows, we will mainly consider *finite graphs*, that is, finite CW complexes of dimension one. Let G be a graph with a fixed cell decomposition $\mathfrak{E} = \{e_\lambda\}_{\lambda\in\Lambda}$. An edge whose two endpoints coincide is called a *loop*. If two distinct vertices are joined by more than one edge, we say that the graph has *multiple edges*. A graph is called *simple* if it has no loops and no multiple edges. In this case, the graph is completely determined by purely combinatorial data, namely by the set V of vertices together with a subset $E \subset \{\{v, w\} \mid v, w \in V,\ v \neq w\}$. Note that a pair (V, E) of a set V and a subset $E \subset V \times V$ corresponds to the notion of a *directed graph*, which is not, at least in a natural way, regarded as a CW complex: Unless otherwise specified, all graphs are assumed to be *undirected* in this note. □

Mini-Exercise I.7.16. Let $n \in \mathbb{Z}_{\geq 0}$. Define the *complex projective space* $\mathbf{P}^n$ as the quotient space

$$\mathbf{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\sim,$$

where "$\sim$" is the relation defined as follows: For $z, w \in \mathbb{C}^{n+1} \setminus \{0\}$, $z \sim w$ if and only if $z = \lambda w$ for some $\lambda \in \mathbb{C}^*$. Show that $\mathbf{P}^n$ admits a CW complex structure consisting of exactly one cell in each even dimension $0, 2, 4, \ldots, 2n$.

Note that it is known that

$$H_q(\mathbf{P}^n) \cong \begin{cases} \mathbb{Z} & \text{if } q \text{ is even and } 0 \leq q \leq 2n, \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

holds for any $n \in \mathbb{Z}_{\geq 0}$, see [Lee-Jo1, Example 13.35].

Mini-Exercise I.7.17. Let $n \in \mathbb{Z}_{\geq 1}$. For

$$S^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + x_1^2 + \cdots + x_n^2 = 1\},$$

we set

$$U_n := S^n \cap \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, x_n > -\frac{1}{100} \right\}$$

and

$$V_n := S^n \cap \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, x_n < \frac{1}{100} \right\}.$$

In this Mini-Exercise, let us show the well-known formula

$$H_q(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0, n \\ 0 & \text{otherwise} \end{cases}. \tag{19}$$

(i) Show that (19) holds for $n = 1$.
(ii) Show that both U_n and V_n are contractible.
(iii) Show that $U_n \cap V_n$ is homotopy equivalent to S^{n-1}.
(iv) By assuming (19) holds for n, show (19) holds for $n + 1$.

MINI-EXERCISE I.7.18. Let $n, m \in \mathbb{Z}_{\geq 1}$. Show that if $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic, then $n = m$.

7.4 Simplicial complex

We say that a locally finite cell complex $(X, \mathfrak{E})$ *admits a simplicial structure* if the characteristic maps of all cells can be chosen so that the following conditions are satisfied:

- For each n-cell $e_\lambda \in \mathfrak{E}$, the characteristic map $\Phi_\lambda \colon \mathrm{Conv}(v_0, v_1, \ldots, v_n) \to X$ is a homeomorphism onto the closure $\overline{e_\lambda}$.
- For each n-cell $e_\lambda \in \mathfrak{E}$, the restriction

  $$\Phi_\lambda|_{\mathrm{Conv}(v_0,\ldots,\widehat{v_\ell},\ldots,v_n)} \colon \mathrm{Conv}(v_0, \ldots, \widehat{v_\ell}, \ldots, v_n) \longrightarrow X$$

  of the characteristic map Φ_λ to each face coincides with the characteristic map of a lower-dimensional cell.
- For any two cells $e_\lambda, e_\mu \in \mathfrak{E}$, if the intersection $\overline{e_\lambda} \cap \overline{e_\mu}$ is non-empty, then it coincides with the closure of a (necessarily unique) cell $e_\nu \in \mathfrak{E}$. Moreover, $\overline{e_\nu} \subset \overline{e_\lambda}$ and $\overline{e_\nu} \subset \overline{e_\mu}$, i.e., e_ν is a common face of e_λ and e_μ.

For a non-empty Hausdorff topological space X, we say that X is *triangulable* if there exists a locally finite cell complex structure $(X, \mathfrak{E})$ on X such that $(X, \mathfrak{E})$ admits a simplicial structure in the above sense.

Assume from now on that $(X, \mathfrak{E})$ is a locally finite cell complex admitting a simplicial structure. For each n-cell $e_\lambda \in \mathfrak{E}$, we fix a characteristic map

$$\Phi_\lambda \colon \mathrm{Conv}(v_0, v_1, \ldots, v_n) \longrightarrow X$$

as above. For each integer $q \in \mathbb{Z}_{\geq 0}$, we define the *$q$-th simplicial chain group* $C_q^{\mathrm{simp}}(X, \mathfrak{E})$ to be the free abelian group generated by the set of q-cells in $\mathfrak{E}$. The boundary homomorphism

$$\partial_q \colon C_q^{\mathrm{simp}}(X, \mathfrak{E}) \longrightarrow C_{q-1}^{\mathrm{simp}}(X, \mathfrak{E})$$

is defined in the same manner as that of the boundary map for singular chains, so that $\partial_{q-1} \circ \partial_q = 0$ holds for any $q \in \mathbb{Z}_{>0}$, see [Ha1, §2.1]. For the obtained chain complex

$$\cdots \longrightarrow C_{q+1}^{\mathrm{simp}}(X, \mathfrak{E}) \xrightarrow{\partial_{q+1}} C_q^{\mathrm{simp}}(X, \mathfrak{E}) \xrightarrow{\partial_q} \cdots \to C_1^{\mathrm{simp}}(X, \mathfrak{E}) \xrightarrow{\partial_1} C_0^{\mathrm{simp}}(X, \mathfrak{E}) \to 0,$$

the homology groups $H_q(C_\bullet^{\mathrm{simp}}(X, \mathfrak{E}))$ (*simplicial homology groups*) are known to be isomorphic to $H_q(X)$ ([Dol, Chapter V §7, 8], [Ha1, Theorem 2.27]).

MINI-EXERCISE I.7.19. Let $v_1 := (1, 0, 0)$, $v_2 := (0, 1, 0)$, $v_3 := (0, 0, 1)$ be points of $\mathbb{R}^3$, and let $X := \mathrm{Conv}(v_1, v_2, v_3) \setminus \mathrm{Conv}^\circ(v_1, v_2, v_3)$. We regard X as a CW complex $(X, \mathfrak{E})$ whose 0-cells and 1-cells are given as follows: the 0-cells are v_1, v_2, v_3, and the 1-cells are

$$e_{12} := \mathrm{Conv}^\circ(v_1, v_2), \quad e_{23} := \mathrm{Conv}^\circ(v_2, v_3), \quad e_{31} := \mathrm{Conv}^\circ(v_3, v_1).$$

We orient each 1-cell e_{ij} by the ordered pair (v_i, v_j). Define characteristic maps by the affine formulas

$$\Phi_{ij}\colon (\text{Simplex})_1 := \{(s,t) \in (\mathbb{R}_{\geq 0})^2 \mid s+t=1\} \to X, \qquad \Phi_{ij}(s,t) := sv_i + tv_j,$$

where $(i,j) \in \{(1,2),(2,3),(3,1)\}$, and let $\Phi_i\colon (\text{Simplex})_0 \to X$ be the constant map with value v_i.
(i) Show that these data define a simplicial structure on the CW complex $(X, \mathfrak{E})$.
(ii) With respect to the ordered bases

$$C_1^{\text{simp}}(X,\mathfrak{E}) = \mathbb{Z}\cdot e_{12} \oplus \mathbb{Z}\cdot e_{23} \oplus \mathbb{Z}\cdot e_{31}, \quad C_0^{\text{simp}}(X,\mathfrak{E}) = \mathbb{Z}\cdot v_1 \oplus \mathbb{Z}\cdot v_2 \oplus \mathbb{Z}\cdot v_3,$$

write down explicitly the boundary homomorphisms

$$\partial_1\colon C_1^{\text{simp}}(X,\mathfrak{E}) \longrightarrow C_0^{\text{simp}}(X,\mathfrak{E})$$

as integer matrices.
(iii) Compute $\operatorname{Ker}\partial_1$, and $\operatorname{Image}\partial_1$ explicitly, and determine the simplicial homology groups $H_q(C_\bullet^{\text{simp}}(X,\mathfrak{E}))$ for $q \in \mathbb{Z}_{\geq 0}$.

Lemma I.7.20. *Let X be a non-empty Hausdorff topological space. Assume that X is triangulable, and fix a locally finite cell complex structure $(X,\mathfrak{E})$ admitting a simplicial structure. Then there exists a locally finite open covering $\{U_v\}_{v\in X^0}$ of X indexed by the set X^0 of 0-cells of $\mathfrak{E}$ such that the following hold:*
(i) For each $v \in X^0$, the open set U_v is contractible.
(ii) For any finite subset $\{v_0, v_1, \ldots, v_k\} \subset X^0$, the intersection $U_{v_0} \cap U_{v_1} \cap \cdots \cap U_{v_k}$ is either empty or contractible.
(iii) For any finite subset $\{v_0, v_1, \ldots, v_k\} \subset X^0$ whose elements are different from each other, the intersection $U_{v_0} \cap U_{v_1} \cap \cdots \cap U_{v_k}$ is non-empty if and only if there exists a k-cell $e_v \in \mathfrak{E}$ such that $\{v_0, v_1, \ldots, v_k\}$ is precisely the set of all 0-cells contained in $\overline{e_v}$.

Proof. For a cell $e \in \mathfrak{E}$, define its *open star* by

$$U_e := \bigcup \{e_\lambda \mid e \subset \overline{e_\lambda}\}.$$

Here the union is taken over open cells, not their closures. The cells not having e as a face form a subcomplex, whose union is closed: it is also the union of their closures, a locally finite family of closed subsets of X. Thus U_e is open.

First let us check that $\{U_v\}_{v\in X^0}$ is a locally finite open covering. Every $x \in X$ belongs to an open cell e_λ, and then $x \in U_v$ for each vertex v of $\overline{e_\lambda}$. To show local finiteness, take $x \in X$. Since $(X,\mathfrak{E})$ is a locally finite cell complex, there exists an open neighborhood W of x meeting only finitely many cells $e_{\lambda_1}, e_{\lambda_2}, \ldots, e_{\lambda_m}$. If $W \cap U_v \neq \emptyset$, then some cell e_{λ_j} meeting W has v as a vertex. Since each of these finitely many cells has only finitely many vertices, only finitely many U_v meet W.

We next show that every U_e is contractible. Let $b_e \in e$ be the barycenter of e, defined using its characteristic map. For a cell e_λ having e as a face, choose the fixed characteristic map

$$\Phi_\lambda \colon \mathrm{Conv}(w_0, w_1, \ldots, w_n) \longrightarrow \overline{e_\lambda},$$

and let $\widetilde{b}_e$ be the barycenter of the face corresponding to e. Consider the straight-line homotopy

$$H_\lambda(x,t) := (1-t)x + t\widetilde{b}_e \qquad (x \in \mathrm{Conv}(w_0, w_1, \ldots, w_n),\ t \in [0,1]).$$

For x in the relative interior of a face containing the face corresponding to e, this homotopy stays in that relative interior for $t < 1$ and ends at $\widetilde{b}_e$ for $t = 1$. After transport by Φ_λ, it therefore preserves $\overline{e_\lambda} \cap U_e$. The characteristic maps agree on common faces, so the transported homotopies agree on overlaps. These subsets form a locally finite closed covering of U_e, and hence the homotopies glue to a continuous deformation retraction of U_e onto b_e. For a vertex v, this proves (i).

Finally, take pairwise distinct vertices $v_0, v_1, \ldots, v_k$. A point x belongs to $U_{v_0} \cap U_{v_1} \cap \cdots \cap U_{v_k}$ exactly when all these vertices belong to the closure of the unique open cell containing x. If the intersection is nonempty, these vertices span a k-face e_ν of that cell. The common-face condition in the simplicial structure makes e_ν independent of the chosen cell. Conversely, if such a cell e_ν exists, its open star is nonempty, and the same description gives

$$U_{v_0} \cap U_{v_1} \cap \cdots \cap U_{v_k} = U_{e_\nu}.$$

This proves (iii), and the contractibility of U_{e_ν} proves (ii). □

7.5 Singular cohomology

Let X be a topological space and G be an abelian group. We define the *singular q-cochain group of X with coefficients in G* by

$$C^q_{\mathrm{sing}}(X, G) := \mathrm{Hom}_{\mathbb{Z}}\big(C_q^{\mathrm{sing}}(X),\, G\big).$$

The *coboundary operator*

$$\delta^q \colon C^q_{\mathrm{sing}}(X, G) \longrightarrow C^{q+1}_{\mathrm{sing}}(X, G)$$

is defined by

$$\delta^q := \partial^*_{q+1},$$

where ∂^*_{q+1} denotes the dual (transpose) of the boundary homomorphism ∂_{q+1}.

By definition, one easily checks that

$$\delta^{q+1} \circ \delta^q = 0 \qquad \text{for all } q \in \mathbb{Z},$$

since $\partial_q \circ \partial_{q+1} = 0$. Therefore

$$0 \to C^0_{\mathrm{sing}}(X, G) \xrightarrow{\delta^0} C^1_{\mathrm{sing}}(X, G) \xrightarrow{\delta^1} \cdots \to C^{q-1}_{\mathrm{sing}}(X, G) \xrightarrow{\delta^{q-1}} C^q_{\mathrm{sing}}(X, G) \xrightarrow{\delta^q} \cdots$$

is a cochain complex of abelian groups. The cohomology group

$$H^q(X,G) := H^q\big(C^\bullet_{\mathrm{sing}}(X,G)\big) = \begin{cases} \mathrm{Ker}\,\delta^0 & \text{if } q=0 \\ \mathrm{Ker}\,\delta^q/\mathrm{Image}\,\delta^{q-1} & \text{if } q>0 \end{cases}$$

is called the *q-th singular cohomology group of X with coefficients in G*. An element of $\mathrm{Ker}\,\delta^q$ is called a *q-cocycle*, and an element of $\mathrm{Image}\,\delta^{q-1}$ is called a *q-coboundary*.

Remark I.7.21. If G is not abelian, the above constructions no longer make sense in general. Indeed, although one may consider the set of group homomorphisms $\mathrm{Hom}_{\mathrm{group}}(C_q^{\mathrm{sing}}(X),G)$, it does not carry a natural group structure, and the coboundary operators fail to be group homomorphisms. As a result, the notions of cochains, cocycles, coboundaries, and their quotients cannot be defined in a natural way (Nevertheless, in low degrees one can still define $H^0(X,G)$ and $H^1(X,G)$ for a non-abelian group G, at least as sets). □

See [Dol, Chapter VI §7], [Ha1, Chapter 3], and [Lee-Jo1, Chapter 13] for further details for the singular cohomology groups. The following properties are fundamental:

- For a continuous map $f\colon X\to Y$ between topological spaces, there is a natural morphism of cochain complexes

  $$f^*\colon C^\bullet_{\mathrm{sing}}(Y,G)\longrightarrow C^\bullet_{\mathrm{sing}}(X,G),\qquad (f^*\varphi)(\sigma):=\varphi(f\circ\sigma),$$

  which induces group homomorphisms $f^*\colon H^\bullet(Y,G)\longrightarrow H^\bullet(X,G)$ (see [Dol, Chapter VI 7.2]).

- If two continuous maps $f,g\colon X\to Y$ are homotopic, then the induced homomorphisms on cohomology coincide; that is, $f^*=g^*$ (see [Dol, Chapter VI 7.4], [Ha1, §3.1 p. 201]).

- For a topological space X and its open subsets $U,V\subset X$ with $X=U\cup V$, there exists a long exact sequence

  $$0\longrightarrow H^0(X,G)\longrightarrow H^0(U,G)\oplus H^0(V,G)\longrightarrow H^0(U\cap V,G)\qquad,$$

  $$\longrightarrow H^1(X,G)\longrightarrow H^1(U,G)\oplus H^1(V,G)\longrightarrow H^1(U\cap V,G)$$

  $$\longrightarrow H^2(X,G)\longrightarrow H^2(U,G)\oplus H^2(V,G)\longrightarrow H^2(U\cap V,G)\longrightarrow\dots$$

  which is called the *Mayer–Vietoris sequence for cohomology* (see [Dol, Chapter VI 7.6], [Ha1, §3.1 p. 203]). In particular, if $U\cap V=\emptyset$, then $H^\bullet(X,G)\cong H^\bullet(U,G)\oplus H^\bullet(V,G)$.

Fix an integer q and apply Lemma I.5.7 to the sequence

$$C^{\mathrm{sing}}_{q+1}(X)\xrightarrow{\partial_{q+1}}C^{\mathrm{sing}}_q(X)\xrightarrow{\partial_q}C^{\mathrm{sing}}_{q-1}(X)$$

of $\mathbb{Z}$-modules. Then the singular homology group $H_q(X)$ coincides with the module H appearing in the lemma, while the quotient

$$\mathfrak{H}_G = \frac{\operatorname{Ker} \partial_{q+1}^*}{\operatorname{Image} \partial_q^*}$$

is precisely the cohomology group $H^q(X, G)$ by definition. The natural homomorphism $\iota_G \colon \mathfrak{H}_G \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H, G)$ constructed in Lemma I.5.7 is identified with the canonical evaluation map

$$H^q(X, G) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_q(X), G).$$

Since each chain group $C_q^{\mathrm{sing}}(X)$ is free abelian by definition, we have

$$\operatorname{Ext}^1_{\mathbb{Z}}(C_q^{\mathrm{sing}}(X), G) = 0 \qquad \text{for all } q \in \mathbb{Z}_{\geq 0}.$$

As $\operatorname{Image} \partial_q$ is a subgroup of a free abelian group $C_{q-1}^{\mathrm{sing}}(X)$ and hence free, we also have

$$\operatorname{Ext}^1_{\mathbb{Z}}(\operatorname{Image} \partial_q, G) = 0 \qquad \text{for all } q \in \mathbb{Z}_{\geq 0}.$$

By considering the short exact sequence

$$0 \longrightarrow H_{q-1}(X) \longrightarrow \operatorname{Coker} \partial_q \longrightarrow \operatorname{Image} \partial_{q-1} \longrightarrow 0,$$

where $\operatorname{Image} \partial_{q-1}$ is free abelian, we also have

$$\operatorname{Ext}^1_{\mathbb{Z}}(H_{q-1}(X), G) \cong \operatorname{Ext}^1_{\mathbb{Z}}(\operatorname{Coker} \partial_q, G).$$

Thus, by Lemma I.5.7 (i) and (ii), we obtain

$$\operatorname{Ker} \iota_G \;\cong\; \operatorname{Ext}^1_{\mathbb{Z}}(H_{q-1}(X), G), \qquad \operatorname{Coker} \iota_G = 0.$$

Consequently, we obtain a natural short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_{q-1}(X), G) \longrightarrow H^q(X, G) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_q(X), G) \longrightarrow 0 \tag{20}$$

(the *universal coefficient theorem for cohomology*). In particular, if G is a divisible abelian group, then $\operatorname{Ext}^1_{\mathbb{Z}}(-, G) = 0$ holds. Therefore, for such G, the short exact sequence (20) reduces to a natural isomorphism

$$H^q(X, G) \;\cong\; \operatorname{Hom}_{\mathbb{Z}}(H_q(X), G) \quad (\text{if } G \text{ is a divisible abelian group}). \tag{21}$$

Mini-Exercise I.7.22. Let X and G be as above. Assume that $G = \mathbb{R}$ or $G = \mathbb{C}$, and $H_\bullet(X)$ is finitely generated for any $\bullet \in \mathbb{Z}_{\geq 0}$. Consider the canonical homomorphism

$$\operatorname{Hom}_{\mathbb{Z}}(H_q(X), \mathbb{Z}) \otimes_{\mathbb{Z}} G \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_q(X), G), \qquad (\varphi \otimes g) \longmapsto (x \mapsto \varphi(x) \cdot g). \tag{22}$$

(i) Show that (22) is an isomorphism.
(ii) Show that there is a natural isomorphism

$$H^q(X, \mathbb{Z}) \otimes_{\mathbb{Z}} G \;\cong\; H^q(X, G) \qquad (G = \mathbb{R}, \mathbb{C}) \tag{23}$$

when $H_\bullet(X)$ is finitely generated for any $\bullet \in \mathbb{Z}_{\geq 0}$.

Mini-Exercise I.7.23. If $H_1(X)$ is finitely generated, show that there is an isomorphism $H^2(X,\mathbb{Z})_{\mathrm{tor}} \cong H_1(X)_{\mathrm{tor}}$.

The following proposition (Proposition I.7.24) is a well-known fact, often referred to as the *Čech–singular correspondence*. Although the construction appearing there is closely related to Čech cohomology, we do not pursue this viewpoint here. A more systematic and general treatment will be given in a later chapter, and we only record the result needed at this point.

Proposition I.7.24. *Let X be a non-empty Hausdorff topological space. Assume that X is triangulable, and fix a locally finite cell complex structure $(X,\mathfrak{E})$ admitting a simplicial structure. Let $\{U_v\}_{v\in X^0}$ be the locally finite open covering as in Lemma I.7.20. Let G be an abelian group. For each integer $q\in\mathbb{Z}_{\geq 0}$, define an abelian group*

$$\check{C}^q(\{U_v\}_{v\in X^0},G) := \prod_{\substack{(v_0,\dots,v_q)\in(X^0)^{q+1}\\ U_{v_0}\cap\cdots\cap U_{v_q}\neq\emptyset}} G,$$

and define a homomorphism

$$\check{\delta}^q\colon \check{C}^q(\{U_v\}_{v\in X^0},G)\longrightarrow \check{C}^{q+1}(\{U_v\}_{v\in X^0},G)$$

by

$$(\check{\delta}^q c)_{v_0,\dots,v_{q+1}} := \sum_{\ell=0}^{q+1}(-1)^\ell\, c_{v_0,\dots,\widehat{v_\ell},\dots,v_{q+1}} \qquad \big(U_{v_0}\cap U_{v_1}\cap\cdots\cap U_{v_{q+1}}\neq\emptyset\big),$$

where we interpret $c_{v_0,\dots,\widehat{v_\ell},\dots,v_{q+1}}$ as 0 if $U_{v_0}\cap\cdots\cap\widehat{U_{v_\ell}}\cap\cdots\cap U_{v_{q+1}}=\emptyset$ (Here the symbol $\widehat{\bullet}$ means that the symbol $\bullet$ is omitted. Thus $c_{v_0,\dots,\widehat{v_\ell},\dots,v_{q+1}} = c_{v_0,\dots,v_{\ell-1},v_{\ell+1},\dots,v_{q+1}}$, for example). Then $\check{\delta}^{q+1}\circ\check{\delta}^q=0$ for all q. Moreover, there exists a natural isomorphism

$$H^q\big(\check{C}^\bullet(\{U_v\}_{v\in X^0},G)\big) \;\cong\; H^q(X,G) \qquad (q\in\mathbb{Z}_{\geq 0}).$$

Proof. Fix a total order on the vertices. Restricting a cochain to tuples $v_0<v_1<\cdots<v_q$ gives the ordered cochain complex $\check{C}^\bullet_<$. This restriction is a cochain homotopy equivalence; see [St, Tag 01FM]. Thus it suffices first to compare the ordered complex with simplicial cochains. We orient each simplex by its increasing vertex order. The unrestricted groups in the statement, which also have components with repeated or permuted indices, are not themselves degreewise isomorphic to the simplicial cochain groups.

For each integer $q\geq 0$, we define a homomorphism

$$\Theta^q\colon \check{C}^q_<(\{U_v\}_{v\in X^0},G)\longrightarrow C^q_{\mathrm{simp}}(X,\mathfrak{E},G) := \mathrm{Hom}_{\mathbb{Z}}\big(C^{\mathrm{simp}}_q(X,\mathfrak{E}),\,G\big)$$

as follows. Let $e_\lambda\in\mathfrak{E}$ be a q-cell, and let

$$\Phi_\lambda\colon \mathrm{Conv}(v_0,v_1,\dots,v_q)\longrightarrow X$$

be the characteristic map with $v_0<v_1<\cdots<v_q$ and this orientation. We define

$$(\Theta^q(c))(e_\lambda) := c_{v_0,\dots,v_q},$$

which uniquely extends as a homomorphism. This is well-defined by Lemma I.7.20 (*iii*), and every generator of $C_q^{\mathrm{simp}}(X,\mathfrak{E})$ arises uniquely in this manner. By construction, Θ^q is an isomorphism of abelian groups for each q. Moreover, by the definitions of the Čech coboundary operator $\check{\delta}^q$ and of the simplicial boundary homomorphism ∂_{q+1} (hence of its dual δ^q_{simp}), one immediately checks that the following diagram commutes:

$$\begin{array}{ccc} \check{C}^q_<(\{U_v\}_{v\in X^0},G) & \xrightarrow{\check{\delta}^q} & \check{C}^{q+1}_<(\{U_v\}_{v\in X^0},G)\,. \\ \Big\downarrow{\scriptstyle \Theta^q} & & \Big\downarrow{\scriptstyle \Theta^{q+1}} \\ C^q_{\mathrm{simp}}(X,\mathfrak{E},G) & \xrightarrow[\delta^q_{\mathrm{simp}}]{} & C^{q+1}_{\mathrm{simp}}(X,\mathfrak{E},G) \end{array}$$

Hence $\Theta^\bullet$ is an isomorphism of cochain complexes. Therefore we obtain a natural isomorphism

$$H^q\big(\check{C}^\bullet(\{U_v\}_{v\in X^0},G)\big) \cong H^q(\check{C}^\bullet_<) \;\cong\; H^q\big(C^\bullet_{\mathrm{simp}}(X,\mathfrak{E},G)\big).$$

It remains to relate $H^q\big(C^\bullet_{\mathrm{simp}}(X,\mathfrak{E},G)\big)$ to the singular cohomology $H^q(X,G)$. As is mentioned in §7.4, the natural chain map $s_\bullet\colon C_\bullet^{\mathrm{simp}}(X,\mathfrak{E}) \longrightarrow C_\bullet^{\mathrm{sing}}(X)$ induces isomorphisms on homology groups:

$$H_q\big(C_\bullet^{\mathrm{simp}}(X,\mathfrak{E})\big) \;\cong\; H_q\big(C_\bullet^{\mathrm{sing}}(X)\big) = H_q(X) \qquad (q\in\mathbb{Z}_{\geq 0}).$$

By taking $\mathbb{Z}$-duals with coefficients in G, we obtain a morphism of cochain complexes

$$s^\bullet\colon C^\bullet_{\mathrm{sing}}(X,G) = \mathrm{Hom}_{\mathbb{Z}}\big(C_\bullet^{\mathrm{sing}}(X),G\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(C_\bullet^{\mathrm{simp}}(X,\mathfrak{E}),G\big) = C^\bullet_{\mathrm{simp}}(X,\mathfrak{E},G),$$

where $s^q(\varphi) := \varphi\circ s_q$. Fix $q\in\mathbb{Z}_{\geq 0}$. Consider the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}\big(H_{q-1}(X),G\big) \longrightarrow H^q(X,G) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(H_q(X),G\big) \longrightarrow 0.$$

(the universal coefficient theorem for cohomology, see (20)). By the same argument as around (20), by applying Lemma I.5.7, one also obtains a natural short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}\Big(H_{q-1}\big(C_\bullet^{\mathrm{simp}}(X,\mathfrak{E})\big),G\Big) \to H^q\big(C^\bullet_{\mathrm{simp}}(X,\mathfrak{E},G)\big) \to \mathrm{Hom}_{\mathbb{Z}}\Big(H_q\big(C_\bullet^{\mathrm{simp}}(X,\mathfrak{E})\big),G\Big) \to 0.$$

By construction, $s^\bullet$ induces a commutative diagram whose left and right vertical arrows are the maps on Ext^1 and Hom induced from the isomorphisms

$$H_{q-1}\big(C_\bullet^{\mathrm{simp}}(X,\mathfrak{E})\big) \cong H_{q-1}(X), \qquad H_q\big(C_\bullet^{\mathrm{simp}}(X,\mathfrak{E})\big) \cong H_q(X).$$

In particular, these left and right vertical arrows are isomorphisms. Therefore the middle vertical arrow

$$H^q(X,G) \longrightarrow H^q\big(C^\bullet_{\mathrm{simp}}(X,\mathfrak{E},G)\big)$$

is also an isomorphism. Consequently,

$$H^q\big(C^\bullet_{\mathrm{simp}}(X,\mathfrak{E},G)\big) \;\cong\; H^q(X,G),$$

which completes the proof. □

REMARK I.7.25. Let N be a non-negative integer and $U \subset \mathbb{R}^N$ be an open set. We say that U is *star-shaped* if there exists a point $x_0 \in U$ such that the line segment $[x_0, x]$ is contained in U for every $x \in U$.

For the covering in Proposition I.7.24, the finite intersections are contractible open stars as described in Lemma I.7.20. An open star in a general simplicial complex need not be an open subset of a Euclidean space. When X is a smooth manifold, these intersections are open submanifolds; their contractibility, together with the smoothing of homotopies in Theorem I.8.13, suffices for the application to de Rham cohomology below. □

Singular cohomology carries further important structures. In particular, it admits a natural graded-commutative product structure (the *cup product*) and satisfies the *Künneth formula* relating the cohomology of a product space to that of its factors. These structures play a fundamental role in many applications, which we do not introduce here. We therefore refer the reader to standard textbooks for these topics ([Dol, Chapter VI] and [Ha1, §3.2] for example).

7.6 Fiber bundles

Let X be a topological space. A continuous surjective map $\pi\colon E \to X$ from a topological space E to X is called a *fiber bundle over* X if there exists a non-empty topological space F such that, for every point $x \in X$, there exist an open neighborhood $U \subset X$ of x and a homeomorphism $\Phi_U\colon \pi^{-1}(U) \to U \times F$ making the following diagram commutative:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times F\,, \\ & {\scriptstyle \pi|_{\pi^{-1}(U)}} \searrow \quad \swarrow {\scriptstyle \mathrm{Pr}_1} & \\ & U & \end{array}$$

where Pr_1 denotes the projection onto the first factor. We also say that the pair (E, π) (or simply E when the projection map π is clear from the context or when it is irrelevant or fixed once and for all) is a fiber bundle over X. The space F is called the *fiber* of the bundle, E the *total space*, X the *base space*, and π the *projection map*. A homeomorphism Φ_U as above is called a *local trivialization of E over U*. Let $A \subset X$ be a subset. A *continuous section of E on A* is a continuous map $\sigma\colon A \to E$ such that $\pi \circ \sigma$ coincides with the identity map on A. In particular, a continuous section defined on the whole space X is sometimes called a *global continuous section* (while a section defined only on a neighborhood of a point is sometimes referred to as a *local continuous section*).

Let U and V be open neighborhoods of a point $x \in X$, and $\Phi_U\colon \pi^{-1}(U) \to U \times F$, $\Phi_V\colon \pi^{-1}(V) \to V \times F$ be local trivializations of E. Then, for each $x \in U \cap V$, there exists a homeomorphism of F, denoted by $\varphi_{UV}(x) \in \mathrm{Homeo}(F)$, where $\mathrm{Homeo}(F)$ denotes the group of all homeomorphisms from F onto itself, such that

$$\Phi_U \circ \Phi_V^{-1}(x, \xi) = \big(x, (\varphi_{UV}(x))(\xi)\big) \qquad (\xi \in F).$$

The map $\varphi_{UV}\colon U \cap V \to \mathrm{Homeo}(F)$ is called the *transition function* associated with the pair of local trivializations Φ_U and Φ_V. Thus φ_{UV} changes fiber coordinates from the V-trivialization to the U-trivialization. In particular, coordinates of the same point of a fiber satisfy $\xi_U = \varphi_{UV}(x)(\xi_V)$.

Let G be a subgroup of $\mathrm{Homeo}(F)$. We say that a fiber bundle E over X *has structure group* G if there exist an open covering $\{U_j\}_{j\in J}$ of X and local trivializations $\Phi_{U_j}\colon \pi^{-1}(U_j) \to U_j \times F$ for each $j \in J$ such that all transition functions $\varphi_{U_jU_k}$ satisfy $\operatorname{Image}\varphi_{U_jU_k} \subset G$. Furthermore, if the transition functions $\varphi_{U_jU_k}$ can be chosen to be G-valued locally constant maps, then we say that the fiber bundle E is *G-flat*. When G is a topological group acting continuously on F, we require these transition functions to be continuous as maps into G. A *G-structure* is a choice of such local trivializations up to refinement and compatible G-valued changes of fiber coordinates; when G has a topology as above, these coordinate changes are also required to be continuous as maps into G.

Two fiber bundles $\pi\colon E \to X$ and $\pi'\colon E' \to X$ over X with the same fiber F and structure group G are said to be *isomorphic as fiber bundles with structure group G* if there exists a homeomorphism $\Psi\colon E \to E'$ such that $\pi' \circ \Psi = \pi$ holds and Ψ preserves the fibers and the G-structure, that is, for each $x \in X$, the induced map $\Psi_x\colon \pi^{-1}(x) \to (\pi')^{-1}(x)$ is given by the action of an element of G under their local trivializations. When G has a topology as above, we also require the local expressions of Ψ to have the form $(x, \xi) \mapsto (x, \psi_j(x)(\xi))$ with continuous maps $\psi_j\colon U_j \to G$. In this case, we will write

$$E \cong_{G\text{-str}} E'.$$

Let E be a fiber bundle over X with fiber F and structure group G, and let $\{U_j\}_{j\in J}$ be an open covering of X together with local trivializations $\Phi_{U_j}\colon \pi^{-1}(U_j) \to U_j \times F$ $(j \in J)$ whose transition functions take values in G. Then the associated transition functions

$$\varphi_{U_jU_k}\colon U_j \cap U_k \to G$$

satisfy the 1-*cocycle condition*: i.e., $\varphi_{U_jU_k}(x)\cdot\varphi_{U_kU_\ell}(x) = \varphi_{U_jU_\ell}(x)$ holds for all $x \in U_j \cap U_k \cap U_\ell$. Conversely, under the assumption that $G \subset \mathrm{Homeo}(F)$ is a topological group (i.e., G is endowed with a topology such that the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are continuous) whose action on F is continuous, for an open covering $\{U_j\}_{j\in J}$ of X a family $\{\varphi_{jk}\colon U_j \cap U_k \to G\}_{j,k\in J}$ of continuous maps that satisfies the 1-cocycle condition (*G-valued continuous Cousin datum*), one can construct a fiber bundle E over X with fiber F and structure group G as follows: Set

$$E := \left(\coprod_{j\in J}(U_j \times F)\right) / \sim,$$

where $\sim$ is the relation defined by

$$(x_j, \xi_j) \sim (x_k, \xi_k) \iff x_j = x_k \in U_j \cap U_k\,(\subset X) \text{ and } \xi_j = (\varphi_{jk}(x_k))(\xi_k)$$

for any $j, k \in J$, $(x_j, \xi_j) \in U_j \times F$, and $(x_k, \xi_k) \in U_k \times F$. We endow each product $U_j \times F$ with the product topology, the disjoint union $\coprod_{j\in J}(U_j \times F)$ with the disjoint union topology,

and the space E with the quotient topology induced by the equivalence relation $\sim$. The first projection $U_j \times F \to U_j$ induces a continuous surjective map

$$\pi\colon E \longrightarrow X,$$

which we regard as the projection map of the resulting fiber bundle.

Mini-Exercise I.7.26. Let X be a topological space, F a non-empty topological space, and let $G \subset \mathrm{Homeo}(F)$ be a subgroup. Assume that G is endowed with a topology making it a topological group whose action on F is continuous. Show that the correspondence described above induces a canonical bijection between the following two quotient sets:

- The quotient of the set of fiber bundles over X with fiber F and structure group G, modulo the equivalence relation of being isomorphic as fiber bundles with structure group G;
- The quotient of the set of G-valued continuous Cousin data, modulo the following equivalence relation: two G-valued continuous Cousin data $\mathfrak{f}$ and $\mathfrak{g}$ are said to be equivalent if, after passing to a common refinement $\{U_j\}_{j\in J}$ of open coverings of X and writing them as $\mathfrak{f} = \{f_{jk}\colon U_j \cap U_k \to G\}_{j,k\in J}$ and $\mathfrak{g} = \{g_{jk}\colon U_j \cap U_k \to G\}_{j,k\in J}$ (see Remark I.7.27 below), there exist continuous maps $\psi_j\colon U_j \to G$ such that $\psi_j(x) \cdot f_{jk}(x) \cdot \psi_k(x)^{-1} = g_{jk}(x)$ holds for all $x \in U_j \cap U_k$.

Remark I.7.27. Let $\{U_j\}_{j\in J}$ be an open covering of X, and $\mathfrak{f} = \{f_{jk}\colon U_j \cap U_k \to G\}_{j,k\in J}$ be a G-valued continuous Cousin datum, where X, F, and G are as in Mini-Exercise I.7.26. If $\{V_\alpha\}_{\alpha\in A}$ is a refinement of $\{U_j\}_{j\in J}$, so that for each $\alpha \in A$ there exists $j(\alpha) \in J$ with $V_\alpha \subset U_{j(\alpha)}$, then $\mathfrak{f}$ induces a G-valued continuous Cousin datum

$$\mathfrak{f}|_{\{V_\alpha\}} = \{f_{j(\alpha)j(\beta)}|_{V_\alpha\cap V_\beta}\colon V_\alpha \cap V_\beta \to G\}_{\alpha,\beta\in A}$$

with respect to the refined covering. This induced Cousin datum defines the same fiber bundle over X (with structure group G) as the original datum $\mathfrak{f}$. In particular, passing to refinements does not change the isomorphism class of the associated fiber bundle. □

We now turn to the case of G-flat fiber bundles. In what follows, we assume that X is connected and locally path-connected.

Let F be a non-empty topological space and $G \subset \mathrm{Homeo}(F)$ a subgroup. We write F_{discrete} and G_{discrete} for the discrete topological spaces associated with F and G, respectively. Then it is clear that G_{discrete} is a topological group and the natural action is continuous. Let $\pi\colon E \to X$ be a fiber bundle over X with fiber F and structure group G. Assume that E is G-flat. By definition, there exist an open covering $\{U_j\}_{j\in J}$ of X and local trivializations $\{\Phi_{U_j}\colon \pi^{-1}(U_j) \to U_j \times F\}_{j\in J}$ such that each transition function φ_{jk} is locally constant. In the discussion of E_{discrete}, monodromy, and G-flat isomorphisms below, we keep this chosen *flat structure*, namely these local trivializations up to refinement and locally constant G-valued changes of fiber coordinates. Since locally constant maps are continuous for the discrete topology, the same family $\{\varphi_{jk}\}$ may also be regarded as a G_{discrete}-valued continuous Cousin

datum with respect to the covering $\{U_j\}_{j\in J}$. Applying the construction explained above to this datum (with F_{discrete} in place of F and G_{discrete} in place of G), we obtain a fiber bundle

$$\pi_{\mathrm{discrete}}\colon E_{\mathrm{discrete}} \longrightarrow X$$

with fiber F_{discrete} and structure group G_{discrete}. Note that, by construction, there is a canonical bijection between the underlying sets of E_{discrete} and E.

LEMMA I.7.28. *In the above setting, the map $\pi_{\mathrm{discrete}}\colon E_{\mathrm{discrete}} \to X$ is a covering map. Moreover, each connected component of E_{discrete} is a connected covering space of X.*

PROOF. By construction, it is clear that π_{discrete} is a covering map. Let C be a connected component of E_{discrete}. Take $\xi_0 \in C$. To show that $\pi_{\mathrm{discrete}}(C) = X$, let us consider the set A of all the points $x \in X$ such that there exists a path in C whose initial point is ξ_0 and whose terminal point is included in $\pi^{-1}_{\mathrm{discrete}}(x)$. As π_{discrete} is a covering map, one can easily show that A is open and closed. As X is connected, one has $X = A$, from which the surjectivity of $\pi_{\mathrm{discrete}}|_C$ follows.

It remains to check that any point of X is evenly covered by $\pi_{\mathrm{discrete}}|_C$. Fix $x \in X$ and choose an evenly covered connected neighborhood U of x for π_{discrete}. Then we have a decomposition

$$\pi^{-1}_{\mathrm{discrete}}(U) = \coprod_{\lambda\in F} W_\lambda$$

into open subsets $W_\lambda \subset E_{\mathrm{discrete}}$ such that $\pi_{\mathrm{discrete}}|_{W_\lambda}\colon W_\lambda \to U$ is a homeomorphism for each λ. Intersecting with C, we obtain

$$(\pi_{\mathrm{discrete}}|_C)^{-1}(U) = C \cap \pi^{-1}_{\mathrm{discrete}}(U) = \coprod_{\lambda\in F} (C \cap W_\lambda),$$

where each $C \cap W_\lambda$ is open in C. Moreover, if $C \cap W_\lambda \neq \emptyset$, then $W_\lambda \subset C$ because W_λ is connected and C is a connected component of E_{discrete}. Hence $\pi_{\mathrm{discrete}}(C \cap W_\lambda) = \pi_{\mathrm{discrete}}(W_\lambda) = U$. Therefore, for each $\lambda \in F$ with $C \cap W_\lambda \neq \emptyset$, the restriction $\pi_{\mathrm{discrete}}|_{C\cap W_\lambda}\colon C \cap W_\lambda \longrightarrow U$ is a homeomorphism, from which the assertion follows. □

EXAMPLE I.7.29. Let $X := \mathrm{U}(1) \subset \mathbb{C}$, and let $q\colon \mathbb{R} \to X$ be the universal covering map as in Example I.7.3. Let $F := \mathbb{C}$ and let $G := \mathrm{U}(1) \subset \mathbb{C}$ act on F by multiplication. For a fixed element $t \in \mathrm{U}(1)$, define an equivalence relation $\sim_t$ on $\mathbb{R} \times \mathbb{C}$ by

$$(s+1, z) \sim_t (s, tz) \qquad (s \in \mathbb{R},\ z \in \mathbb{C}),$$

and set

$$L_t := (\mathbb{R} \times \mathbb{C})/\sim_t .$$

Let

$$\pi_t\colon L_t \longrightarrow X, \qquad \pi_t([s, z]_t) := q(s),$$

be the induced map. Then π_t is a fiber bundle over X with fiber $\mathbb{C}$ and structure group $\mathrm{U}(1)$. Moreover, since the transition function is constant and equal to t, the bundle π_t is $\mathrm{U}(1)$-flat. We now consider the associated map

$$(\pi_t)_{\mathrm{discrete}} \colon (L_t)_{\mathrm{discrete}} \longrightarrow X.$$

Write $t = e^{2\pi\sqrt{-1}\theta}$ with $\theta \in \mathbb{R}$. Then, for the connected component C that includes $[(0, z_0)] \in (L_t)_{\mathrm{discrete}}$ ($z_0 \in \mathbb{C}$), the following holds: the restriction $(\pi_t)_{\mathrm{discrete}}|_C$ is a homeomorphism when $z_0 = 0$, and when $z_0 \neq 0$,

- if $\theta \in \mathbb{Q}$, $(\pi_t)_{\mathrm{discrete}}|_C$ is a connected covering of degree n and it is isomorphic to the covering $z \mapsto z^n$ in Example I.7.6, where $n \geq 1$ is the smallest integer such that $t^n = 1$;
- if $\theta \notin \mathbb{Q}$, $(\pi_t)_{\mathrm{discrete}}|_C$ is isomorphic to the universal covering map as in Example I.7.3.

□

Fix a base point $x_0 \in X$, and take a point $\xi_0 \in E_{\mathrm{discrete}}$ with $\pi_{\mathrm{discrete}}(\xi_0) = x_0$. Let C be a connected component of E_{discrete} such that $\xi_0 \in C$. Since X is connected and locally path-connected, Theorem I.7.4 applies to the connected covering $\pi_{\mathrm{discrete}}|_C \colon C \to X$. In particular, for any loop $\gamma \colon [0,1] \to X$ based at x_0, there exists a unique lift $\widetilde{\gamma} \colon [0,1] \to C$ such that $\widetilde{\gamma}(0) = \xi_0$ and $\pi_{\mathrm{discrete}} \circ \widetilde{\gamma} = \gamma$. We define

$$F_{\xi_0}(\gamma) := \widetilde{\gamma}(1) \in \pi_{\mathrm{discrete}}^{-1}(x_0) \cap C.$$

Again by applying Theorem I.7.4 to the homotopy maps, it follows that this assignment depends only on the homotopy class $[\gamma] \in \pi_1(X, x_0)$. By construction, $F_{\xi_0}(\gamma)$ belongs to the G-orbit $G \cdot \xi_0$. Indeed, recalling the local construction of the lift, $F_{\xi_0}(\gamma)$ is computed by successively composing the values of the G-valued Cousin datum along the path γ. In particular, the correspondence

$$\pi_1(X, x_0) \times \left(\pi_{\mathrm{discrete}}^{-1}(x_0) \cap C\right) \longrightarrow \pi_{\mathrm{discrete}}^{-1}(x_0) \cap C, \qquad ([\gamma], \xi) \longmapsto F_\xi(\gamma),$$

defines a natural right action of the group $\pi_1(X, x_0)$ on the set $\pi_{\mathrm{discrete}}^{-1}(x_0) \cap C$.

From now on, we additionally assume that the action of G on F is free and transitive (i.e., for any $f_1, f_2 \in F$, there uniquely exists $g \in G$ such that $f_2 = g(f_1)$). Fix an identification of the fiber $\pi^{-1}(x_0)$ with F induced by a local trivialization near x_0, and regard ξ_0 as a point of F via this identification. Since $F_{\xi_0}(\gamma) \in G \cdot \xi_0$ for each $[\gamma] \in \pi_1(X, x_0)$ and the G-action on F is free and transitive, there uniquely exists an element $\rho_E([\gamma]) \in G$ such that

$$F_{\xi_0}(\gamma) = \rho_E([\gamma])(\xi_0).$$

Our product $[\gamma_1][\gamma_2]$ traverses γ_1 first. Consequently endpoint transport satisfies

$$\rho_E([\gamma_1][\gamma_2]) = \rho_E([\gamma_2])\rho_E([\gamma_1]).$$

For non-abelian G we must keep this order: write $\pi_1(X,x_0)^{\mathrm{op}}$ for the same set with reversed multiplication. Then the assignment

$$\rho_E\colon \pi_1(X,x_0)^{\mathrm{op}} \longrightarrow G, \qquad [\gamma] \longmapsto \rho_E([\gamma]),$$

is a group homomorphism. Equivalently, $[\gamma] \mapsto \rho_E([\gamma])^{-1}$ is a homomorphism from the usual fundamental group. We retain endpoint transport as our convention; for abelian G it is itself a homomorphism from the usual fundamental group. We call ρ_E the *monodromy representation* of the G-flat fiber bundle E. Note that the monodromy representation ρ_E depends on the choice of the identification of the fiber $\pi^{-1}(x_0)$ with F, and different choices lead to conjugate representations (see Mini-Exercise I.7.30 (iv) below).

Mini-Exercise I.7.30. Keep the setting above. Let $\{U_j\}_{j\in J}$ be an open covering of X and $\{\Phi_{U_j}\colon \pi^{-1}(U_j) \to U_j \times F\}_{j\in J}$ be local trivializations such that transition functions $\varphi_{U_jU_k}$ are G-valued locally constant maps, by which E is regarded as G-flat fiber bundle. Fix a base point $x_0 \in X$ and an index $j_0 \in J$ such that $x_0 \in U_{j_0}$. For $\xi \in \pi^{-1}_{\mathrm{discrete}}(x_0)$, denote by

$$\rho_{E,j_0,\xi}\colon \pi_1(X,x_0)^{\mathrm{op}} \to G$$

the monodromy representation determined by the choice of ξ, where we identify the fiber $\pi^{-1}(x_0)$ with F using $\Phi_{U_{j_0}}$.
(i) Let $\gamma\colon [0,1] \to X$ be a loop based at x_0. Show that there exist a subdivision $0 = t_0 < t_1 < \cdots < t_{N+1} = 1$ of $[0,1]$ and indices $j_0, j_1, \ldots, j_N \in J$ with $x_0 \in U_{j_0}$ and $j_N = j_0$ such that $\gamma([t_\nu, t_{\nu+2}]) \subset U_{j_\nu}$ holds for each $\nu \in \{0,1,\ldots,N-1\}$, and $\gamma([t_N, t_{N+1}]) \subset U_{j_N}$.
(ii) Let $\gamma\colon [0,1] \to X$ be a loop based at x_0. For a subdivision $0 = t_0 < t_1 < \cdots < t_{N+1} = 1$ and indices $j_0, j_1, \ldots, j_N \in J$ as in (i), show that

$$\rho_{E,j_0,\xi_0}([\gamma]) = \varphi_{U_{j_N}U_{j_{N-1}}}(\gamma(t_N)) \circ \varphi_{U_{j_{N-1}}U_{j_{N-2}}}(\gamma(t_{N-1})) \circ \cdots \circ \varphi_{U_{j_1}U_{j_0}}(\gamma(t_1)).$$

(iii) Let $\xi_0, \xi_1 \in \pi^{-1}_{\mathrm{discrete}}(x_0)$. Show that $\rho_{E,j_0,\xi_1} = \rho_{E,j_0,\xi_0}$ holds.
(iv) Let $k \in J$ be an index such that $x_0 \in U_k$. Show that there exists an element $g \in G$ such that

$$\rho_{E,j_0,\xi_0}([\gamma]) = g \circ \rho_{E,k,\xi_0}([\gamma]) \circ g^{-1}$$

holds for all $[\gamma] \in \pi_1(X,x_0)$.

Conversely, when X is connected, and locally path-connected, and locally simply connected, one can construct a G-flat fiber bundle E_ρ over X from a group homomorphism $\rho\colon \pi_1(X,*)^{\mathrm{op}} \to G$ (*suspension construction*) as follows: Let $\widetilde{X}$ denote the universal covering space of X. We define a G-flat fiber bundle E_ρ over X as the quotient of $\widetilde{X} \times F$ by the equivalence relation

$$([\gamma]\cdot\tilde{x}, \xi) \sim (\tilde{x}, \rho([\gamma])(\xi)) \qquad ([\gamma] \in \pi_1(X,*),\ \tilde{x} \in \widetilde{X},\ \xi \in F),$$

where $\pi_1(X,*)$ acts on $\widetilde{X}$ by deck transformations, using the path model $[\gamma]\cdot[\eta] = [\gamma *_{\mathrm{path}} \eta]$. The reversed multiplication law for ρ makes the displayed identifications compatible with

successive deck transformations. The resulting space is a fiber bundle over X with fiber F and structure group G, and it is G-flat by construction.

In the following Lemma I.7.31, under the following equivalence relation $\cong_{G-\mathrm{flat}}$ on G-flat fiber bundles, we show that the correspondence $E \mapsto \rho_E$ and $\rho \mapsto E_\rho$ induce a canonical bijection between the set of classes of G-flat fiber bundles and conjugate classes of G-representations of $\pi_1(X,*)^{\mathrm{op}}$: For G-flat fiber bundles $\pi\colon E \to X$ and $\pi'\colon E' \to X$ over X with the same fiber F and structure group G, we write $E \cong_{G-\mathrm{flat}} E'$ if there exists an isomorphism $\Psi\colon E \to E'$ as fiber bundles with structure group G such that the induced map $\Psi_{\mathrm{discrete}}\colon E_{\mathrm{discrete}} \to E'_{\mathrm{discrete}}$ is also continuous (and hence it is a homeomorphism), where E_{discrete} and E'_{discrete} are coverings of X constructed from E and E', respectively, in the same manner as explained above.

Lemma I.7.31. *Let X be a connected, locally path-connected, and locally simply connected topological space, and fix a base point $x_0 \in X$. Let F be a non-empty topological space and $G \subset \mathrm{Homeo}(F)$ a subgroup. Assume that G is endowed with a topology making it a topological group whose action on F is continuous. Assume also that the action of G on F is free and transitive. Let $\pi\colon E \to X$ and $\pi'\colon E' \to X$ be G-flat fiber bundles over X with the same fiber F and structure group G. Then the following are equivalent:*
(i) $E \cong_{G-\mathrm{flat}} E'$.
(ii) There exists an element $g \in G$ such that $\rho_{E'}(\gamma) = g\,\rho_E(\gamma)\,g^{-1}$ for all $\gamma \in \pi_1(X,x_0)$, where ρ_E and $\rho_{E'}$ are monodromies of E and E', respectively.
Moreover, the correspondence $E \mapsto \rho_E$ and $\rho \mapsto E_\rho$ explained above induce a canonical bijection between the following two quotient sets

- *The quotient of the set of G-flat fiber bundles over X with fiber F and structure group G, modulo $\cong_{G-\mathrm{flat}}$;*
- $\mathrm{Hom}_{\mathrm{group}}\big(\pi_1(X,x_0)^{\mathrm{op}}, G\big)/G$, *where G acts on* $\mathrm{Hom}_{\mathrm{group}}(\pi_1(X,x_0)^{\mathrm{op}}, G)$ *by conjugation.*

Proof. Assume that $E \cong_{G-\mathrm{flat}} E'$. By definition, there exists an isomorphism $\Psi\colon E \to E'$ as fiber bundles with structure group G such that the induced map

$$\Psi_{\mathrm{discrete}}\colon E_{\mathrm{discrete}} \longrightarrow E'_{\mathrm{discrete}}$$

is continuous (hence a homeomorphism) and satisfies $\pi'_{\mathrm{discrete}} \circ \Psi_{\mathrm{discrete}} = \pi_{\mathrm{discrete}}$. Fix a base point $x_0 \in X$ and take $\xi_0 \in E_{\mathrm{discrete}}$ with $\pi_{\mathrm{discrete}}(\xi_0) = x_0$. Let $C \subset E_{\mathrm{discrete}}$ be the connected component containing ξ_0, and let $C' \subset E'_{\mathrm{discrete}}$ be the connected component containing $\xi'_0 := \Psi_{\mathrm{discrete}}(\xi_0)$. Then $\Psi_{\mathrm{discrete}}|_C\colon C \to C'$ is a homeomorphism over X. In particular, for any loop $\gamma\colon [0,1] \to X$ based at x_0, if $\widetilde{\gamma}$ denotes the lift of γ to C with $\widetilde{\gamma}(0) = \xi_0$, then $\Psi_{\mathrm{discrete}} \circ \widetilde{\gamma}$ is the lift of γ to C' starting at ξ'_0, from which (ii) follows.

Conversely, assume that there exists $g \in G$ such that $\rho_{E'}([\gamma]) = g\,\rho_E([\gamma])\,g^{-1}$ for all $[\gamma] \in \pi_1(X,x_0)$. Let $\widetilde{X}$ be the universal covering space of X, and consider the suspension constructions E_{ρ_E} and $E_{\rho_{E'}}$ associated with ρ_E and $\rho_{E'}$, respectively:

$$E_{\rho_E} := (\widetilde{X} \times F)/\sim_{\rho_E}, \qquad E_{\rho_{E'}} := (\widetilde{X} \times F)/\sim_{\rho_{E'}},$$

where

$$([\gamma]\cdot\widetilde{x},\xi)\sim_{\rho_E}(\widetilde{x},\rho_E([\gamma])(\xi)),\qquad([\gamma]\cdot\widetilde{x},\xi)\sim_{\rho_{E'}}(\widetilde{x},\rho_{E'}([\gamma])(\xi)).$$

Define a map

$$\Theta\colon\widetilde{X}\times F\longrightarrow\widetilde{X}\times F,\qquad\Theta(\widetilde{x},\xi):=(\widetilde{x},g(\xi)).$$

Then Θ descends to a well-defined homeomorphism

$$\overline{\Theta}\colon E_{\rho_E}\longrightarrow E_{\rho_{E'}},$$

from which (i) follows.

Let us denote by $\mathcal{F}(X;F,G)$ the set of isomorphism classes of G-flat fiber bundles over X with fiber F and structure group G, with respect to $\cong_{G-\mathrm{flat}}$. By construction, the correspondence $E\mapsto\rho_E$ defines a map

$$\mathcal{F}(X;F,G)\longrightarrow\mathrm{Hom}_{\mathrm{group}}(\pi_1(X,x_0)^{\mathrm{op}},G)/G,$$

because changing the identification of $\pi^{-1}(x_0)$ with F changes ρ_E by conjugation (Mini-Exercise I.7.30 (iv)), and by $(i)\Rightarrow(ii)$ the class of ρ_E depends only on the class of E. Conversely, given $\rho\in\mathrm{Hom}_{\mathrm{group}}(\pi_1(X,x_0)^{\mathrm{op}},G)$, the suspension construction produces a G-flat fiber bundle E_ρ, and by $(ii)\Rightarrow(i)$ its class depends only on the conjugacy class of ρ. Finally, $E\mapsto\rho_E$ and $\rho\mapsto E_\rho$ are inverse to each other by construction, hence the claimed bijection follows. □

Remark I.7.32. Keep the setting of Lemma I.7.31. Assume in addition that the group G is abelian. Then the same maps are homomorphisms from $\pi_1(X,x_0)$ and from $\pi_1(X,x_0)^{\mathrm{op}}$, so we omit the superscript here. The conjugation action of G on $\mathrm{Hom}_{\mathrm{group}}(\pi_1(X,x_0),G)$ is trivial, and hence Lemma I.7.31 yields a canonical bijection between the quotient set of G-flat fiber bundles over X with fiber F and structure group G, modulo $\cong_{G-\mathrm{flat}}$, and $\mathrm{Hom}_{\mathrm{group}}(\pi_1(X,x_0),G)$. Moreover, since G is abelian, we have

$$\mathrm{Hom}_{\mathrm{group}}(\pi_1(X,x_0),G)=\mathrm{Hom}_{\mathbb{Z}}(\pi_1(X,x_0)^{\mathrm{ab}},G)\cong\mathrm{Hom}_{\mathbb{Z}}(H_1(X),G),$$

as $\pi_1(X,x_0)^{\mathrm{ab}}$ is naturally isomorphic to $H_1(X)$ (see Example I.7.12). As $H_0(X)$ is a free abelian group, from the universal coefficient theorem for cohomology (20) it follows that $\mathrm{Hom}_{\mathbb{Z}}(H_1(X),G)$ is naturally isomorphic to $H^1(X,G)$. Therefore, G-flat fiber bundles over X with fiber F and structure group G, modulo $\cong_{G-\mathrm{flat}}$, can be classified by considering the singular cohomology $H^1(X,G)$. □

Remark I.7.33. Example I.7.29 also illustrates that the equivalence relation $\cong_{\mathrm{U}(1)-\mathrm{flat}}$ is in general strictly finer than the usual notion of isomorphism of fiber bundles with structure group $\mathrm{U}(1)$, where one forgets the flat structure. Indeed, for $t\in\mathrm{U}(1)$, the $\mathrm{U}(1)$-flat fiber bundle $\pi_t\colon L_t\to X$ as in Example I.7.29 has monodromy representation

$$\rho_t\colon\pi_1(X,x_0)\cong\mathbb{Z}\longrightarrow\mathrm{U}(1),\qquad1\longmapsto t.$$

Here the monodromy is that of the unit-circle subbundle of L_t, whose fiber carries a free and transitive $\mathrm{U}(1)$-action. A flat isomorphism between these subbundles extends fiberwise

by complex linearity to the corresponding bundles L_t. Since $\mathrm{U}(1)$ is abelian, it follows from Lemma I.7.31, applied to these unit-circle subbundles, that

$$L_t \cong_{\mathrm{U}(1)-\mathrm{flat}} L_{t'} \quad \Longleftrightarrow \quad t = t'.$$

On the other hand, all bundles L_t are isomorphic to each other as fiber bundles with structure group $\mathrm{U}(1)$. To see this explicitly, write $t = e^{2\pi\sqrt{-1}\theta}$ with $\theta \in [0,1)$, and define a map

$$\sigma_t \colon X \longrightarrow L_t, \qquad \sigma_t\big(q(s)\big) := \Big[\, s, e^{-2\pi\sqrt{-1}\theta s} \,\Big]_t,$$

where $[\bullet]_t$ denotes the equivalence class of $\sim_t$. This map is well-defined, since

$$\Big(s+1, e^{-2\pi\sqrt{-1}\theta(s+1)}\Big) = \Big(s+1, t^{-1}e^{-2\pi\sqrt{-1}\theta s}\Big) \sim_t \Big(s, e^{-2\pi\sqrt{-1}\theta s}\Big),$$

and it defines a continuous global section of π_t. Using this section, we define a map

$$\Phi_t \colon X \times \mathbb{C} \longrightarrow L_t, \qquad \Phi_t\big(q(s), z\big) := [\, s, e^{-2\pi\sqrt{-1}\theta s} z \,]_t.$$

One checks directly that Φ_t is well-defined, continuous, fiber-preserving, and $\mathrm{U}(1)$-equivariant. Moreover, Φ_t is a homeomorphism with inverse

$$[\, s, w \,]_t \longmapsto \big(q(s), e^{2\pi\sqrt{-1}\theta s} w\big).$$

Hence Φ_t is an isomorphism of fiber bundles with structure group $\mathrm{U}(1)$, and in particular

$$L_t \cong_{\mathrm{U}(1)-\mathrm{str}} X \times \mathbb{C} \cong_{\mathrm{U}(1)-\mathrm{str}} L_1.$$

□

In this note, the following classes of fiber bundles play particularly important roles:

- *Vector bundles*, where the fiber F is a vector space and the structure group G is a group of linear transformations. Refer to the next subsection for vector bundles.
- *Affine bundles*, where the fiber F is an affine space (a vector space without a distinguished origin) and G is the group of affine transformations.
- *Projective space bundles*, where the fiber F is a projective space and G is a group of projective linear transformations.

Up to this point, we have discussed fiber bundles over a fixed base space. Before closing this subsection, we briefly introduce the notion of the pull-back of a fiber bundle, which will be used repeatedly in what follows.

Let $\pi \colon E \to X$ be a fiber bundle over a topological space X with fiber F and structure group G, and let $f \colon Y \to X$ be a continuous map from another topological space Y. We define the *pull-back bundle* of E along f as

$$f^*E := \{(y, \xi) \in Y \times E \mid f(y) = \pi(\xi)\},$$

endowed with the subspace topology inherited from $Y \times E$. The projection

$$\pi_{f^*E} \colon f^*E \longrightarrow Y, \qquad (y, \xi) \longmapsto y,$$

is continuous and surjective, and makes f^*E into a fiber bundle over Y with the same fiber F and the same structure group G. We call f^*E the *pull-back of E by f*.

If $\{U_j\}_{j\in J}$ is an open covering of X with local trivializations $\Phi_{U_j} \colon \pi^{-1}(U_j) \to U_j \times F$, then $\{f^{-1}(U_j)\}_{j\in J}$ is an open covering of Y, and f^*E admits induced local trivializations

$$f^{-1}(U_j) \times F \longrightarrow \pi_{f^*E}^{-1}\big(f^{-1}(U_j)\big), \qquad (y, \xi) \longmapsto (y, \Phi_{U_j}^{-1}(f(y), \xi)),$$

whose transition functions are $\varphi_{jk} \circ f$, the pull-backs of those of E. In particular, if E has structure group G, then so does f^*E. Moreover, if E is G-flat, then the pull-back bundle f^*E is also G-flat. If $\Psi \colon E \to E'$ is an isomorphism of fiber bundles with structure group G, then it induces a canonical isomorphism

$$f^*\Psi \colon f^*E \longrightarrow f^*E'$$

of pull-back bundles over Y.

When Y is a subspace of X and $f \colon Y \hookrightarrow X$ is the inclusion map, the pull-back bundle f^*E is often denoted by $E|_Y$ and is called the *restriction of E to Y*.

7.7 Vector bundles

We now specialize the general theory of fiber bundles to the most important case for this note, namely *vector bundles*.

Throughout this note, unless otherwise stated, a vector bundle means a finite-rank complex vector bundle. More precisely, a vector bundle of rank $r \in \mathbb{Z}_{\geq 0}$ over a topological space X is a fiber bundle $\pi \colon E \to X$ with fiber $F = \mathbb{C}^r$ and structure group $\mathrm{GL}_r(\mathbb{C})$, the group of all invertible complex $r \times r$ matrices, acting on $\mathbb{C}^r$ by the standard linear action. The integer r is called the *rank* of the vector bundle. For a vector bundle, a local trivialization

$$\Phi_U \colon \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^r$$

is equivalently described by a choice of r continuous sections

$$e_1, e_2, \ldots, e_r \colon U \longrightarrow E$$

such that, for each $x \in U$, the vectors $e_1(x), \ldots, e_r(x)$ form a basis of the vector space $\pi^{-1}(x)$. Such a family of sections is called a *local frame* of E over U. The vector bundle E admits a canonical continuous section $0_X \colon X \longrightarrow E$, called the *zero section*, which assigns to each point $x \in X$ the zero vector in the fiber $\pi^{-1}(x) \cong \mathbb{C}^r$. By abuse of notation, we also refer to the image of the map 0_X as the *zero section*, and denote it by

$$(\text{Zero Section})_E \subset E.$$

An isomorphism of vector bundles over X is, by definition, an isomorphism of fiber bundles with structure group $\mathrm{GL}_r(\mathbb{C})$. In what follows, we write

$$E \cong_{\mathrm{top}} E'$$

to denote such an isomorphism (instead of $\cong_{\mathrm{GL}_r(\mathbb{C})\text{-str}}$), and refer to it simply as a *topological isomorphism of vector bundles*. In this situation, we also say that E and E' are *topologically isomorphic*. If a vector bundle E over X is topologically isomorphic to the product bundle $X \times \mathbb{C}^r$ (the *trivial vector bundle of rank* r), we say that E is *topologically trivial*.

Since vector bundles are fiber bundles with structure group $\mathrm{GL}_r(\mathbb{C})$, the linear structure of the fibers is preserved under transition maps. Accordingly, many isomorphism-invariant notions from linear algebra can be transported to the setting of vector bundles by applying them fiberwise and checking compatibility of transition functions. Typical examples include the following for vector bundles E and E' over the same topological space:

- The *direct sum* $E \oplus E'$ is defined by taking the direct sum of transition functions.
- The *dual bundle* E^* is obtained by taking the dual representations of the transition functions.
- The *complex conjugate* $\overline{E}$, the *tensor product* $E \otimes E'$, the *wedge product* $\bigwedge^k E$, the *determinant bundle* $\det E := \bigwedge^{\operatorname{rank} E} E$, and the *Hom bundle* $\operatorname{Hom}(E, E')$ are defined similarly.
- *Subbundles* and *quotient bundles* are defined fiberwise, provided the corresponding compatibility conditions are satisfied.
- A *continuous Hermitian metric* on E is a fiberwise positive-definite Hermitian form varying continuously with the base point.

A small caution is in order concerning morphisms between vector bundles. When we say that a map

$$f \colon E \longrightarrow E'$$

between vector bundles $\pi \colon E \to X$ and $\pi' \colon E' \to X$ over the same base space X is a *(continuous) morphism of vector bundles*, we always mean that f satisfies the following conditions:

- f is continuous and fiber-preserving: i.e., $\pi' \circ f = \pi$;
- for each $x \in X$, the restriction $f_x \colon \pi^{-1}(x) \to (\pi')^{-1}(x)$ is linear;
- the rank of f_x is locally constant (in particular, constant on each connected component of X).

The last condition ensures that the kernel $\operatorname{Ker} f$, the image $\operatorname{Image} f$, and the cokernel $\operatorname{Coker} f$ naturally define vector subbundles or quotient bundles on each open and closed subset of X on which the rank of f_x is constant. Based on this notion of morphism, one can define *(short) exact sequences* of vector bundles fiberwise in the usual manner.

Lemma I.7.34. *Let X be a paracompact Hausdorff topological space. Then any vector bundle E over X admits a continuous Hermitian metric.*

Proof. Let $\{U_j\}_{j\in J}$ be an open covering of X such that $E|_{U_j}$ is trivial for each j. As X is paracompact, we may assume that $\{U_j\}_{j\in J}$ is locally finite. Denote by h_j the continuous Hermitian metric on $E|_{U_j}$ obtained by identifying $E|_{U_j}$ with $U_j\times\mathbb{C}^r$ and equipping each fiber with the standard Hermitian metric on $\mathbb{C}^r$. Take a partition of unity $\{\varphi_j\}_{j\in J}$ subordinate to $\{U_j\}_{j\in J}$ (see Mini-Exercise I.3.14 (*ii*)). Then $h:=\sum_{j\in J}\varphi_j\cdot h_j$ defines a continuous Hermitian metric on E. □

Let X be a connected, locally path-connected, and locally simply connected topological space, and let $G:=\mathrm{U}(r)$ or $G:=\mathrm{GL}_r(\mathbb{C})$. A vector bundle on X is said to be a *G-flat vector bundle* over X if it admits a structure as G-flat fiber bundle. Note that, when $G=\mathrm{U}(r)$, the fiberwise Euclidean inner product defines a continuous Hermitian metric on E, which is called the *flat metric* on E.

In what follows, for simplicity, we restrict ourselves to the case of rank one. Note that the vector bundles of rank one are called *line bundles*. Since the action of $G=\mathrm{GL}_1(\mathbb{C})=\mathbb{C}^*$ or $\mathrm{U}(1)$ on $\mathbb{C}$ is not free and transitive, the monodromy construction does not apply directly to the whole line bundle E. However, as

- $\mathbb{C}^*$ acts freely and transitively on $\mathbb{C}^*$, and
- $\mathrm{U}(1)$ acts freely and transitively on the unit circle of $\mathbb{C}$,

we define the *monodromy* of E to be the monodromy of the $\mathbb{C}^*$-flat fiber bundle $E\setminus(\text{Zero Section})_E$ (resp. the $\mathrm{U}(1)$-flat fiber bundle $\{\xi\in E\mid|\xi|_{\mathrm{flat}}=1\}$, where $|\bullet|_{\mathrm{flat}}$ denotes the (fiberwise) norm of the flat metric).

The two general classification results in the previous subsection will be used repeatedly in this note, especially in the important case of vector bundles. For the reader's convenience, we record their vector-bundle versions here.

Lemma I.7.35. *Let X be a topological space and r be a non-negative integer. Then the correspondence described in Mini-Exercise I.7.26 induces a canonical bijection between the following two quotient sets:*

- *The quotient of the set of vector bundles of rank r over X, modulo $\cong_{\mathrm{top}}$;*
- *The quotient of the set of $\mathrm{GL}_r(\mathbb{C})$-valued continuous Cousin data, modulo the following equivalence relation: two $\mathrm{GL}_r(\mathbb{C})$-valued continuous Cousin data $\mathfrak{f}$ and $\mathfrak{g}$ are said to be equivalent if, after passing to a common refinement $\{U_j\}_{j\in J}$ of open coverings of X and writing them as $\mathfrak{f}=\{f_{jk}\colon U_j\cap U_k\to\mathrm{GL}_r(\mathbb{C})\}_{j,k\in J}$ and $\mathfrak{g}=\{g_{jk}\colon U_j\cap U_k\to\mathrm{GL}_r(\mathbb{C})\}_{j,k\in J}$, there exist continuous maps $\psi_j\colon U_j\to\mathrm{GL}_r(\mathbb{C})$ such that $\psi_j(x)\cdot f_{jk}(x)\cdot\psi_k(x)^{-1}=g_{jk}(x)$ holds for all $x\in U_j\cap U_k$.*

Proof. This is the special case of Mini-Exercise I.7.26 for $F=\mathbb{C}^r$ and $G=\mathrm{GL}_r(\mathbb{C})$. □

Lemma I.7.36. *Let X be a connected, locally path-connected, and locally simply connected topological space. Let $G := \mathrm{U}(1)$ or $G := \mathbb{C}^*$ act on $F := \mathbb{C}$ by the standard action. Then the monodromy representation and the suspension construction induce canonical bijections between the following three sets:*

- *The quotient of the set of G-flat line bundles, modulo $\cong_{G-\mathrm{flat}}$;*
- $\mathrm{Hom}_{\mathrm{group}}(\pi_1(X, *), G)$;
- $H^1(X, G)$.

Proof. Apply Lemma I.7.31 and Remark I.7.32 to the present setting (Note that both $\mathrm{U}(1)$ and $G = \mathbb{C}^*$ are abelian). □

Remark I.7.37. We note that the statement of Lemma I.7.35 can be considerably streamlined once *Čech cohomology* is introduced, which will be done in the next chapter. In particular, for line bundles, the classification becomes especially transparent from the cohomological viewpoint.

In what follows, we shall restrict ourselves to triangulable spaces. On such a space X, it can be shown that the set of isomorphism classes of line bundles over X, modulo $\cong_{\mathrm{top}}$, is in canonical bijection with $H^2(X, \mathbb{Z})$, which will also be explained in the next chapter. However, it is worth emphasizing that this classification can already be understood using only the material developed so far: Indeed, one classical approach is to use the fact that a CW complex $\mathbf{P}^\infty := \varinjlim_n \mathbf{P}^n$ carries a distinguished line bundle, often called the *universal line bundle*, such that every line bundle over a paracompact space X can be realized as the pullback of this universal line bundle along a suitable continuous map $X \to \mathbf{P}^\infty$, which leads to the classification of line bundles by $H^2(X, \mathbb{Z})$. We refer the reader to [Ha2, Chapter 1] for a detailed exposition of this approach and its relationship with cohomology. □

Most of the notions and constructions discussed in this subsection admit straightforward analogues for *real vector bundles*, namely fiber bundles whose fibers are finite-dimensional real vector spaces and whose structure groups are $\mathrm{GL}_r(\mathbb{R})$ or its subgroups. In particular, concepts such as metrics on real vector bundles, as well as their dual bundles, tensor bundles, and wedge product bundles, can be developed in complete parallel with the complex case. These real vector bundle versions will be discussed in the next subsection. However, from the next chapter onward, the main objects of study in this note will be complex vector bundles. Accordingly, when we wish to emphasize the real case, we will explicitly use the terms "real vector bundle" or "real line bundle": As already stated at the beginning of this subsection, unless otherwise stated, the term "vector bundle" (resp. line bundle) will always mean a complex vector bundle (resp. complex line bundle).

8 From differential geometry

In this section, we collect some basic facts and conventions on C^∞ manifolds.

Unless otherwise specified, the term *manifold* always means a Hausdorff second-countable manifold without boundary. In particular, throughout this note, a *closed manifold* means a compact manifold without boundary; hence, under our standing convention "without boundary", compactness is equivalent to closedness.

By a C^∞ *manifold* we mean a real differentiable manifold X in the usual sense, namely a manifold which is locally C^∞-diffeomorphic to an open subset of a finite-dimensional real Euclidean space (that is Hausdorff, second countable, and without boundary). The term "smooth" is often used synonymously with "C^∞"; however, when confusion with the word "smooth" in complex/algebraic geometry (e.g., "non-singular") may arise, we will avoid the word "smooth" and say "C^∞" instead.

For a C^∞ manifold X, its real dimension is denoted by $\dim_{\mathbb{R}} X$ (We reserve the notation dim for the complex dimension of complex vector spaces and complex manifolds, or the Krull dimension of a ring). We do *not* assume that a C^∞ manifold is connected. Thus, strictly speaking, the real dimension is defined pointwise: for $x \in X$ we write $\dim_{\mathbb{R},x} X$ for the local dimension at x, which is well-defined (Refer to Mini-Exercise I.7.18). When we say that X is of real dimension n (or write $\dim_{\mathbb{R}} X = n$) for an integer $n \in \mathbb{Z}_{\geq 0}$, we mean that $\dim_{\mathbb{R},x} X = n$ holds for every $x \in X$. For the later convention on complex dimensions, which distinguishes dimension from pure dimension, see Remark III.2.41 in Chapter III.

For a C^∞ manifold X, the following topological properties are fundamental:

- X is locally compact, locally connected, and locally simply connected (since $\mathbb{R}^n$ has these properties; see, for instance, [Lee-Jo2, Propositions 1.11 and 1.12]);
- X is paracompact (it follows from second countability; see [Lee-Jo2, Theorem 1.15]).

We assume that the reader is familiar with the basic notions such as *charts* and *local coordinates*, *partitions of unity* (Recall that on a smooth manifold, any partition of unity can be chosen to be C^∞; accordingly, whenever we speak of a partition of unity on a C^∞ manifold, we will always mean a C^∞ one), *tangent spaces* T_xX $(x \in X)$, *the tangent bundle* TX, *orientations*, and C^∞ *maps* between manifolds (see [Lee-Je], [Lee-Jo2], [Lee-Jo3] for example). By a *submanifold* of a C^∞ manifold, we always mean a properly embedded submanifold, unless explicitly stated otherwise. When a submanifold is only given as the image of an immersion and is not necessarily embedded, we will explicitly refer to it as an *immersed submanifold*. For a C^∞ map $f\colon X \to Y$ between C^∞ manifolds, the induced map on tangent bundles is usually denoted by

$$f_*\colon TX \longrightarrow TY,$$

whose fiberwise restriction $(f_*)_x\colon T_xX \to T_{f(x)}Y$ can be identified in local coordinates with the Jacobian matrix $Df(x)$. Accordingly, one also writes $(Df)_x$ or D_xf in some texts; however, in contexts where the symbol D is reserved for connections, we will avoid this notation. One also encounters the notation $(df)_x$, but we will basically avoid it as well, to prevent confusion with the exterior derivative. For an open interval $I \subset \mathbb{R}$ and a C^∞ curve $\gamma\colon I \to X$, we denote by

$$\dot{\gamma}(t_0) := (\gamma_*)_{t_0}\left(\frac{d}{dt}\right) \in T_{\gamma(t_0)}X$$

its *velocity vector* at $t_0 \in I$, where t is the Euclidean coordinate of $\mathbb{R}$.

The dual vector space of the tangent space T_xX at a point $x \in X$ (the *cotangent space* of X at x) is denoted by T^*_xX, and the dual vector bundle of TX (the *cotangent bundle* of X) is denoted by T^*X. For a C^∞ map $f\colon X \to Y$ between C^∞ manifolds and each $x \in X$, the induced map on cotangent spaces (the *pull-back*) is

$$f^*_x\colon T^*_{f(x)}Y \longrightarrow T^*_xX, \qquad f^*_x(\eta) := \eta \circ (f_*)_x.$$

We shall use the same notation f^* for the induced pull-back on sections of T^*Y and on its exterior powers (We also assume that the reader is familiar with the basic properties of pull-backs of differential forms, the exterior derivative d, and Stokes' theorem on smooth manifolds, as can be found, for example, in [Lee-Jo2, Chapters 14 and 16] or [Sp]).

We also recall that every C^∞ manifold admits a (proper) C^∞ *exhaustion function*: there exists a C^∞ function $h\colon X \to \mathbb{R}_{>0}$ such that the *sublevel set* $\{x \in X \mid h(x) \leq c\}$ is compact for every $c \in \mathbb{R}$ (see [Lee-Jo2, Proposition 2.28] for example).

Finally, in the previous section we introduced (real) vector bundles, their sections, and (Hermitian) metrics over a general topological space. When the base space is a C^∞ manifold, it is often natural to work with the corresponding C^∞ notions: C^∞ (real) vector bundles, C^∞ sections, and C^∞ (Hermitian) metrics. We will use these notions freely in what follows. Note that any C^∞ real vector bundle E over a C^∞ manifold X admits a C^∞ metric, and any C^∞ vector bundle E over X admits a C^∞ Hermitian metric: Indeed, such metrics can be shown in the same manner as in the proof of Lemma I.7.34, just by replacing a partition of unity used in the argument there with a C^∞ one. Note also that TX gives an example of a C^∞ real vector bundle over a C^∞ manifold X, and that a C^∞ metric on TX is called a *Riemannian metric* on X.

8.1 Some basic analytic facts on C^∞ manifolds

Let X be a C^∞ manifold of real dimension n. By definition, any notion that is defined on open subsets of $\mathbb{R}^n$ and is invariant under C^∞ coordinate changes naturally gives rise to a corresponding notion on X. As a typical example, recall the following classical fact: for each non-negative integer n, if $U \subset \mathbb{R}^n$ is an open subset, $A \subset U$ is a subset of Lebesgue measure zero, and $f\colon U \to \mathbb{R}^n$ is a C^∞ map, then the image $f(A)$ is again a subset of Lebesgue measure zero (see [Lee-Jo2, Proposition 6.5]). Since X can be covered by at most countably many coordinate neighborhoods (which follows from the second countability of X), it follows that the property "being of *Lebesgue measure zero*" is well-defined for subsets of X (We emphasize that this does *not* mean that there is a canonical notion of "the Lebesgue measure on X": Indeed, as will be recalled later, once a Riemannian metric or a volume form on X is chosen, one obtains a specific measure on X, and different choices lead to different measures. However, the property of a subset being of measure zero is independent of such choices, and hence is intrinsic to the smooth structure of X).

One of the most important results related to subsets of measure zero is *Sard's theorem*: For a C^∞ map $f\colon X \to Y$ between C^∞ manifolds, the set of critical values of f is a subset of Y of Lebesgue measure zero (see [Lee-Jo2, Theorem 6.10]. Recall that a point $x \in X$ is called

a *critical point* of f if the differential $(f_*)_x \colon T_xX \longrightarrow T_{f(x)}Y$ is not surjective; otherwise, x is called a *regular point*. A point $y \in Y$ is called a *critical value* of f if there exists a critical point $x \in X$ such that $f(x) = y$; otherwise, y is called a *regular value*).

MINI-EXERCISE I.8.1. Let $f \colon X \to Y$ be a C^∞ map between C^∞ manifolds.
(*i*) For each regular point $p \in X$ of f, show that there exist local coordinates $(x^1, x^2, \dots, x^m)$ on a neighborhood of p in X and $(y^1, y^2, \dots, y^n)$ on a neighborhood of $f(p)$ in Y such that $p = (0, 0, \dots, 0)$, $f(p) = (0, 0, \dots, 0)$, and with respect to these coordinates, the map f is written as

$$f(x^1, x^2, \dots, x^m) = (x^1, x^2, \dots, x^n),$$

where $m := \dim_{\mathbb{R},p} X$ and $n := \dim_{\mathbb{R},f(p)} Y$.
(*ii*) For each regular value $q \in Y$ of f, show that the fiber $f^{-1}(q)$ is a C^∞ submanifold of X.

Corollary I.4.19 immediately implies the following global formulation on C^∞ manifolds.

PROPOSITION I.8.2. *Let X be a C^∞ manifold and let v be a C^∞* vector field *on X (i.e., v is a C^∞ global section of TX). Then there exist an open neighborhood $U \subset \mathbb{R} \times X$ of $\{0\} \times X$ and a C^∞ map $\varphi \colon U \to X$ such that the following properties hold.*

- *For every $x \in X$, the set $U \cap (\mathbb{R} \times \{x\})$ is connected.*
- *For every $x \in X$, one has $\varphi(0, x) = x$.*
- *For any $x \in X$ and $s, t \in \mathbb{R}$, if $(t, x), (s + t, x) \in U$ and if $(s, \varphi(t, x)) \in U$, then*

$$\varphi(s, \varphi(t, x)) = \varphi(s + t, x).$$

- *For any $x \in X$ and any $t_0 \in \mathbb{R}$ with $(t_0, x) \in U$, the curve $t \mapsto \varphi(t, x)$ satisfies*

$$\left.\frac{d}{dt}\right|_{t=t_0} \rho(\varphi(t, x)) = v_{\varphi(t_0,x)}(\rho)$$

  for every C^∞ function $\rho \colon X \to \mathbb{R}$. Equivalently, the velocity vector of the curve $t \mapsto \varphi(t, x)$ at $t = t_0$ coincides with the vector $v_{\varphi(t_0,x)} \in T_{\varphi(t_0,x)}X$.

Moreover, if X is compact, then there exists a positive number $\varepsilon > 0$ such that one can take $U = (-\varepsilon, \varepsilon) \times X$.

PROOF. Fix a point $x_0 \in X$, and take a coordinate chart

$$\kappa \colon W \longrightarrow \widetilde{W} \subset \mathbb{R}^m \qquad (m := \dim_{\mathbb{R},x_0} X)$$

with $x_0 \in W$. Let $\widetilde{v}$ be the push-forward of $v|_W$ to $\widetilde{W}$, namely, the C^∞ vector field on $\widetilde{W}$ defined by

$$\widetilde{v}_p := (\kappa_*)_{\kappa^{-1}(p)}\big(v_{\kappa^{-1}(p)}\big) \in T_p\widetilde{W} \cong \mathbb{R}^m \qquad (p \in \widetilde{W}).$$

By Corollary I.4.19 applied to $\widetilde{v}$ on the open set $\widetilde{W} \subset \mathbb{R}^m$, there exist a positive number $\varepsilon_{x_0} > 0$, an open neighborhood $\widetilde{W}_{x_0} \subset \widetilde{W}$ of $\kappa(x_0)$, and a C^∞ map

$$\widetilde{\varphi}_{x_0} \colon I_{\varepsilon_{x_0}} \times \widetilde{W}_{x_0} \to \widetilde{W}$$

which is a local flow of $\widetilde{v}$. Define a C^∞ map

$$\varphi_{x_0} \colon I_{\varepsilon_{x_0}} \times W_{x_0} \to W$$

by

$$\varphi_{x_0}(t, x) := \kappa^{-1}\big(\widetilde{\varphi}_{x_0}(t, \kappa(x))\big),$$

where $W_{x_0} := \kappa^{-1}(\widetilde{W}_{x_0}) \subset W$. Then φ_{x_0} is a local flow of v on W_{x_0}, i.e., it satisfies $\varphi_{x_0}(0, x) = x$, the flow property whenever both sides are defined, and the ODE condition

$$\left.\frac{d}{dt}\right|_{t=t_0} \rho(\varphi_{x_0}(t, x)) = v_{\varphi_{x_0}(t_0, x)}(\rho)$$

for every C^∞ function $\rho \colon X \to \mathbb{R}$ and every $(t_0, x) \in I_{\varepsilon_{x_0}} \times W_{x_0}$ (indeed, this can be checked in local coordinates).

Consider the family of pairs

$$\big\{\big(I_{\varepsilon_{x_0}} \times W_{x_0},\ \varphi_{x_0}\big)\big\}_{x_0 \in X}.$$

If $x_0, x_1 \in X$ and the domains $I_{\varepsilon_{x_0}} \times W_{x_0}$ and $I_{\varepsilon_{x_1}} \times W_{x_1}$ overlap, then on the overlap the two maps φ_{x_0} and φ_{x_1} solve the same ODE with the same initial condition. By the local uniqueness statement in Corollary I.4.19, for each $x \in W_{x_0} \cap W_{x_1}$ the two curves coincide on their whole common time interval, which contains 0 (The coincidence set is non-empty, closed by continuity, and open by local uniqueness). Thus the two maps coincide on the overlap. Hence the family $(\varphi_{x_0})_{x_0 \in X}$ glues to a well-defined C^∞ map $\varphi \colon U \to X$ on an open neighborhood $U \subset \mathbb{R} \times X$ of $\{0\} \times X$, and this map satisfies the stated properties. The connectedness of $U \cap (\mathbb{R} \times \{x\})$ follows from the fact that, for each fixed x, the set of times t for which $\varphi(t, x)$ is defined is an open interval containing 0.

Finally, assume that X is compact. By the argument above, for each $x \in X$ there exist $\varepsilon_x > 0$ and an open neighborhood $W_x \subset X$ of x such that the flow is defined on $I_{\varepsilon_x} \times W_x$. By compactness, we can choose finitely many points $x_1, \ldots, x_N \in X$ such that $X = \bigcup_{j=1}^N W_{x_j}$. Set $\varepsilon := \min_{1 \le j \le N} \varepsilon_{x_j} > 0$. Then, by the gluing argument as above, the flow φ is defined on $(-\varepsilon, \varepsilon) \times X$. □

For a map φ as in Proposition I.8.2, the value $\varphi(t, x)$ is often denoted by $\varphi_t(x)$. In some contexts (for instance, when X is a Lie group and v is a left-invariant vector field), one also writes $\varphi(t, x) = \exp(tv)(x)$ to emphasize that φ is the *(local) flow* generated by the vector field v, that is, a *(local) one-parameter group action* of $\mathbb{R}$ on X. See [Lee-Jo2, Chapter 9] for further details.

At this point, in order to avoid possible confusion concerning the notation exp, let us recall another notion of exponential map arising in Riemannian geometry.

Suppose that a C^∞ Riemannian metric g on X is fixed. Then g determines a unique *Levi–Civita connection*, and hence a well-defined notion of *geodesics* on X: For each point $x \in X$ and each tangent vector $v \in T_xX$, there exists a unique geodesic $\gamma_{x,v}\colon (-\varepsilon,\varepsilon) \to X$ such that $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = v$, namely $\gamma_{x,v}$ is the unique C^∞ curve determined by these initial conditions and the *geodesic equation*

$$\nabla_{\dot\gamma_{x,v}(t)}\dot\gamma_{x,v}(t) = 0,$$

where ∇ denotes the Levi–Civita connection of g. The (Riemannian) exponential map at x is then defined by

$$\exp_x^g(v) := \gamma_{x,v}(1),$$

whenever this value is defined. We emphasize that this exponential map depends on the choice of the Riemannian metric g (or equivalently, on the associated Levi–Civita connection), and that the vector v appearing here is a tangent vector in T_xX, not a vector field on X.

Mini-Exercise I.8.3. Let X be a C^∞ manifold and g be a C^∞ Riemannian metric on X. Show that there exists a C^∞ vector field G on the total space of the tangent bundle TX with the following property (the *geodesic spray*): For any $\varepsilon \in \mathbb{R}_{>0}$ and any C^∞ curve

$$(-\varepsilon,\varepsilon) \ni t \mapsto (\gamma(t),\delta(t)) \in TX,$$

it is an integral curve of G if and only if the projected curve γ on X is a geodesic with respect to g and $\delta(t) = \dot\gamma(t)$ (i.e., $\delta(t)$ is the velocity vector of γ at $\gamma(t)$) for any $t \in (-\varepsilon,\varepsilon)$.

Note that the (local) flow of G on TX as in Mini-Exercise I.8.3 is called the *geodesic flow* associated with g. Note also that, by combining Mini-Exercise I.8.3 and Proposition I.8.2 (applied to the C^∞ vector field G on the manifold TX), it follows that, for any $x \in X$ and any $v \in T_xX$, there exist $\varepsilon \in \mathbb{R}_{>0}$ and a unique geodesic $\gamma_{x,v}\colon (-\varepsilon,\varepsilon) \to X$ such that $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = v$.

8.2 Tubular neighborhood theorem

One geometric application of the Riemannian exponential map introduced in the previous subsection is the following classical result.

Theorem I.8.4 (Tubular neighborhood theorem). *Let X be a C^∞ manifold and $Y \subset X$ be a C^∞ submanifold. Set $N := TX|_Y/TY$. Then there exist an open neighborhood $U \subset N$ of* (Zero Section)$_N$ *and an open neighborhood $V \subset X$ of Y, together with a C^∞ diffeomorphism $F\colon U \to V$ such that $F|_{(\text{Zero Section})_N}$ coincides with the restriction of the projection $N \to Y$ to* (Zero Section)$_N$.

Note that the C^∞ real vector bundle $N \to Y$ as in Theorem I.8.4 is called the C^∞ *normal bundle* of Y in X, and an open neighborhood $V \subset X$ of Y as in the conclusion is called a C^∞ *tubular neighborhood* of Y in X.

Before proving this theorem, let us first record a basic splitting lemma for short exact sequences of C^∞ vector bundles.

LEMMA I.8.5. *Let Y be a C^∞ manifold, and let*

$$0 \longrightarrow E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \longrightarrow 0$$

be a short exact sequence of C^∞ real vector bundles over Y. Then there exists a C^∞ vector bundle morphism $s\colon E'' \to E$ over id_Y such that $\beta \circ s = \mathrm{id}_{E''}$. Equivalently, E is C^∞-isomorphic to $E' \oplus E''$.

PROOF. Fix a C^∞ metric on E. Let $E'^{\perp} \subset E$ be the orthogonal complement of $\alpha(E')$ with respect to this metric. Then for each $y \in Y$ one has the orthogonal direct sum decomposition

$$E_y = \alpha(E'_y) \oplus E'^{\perp}_y,$$

and the restriction $\beta|_{E'^{\perp}}\colon E'^{\perp} \to E''$ is an isomorphism of vector bundles over Y. Hence we may define

$$s\colon E'' \longrightarrow E, \qquad s := \big(\beta|_{E'^{\perp}}\big)^{-1},$$

which is a C^∞ bundle map and satisfies $\beta \circ s = \mathrm{id}_{E''}$. □

Proof of Theorem I.8.4. Since the assertion is local near Y and can be checked componentwise, we may assume that X and Y are connected.

Fix a C^∞ Riemannian metric g on X. By Lemma I.8.5 applied to the normal exact sequence $0 \to TY \to TX|_Y \to N \to 0$, we obtain a C^∞ splitting

$$TX|_Y \cong TY \oplus N,$$

which we choose so that N is identified with the g-orthogonal complement of TY in $TX|_Y$. In particular, we view N as a C^∞ subbundle of $TX|_Y$.

Let G be the geodesic spray on TX associated with g as in Mini-Exercise I.8.3. By applying Proposition I.8.2 to the C^∞ vector field G on the manifold TX, we obtain a (local) flow

$$\Phi\colon \Omega \longrightarrow TX$$

of G defined on an open neighborhood Ω of $\{0\} \times TX$. We extend this flow near the zero section to include time 1. Indeed, G vanishes at every zero vector 0_y, so its integral curve there is constant. Choose a positive time a smaller than the local existence time near 0_y, and an integer k with $ka > 1$. The first k successive local time-a maps are all defined on some neighborhood of 0_y: each fixes 0_y, so this follows by shrinking that neighborhood finitely many times. The intervening local flows give smooth solutions on a time interval containing $[0, 1]$ for those initial vectors. Uniqueness makes these extensions agree with the original flow and with one another. We continue to write Φ and Ω for the extended flow and its open domain. For $(y, v) \in TX$, write $\Phi(t, (y, v)) = (\gamma(t), \delta(t))$. By the defining property of G, the projected curve $\gamma(t)$ is the unique geodesic with $\gamma(0) = y$ and $\dot\gamma(0) = v$, and moreover $\delta(t) = \dot\gamma(t)$. Define the exponential map associated with g by

$$\exp^g_y(v) := \gamma(1)$$

whenever $(1,(y,v)) \in \Omega$. Since Φ is C^∞ and the projection $TX \to X$ is C^∞, the map $(y,v) \mapsto \exp_y^g(v)$ is C^∞ on its domain of definition.

Now consider the map

$$F\colon U_0 \longrightarrow X, \qquad F(v) := \exp^g_{\pi(v)}(v),$$

where $\pi\colon N \to Y$ denotes the bundle projection, and where $U_0 \subset N$ is an open neighborhood of $(\text{Zero Section})_N$ on which $\exp^g_{\pi(v)}(v)$ is defined: Indeed, such a neighborhood U_0 can be constructed as follows. Consider the slice

$$\Omega_1 := \{\, \xi \in TX \mid (1,\xi) \in \Omega \,\} \subset TX.$$

Since Ω is open in $\mathbb{R} \times TX$, the set Ω_1 is open in TX. The extension just constructed gives $(1,0_y) \in \Omega$ for every $y \in X$, where 0_y is the zero of T_yX. Thus Ω_1 contains the zero section. By restricting to $Y \subset X$ and considering the submanifold $N \subset TX|_Y \subset TX$, we have that $U_0 := \Omega_1 \cap N$ satisfies the desired property.

We next show that F is a local diffeomorphism on a neighborhood of $(\text{Zero Section})_N$. Recall that, for each $y \in X$, by the definition of the Riemannian exponential map $\exp^g$ via the geodesic flow, the map

$$E\colon \Omega_1 \longrightarrow X \times X, \qquad E(v) := \big(\pi(v), \exp^g_{\pi(v)}(v)\big),$$

where $\pi\colon TX \to X$ denotes the projection, is well-defined and C^∞ on a neighborhood of the zero section. For $y \in X$, consider the point $0_y \in T_yX \subset TX$. Then the induced map on tangent spaces

$$(E_*)_{0_y}\colon T_{0_y}(TX) \longrightarrow T_{(y,y)}(X \times X) = T_yX \oplus T_yX \tag{24}$$

is given by

$$(E_*)_{0_y}(u,v) = (u, u+v).$$

Here we use the canonical identification

$$T_{0_y}(TX) \cong T_yX \oplus T_yX,$$

whose first summand is tangent to the zero section and whose second summand is tangent to the fiber. Indeed, $E(0_y) = (y,y)$, while $\frac{d}{dt}\big|_{t=0} \exp^g_y(tv) = v$, by rescaling the geodesic with initial velocity v. On the other hand, since $N \to Y$ is a C^∞ vector bundle, the tangent space of N at the zero vector 0_y admits the canonical splitting $T_{0_y}N \cong T_yY \oplus N_y$, where T_yY corresponds to the tangent directions along the zero section and N_y to the vertical tangent directions. Restricting the second component of (24) to this subspace gives

$$(F_*)_{0_y}\colon T_yY \oplus N_y \longrightarrow T_yX, \qquad (u,v) \longmapsto u+v.$$

This is an isomorphism by the splitting $TX|_Y \cong TY \oplus N$ fixed above.

By the inverse function theorem, for each $y \in Y$ there exist an open neighborhood $U_y \subset N$ of 0_y and an open neighborhood $V_y \subset X$ of y such that $F|_{U_y}\colon U_y \to V_y$ is a C^∞ diffeomorphism. Their union $U_1 := \bigcup_{y\in Y} U_y$ is an open neighborhood of $(\text{Zero Section})_N$ on which F is a local diffeomorphism.

We give the shrinking argument also when Y is non-compact. Fix a distance function d inducing the topology of X, and write $|v|_g$ for the norm of $v \in N$. For $y \in Y$ and $s > 0$, put

$$T(y,s) := \{v \in N \mid d(\pi(v), y) < s,\ |v|_g < s\}.$$

Let $r(y)$ be the supremum of the numbers $s \in (0,1]$ such that $T(y,s) \subset U_1$ and F is injective on $T(y,s)$. Local bundle coordinates and the preceding inverse function theorem give $r(y) > 0$. Moreover, $|r(y) - r(y')| \le d(y,y')$: if $s > d(y,y')$, then $T(y', s - d(y,y')) \subset T(y,s)$, and one applies this inclusion to admissible s and then interchanges y, y'. In particular, r is continuous. Define

$$U := \{v \in U_1 \mid |v|_g < r(\pi(v))/8,\ d(F(v), \pi(v)) < r(\pi(v))/8\}.$$

This is an open neighborhood of the zero section. If $v, v' \in U$ and $F(v) = F(v')$, put $y = \pi(v)$ and $y' = \pi(v')$. Then

$$d(y,y') < \frac{r(y) + r(y')}{8} \le \frac{2r(y) + d(y,y')}{8},$$

so $d(y,y') < 2r(y)/7$. Also $|v'|_g < r(y')/8 \le (r(y) + d(y,y'))/8 < r(y)/2$. Thus both v and v' belong to $T(y, r(y)/2)$, where F is injective by the definition of $r(y)$. Hence $v = v'$. Consequently $F|_U$ is a C^∞ diffeomorphism onto the open set $V := F(U)$. By construction, its restriction to the zero section is the inclusion $Y \hookrightarrow X$, as required. □

Remark I.8.6. When Y consists of a single point $y \in X$, the above construction gives a diffeomorphism from a neighborhood of $0_y \in T_yX$ onto a neighborhood of y in X via the map $v \mapsto \exp_y^g(v)$. Such neighborhoods are called *normal neighborhoods*, and the corresponding coordinate systems are called *normal coordinates* at y. □

8.3 Ehresmann's fibration theorem

Let X and Y be C^∞ manifolds, and let $f\colon X \to Y$ be a C^∞ map. Recall that f is called a *submersion* if every point $x \in X$ is a regular point of f, that is, if the differential $f_*\colon T_xX \to T_{f(x)}Y$ is surjective for all $x \in X$. In this case one necessarily has $\dim_{\mathbb{R}} X \ge \dim_{\mathbb{R}} Y$.

For proper surjective submersions, the following fundamental result holds.

Theorem I.8.7 (Ehresmann's fibration theorem). *Let X and Y be C^∞ manifolds, and $f\colon X \to Y$ be a proper surjective C^∞ submersion. Then f is a locally trivial C^∞ fibration: i.e., For every point $y_0 \in Y$, there exist an open neighborhood $V \subset Y$ of y_0 and a C^∞ diffeomorphism*

$$\Phi\colon f^{-1}(V) \longrightarrow V \times X_0, \qquad X_0 := f^{-1}(y_0),$$

such that the diagram

$$\begin{array}{ccc} f^{-1}(V) & \xrightarrow{\ \Phi\ } & V \times X_0 \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \mathrm{Pr}_1} & \\ & V & \end{array}$$

commutes, where Pr_1 *denotes the projection onto the first factor.*

Note that X_0 in Theorem I.8.7 is a compact submanifold of X, since f is proper and y_0 is a regular value of f (refer to Mini-Exercise I.8.1 (ii)).

In this subsection, in order to keep the exposition as simple as possible, we explain the proof of Theorem I.8.7 only in the case where the base manifold Y is real one-dimensional (The same idea works in higher dimensions, see [E]).

From now on, we assume that Y is a one-dimensional manifold. Since the assertion is local on Y, it suffices to prove it in a neighborhood of a fixed point $y_0 \in Y$. We may therefore assume that

$$Y = (-1, 1), \qquad y_0 = 0,$$

and we write y for the standard coordinate on $(-1, 1)$. Replacing X by a connected component of $f^{-1}((-1, 1))$, we may further assume that $f\colon X \to (-1, 1)$ is a proper surjective C^∞ submersion, where X is a real m-dimensional C^∞ manifold.

We first construct a C^∞ vector field v on X satisfying

$$f_* v = \frac{\partial}{\partial y}.$$

Take a locally finite open covering $\{W_j\}$ of X by coordinate neighborhoods. Since f is a submersion, for each j we may choose local coordinates

$$x_j = (x_j^1, x_j^2, \dots, x_j^m) \quad \text{on } W_j$$

such that

$$f(x_j^1, x_j^2, \dots, x_j^m) = x_j^1$$

(see Mini-Exercise I.8.1 (i)). Let $\{(W_j, \rho_j)\}$ be a C^∞ partition of unity subordinate to $\{W_j\}$. Define

$$v := \sum_j \rho_j(x_j) \frac{\partial}{\partial x_j^1}.$$

Then, by construction and the linearity of f_*, one has $f_* v = \frac{\partial}{\partial y}$.

Let $\varphi\colon U \to X$ be the (local) flow of the vector field v as in Proposition I.8.2, defined on an open neighborhood $U \subset \mathbb{R} \times X$ of $\{0\} \times X$.

Lemma I.8.8. *With the above notation, for any $(t, p) \in U$ one has*

$$f(\varphi(t, p)) = f(p) + t,$$

where the right-hand side is interpreted using the coordinate y on $(-1, 1)$.

Proof. Fix $p \in X$. Since $\varphi(0, \cdot)$ is the identity map on X, one has $f(\varphi(0, p)) = f(p)$. On the other hand, for any $t_0 \in \mathbb{R}$ with $(t_0, p) \in U$,

$$\frac{d}{dt}\bigg|_{t=t_0} f(\varphi(t, p)) = f_*\big(v_{\varphi(t_0, p)}\big) = \frac{\partial}{\partial y}.$$

Applying the coordinate function $y\colon (-1, 1) \to \mathbb{R}$, we see that $f(\varphi(t, p))$ is an affine function of t whose derivatives are identically equal to 1. The initial condition at $t = 0$ then implies $f(\varphi(t, p)) = f(p) + t$. □

Let $X_0 := f^{-1}(0)$. Since f is proper, the set $f^{-1}([-\varepsilon,\varepsilon])$ is compact for every $\varepsilon \in (0,1)$. By Proposition I.8.2, it follows that, for $\varepsilon > 0$ sufficiently small, the flow $\varphi(t,p)$ is defined for all $p \in f^{-1}([-\varepsilon,\varepsilon])$ and all $t \in [-\varepsilon,\varepsilon]$. Fix such a positive number ε, and set $V := (-\varepsilon,\varepsilon) \subset (-1,1)$. Define a map

$$\Phi\colon f^{-1}(V) \longrightarrow V \times X_0$$

by

$$\Phi(p) := \big(f(p),\ \varphi(-f(p),p)\big).$$

By Lemma I.8.8, one has $f(\varphi(-f(p),p)) = 0$, hence $\varphi(-f(p),p) \in X_0$ and Φ is well-defined. The map Φ is C^∞ as a composition of C^∞ maps.

Define also a map

$$\Psi\colon V \times X_0 \longrightarrow f^{-1}(V)$$

by

$$\Psi(y,p) := \varphi(y,p).$$

Again by Lemma I.8.8, one has $f(\Psi(y,p)) = y$, so Ψ indeed maps into $f^{-1}(V)$.

We now verify that Φ and Ψ are inverse to each other. Let $(y,p) \in V \times X_0$. Using the flow property of φ, we compute

$$\Phi(\Psi(y,p)) = \Phi(\varphi(y,p)) = \big(f(\varphi(y,p)),\ \varphi(-f(\varphi(y,p)),\varphi(y,p))\big).$$

By Lemma I.8.8, one has $f(\varphi(y,p)) = f(p) + y = y$, since $p \in X_0 = f^{-1}(0)$. Hence, using again the flow property,

$$\varphi(-f(\varphi(y,p)),\varphi(y,p)) = \varphi(-y,\varphi(y,p)) = \varphi(0,p) = p.$$

Therefore $\Phi(\Psi(y,p)) = (y,p)$. Conversely, let $p \in f^{-1}(V)$. Then

$$\Psi(\Phi(p)) = \Psi\big(f(p),\varphi(-f(p),p)\big) = \varphi\big(f(p),\varphi(-f(p),p)\big) = \varphi(0,p) = p,$$

again by the flow property. Thus Φ and Ψ are inverse to each other.

Therefore Φ is a C^∞ diffeomorphism satisfying $\mathrm{Pr}_1 \circ \Phi = f$, which proves the theorem in the case $\dim_{\mathbb{R}} Y = 1$. □

8.4 Topological aspects of C^∞ manifolds

Before proceeding further, let us briefly summarize some basic and well-known facts concerning the topology of a C^∞ manifold X.

As already mentioned, by its local Euclidean nature, a C^∞ manifold X is locally compact, locally connected, and locally simply connected (see [Lee-Jo2, Propositions 1.11 and 1.12]). Moreover, since X is assumed to be second-countable, it is paracompact (see [Lee-Jo2, Theorem 1.15]). As a consequence, all standard topological arguments concerning paths, fundamental groups, and covering spaces—such as the coincidence of path-connected components with connected components, the existence of universal covering spaces, and the classification

of covering spaces—apply to X (within each connected component, if necessary), as discussed in §7.1.

Now fix a C^∞ Riemannian metric g on X. Then each connected component of X naturally acquires the structure of a metric space. For simplicity, we assume in what follows that X is connected. Let $\mathrm{Path}_{\mathrm{pws}}(X)$ denote the set of all *piecewise smooth* paths $\gamma\colon [0,1] \to X$; that is, γ is continuous, and there exists a partition

$$0 = t_0 < t_1 < \cdots < t_N = 1$$

such that, for each $\nu \in \{1, 2, \ldots, N\}$, the restriction $\gamma|_{(t_{\nu-1}, t_\nu)}$ is C^∞. For such a path γ and a partition as above, the *length* of γ with respect to g is defined by

$$\mathrm{length}_g(\gamma) := \sum_{\nu=1}^{N} \int_{t_{\nu-1}}^{t_\nu} |\dot\gamma(t)|_g \, dt.$$

For points $p, q \in X$, we define

$$\mathrm{dist}_g(p,q) := \inf\bigl\{\mathrm{length}_g(\gamma) \;\big|\; \gamma \in \mathrm{Path}_{\mathrm{pws}}(X),\ \gamma(0) = p,\ \gamma(1) = q\bigr\}.$$

Then dist_g is a distance function on X, and it is known that the topology induced by dist_g coincides with the original manifold topology of X (see [Lee-Je, Theorem 13.87]).

As a fundamental result describing the relationship between the metric properties of (X, dist_g) and the differential-geometric properties of the Riemannian manifold (X, g), here we review the statement of *Hopf–Rinow Theorem* (see [Lee-Je, Theorem 13.139] for example): For a connected Riemannian manifold (X, g), the following conditions are equivalent:

- (X, dist_g) is a complete metric space;
- Every closed and bounded subset of (X, dist_g) is compact;
- There exists a point $p \in X$ such that (X, g) is *geodesically complete at* p, meaning that every geodesic starting at p can be extended to one defined on all of $\mathbb{R}$ (Equivalently, the Riemannian exponential map $\exp_p^g$ is defined on the whole tangent space T_pX);
- (X, g) is geodesically complete at every point $p \in X$.

Moreover, under these equivalent conditions, for any two points $p, q \in X$, it is known that there exists a geodesic connecting p and q whose length realizes the distance $\mathrm{dist}_g(p, q)$ (see [Lee-Je, Theorem 13.138] and [Mi1, Theorem 10.9]).

From this point on, unless otherwise stated, we again allow X to be an arbitrary (not necessarily connected) C^∞ manifold.

One important consequence of the smooth structure of X is that X has the homotopy type of a CW complex: a standard way to establish this fact is to apply *Morse theory*. Let $f\colon X \to \mathbb{R}$ be a C^∞ function. For a critical point $p \in X$ of f, choosing local coordinates $x = (x^1, x^2, \ldots, x^n)$ on a neighborhood of p with $x(p) = 0$ ($n := \dim_{\mathbb{R},p} X$), consider the Hessian matrix $\mathrm{Hess}_p(f)$ of f at p. The critical point p is said to be *non-degenerate* if $\mathrm{Hess}_p(f)$

is non-degenerate: this definition is independent of the choice of local coordinates by Mini-Exercise I.8.10 below together with Sylvester's law of inertia. For the same reason, if $p \in X$ is a non-degenerate critical point of f, the numbers of positive and negative eigenvalues of $\mathrm{Hess}_p(f)$ are independent of the choice of local coordinates. The number of negative eigenvalues of $\mathrm{Hess}_p(f)$ is called the *index* of f at p. The following two basic facts form the foundation of Morse theory:

- There exists a C^∞ exhaustion function $f\colon X \to \mathbb{R}$ whose critical points are all non-degenerate (see [Mi1, Corollary 6.7]);
- Let $p \in X$ be a non-degenerate critical point of a C^∞ function $f\colon X \to \mathbb{R}$, and let λ be the index of f at p. Then there exist local coordinates $(x^1, x^2, \ldots, x^n)$ with $x(p) = 0$ such that, in a neighborhood of p, the function f can be written as

$$f(x^1, x^2, \ldots, x^n) = f(p) - \sum_{j=1}^{\lambda} (x^j)^2 + \sum_{j=\lambda+1}^{n} (x^j)^2,$$

  where $n = \dim_{\mathbb{R},p} X$ (the *Morse lemma*, see [Mi1, Lemma 2.2]). In particular, non-degenerate critical points of f are isolated ([Mi1, Corollary 2.3]).

Fix once and for all a C^∞ exhaustion function $f\colon X \to \mathbb{R}$ whose critical points are all non-degenerate. For $c \in \mathbb{R}$, we set

$$X_c := \{\, x \in X \mid f(x) < c \,\},$$

and study how the topology of X_c changes as the parameter c increases. For a regular value c, the compact sublevel set $\overline{X_c} = \{x \in X \mid f(x) \leq c\}$ is a manifold with boundary whose interior is X_c and has the same homotopy type. We describe handle attachments using these compact sublevel sets.

The case $X = \emptyset$ is immediate, so assume in this construction that $X \neq \emptyset$. Since f is continuous and exhaustive, it attains a minimum value on X. Let $c_0 := \min_X f$. Then every point of $f^{-1}(c_0)$ is clearly a critical point of f. By our assumption, all such critical points are non-degenerate, and hence isolated by the Morse lemma. In particular, since the level set $f^{-1}(c_0)$ is compact, it consists of finitely many points. Moreover, as c_0 is a minimum value, the index of f at each of these critical points is equal to 0. It follows again from the Morse lemma that, for sufficiently small $\varepsilon > 0$, the sublevel set $X_{c_0+\varepsilon}$ is diffeomorphic to a finite disjoint union of open balls.

As c increases further, the topology of X_c can change only when c passes through a critical value of f. More precisely, let $a, b \in \mathbb{R}_{>c_0}$ with $a < b$. If the set $f^{-1}([a, b])$ contains no critical point of f, then X_a and X_b are C^∞ diffeomorphic (see [Mi1, Theorem 3.1]; compare also Ehresmann's fibration theorem, Theorem I.8.7).

Since the critical points of f are isolated and the closed sublevel sets of f are compact, it suffices to consider the following situation: Let $\alpha < \beta < \gamma$ be real numbers greater than c_0 such that the set $f^{-1}([\alpha, \gamma])$ contains exactly finitely many critical points $p_1, p_2, \ldots, p_N \in$

$f^{-1}(\beta)$. For each $\nu \in \{1, 2, \ldots, N\}$, let λ_ν denote the index of f at the critical point p_ν. Then $\overline{X_\gamma}$ is obtained from $\overline{X_\alpha}$ by attaching N *handles*, one for each critical point p_ν, of index λ_ν. In particular, the homotopy type of X_γ is obtained from that of X_α by attaching a λ_1-cell, a λ_2-cell, ..., and a λ_N-cell (see [Mi1, Theorem 3.2 and Remark 3.3]).

From these considerations, we conclude that X has the homotopy type of a CW complex whose cells are in one-to-one correspondence with the critical points of f, and whose cell dimensions are given by the indices of f at those points (see [Mi1, Theorem 3.5]).

Example I.8.9. Let

$$X := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

be the unit circle, viewed as a C^∞ submanifold of $\mathbb{R}^2$. Define a C^∞ function $f\colon X \to \mathbb{R}$ by restricting the projection $(x, y) \mapsto y$ to X. Then the critical points of f are precisely the two points

$$p := (0, -1), \qquad q := (0, 1)$$

and, near each of these points, the coordinate x serves as a local coordinate on X. More concretely, if we write y as a function of x on X, then near p we have

$$f(x) = y = -\sqrt{1 - x^2},$$

while near q we have

$$f(x) = y = \sqrt{1 - x^2}.$$

A direct differentiation yields

$$f'(x) = \frac{x}{\sqrt{1 - x^2}}, \qquad f''(x) = \frac{1}{(1 - x^2)^{3/2}} \quad \text{near } p,$$

and

$$f'(x) = -\frac{x}{\sqrt{1 - x^2}}, \qquad f''(x) = -\frac{1}{(1 - x^2)^{3/2}} \quad \text{near } q.$$

In particular,

$$f'(0) = 0, \quad f''(0) = 1 \text{ near } p, \qquad f'(0) = 0, \quad f''(0) = -1 \text{ near } q,$$

so both p and q are non-degenerate critical points. Moreover, the index of f at p is 0 (a local minimum), whereas the index at q is 1 (a local maximum).

Let us consider the sublevel sets $X_c := \{\, x \in X \mid f(x) < c \,\}$. If $c < -1$, then $X_c = \emptyset$. If $-1 < c < 1$, then X_c is diffeomorphic to an open interval, hence has the homotopy type of a point. Finally, if $c > 1$, then $X_c = X$.

Thus, as c passes the critical value $-1 = f(p)$, a 0-cell is created. As c further passes the critical value $1 = f(q)$, a 1-cell is attached. In particular, X has the homotopy type of a CW complex obtained by attaching a 1-cell to a 0-cell. □

MINI-EXERCISE I.8.10. Let X be a C^∞ manifold, $f\colon X \to \mathbb{R}$ a C^∞ function, and $p \in X$ a critical point of f. Let $x = (x^1, x^2, \ldots, x^n)$ and $u = (u^1, u^2, \ldots, u^n)$ be two systems of local coordinates on a neighborhood of p such that $x(p) = 0$ and $u(p) = 0$ ($n := \dim_{\mathbb{R},p} X$). Show that the Hessian matrices of f at p computed in the x-coordinates and in the u-coordinates are congruent.

MINI-EXERCISE I.8.11. Let $X \subset \mathbb{R}^3$ be the C^∞ submanifold defined by

$$X := \big\{ \big((2+\cos v)\cos u,\ (2+\cos v)\sin u,\ \sin v\big) \in \mathbb{R}^3 \ \big|\ u, v \in \mathbb{R} \big\}.$$

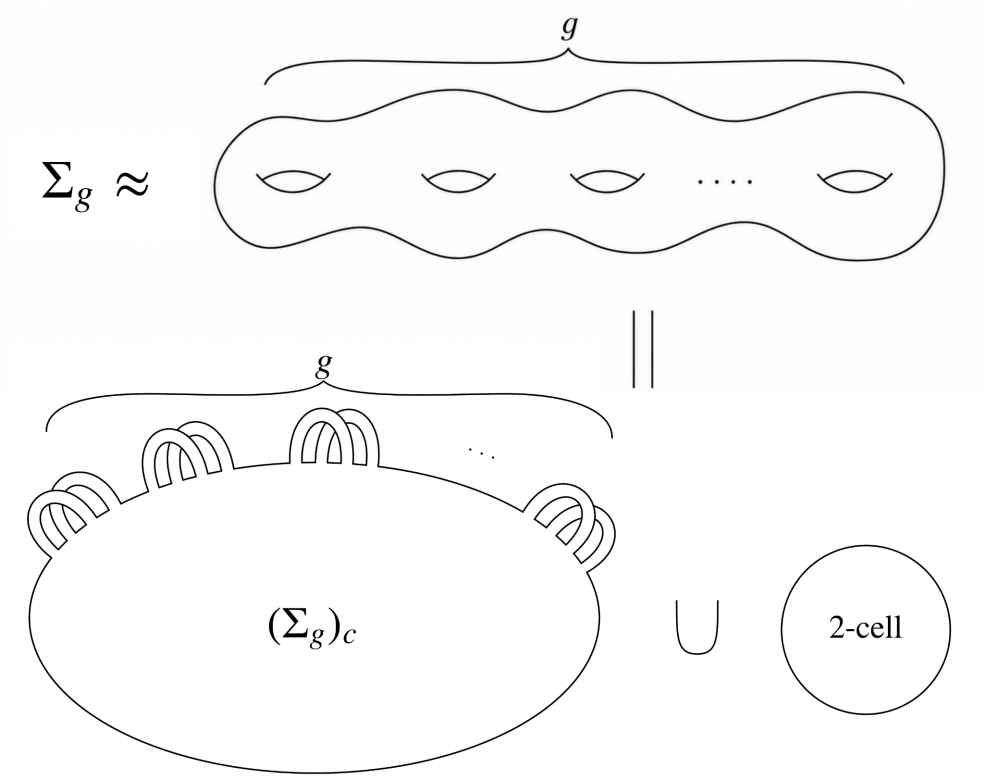
Define a C^∞ function $f\colon X \to \mathbb{R}$ as the restriction to X of the projection $(x, y, z) \mapsto x$ on $\mathbb{R}^3$.
(i) Determine all critical points of f.
(ii) Show that all critical points of f are non-degenerate, and compute the index of f at each critical point.
(iii) Following the discussion above, describe the handle decomposition and the CW structure of X induced by the function f.

The observations made in Mini-Exercise I.8.11 (and also in [Mi1, Chapter 1, §1]) admit a more general and visually clear interpretation for connected, compact, orientable C^∞ manifolds X of real dimension 2. According to the topological classification of such surfaces (see [Lee-Jo1, Proposition 6.20, Theorem 10.22]), any such manifold X is homeomorphic either to the sphere or to a connected sum Σ_g of g tori, for some $g \in \mathbb{Z}_{\geq 1}$. The integer g is called the *genus* of X (by convention, the sphere is said to have genus 0). Since there is nothing further to discuss in the case of the sphere, we assume in what follows that the genus g of X is positive.

By realizing X (as in Mini-Exercise I.8.11 when $g = 1$) as a suitable embedded surface in $\mathbb{R}^3$, and by considering a C^∞ function $f\colon X \to \mathbb{R}$ obtained as the restriction of a coordinate projection, one may arrange that f has the following properties:

- The function f has exactly $2g + 2$ critical points $p_0, p_1, p_2, \ldots, p_{2g}, p_{2g+1}$, all of which are non-degenerate;
- The minimum of f is attained only at p_0, and the maximum only at p_{2g+1} (Their indices are 0 and 2, respectively);
- The remaining $2g$ critical points $p_1, p_2, \ldots, p_{2g}$ all have index 1.

For such a function f, consider the handle decomposition described above. Let $c \in \mathbb{R}$ be a regular value slightly smaller than the maximum value $f(p_{2g+1})$. Then the surface $X \cong \Sigma_g$ is obtained from the compact sublevel set $\overline{X_c}$ by attaching a closed 2-disk along its boundary circle. The space $\overline{X_c}$ itself is obtained from a closed 2-disk by attaching $2g$ many 1-handles (each homeomorphic to $[0,1] \times [0,1]$), corresponding to the $2g$ critical points $p_1, p_2, \ldots, p_{2g}$. Each handle is attached along its two opposite edges to suitable disjoint intervals in the boundary of the surface already constructed. The interior X_c has the same homotopy type



as $\overline{X_c}$, namely that of a finite graph with one vertex and $2g$ edges. From this description, together with the attaching of the final 2-cell and the Seifert–Van Kampen theorem, one obtains that the fundamental group $\pi_1(X, *)$ is generated by $2g$ elements

$$\alpha_1, \alpha_2, \ldots, \alpha_g,\ \beta_1, \beta_2, \ldots, \beta_g,$$

subject to the single non-trivial relation

$$\prod_{j=1}^{g} \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} = 1.$$

The homology groups of X are also easily computed from this decomposition:

$$H_q(X) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0, 2, \\ \mathbb{Z}^{2g} & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

For other approaches to these topological invariants of Σ_g, see for example [Lee-Jo1, Corollary 13.15] and [Ha1, Example 2.36].

Note that, by combining the handle decomposition arising from Morse theory with the general facts on CW complexes recalled in §7.3, we immediately obtain the following important consequences:

- For any compact C^∞ manifold X, all singular homology groups $H_q(X)$ are finitely generated (since X has the homotopy type of a finite CW complex). As a consequence, for compact C^∞ manifolds, Betti numbers and the Euler–Poincaré characteristic are well-defined finite integers.

- Let X be a C^∞ manifold, and fix $q \in \mathbb{Z}_{\geq 0}$. Assume that there exists a C^∞ exhaustion function $f\colon X \to \mathbb{R}$ (i.e., $\{x \in X \mid f(x) \leq c\}$ is compact for every $c \in \mathbb{R}$) whose critical points are all non-degenerate, and such that none of these critical points has index q. Then X is homotopy equivalent to a CW complex with no q-cells, and hence $H_q(X) = 0$.

Whereas we have so far focused on the CW homotopy type of C^∞ manifolds, one can say more precisely that every C^∞ manifold is C^∞ *triangulable* in the following sense (see, for instance, [Mu1, Theorem 10.6]): We say that X is C^∞ triangulable if there exists a locally finite cell complex structure $(X, \mathfrak{E})$ admitting a simplicial structure in the sense of §7.4, such that all characteristic maps

$$\Phi_\lambda\colon \mathrm{Conv}(v_0, \ldots, v_n) \longrightarrow X$$

can be chosen so that, for each λ, the restriction

$$\Phi_\lambda|_{\mathrm{Conv}^\circ(v_0,\ldots,v_n)}\colon \mathrm{Conv}^\circ(v_0, \ldots, v_n) \longrightarrow X$$

is a C^∞ embedding.

Note that this fact is also compatible with the picture provided by Morse theory, at least in lower-dimensional cases: starting from a handle decomposition of X and using the C^∞ triangulability of each handle, one is naturally led to a C^∞ triangulation of X (It seems plausible that, even in arbitrary dimension, the C^∞ triangulability of C^∞ manifolds can be proved along this line by an induction on the dimension. However, the author has not been able to find a reference in which such an argument is written out in a complete and self-contained manner. Another possible approach, which is at least well known at a folklore level, is to fix a Riemannian metric g on a C^∞ manifold X, take sufficiently small normal neighborhoods around each point, and consider suitable star-shaped open subsets in the tangent bundle. By constructing an appropriate Voronoi-type decomposition with respect to the distance function dist_g, one may hope to obtain a cellular decomposition. To the author's knowledge, however, a detailed proof of this strategy for general C^∞ manifolds does not seem to be available in the literature).

In any case, since C^∞ manifolds are triangulable, the Čech–singular correspondence holds in this setting: i.e., for any C^∞ manifold X and any abelian group G, there exists a suitably chosen locally finite open covering $\{U_v\}_v$ of X (as in Lemma I.7.20) such that, with the notation of Proposition I.7.24, one has a natural isomorphism

$$H^q\big(\check{C}^\bullet(\{U_v\}_v, G)\big) \;\cong\; H^q(X, G) \qquad (q \in \mathbb{Z}_{\geq 0}). \tag{26}$$

Finally, we note that the C^∞ triangulability of compact, orientable C^∞ manifolds provides a particularly clear framework for proving classical results such as the *Gauss–Bonnet theorem* and the *Poincaré–Hopf theorem*. This viewpoint is widely known; see, for instance, [Doc, §4–5] and [Mi2, §6]. For Gauss–Bonnet, see also [J, §2.5].

8.5 Whitney approximation theorem and its applications

Concerning the approximation of continuous maps between C^∞ manifolds, the following fundamental result is of central importance.

THEOREM I.8.12 (Whitney approximation theorem). *Let X and Y be C^∞ manifolds, let $Z \subset X$ be a closed C^∞ submanifold, and let $f\colon X \to Y$ be a continuous map. Assume that the restriction $f|_Z\colon Z \to Y$ is of class C^∞. Then there exists a C^∞ map $\widetilde{f}\colon X \to Y$ homotopic to f such that $\widetilde{f}|_Z = f|_Z$.*

PROOF. The details can be found in the proof of [Lee-Jo2, Theorem 6.26]; here we only outline the main ideas.

The first key step is to establish the following special case, mainly by patching local approximation arguments using a partition of unity. Assume in addition that the target manifold Y is a Euclidean space $\mathbb{R}^k$. Then, for any given positive continuous function $\delta\colon X \to \mathbb{R}_{>0}$, there exists a C^∞ map $\widetilde{f}\colon X \to \mathbb{R}^k$ which is δ-close to f and satisfies $\widetilde{f}|_Z = f|_Z$. This is the *Whitney approximation theorem for functions*; see [Lee-Jo2, Theorem 6.21].

In the general case, we may work separately on the connected components of Y and their open and closed inverse images in X. To reduce to the Euclidean situation, one applies the Whitney embedding theorem (see [Lee-Jo2, Theorem 6.15]) and realizes the manifold Y as a C^∞ properly embedded submanifold of some Euclidean space $\mathbb{R}^k$. By composing f with this embedding, we may regard f as a continuous map from X to $\mathbb{R}^k$ and apply the Whitney approximation theorem for functions. In this way, f can be modified, while keeping its values fixed on Z, to a C^∞ map $g\colon X \to \mathbb{R}^k$ which is arbitrarily close to f. At this stage, the image of g may no longer be contained in Y. This difficulty can be overcome by applying the tubular neighborhood theorem (Theorem I.8.4) to the C^∞ submanifold $Y \subset \mathbb{R}^k$. Indeed, a C^∞ tubular neighborhood U of Y in $\mathbb{R}^k$ comes equipped with a smooth retraction $U \to Y$ induced by the projection along the normal bundle. Roughly speaking, by composing g with this retraction, one obtains a C^∞ map $\widetilde{f}\colon X \to Y$ which is homotopic to f and agrees with f on Z. □

One of the most important applications of the Whitney approximation theorem is the following:

THEOREM I.8.13 ([Lee-Jo2, Theorem 6.29]). *Let X and Y be C^∞ manifolds, and let $f, g\colon X \to Y$ be C^∞ maps. Assume that f and g are homotopic. Then they are C^∞ homotopic: i.e., there exist an open neighborhood U of $[0,1] \times X$ in $\mathbb{R} \times X$ and a C^∞ map $H\colon U \to Y$ such that $H(0,-) = f$ and $H(1,-) = g$.*

PROOF. By assumption, there exists a continuous map $h_0\colon [0,1] \times X \to Y$ such that $h_0(0,-) = f$ and $h_0(1,-) = g$. Using this, define a continuous map $h\colon \mathbb{R} \times X \to Y$ by

$$h(t,x) := \begin{cases} f(x) & \text{if } t < 0, \\ h_0(t,x) & \text{if } 0 \le t \le 1, \\ g(x) & \text{if } t > 1. \end{cases}$$

Then h is continuous, and its restriction to $Z := \{0,1\} \times X \subset \mathbb{R} \times X$ is of class C^∞ by construction. Therefore, by Theorem I.8.12, there exists a C^∞ map $H\colon \mathbb{R} \times X \to Y$ homotopic to h such that $H|_{\{0,1\}\times X} = h|_{\{0,1\}\times X}$. The assertion follows immediately. □

Another important application of the Whitney approximation theorem concerns the smoothing of representatives of singular homology classes. Let X be a C^∞ manifold. For each integer $q \in \mathbb{Z}_{\geq 0}$, we denote by $C_q^{\mathrm{smooth}}(X)$ the free abelian group generated by the set of all C^∞ maps $\sigma\colon (\mathrm{Simplex})_q \to X$ (C^∞ *q-simplices* of X), and call its elements C^∞ *q-chains* of X: Here, regarding $(\mathrm{Simplex})_q$ as $\mathrm{Conv}(v_0, v_1, \ldots, v_q)$, a map $\sigma\colon \mathrm{Conv}(v_0, v_1, \ldots, v_q) \to X$ is said to be of class C^∞ if there exist an open neighborhood U of $\mathrm{Conv}(v_0, v_1, \ldots, v_q)$ in the affine hyperplane

$$\left\{ \sum_{j=0}^{q} t_j v_j \;\middle|\; (t_0, t_1, \ldots, t_q) \in \mathbb{R}^{q+1},\ \sum_{j=0}^{q} t_j = 1 \right\}$$

and a C^∞ map $\widetilde{\sigma}\colon U \to X$ such that $\widetilde{\sigma}|_{\mathrm{Conv}(v_0,v_1,\ldots,v_q)} = \sigma$.

For each $q \in \mathbb{Z}_{>0}$, the boundary homomorphism $\partial_q\colon C_q^{\mathrm{sing}}(X) \to C_{q-1}^{\mathrm{sing}}(X)$ restricts to give a boundary map between $C_\bullet^{\mathrm{smooth}}(X)$. Thus we obtain a chain complex of $\mathbb{Z}$-modules

$$\cdots \longrightarrow C_{q+1}^{\mathrm{smooth}}(X) \xrightarrow{\partial_{q+1}} C_q^{\mathrm{smooth}}(X) \xrightarrow{\partial_q} C_{q-1}^{\mathrm{smooth}}(X) \longrightarrow \cdots \longrightarrow C_0^{\mathrm{smooth}}(X) \longrightarrow 0.$$

By construction, the inclusion $C_\bullet^{\mathrm{smooth}}(X) \hookrightarrow C_\bullet^{\mathrm{sing}}(X)$ induces a natural homomorphism $H_q\big(C_\bullet^{\mathrm{smooth}}(X)\big) \to H_q(X)$. It is known that, as an application of the Whitney approximation theorem, this homomorphism can be shown to be an isomorphism for every $q \in \mathbb{Z}_{\geq 0}$ (see [Lee-Jo2, Theorem 18.7]):

$$H_q\big(C_\bullet^{\mathrm{smooth}}(X)\big) \;\cong\; H_q(X). \tag{27}$$

8.6 Spaces of sections of C^∞ vector bundles

Let X be a C^∞ manifold and $E \to X$ be a C^∞ vector bundle of rank r. In this subsection, we introduce the notation for various spaces of sections of E with prescribed regularity or support conditions that will be used throughout this note, and explain how to endow them with natural structures of LCTVSs.

For simplicity of exposition, in this subsection we assume that $\dim_{\mathbb{R}} X = n$ for some non-negative integer n. In the general case, one may apply the constructions below to the union of connected components of X with fixed real dimension, and then define the corresponding spaces of sections as the TVS direct sum of the resulting spaces. Note that, strictly speaking, one should take (topological) direct products rather than direct sums in full generality. However, in this note we will essentially encounter only C^∞ manifolds for which the function $X \ni x \mapsto \dim_{\mathbb{R},x} X$ is bounded from above. In such situations, the distinction between these two constructions is harmless for our purposes, and working with TVS direct sums will be sufficient in practice. Note also that similar arguments and constructions below make sense for C^∞ real vector bundles (in which case the resulting spaces are real vector spaces).

Throughout this subsection, fix a locally finite open covering $\{U_j\}_{j\in J}$ of X such that J is at most countable, each U_j is a coordinate neighborhood, and $E|_{U_j}$ is C^∞-trivial. More precisely, for each $j \in J$ we fix a C^∞ chart $\psi_j\colon U_j \to \Omega_j \subset \mathbb{R}^n$ and a C^∞ trivialization

$$\Phi_j\colon E|_{U_j} \xrightarrow{\cong} U_j \times \mathbb{C}^r$$

(Existence of such a covering follows from the second countability and paracompactness of X; we shall use this covering throughout this subsection).

The Fréchet space $C^\infty(X, E)$

For each $j \in J$, via ψ_j and Φ_j we identify the space $C^\infty(E|_{U_j})$ of smooth sections on U_j with an r-fold TVS direct sum of smooth functions on Ω_j:

$$C^\infty(E|_{U_j}) \;\cong\; \big(C^\infty(\Omega_j)\big)^{\oplus r},$$

where $C^\infty(\Omega_j)$ are endowed with the Fréchet space structure as in Example I.6.41. For $j, k \in J$, restriction defines continuous linear maps

$$\mathrm{res}_{jk}\colon C^\infty(E|_{U_j}) \longrightarrow C^\infty(E|_{U_j\cap U_k}), \qquad s \longmapsto s|_{U_j\cap U_k},$$

where $C^\infty(E|_{U_j\cap U_k})$ is endowed with the same type of Fréchet topology as above. Define a continuous linear map

$$\delta\colon \prod_{j\in J} C^\infty(E|_{U_j}) \longrightarrow \prod_{(j,k)\in J\times J} C^\infty(E|_{U_j\cap U_k})$$

by

$$\delta\big((s_j)_{j\in J}\big) := \big(s_j|_{U_j\cap U_k} - s_k|_{U_j\cap U_k}\big)_{(j,k)\in J\times J}.$$

We then define

$$C^\infty(X, E) := \operatorname{Ker}\delta.$$

Since J is at most countable, the product $\prod_{j\in J} C^\infty(E|_{U_j})$ is a Fréchet space (Mini-Exercise I.6.42 (iv)), and $\operatorname{Ker}\delta$ is a closed vector subspace. Hence $C^\infty(X, E)$ is a Fréchet space (Mini-Exercise I.6.42 (iii)). By construction, $C^\infty(X, E)$ is canonically identified with the usual vector space of global C^∞ sections of E.

Remark I.8.14. The Fréchet space topology on $C^\infty(X, E)$ defined above is independent of the choice of the trivializing covering. To see this, let $\{U_j\}_{j\in J}$ and $\{V_\alpha\}_{\alpha\in A}$ be two locally finite, at most countable open coverings of X by coordinate neighborhoods such that E is C^∞-trivial over each U_j and each V_α. By passing to a common refinement, we may and shall assume that $\{V_\alpha\}_{\alpha\in A}$ is a refinement of $\{U_j\}_{j\in J}$; that is, for each $\alpha \in A$ there exists $j = j(\alpha) \in J$ such that $V_\alpha \subset U_{j(\alpha)}$. Let $C^\infty(\{U_j\}, E)$ (resp. $C^\infty(\{V_\alpha\}, E)$) denote the Fréchet space constructed from the covering $\{U_j\}$ (resp. $\{V_\alpha\}$). Restriction of sections defines a canonical linear map

$$R\colon C^\infty(\{U_j\}, E) \longrightarrow C^\infty(\{V_\alpha\}, E), \qquad (s_j)_{j\in J} \longmapsto (s_{j(\alpha)}|_{V_\alpha})_{\alpha\in A}.$$

It is straightforward to check that R is well-defined, linear, and continuous. Moreover, R is bijective, since a compatible family of local sections on the refinement $\{V_\alpha\}$ uniquely determines a compatible family on the coarser covering $\{U_j\}$. Since both $C^\infty(\{U_j\}, E)$ and $C^\infty(\{V_\alpha\}, E)$ are Fréchet spaces, the open mapping theorem (Theorem I.6.46) implies that R is a TVS isomorphism. This shows that the Fréchet space topology on $C^\infty(X, E)$ is independent of the choice of the trivializing covering. In particular, it does not depend on the choices of local coordinates or local trivializations of E. □

Remark I.8.15. Let $p \in \mathbb{Z}_{>0} \cup \{\infty\}$. For each $j \in J$, denote by $\mathcal{M}(\Omega_j)$ the vector space of complex-valued functions on Ω_j that are measurable with respect to the Lebesgue measure in the chosen local coordinates, and by $L^p_{\mathrm{loc}}(\Omega_j)$ the vector space of complex-valued functions on Ω_j that are locally L^p. At this stage, we regard these spaces merely as vector spaces, without endowing them with any topology. Strictly speaking, their elements are equivalence classes modulo equality almost everywhere; however, following the usual convention, we shall treat them as functions whenever no confusion is likely to arise.

Via the local chart ψ_j and the trivialization Φ_j, we set $\mathcal{M}(E|_{U_j}) := \big(\mathcal{M}(\Omega_j)\big)^{\oplus r}$ and $L^p_{\mathrm{loc}}(E|_{U_j}) := \big(L^p_{\mathrm{loc}}(\Omega_j)\big)^{\oplus r}$. Exactly as in the construction of $C^\infty(X, E)$ above, we define $\mathcal{M}(X, E)$ and $L^p_{\mathrm{loc}}(X, E)$ as the kernels of the corresponding maps defined in the same manner as δ above, obtained by gluing these local spaces via restriction. Elements of these spaces are thus equivalence classes of sections of E over X, which we shall simply call *measurable sections* and *locally L^p sections* of E, respectively.

By the same refinement argument as in Remark I.8.14, these constructions are independent of the choice of the covering $\{U_j\}_{j\in J}$. Although the notion of being locally L^p might at first glance appear to depend on a choice of a C^∞ Hermitian metric on E, this is not the case: locally on X, the norms induced by any two such metrics are equivalent up to a multiplicative constant, and hence the property of being locally L^p is independent of the chosen Hermitian metric. □

The strict countable LF-space $C^\infty_c(X, E)$

For a compact subset $K \subset X$, set

$$C^\infty_K(X, E) := \{\, s \in C^\infty(X, E) \mid \operatorname{Supp} s \subset K \,\},$$

which is endowed with the subspace topology. As $C^\infty_K(X, E)$ is a closed vector subspace of the Fréchet space $C^\infty(X, E)$, $C^\infty_K(X, E)$ is also a Fréchet space (Mini-Exercise I.6.42 (*iii*)). Note that, by Remark I.8.14, this topology is independent of all auxiliary choices.

As mentioned earlier in this section, X admits a C^∞ exhaustion function $f\colon X \to \mathbb{R}$ (i.e., $\{x \in X \mid f(x) \le c\}$ is compact for every $c \in \mathbb{R}$). In particular, one may take an increasing sequence of compact subsets $\{K_\nu\}_{\nu\in\mathbb{Z}_{\ge 0}}$ such that

$$K_\nu \subset \operatorname{Int} K_{\nu+1}, \qquad X = \bigcup_{\nu=0}^{\infty} K_\nu,$$

for example by taking sublevel sets of such an exhaustion function. Let

$$\iota_\nu \colon C^\infty_{K_\nu}(X,E) \hookrightarrow C^\infty_{K_{\nu+1}}(X,E)$$

be the inclusion map. Then $\{\iota_\nu \colon C^\infty_{K_\nu}(X,E) \to C^\infty_{K_{\nu+1}}(X,E)\}_{\nu\in\mathbb{Z}_{\geq 0}}$ is a strict inductive sequence of Fréchet spaces, and we define

$$C^\infty_c(X,E) := \varinjlim_\nu C^\infty_{K_\nu}(X,E),$$

endowed with the locally convex final topology. Thus $C^\infty_c(X,E)$ is a strict countable LF-space (refer also to the construction in Example I.6.44).

Remark I.8.16. The strict countable LF-space structure on $C^\infty_c(X,E)$ is independent of the choice of the trivializing covering and of the exhaustion $\{K_\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$. Indeed, independence of the choice of the trivializing covering follows by the argument in Remark I.8.14. To see the independence of the strict countable LF-space structure on $C^\infty_c(X,E)$, let $\{K_\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ and $\{L_\mu\}_{\mu\in\mathbb{Z}_{\geq 0}}$ be two exhaustions of X by compact subsets such that

$$K_\nu \subset \operatorname{Int} K_{\nu+1}, \qquad L_\mu \subset \operatorname{Int} L_{\mu+1}, \qquad X = \bigcup_\nu K_\nu = \bigcup_\mu L_\mu.$$

We write

$$C^\infty_c(E,\{K_\nu\}) := \varinjlim_\nu C^\infty_{K_\nu}(X,E), \qquad C^\infty_c(E,\{L_\mu\}) := \varinjlim_\mu C^\infty_{L_\mu}(X,E)$$

for the corresponding strict countable LF-spaces. For each ν there exists μ such that $K_\nu \subset L_\mu$, and the inclusion $C^\infty_{K_\nu}(X,E) \hookrightarrow C^\infty_{L_\mu}(X,E)$ is a continuous linear map between Fréchet spaces. By the universal property of inductive limits, these maps induce a unique continuous linear map $C^\infty_c(E,\{K_\nu\}) \to C^\infty_c(E,\{L_\mu\})$. By construction, it is the identity map on the underlying vector space $C^\infty_c(E)$, and in particular it is bijective. Since both $C^\infty_c(E,\{K_\nu\})$ and $C^\infty_c(E,\{L_\mu\})$ are strict countable LF-spaces, the open mapping theorem (Theorem I.6.46) implies that it is a TVS isomorphism. □

Remark I.8.17. Let $K \subset X$ be compact. Although $C^\infty_K(X,E)$ was defined above as a closed subspace of the Fréchet space $C^\infty(X,E)$, it is often convenient to work with an explicit family of C^m-type seminorms on $C^\infty_K(X,E)$, whose concrete definition can be described by using the fixed locally finite trivializing covering $\{U_j\}_{j\in J}$ together with charts $\psi_j \colon U_j \to \Omega_j \subset \mathbb{R}^n$ and trivializations Φ_j. If $K \neq \emptyset$ and $r > 0$, choose a finite subset $J(K) \subset J$ and non-empty compact subsets $K_j \subset K \cap U_j$ such that $K = \bigcup_{j\in J(K)} K_j$ (Such a choice follows by shrinking the finite open covering of the compact Hausdorff space K). Set $\widetilde{K}_j := \psi_j(K_j) \subset \Omega_j$. Via ψ_j and Φ_j, the restriction $s|_{U_j}$ of a section $s \in C^\infty_K(X,E)$ is identified with an r-tuple $(s_{j,1}, s_{j,2}, \ldots, s_{j,r})$ of smooth functions on Ω_j. We evaluate these functions on $\widetilde{K}_j$; their supports need not be contained in $\widetilde{K}_j$. For each $m \in \mathbb{Z}_{\geq 0}$, we define the C^m-seminorm of s on K by

$$\|s\|_{C^m(K)} := \max_{j\in J(K)} \max_{1\leq\lambda\leq r} \max_{|\alpha|\leq m} \sup_{x\in\widetilde{K}_j} \left|D^\alpha_{\mathbb{R}^n} s_{j,\lambda}(x)\right|, \tag{28}$$

where $D^{\alpha}_{\mathbb{R}^n}$ denotes the ordinary partial derivative in the coordinates of Ω_j, as in Example I.6.44. When $K = \emptyset$ or $r = 0$, the space is zero and we set this seminorm equal to zero. The same formula also gives a continuous seminorm on $C^\infty(X, E)$, without a support condition.

The family of seminorms $\{\|\cdot\|_{C^m(K)}\}_{m\in\mathbb{Z}_{\geq 0}}$ defines a Fréchet space topology on $C^\infty_K(X, E)$, and this topology coincides with the subspace topology induced from $C^\infty(X, E)$.

The numerical value of $\|s\|_{C^m(K)}$ depends on the auxiliary choices (charts, trivializations, and the finite index set $J(K)$). However, if one replaces these choices by another such system and defines seminorms $\|\cdot\|'_{C^m(K)}$ in the same manner, then the two families of seminorms are equivalent: they generate the same locally convex topology on $C^\infty_K(X, E)$. Consequently, notions depending only on the topology—such as convergence, continuity, or boundedness of subsets of $C^\infty_K(X, E)$—are independent of all auxiliary choices. □

Spaces of forms

All spaces of forms below are understood to be zero in degrees less than 0 or greater than n. For each $q \in \mathbb{Z}_{\geq 0}$, set

$$\mathcal{A}^q(X, E) := C^\infty\Big(X,\ E \otimes \bigwedge^q T^*X\Big), \qquad \mathcal{A}^q_c(X, E) := C^\infty_c\Big(X,\ E \otimes \bigwedge^q T^*X\Big).$$

Elements of $\mathcal{A}^q(X, E)$ (resp. $\mathcal{A}^q_c(X, E)$) are called *E-valued q-forms* (resp. *compactly supported E-valued q-forms*). When E is the trivial line bundle $X \times \mathbb{C}$, we write

$$\mathcal{A}^q(X) := \mathcal{A}^q(X, X \times \mathbb{C}), \qquad \mathcal{A}^q_c(X) := \mathcal{A}^q_c(X, X \times \mathbb{C}),$$

and call their elements (compactly supported) q-forms on X. Note that $\mathcal{A}^0(X, E) = C^\infty(E)$ and $\mathcal{A}^0_c(X, E) = C^\infty_c(E)$. Also, $\mathcal{A}^0(X)$ and $\mathcal{A}^0_c(X)$ are often written simply as $C^\infty(X)$ and $C^\infty_c(X)$, respectively.

Similarly, using the notation introduced in Remark I.8.15, we define spaces of E-valued forms with weaker regularity: For each $q \in \mathbb{Z}_{\geq 0}$, we set

$$\mathcal{M}^q(X, E) := \mathcal{M}\Big(X,\ E \otimes \bigwedge^q T^*X\Big), \qquad L^q_{(p),\mathrm{loc}}(X, E) := L^p_{\mathrm{loc}}\Big(X,\ E \otimes \bigwedge^q T^*X\Big).$$

At this stage, these spaces are regarded simply as complex vector spaces, without specifying any topology. Elements of $\mathcal{M}^q(X, E)$ (resp. $L^q_{(p),\mathrm{loc}}(X, E)$) are called *measurable E-valued q-forms* (resp. *locally L^p E-valued q-forms*). As in the smooth case, when E is the trivial line bundle $X \times \mathbb{C}$, we write

$$\mathcal{M}^q(X) := \mathcal{M}^q(X, X \times \mathbb{C}), \qquad L^q_{(p),\mathrm{loc}}(X) := L^q_{(p),\mathrm{loc}}(X, X \times \mathbb{C}),$$

and call their elements measurable q-forms and locally L^p q-forms on X, respectively. Note that $\mathcal{M}^0(X, E) = \mathcal{M}(X, E)$ and $L^0_{(p),\mathrm{loc}}(X, E) = L^p_{\mathrm{loc}}(X, E)$.

Spaces of currents

For $q \in \mathbb{Z}_{\geq 0}$, define

$$\mathcal{K}^q(X,E) := \big(\mathcal{A}_c^{n-q}(X,E^*)\big)^\vee, \qquad \mathcal{K}_c^q(X,E) := \big(\mathcal{A}^{n-q}(X,E^*)\big)^\vee.$$

We likewise set the spaces of currents equal to zero outside degrees $0, 1, \ldots, n$. By our convention in §6.3, the duals are always endowed with the strong topologies $\beta(\mathcal{K}^q(X,E), \mathcal{A}_c^{n-q}(X,E^*))$ and $\beta(\mathcal{K}_c^q(X,E), \mathcal{A}^{n-q}(X,E^*))$ unless explicitly stated otherwise. When we consider these spaces with the corresponding weak-$*$ topologies $\sigma(\mathcal{K}^q(X,E), \mathcal{A}_c^{n-q}(X,E^*))$ and $\sigma(\mathcal{K}_c^q(X,E), \mathcal{A}^{n-q}(X,E^*))$, we shall indicate this explicitly. Elements of $\mathcal{K}^q(X,E)$ (resp. $\mathcal{K}_c^q(X,E)$) are called *q-currents with values in E* (resp. *compactly supported q-currents with values in E*).

When $E = X \times \mathbb{C}$, we write $\mathcal{K}^q(X) := \mathcal{K}^q(X,E)$ and $\mathcal{K}_c^q(X) := \mathcal{K}_c^q(X,E)$. In particular, $\mathcal{K}^0(X)$ and $\mathcal{K}_c^0(X)$ are often denoted by $\mathcal{K}(X)$ and $\mathcal{K}_c(X)$, respectively, and their elements are also called *(Schwartz) distributions*.

In what follows, we give some typical examples of currents. Before proceeding, let us explain a general pattern that underlies the continuity arguments presented below. By definition of the locally convex topology on $\mathcal{A}_c^k(X,E)$, which makes this space a strict countable LF-space, a subset $B \subset \mathcal{A}_c^k(X,E)$ is TVS bounded if and only if there exists a compact subset $K \subset X$ such that

$$\operatorname{Supp}\eta \subset K \quad \text{for all } \eta \in B,$$

and moreover, for each non-negative integer m, the C^m-seminorms of η on K defined by (28) are uniformly bounded over $\eta \in B$. In general, to prove continuity of a linear functional

$$T\colon \mathcal{A}_c^k(X,E) \longrightarrow \mathbb{C},$$

it suffices to show that for each compact $K \subset X$ there exist $m_K \in \mathbb{Z}_{\geq 0}$ and $C_K > 0$ such that

$$|T(\eta)| \leq C_K \|\eta\|_{C^{m_K}(K)} \qquad (\operatorname{Supp}\eta \subset K).$$

These estimates give continuity on each fixed-support Fréchet space, and hence continuity on $\mathcal{A}_c^k(X,E)$ by the universal property of the locally convex final topology.

However, in the examples presented below, the functionals T are all of *order zero* in the sense that they depend only on the pointwise values of the test forms and do not involve any derivatives of them (In particular, no differential operator is applied to the test forms). As a consequence, for these examples it suffices to obtain a uniform C^0-estimate on compact subsets: for each compact subset $K \subset X$, it is enough to show that there exists a constant $C_K > 0$ such that

$$|T(\eta)| \leq C_K \cdot \|\eta\|_{C^0(K)} \qquad \text{for all } \eta \in \mathcal{A}_c^k(X,E) \text{ with } \operatorname{Supp}\eta \subset K.$$

In other words, in these cases continuity reduces to a uniform C^0-bound on compact subsets of X.

Example I.8.18. Let μ be a Radon measure on X. Then μ defines a continuous linear functional on $\mathcal{A}^0_c(X) = C^\infty_c(X)$ by

$$\mathcal{A}^0_c(X) \ni \varphi \longmapsto \int_X \varphi \, d\mu \in \mathbb{C}.$$

Hence μ can be regarded as an n-current. Note that the continuity can be shown as follows: Let $B \subset C^\infty_c(X)$ be TVS bounded. Then there exists a compact set $K \subset X$ such that $\operatorname{Supp} \varphi \subset K$ for all $\varphi \in B$, and moreover $\sup_{\varphi \in B} \|\varphi\|_{C^0(K)} < \infty$. Therefore

$$\sup_{\varphi \in B} \left| \int_X \varphi \, d\mu \right| \leq |\mu|(K) \cdot \sup_{\varphi \in B} \|\varphi\|_{C^0(K)} < \infty,$$

which shows that $\varphi \mapsto \int_X \varphi \, d\mu$ is continuous on $C^\infty_c(X)$ with its strict LF-space topology. □

For the integration pairings below, assume that X is orientable and fix an orientation.

Example I.8.19. Let $q \in \mathbb{Z}_{\geq 0}$ and let $\alpha \in L^q_{(1),\mathrm{loc}}(X, E)$. Then α defines a q-current with values in E by

$$\mathcal{A}^{n-q}_c(X, E^*) \ni \eta \longmapsto \int_X \alpha \wedge \eta \in \mathbb{C}.$$

Here we regard the coefficients of $\alpha \wedge \eta$ as complex-valued functions via the natural pairing $E \otimes E^* \to \mathbb{C}$, and then integrate the resulting (measurable) top-degree form. In this way we obtain a canonical linear map

$$L^q_{(1),\mathrm{loc}}(X, E) \longrightarrow \mathcal{K}^q(X, E).$$

One can show that this map is injective by a standard (local) argument. Thus it is often harmless to regard $L^q_{(1),\mathrm{loc}}(X, E)$ as a subspace of $\mathcal{K}^q(X, E)$ (as vector spaces). Note that the continuity can be shown as follows. Fix a compact subset $K \subset X$ and choose finitely many smooth functions χ_j with compact support in coordinate neighborhoods U_j, such that $\sum_j \chi_j = 1$ near K. For $\operatorname{Supp} \eta \subset K$, we have

$$\int_X \alpha \wedge \eta = \sum_j \int_{U_j} \chi_j \alpha \wedge \eta.$$

The coefficients of α are integrable on the compact support of χ_j. The pointwise coefficients of η there, wherever they are non-zero, are bounded by a constant times $\|\eta\|_{C^0(K)}$ using the finite compact pieces in Remark I.8.17 and the transition matrices on their compact overlaps. Consequently, for a constant $C_{\alpha,K} > 0$,

$$\left| \int_X \alpha \wedge \eta \right| \leq C_{\alpha,K} \|\eta\|_{C^0(K)} \qquad (\operatorname{Supp} \eta \subset K).$$

This proves continuity on each fixed-support space, and hence on $\mathcal{A}^{n-q}_c(X, E^*)$. □

EXAMPLE I.8.20. Let $q \in \{0, 1, \dots, n\}$ and let $\sigma\colon (\mathrm{Simplex})_{n-q} \to X$ be a C^∞ simplex. Then σ defines a compactly supported q-current

$$T_\sigma \in \mathcal{K}^q_c(X)$$

by

$$T_\sigma(\eta) := \int_{(\mathrm{Simplex})_{n-q}} \sigma^*\eta, \qquad \eta \in \mathcal{A}^{n-q}(X).$$

This is well-defined since $(\mathrm{Simplex})_{n-q}$ is compact. Reversing the orientation of $(\mathrm{Simplex})_{n-q}$ multiplies T_σ by -1. By linearity, any C^∞ singular $(n-q)$-chain $\sigma = \sum_\nu c_\nu \sigma_\nu \in C^{\mathrm{smooth}}_{n-q}(X)$ defines a compactly supported q-current $T_\sigma := \sum_\nu c_\nu T_{\sigma_\nu}$, hence a linear map

$$T_\bullet\colon C^{\mathrm{smooth}}_{n-q}(X) \to \mathcal{K}^q_c(X).$$

Note that the continuity follows from the estimate

$$|T_\sigma(\eta)| \le \mathrm{Vol}_{\mathrm{Euc}}((\mathrm{Simplex})_{n-q}) \|\sigma^*\eta\|_{C^0((\mathrm{Simplex})_{n-q})} \le C_\sigma \|\eta\|_{C^0(\sigma((\mathrm{Simplex})_{n-q}))}.$$

Here the first norm is taken on the standard simplex, and the second inequality follows from the boundedness of the derivatives of the fixed smooth map σ on that compact simplex. The last seminorm is continuous on $\mathcal{A}^{n-q}(X)$. □

EXAMPLE I.8.21. Let $Y \subset X$ be an oriented C^∞ submanifold of codimension $q \in \mathbb{Z}_{\ge 0}$, i.e. $\dim_{\mathbb{R}} Y = n - q$. Let $\iota\colon Y \to X$ be the inclusion map, which is a proper embedding by our convention. Then Y defines a q-current

$$T_Y \in \mathcal{K}^q(X)$$

by

$$T_Y(\eta) := \int_Y \eta := \int_Y \iota^*\eta, \qquad \eta \in \mathcal{A}^{n-q}_c(X).$$

The integral is well-defined since $\mathrm{Supp}\,\eta \cap Y$ is compact in Y. For the same reason, T_Y is well-defined as an element of $\mathcal{K}^q_c(X)$ whenever Y is compact. The definition of T_Y is independent of the choice of local coordinates or local parametrizations of Y, and it depends only on the given orientation of Y (see [Lee-Jo2, Proposition 16.5] and [Sp, §5-4]). Note that the continuity can be shown in the same manner as in Example I.8.20. □

Note that the exterior derivative defines a continuous linear map

$$d\colon \mathcal{A}^{n-q-1}_c(X) \longrightarrow \mathcal{A}^{n-q}_c(X),$$

which can be extended canonically to an operator on currents within the framework of Proposition I.6.49. More generally, let E be a C^∞ complex vector bundle on X, and let

$$D\colon \mathcal{A}^k(X, E) \longrightarrow \mathcal{A}^{k+1}(X, E)$$

be a C^∞ connection on E. We explain how such an operator extends naturally to an operator on E-valued currents, and then recover the case of the exterior derivative as a special case.

In the notation of Proposition I.6.49, we set

$$A^+ := \mathcal{A}^q(X,E), \qquad A^- := \mathcal{A}^{q+1}(X,E),$$

and

$$A_c^+ := \mathcal{A}_c^{n-q}(X,E^*), \qquad A_c^- := \mathcal{A}_c^{n-q-1}(X,E^*).$$

Using the compact exhaustion $\{K_\nu\}_\nu$ chosen above, take

$$A_\nu^+ := \{\varphi \in \mathcal{A}^{n-q}(X,E^*) \mid \operatorname{Supp}\varphi \subset K_\nu\}, \qquad A_\nu^- := \{\psi \in \mathcal{A}^{n-q-1}(X,E^*) \mid \operatorname{Supp}\psi \subset K_\nu\}.$$

With the induced Fréchet topologies and the inclusion maps, these form strict inductive sequences with limits A_c^+ and A_c^-, respectively. The natural bilinear pairings

$$\langle -,-\rangle_+ \colon \mathcal{A}^q(X,E) \times \mathcal{A}_c^{n-q}(X,E^*) \longrightarrow \mathbb{C}, \qquad \langle -,-\rangle_- \colon \mathcal{A}^{q+1}(X,E) \times \mathcal{A}_c^{n-q-1}(X,E^*) \longrightarrow \mathbb{C},$$

are defined by integration, namely

$$\langle \alpha,\varphi\rangle_+ := \int_X \alpha\wedge\varphi, \qquad \langle \beta,\psi\rangle_- := \int_X \beta\wedge\psi,$$

where the wedge products are regarded as top-degree forms by applying the natural pairing $E\otimes E^* \to \mathbb{C}$ to the coefficients. For each fixed ν, a finite collection of coordinate neighborhoods trivializing E near K_ν gives an estimate for each integral by a constant times the product of continuous seminorms of its two arguments. Thus the pairings are continuous on $A^+\times A_\nu^+$ and $A^-\times A_\nu^-$, respectively (If $K_\nu=\emptyset$, the corresponding test spaces are zero). Let

$$D^* \colon \mathcal{A}^k(X,E^*) \longrightarrow \mathcal{A}^{k+1}(X,E^*)$$

be the dual connection induced by D. The connection D defines a continuous linear map

$$T := D \colon A^+ = \mathcal{A}^q(X,E) \longrightarrow A^- = \mathcal{A}^{q+1}(X,E).$$

With respect to the above pairings, the formal adjoint of T is the continuous map

$$T^*_{\text{formal}} = (-1)^{q+1}D^* \colon A_c^- = \mathcal{A}_c^{n-q-1}(X,E^*) \longrightarrow A_c^+ = \mathcal{A}_c^{n-q}(X,E^*).$$

Indeed, for any $\alpha\in\mathcal{A}^q(X,E)$ and $\eta\in\mathcal{A}_c^{n-q-1}(X,E^*)$, one has

$$\int_X (D\alpha)\wedge\eta = (-1)^{q+1}\int_X \alpha\wedge(D^*\eta),$$

which follows from the Leibniz rule for the natural pairing and Stokes' theorem.

The transpose construction of Proposition I.6.49 then gives a continuous linear operator

$$\widetilde{D} \colon \mathcal{K}^q(X,E) = (A_c^+)^\vee \longrightarrow \mathcal{K}^{q+1}(X,E) = (A_c^-)^\vee$$

defined by

$$(\widetilde{D}S)(\eta) := (-1)^{q+1} S\big(D^*\eta\big), \qquad S \in \mathcal{K}^q(X,E),\ \eta \in \mathcal{A}_c^{n-q-1}(X,E^*).$$

By a standard abuse of notation, and as long as no confusion is likely to arise, we shall denote the induced operator $\widetilde{D}$ on currents by the same symbol as the original connection and simply write $D := \widetilde{D}$.

Since the dual connection D^* preserves compact supports, the same argument applies verbatim to compactly supported currents and yields a continuous linear operator

$$D\colon \mathcal{K}_c^q(X,E) \longrightarrow \mathcal{K}_c^{q+1}(X,E).$$

In the special case where $E = X \times \mathbb{C}$ is the trivial line bundle and $D = d$ is the exterior derivative, one has $D^* = d$, and the above definition recovers the usual exterior derivative on currents:

$$(dS)(\eta) = (-1)^{q+1} S(d\eta).$$

Remark I.8.22. Let $q \in \mathbb{Z}_{\geq 0}$ and let $\sigma\colon (\mathrm{Simplex})_{n-q} \to X$ be a C^∞ simplex, and let $T_\sigma \in \mathcal{K}_c^q(X)$ be the associated integration current defined in Example I.8.20. Then, for any $\eta \in \mathcal{A}^{n-q-1}(X)$, we have

$$(dT_\sigma)(\eta) = (-1)^{q+1} T_\sigma(d\eta) = (-1)^{q+1} \int_{(\mathrm{Simplex})_{n-q}} d(\sigma^*\eta).$$

By the classical Stokes theorem on the oriented simplex $(\mathrm{Simplex})_{n-q}$, this equals

$$(-1)^{q+1} \int_{\partial(\mathrm{Simplex})_{n-q}} \sigma^*\eta = (-1)^{q+1} T_{\partial\sigma}(\eta).$$

Hence

$$dT_\sigma = (-1)^{q+1} T_{\partial\sigma} \quad \text{in } \mathcal{K}_c^{q+1}(X).$$

Similarly, if one allows Y to be an oriented C^∞ submanifold with non-empty boundary ∂Y (oriented in the usual way) in Example I.8.21, the same computation shows that

$$dT_Y = (-1)^{q+1} T_{\partial Y} \quad \text{in } \mathcal{K}^{q+1}(X).$$

□

At this point we have explained how the exterior derivative and, more generally, a C^∞ connection on a vector bundle can be extended to operators on currents by means of Proposition I.6.49. The same functional-analytic framework applies to many other operators that naturally act on smooth forms. Typical examples include the following:

- *Multiplication by a smooth function.* For $f \in C^\infty(X)$, the operator

  $$T \longmapsto fT$$

  on currents is defined by

  $$(fT)(\eta) := T(f\eta).$$

- *Wedge product with a smooth differential form.* For $\alpha \in \mathcal{A}^p(X)$ and $T \in \mathcal{K}^q(X)$, one defines
$$T \longmapsto \alpha \wedge T$$
by
$$(\alpha \wedge T)(\eta) := (-1)^{pq} T(\alpha \wedge \eta) = (-1)^{p(n-p)} T(\eta \wedge \alpha),$$
where $\eta \in \mathcal{A}_c^{n-p-q}(X)$.

- *Push-forward by a smooth map.* Let Y be a C^∞ manifold of pure real dimension m, and recall that X has pure real dimension n. If $f\colon X \to Y$ is a C^∞ proper map, the push-forward
$$f_*\colon \mathcal{K}^q(X) \longrightarrow \mathcal{K}^{q+m-n}(Y)$$
is defined by
$$(f_*T)(\eta) := T(f^*\eta), \qquad \eta \in \mathcal{A}_c^{n-q}(Y).$$
Here $\mathcal{K}^r(Y) = 0$ when $r < 0$ or $r > m$. Thus push-forward preserves the dimension $n-q$ of a current, rather than its degree. With the convention $(dT)(\eta) = (-1)^{q+1} T(d\eta)$ above, one has $d(f_*T) = (-1)^{m-n} f_*(dT)$; in particular, these operations commute when both real dimensions are even.

In what follows, we shall freely extend various operators to spaces of currents whenever such an extension is well-defined. In practice, the validity of such extensions can be checked by verifying the assumptions listed before and after Proposition I.6.49, and no difficulty will arise in the situations considered in this note. However, the reader should keep in mind the following important caveats:

- The wedge product of two arbitrary currents is *not* well-defined in general;
- The pull-back of currents by a general smooth map is not defined in general.

Sobolev spaces

So far, we have mainly worked with spaces of (compactly supported) C^∞ sections endowed with their natural locally convex topologies, and with their continuous duals, namely spaces of currents, which depend only on the C^∞ structure of X and E. In many analytic problems, however, it is essential to work with Hilbert or Banach space structures defined by integration, and with completions of spaces of smooth sections with respect to norms defined by integration.

From now on, we assume that X is orientable and compact (and hence $\#J < \infty$) and fix a family of non-negative functions $\{\chi_j\}_{j\in J} \subset C^\infty(X)$ satisfying
$$\operatorname{Supp} \chi_j \subset U_j, \qquad \sum_{j\in J} \chi_j^2 = 1 \quad \text{on } X$$
(Such a family can always be obtained from an ordinary partition of unity by a smooth normalization).

Let $k \in \mathbb{Z}_{\geq 0}$. For a smooth section $s \in C^\infty(X, E)$, we define its Sobolev $W^{k,2}$-norm by

$$\|s\|^2_{W^{k,2}(X,E)} := \sum_{j \in J} \sum_{|\alpha| \leq k} \int_{\Omega_j} \left| D^\alpha_{\mathbb{R}^n} \big(\chi_j s\big)_j \right|^2 \, d\lambda_{\Omega_j},$$

where:

- $(\chi_j s)_j = ((\chi_j s)_{j,1}, \ldots, (\chi_j s)_{j,r})$ denotes the expression of $\chi_j s$ in the fixed local trivialization Φ_j and local coordinates ψ_j;
- $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ is a multi-index, with $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;
- $D^\alpha_{\mathbb{R}^n}$ denotes the ordinary partial derivative on $\Omega_j \subset \mathbb{R}^n$: i.e.,

  $$D^\alpha_{\mathbb{R}^n} := \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} (\partial x^2)^{\alpha_2} \cdots (\partial x^n)^{\alpha_n}},$$

  where $(x^1, x^2, \ldots, x^n)$ denotes the standard coordinate system of $\mathbb{R}^n$;
- $d\lambda_{\Omega_j}$ denotes the Lebesgue measure on Ω_j.

We define the *Sobolev space* $W^{k,2}(X, E)$ to be the completion of $C^\infty(X, E)$ with respect to the norm $\| \cdot \|_{W^{k,2}(X,E)}$. The inner product

$$(s, t)_{W^{k,2}(X,E)} := \sum_{j \in J} \sum_{|\alpha| \leq k} \int_{\Omega_j} \langle D^\alpha_{\mathbb{R}^n} (\chi_j s)_j, \, D^\alpha_{\mathbb{R}^n} (\chi_j t)_j \rangle \, d\lambda_{\Omega_j}$$

makes $W^{k,2}(X, E)$ a Hilbert space.

Remark I.8.23. The norm $\| \cdot \|_{W^{k,2}(X,E)}$ depends on the choice of the covering $\{U_j\}_{j \in J}$, the local trivializations Φ_j, and the functions $\{\chi_j\}$. However, different choices lead to equivalent norms (see the arguments in [Kod, p. 421–425]). In particular, the resulting Sobolev space $W^{k,2}(X, E)$ is well-defined as a TVS, independently of all auxiliary choices (when X is compact). □

Note also that, since X is compact, the underlying vector space of $W^{0,2}(X, E)$ can be naturally identified, as a subspace of $\mathcal{M}(X, E)$, with $L^2_{\mathrm{loc}}(X, E)$. When X is compact, it is therefore natural to write $L^2(X, E)$ instead of $L^2_{\mathrm{loc}}(X, E)$. Similarly, when X is compact, we shall write $L^q_{(2)}(X, E)$ for $L^q_{(2),\mathrm{loc}}(X, E)$. The Hilbert space structure on these spaces (which induces the same underlying TVS topology, although the inner product itself may differ) can also be described simply by fixing a Riemannian metric g on X and a C^∞ Hermitian metric h on E: More precisely, by regarding $L^q_{(2)}(X, E)$ as the completion $L^q_{(2)}(X, E; g, h)$ of $\mathcal{A}^q(X, E)$ with respect to the norm induced by the L^2-inner product

$$(s, t)_{L^q_{(2)}(X,E;g,h)} := \int_X \langle s, \, t \rangle_{g,h} \, dV_g.$$

Here dV_g denotes the volume form on X determined by g, and $\langle s,t\rangle_{g,h}$ denotes the pointwise inner product computed fiberwise using the Hermitian metric h on the coefficients in E and the Riemannian metric g on the form components. For the evident reason, when $q=0$ this fiberwise inner product does not depend on g, and one simply has

$$(s,t)_{L^q_{(2)}(X,E;g,h)} = \int_X \langle s,\, t\rangle_h \, dV_g.$$

When $E = X\times\mathbb{C}$ is the trivial line bundle, we shall often write $L^q_{(2)}(X;g,h)$ instead of $L^q_{(2)}(X,X\times\mathbb{C};g,h)$. In this case, a C^∞ Hermitian metric h on E is simply given by a positive C^∞ function on X. If $h\equiv 1$, we further simplify the notation and write $L^q_{(2)}(X;g)$ for $L^q_{(2)}(X;g,1)$.

Remark I.8.24. The Hilbert space $L^q_{(2)}(X,E;g,h)$ and the inner product defined above also make sense when X is not compact: in this case we take the completion of $\mathcal{A}^q_c(X,E)$ with respect to the indicated L^2-norm. However, when X is non-compact, one should keep in mind that the choice of the metrics g and h affects not only the TVS structure of $L^q_{(2)}(X,E;g,h)$, but even the underlying vector subspace of $L^q_{(2),\mathrm{loc}}(X,E)$.

For instance, if one modifies the Hermitian metric h by multiplying it by a real-valued function that grows sufficiently fast at infinity on X, the L^2-integrability condition becomes more restrictive, and $L^q_{(2)}(X,E;g,h)$ becomes a strictly smaller vector subspace of $L^q_{(2),\mathrm{loc}}(X,E)$. The dependence on the Riemannian metric g is more subtle, since it affects both the fiberwise inner product and the volume form dV_g.

Moreover, when working on $L^q_{(2)}(X,E;g,h)$ in the non-compact case, additional care is required when discussing analytic properties of operators, such as whether $\mathcal{A}^q_c(X,E)$ is dense in the domain or range of a given operator. In many situations, this density can be recovered under the assumption that the metric space (X,dist_g) is complete, by an application of the Hopf–Rinow theorem and the existence of suitable cutoff functions (see [De4, Chapter VIII, (2.4), (3.2)]).

In view of these metric-dependence issues, throughout this note we shall always specify explicitly how the metrics g and h are fixed whenever X is non-compact. □

Let $\ell\in\mathbb{Z}_{\geq 0}$. We denote by $C^\ell(X,E)$ the vector space of sections $s\colon X\to E$ which are ℓ times continuously differentiable. We endow $C^\ell(X,E)$ with the Banach space topology defined by the norm

$$\|s\|_{C^\ell(X,E)} := \sum_{m=0}^{\ell} \|s\|_{C^m(X,E)},$$

where $\|s\|_{C^m(X,E)}$ denotes the global C^m-norm defined by (28) with $K=X$ (Recall that we are assuming that X is compact; If X is not compact, we regard $C^\ell(X,E)$ merely as a vector space). By compactness of X, this norm is finite and independent (up to equivalence) of all auxiliary choices.

Under the above assumptions, namely that X is a compact C^∞ manifold of real dimension n and E is a C^∞ complex vector bundle on X, the Sobolev spaces $W^{k,2}(X,E)$ enjoy a number of fundamental analytic properties. The following facts are classical:

- For any integers $k > \ell \in \mathbb{Z}_{\geq 0}$, the natural inclusion
$$W^{k,2}(X,E) \hookrightarrow W^{\ell,2}(X,E)$$
is a compact linear mapping (*Rellich's theorem*, see [Kod, Appendix Theorem 3.1] for example).

- Let $k, \ell \in \mathbb{Z}_{\geq 0}$. If
$$k > \ell + \frac{n}{2},$$
then the inclusion map
$$W^{k,2}(X,E) \hookrightarrow C^{\ell}(X,E),$$
defined by considering the underlying vector spaces as subspaces of $\mathcal{M}(X,E)$, induces a continuous embedding between them as TVSs (*Sobolev's embedding theorem*, see [Kod, Appendix Theorem 3.2]).

In [Kod, Appendix], the proofs of Rellich's theorem and Sobolev's embedding theorem are first carried out in the special case where X is a compact torus $\mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$. In this situation, the arguments become particularly transparent, since Sobolev norms admit a concrete description in terms of Fourier series (see [Kod, Appendix Theorems 1.3 and 1.10]. See also Example I.8.25 below for the one-dimensional case; the argument is similar in higher dimensions). The general case of a compact C^{∞} manifold is then reduced to this model situation by a standard localization argument: roughly speaking, thanks to the definition of Sobolev norms using the cutoff functions $\{\chi_j\}$, Sobolev norms on X can be compared with Sobolev norms on open subsets $\Omega_j \subset \mathbb{R}^n$, and hence with those on tori.

Example I.8.25. Here we consider the case
$$X = \mathrm{U}(1), \qquad E = X \times \mathbb{C}, \qquad q = 0.$$
In this example, we write $W^{k,2}(\mathrm{U}(1)) := W^{k,2}(\mathrm{U}(1), \mathrm{U}(1) \times \mathbb{C})$. Let z be the complex coordinate on $\mathbb{C}$. We denote by g_{Euc} the Riemannian metric on $\mathrm{U}(1)$ induced by the Euclidean metric on $\mathbb{C}$. The associated volume form is given by
$$dV_{g_{\mathrm{Euc}}} = |dz|,$$
and under the parametrization $z = e^{\sqrt{-1}\theta}$ one has $dV_{g_{\mathrm{Euc}}} = d\theta$. Fix $\varepsilon > 0$ sufficiently small and set
$$U_1 = \{e^{\sqrt{-1}\theta} \mid -\varepsilon < \theta < \pi + \varepsilon\}, \qquad U_2 = \{e^{\sqrt{-1}\theta} \mid \pi - \varepsilon < \theta < 2\pi + \varepsilon\}.$$
Define coordinate maps
$$\psi_j \colon U_j \to \Omega_j \subset \mathbb{R}, \qquad \psi_j(e^{\sqrt{-1}\theta}) = \theta.$$
Choose smooth functions $\chi_1, \chi_2 \in C^{\infty}(\mathrm{U}(1))$ such that
$$\operatorname{Supp} \chi_j \subset U_j, \qquad \chi_1^2 + \chi_2^2 = 1.$$

From now on, by identifying $\mathrm{U}(1)$ with the interval $[0, 2\pi)$ via $z = e^{\sqrt{-1}\theta}$, we shall regard sections on $\mathrm{U}(1)$ simply as 2π-periodic functions of θ, and omit writing the inverse coordinate maps ψ_j^{-1} whenever no confusion is likely to arise.

By definition of the Sobolev norm, for $s \in C^\infty(\mathrm{U}(1))$ one has

$$\|s\|^2_{W^{k,2}(\mathrm{U}(1))} = \sum_{j=1}^{2} \sum_{m=0}^{k} \int_{\Omega_j} \left| \frac{d^m}{d\theta^m}(\chi_j s)(\theta) \right|^2 d\theta.$$

Since each χ_j is a fixed smooth function, we may expand derivatives of the product by the Leibniz rule:

$$\frac{d^m}{d\theta^m}(\chi_j s) = \sum_{\ell=0}^{m} \binom{m}{\ell} \left(\frac{d^\ell \chi_j}{d\theta^\ell} \right) \left(\frac{d^{m-\ell} s}{d\theta^{m-\ell}} \right).$$

As χ_j and all its derivatives are bounded, it follows that for each m the integral

$$\int_{\Omega_j} \left| \frac{d^m}{d\theta^m}(\chi_j s) \right|^2 d\theta$$

can be estimated from above by finite linear combinations of

$$\int_0^{2\pi} \left| \frac{d^{m-\ell} s}{d\theta^{m-\ell}} \right|^2 d\theta, \qquad \ell = 0, 1, \ldots, m,$$

with constants independent of s. For the reverse estimate, use $s = \sum_j \chi_j(\chi_j s)$, which follows from $\sum_j \chi_j^2 = 1$, and apply the Leibniz rule once more. After summing over the finitely many charts and derivative orders, the global integrals are bounded by a constant times the sum of the local integrals above. Motivated by this, we introduce the norm

$$\|s\|_k^2 := \sum_{m=0}^{k} \int_0^{2\pi} \left| \frac{d^m}{d\theta^m} s(e^{\sqrt{-1}\theta}) \right|^2 d\theta.$$

By the above estimates, the Sobolev norm $\|\cdot\|_{W^{k,2}(\mathrm{U}(1))}$ and the norm $\|\cdot\|_k$ are equivalent on $C^\infty(\mathrm{U}(1))$. Hence $W^{k,2}(\mathrm{U}(1))$ may be identified (as a TVS) with the completion of $C^\infty(\mathrm{U}(1))$ with respect to $\|\cdot\|_k$. Equivalently, $W^{k,2}(\mathrm{U}(1))$ is a Hilbert space with inner product

$$(s, t)_k := \sum_{m=0}^{k} \int_0^{2\pi} \left(\frac{d^m}{d\theta^m} s(e^{\sqrt{-1}\theta}) \right) \overline{\left(\frac{d^m}{d\theta^m} t(e^{\sqrt{-1}\theta}) \right)} \, d\theta.$$

For $n \in \mathbb{Z}$, define

$$\mathfrak{e}_n(z) := \frac{z^n}{\sqrt{2\pi}}.$$

Then $\{\mathfrak{e}_n\}_{n\in\mathbb{Z}}$ is a CONS of $W^{0,2}(\mathrm{U}(1))$. For $s \in W^{0,2}(\mathrm{U}(1))$ and $n \in \mathbb{Z}$, we write

$$\widehat{s}(n) := (s,\, \mathfrak{e}_n)_0$$

so that

$$s(z) = \sum_{n\in\mathbb{Z}} \widehat{s}(n)\, \mathfrak{e}_n(z)$$

holds. Since

$$\frac{d^m}{d\theta^m}\mathfrak{e}_n(e^{\sqrt{-1}\theta}) = (\sqrt{-1}n)^m \mathfrak{e}_n(e^{\sqrt{-1}\theta}),$$

it follows that $s \in W^{k,2}(\mathrm{U}(1))$ if and only if

$$\sum_{n\in\mathbb{Z}} \left(\sum_{m=0}^{k} |n|^{2m}\right) |\widehat{s}(n)|^2 < \infty.$$

To show Rellich's theorem in our setting, fix integers $k > \ell \geq 0$ and let $\{s_\nu\}_{\nu\geq 0}$ be a bounded sequence in $W^{k,2}(\mathrm{U}(1))$. Then there exists $M > 0$ such that for all ν

$$\sum_{n\in\mathbb{Z}} \left(\sum_{m=0}^{k} |n|^{2m}\right) |\widehat{s_\nu}(n)|^2 \leq M. \tag{29}$$

Since $k > \ell$, for $|n| \geq 1$ we have

$$\sum_{m=0}^{\ell} |n|^{2m} \leq |n|^{-2(k-\ell)} \sum_{m=0}^{k} |n|^{2m}.$$

Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{Z}_{>0}$ so large that

$$\sum_{|n|>N} |n|^{-2(k-\ell)} < \varepsilon.$$

Then, by the uniform bound (29), we have

$$\sum_{|n|>N} \left(\sum_{m=0}^{\ell} |n|^{2m}\right) |\widehat{s_\nu}(n)|^2 \leq \varepsilon\, M \qquad (\text{for all } \nu \in \mathbb{Z}_{\geq 0}). \tag{30}$$

On the other hand, the finite-dimensional vectors $(\widehat{s_\nu}(n))_{|n|\leq N} \in \mathbb{C}^{2N+1}$ form a bounded sequence. By a diagonal subsequence argument over $N \in \mathbb{Z}_{\geq 0}$, we may therefore assume that $\widehat{s_\nu}(n)$ converges for every $n \in \mathbb{Z}$. Combining this convergence on the low-frequency part with the uniform estimate (30) on the high-frequency tail, we conclude that $\{s_\nu\}$ admits a Cauchy subsequence in $W^{\ell,2}(\mathrm{U}(1))$. This proves that the inclusion

$$W^{k,2}(\mathrm{U}(1)) \hookrightarrow W^{\ell,2}(\mathrm{U}(1))$$

is compact.

Next, to show Sobolev's embedding theorem in our setting, take $k, \ell \in \mathbb{Z}_{\geq 0}$ with $k > \ell + \frac{1}{2}$. For $0 \leq m \leq \ell$, we have

$$\frac{d^m}{d\theta^m} s(e^{\sqrt{-1}\theta}) = \sum_{n\in\mathbb{Z}} (\sqrt{-1}n)^m \widehat{s}(n) \mathfrak{e}_n(e^{\sqrt{-1}\theta}).$$

Applying the Cauchy–Schwarz inequality yields

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^m |\widehat{s}(n)| \le \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{2k} |\widehat{s}(n)|^2 \right)^{1/2} \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{-2(k-m)} \right)^{1/2}.$$

Since $k > m + \frac{1}{2}$, the second series converges. The zero-frequency term satisfies $|\widehat{s}(0)| \le \|s\|_0$ and disappears when $m \ge 1$. Hence the Fourier series for $\frac{d^m}{d\theta^m} s$ converges absolutely and uniformly, so $\frac{d^m}{d\theta^m} s$ is continuous. Therefore $s \in C^\ell(\mathrm{U}(1))$, and the inclusion

$$W^{k,2}(\mathrm{U}(1)) \hookrightarrow C^\ell(\mathrm{U}(1))$$

is continuous. □

8.7 Linear partial differential operators and ellipticity

Let X be a C^∞ manifold of real dimension n, and let $E \to X$ be a C^∞ complex vector bundle of rank r. Throughout this subsection, we adopt exactly the same local data and notation as in the preceding subsection. In particular, we fix once and for all a locally finite open covering $\{U_j\}_{j\in J}$ of X, and a C^∞ chart $\psi_j \colon U_j \to \Omega_j \subset \mathbb{R}^n$ and a C^∞ trivialization $\Phi_j \colon E|_{U_j} \to U_j \times \mathbb{C}^r$ for each $j \in J$, as in the previous subsection. Via ψ_j and Φ_j, smooth sections of E over U_j are identified with $\mathbb{C}^r$-valued smooth functions on Ω_j. All local expressions below are to be understood with respect to these fixed choices.

Linear partial differential operators

Let $\ell \in \mathbb{Z}_{\ge 0}$. A *linear partial differential operator of order at most* ℓ acting on sections of E is a linear map

$$P \colon C^\infty(X, E) \longrightarrow C^\infty(X, E)$$

with the following property: For every $j \in J$, there exist C^∞ functions

$$a_{j,\alpha} \colon \Omega_j \longrightarrow \mathbb{C}^{r\times r}, \qquad \alpha \in \mathbb{Z}^n_{\ge 0},\ |\alpha| \le \ell,$$

such that, for any $s \in C^\infty(X, E)$ and any $x \in U_j$, the restriction of Ps to U_j is given in local coordinates by

$$(Ps)_j(\psi_j(x)) = \sum_{|\alpha|\le\ell} a_{j,\alpha}\big(\psi_j(x)\big)\, D^\alpha_{\mathbb{R}^n}\big(s_j\big)\big(\psi_j(x)\big), \tag{31}$$

where:

- s_j denotes the $\mathbb{C}^r$-valued function corresponding to $s|_{U_j}$ under ψ_j and Φ_j;
- $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_{\ge 0}$ is regarded as a multi-index, with $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$;
- each value of $a_{j,\alpha}$ is regarded as a square matrix of order r;

- $D^{\alpha}_{\mathbb{R}^n}$ denotes the ordinary partial derivative on $\Omega_j \subset \mathbb{R}^n$: i.e.,

$$D^{\alpha}_{\mathbb{R}^n} := \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}(\partial x^2)^{\alpha_2}\cdots(\partial x^n)^{\alpha_n}},$$

  where $(x^1, x^2, \ldots, x^n)$ denotes the standard coordinate system of $\mathbb{R}^n$.

If, in the above expression, at least one coefficient $a_{j,\alpha}$ with $|\alpha| = \ell$ is non-zero on some U_j, we say that P is of *order* ℓ.

Assume in this paragraph that X is compact. If P is of order ℓ, then for every $\mu \in \mathbb{Z}_{\geq 0}$ there exists a constant $C_{P,\mu} > 0$ such that

$$\|Ps\|_{W^{\mu,2}(X,E)} \leq C_{P,\mu}\, \|s\|_{W^{\mu+\ell,2}(X,E)} \qquad \text{for all } s \in C^{\infty}(X,E) \tag{32}$$

(see [Kod, Lemma 3.3]). Consequently, P extends uniquely to a continuous linear map

$$P\colon W^{\mu+\ell,2}(X,E) \longrightarrow W^{\mu,2}(X,E).$$

In this case, we also regard P as an operator

$$P\colon W^{\mu,2}(X,E) \xrightarrow{\text{op}} W^{\mu,2}(X,E)$$

by letting $\mathrm{Dom}(P) := W^{\mu+\ell,2}(X,E) \subset W^{\mu,2}(X,E)$. Note that $P\colon W^{\mu,2}(X,E) \xrightarrow{\text{op}} W^{\mu,2}(X,E)$ is densely defined, since $C^{\infty}(X,E)$ is dense in $W^{\mu,2}(X,E)$ and is contained in $\mathrm{Dom}(P) = W^{\mu+\ell,2}(X,E)$.

Principal part and principal symbol

Let P be a linear partial differential operator of order ℓ on E whose local expression is given by (31). For each $j \in J$, we define the *principal part* of P on U_j by

$$P_{\ell,j} := \sum_{|\alpha|=\ell} a_{j,\alpha}\, D^{\alpha}_{\mathbb{R}^n}.$$

Let $x \in U_j$, and let

$$\xi = \sum_{\lambda=1}^{n} \xi_{j,\lambda}\, dx^{\lambda}_j \in T^*_x X,$$

where $(x^1_j, x^2_j, \ldots, x^n_j)$ denote the Euclidean coordinates on $\Omega_j \subset \mathbb{R}^n$. We set

$$\sigma_P(x,\xi) := \sum_{|\alpha|=\ell} \xi^{\alpha}_j \cdot a_{j,\alpha}\big(\psi_j(x)\big) \;\in\; \mathbb{C}^{r\times r}, \tag{33}$$

where

$$\xi^{\alpha}_j := \prod_{\lambda=1}^{n} \xi^{\alpha_\lambda}_{j,\lambda} \qquad \big(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}\big),$$

and we regard $\sigma_P(x,\xi)$ as a square matrix of order r acting on $\mathbb{C}^r$ via the fixed local trivialization.

Although the expression (33) depends on the choice of local coordinates and local trivializations, in the sense made precise in Lemma I.8.26 below, the resulting object defines a well-defined C^∞ fiber-preserving map

$$\sigma_P\colon T^*X \longrightarrow \mathrm{Hom}(E,E),$$

which is called the *principal symbol* of P. Note that σ_P is *not* a morphism of vector bundles in general: As is clear from the construction, σ_P is homogeneous of degree ℓ in the cotangent variable ξ, and unless $\ell = 1$, it is *not* linear with respect to ξ.

LEMMA I.8.26. *With the above notation, let $x \in U_j$ and $\xi = \sum_{k=1}^n \xi_k\, dx^k \in T^*_xX$, and let $e \in E_x$. Define $(\sigma_P(\xi))(e) \in E_x$ by*

$$(\sigma_P(\xi))\,(e) = \Phi_j^{-1}\,(x,\ \sigma_P(x,\xi)\widetilde{e}\,),$$

where $\widetilde{e} \in \mathbb{C}^r$ is the unique vector satisfying $\Phi_j(e) = (x,\widetilde{e})$. Then the map $e \mapsto (\sigma_P(\xi))(e)$ defines a linear endomorphism of E_x. Moreover, this endomorphism is independent of the choice of the index $j \in J$ such that $x \in U_j$, and depends only on the data (X,E,P,x,ξ). In particular, the map

$$\sigma_P\colon T^*X \longrightarrow \mathrm{Hom}(E,E)$$

is well-defined as a C^∞ fiber-preserving map, and does not depend on the choice of the open covering $\{U_j\}_{j\in J}$, nor on the associated local coordinates or local trivializations.

PROOF. Since the second assertion follows from the same argument (after passing to a common refinement of two given coverings if necessary), it suffices to prove the first assertion, namely the independence of the definition of $(\sigma_P(\xi))(e)$ from the choice of the index $j \in J$ such that $x \in U_j$.

Fix a point $x \in X$, $\xi \in T^*_xX$, and an element $e \in E_x$. Take a section $s \in C^\infty(X,E)$ satisfying $s(x) = e$, whose existence follows immediately from a standard bump function argument. Similarly, take a C^∞ function $f\colon X \to \mathbb{C}$ such that $f(x) = 0$ and $(df)_x = \xi$. For each $t > 0$, define a C^∞ section $s^{(t)} \in C^\infty(X,E)$ by

$$s^{(t)} := \exp\bigl(\sqrt{-1}\,t\,f\bigr)\,s.$$

Choose an index $j \in J$ such that $x \in U_j$. Let $(x^1,x^2,\ldots,x^n)$ be the Euclidean coordinates on $\Omega_j = \psi_j(U_j) \subset \mathbb{R}^n$. Using these coordinates, we identify ξ with its coordinate representation $\xi = \sum_{k=1}^n \xi_k\, dx^k$. Let $\Phi_j(e) = (x,\widetilde{e})$, where $\widetilde{e} \in \mathbb{C}^r$.

We claim that, for each multi-index

$$\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathbb{Z}^n_{\geq 0},$$

one has

$$D^\alpha_{\mathbb{R}^n}\bigl((s^{(t)})_j\bigr)(\psi_j(x)) = (\sqrt{-1}t)^{|\alpha|}\,\xi^\alpha\,\widetilde{e}\ +\ O\bigl(t^{|\alpha|-1}\bigr) \qquad (t \to +\infty), \tag{34}$$

where $\xi^\alpha = \prod_{\lambda=1}^n \xi_\lambda^{\alpha_\lambda}$. Put $f_j := f \circ \psi_j^{-1}$. By repeated application of the Leibniz rule,

$$\exp(-\sqrt{-1}tf_j)\, D^\alpha_{\mathbb{R}^n}\big(\exp(\sqrt{-1}tf_j)s_j\big)$$

is a polynomial in t of degree at most $|\alpha|$, with smooth coefficients on Ω_j and leading term

$$(\sqrt{-1}t)^{|\alpha|} \prod_{\lambda=1}^n \left(\frac{\partial f_j}{\partial x^\lambda}\right)^{\alpha_\lambda} s_j.$$

Indeed, at each differentiation the degree in t increases only when the exponential is differentiated. Evaluating this identity at $\psi_j(x)$, where $f_j(\psi_j(x)) = 0$, $(df)_x = \xi$, and $s_j(\psi_j(x)) = \widetilde{e}$, gives (34); when $\alpha = 0$ the identity is exact.

Substituting this into the local expression (31) of P on U_j, we obtain

$$(Ps^{(t)})_j(\psi_j(x)) = (\sqrt{-1}t)^\ell \left(\sum_{|\alpha|=\ell} \xi^\alpha\, a_{j,\alpha}(\psi_j(x))\right) \widetilde{e} \;+\; O\big(t^{\ell-1}\big) \qquad (t \to +\infty).$$

Since $Ps^{(t)}$ is a globally defined section of E, the element $(Ps^{(t)})(x) \in E_x$ is intrinsic and independent of the choice of local coordinates and local trivializations. Therefore the assertion follows, since this leading term is precisely $\Phi_j^{-1}(x,\ \sigma_P(x,\xi)\widetilde{e})$. □

Example I.8.27. Here, using the notation and conventions of Example I.8.25, we consider the case where $X = \mathrm{U}(1)$ and E is the trivial line bundle. Let

$$P := F(\theta)\,\frac{d^2}{d\theta^2} \;+\; G(\theta)\,\frac{d}{d\theta} \;+\; H(\theta),$$

where $F, G, H \in C^\infty(X)$.

First, assume that $F \equiv 0$ and $G \not\equiv 0$. Then P is a linear partial differential operator of order 1. Identifying the cotangent bundle T^*X with $X \times \mathbb{R}$ via the coordinate θ, in this case we have

$$\sigma_P(\theta, \xi\, d\theta) = \xi\, G(\theta),$$

which is homogeneous of degree 1 in ξ.

Next, assume that $F \not\equiv 0$. Then P is a linear partial differential operator of order 2. In this case, the principal symbol of P is given by

$$\sigma_P(\theta, \xi\, d\theta) = \xi^2\, F(\theta),$$

which is homogeneous of degree 2 in ξ. □

Example I.8.28. Here, again by using the notation and conventions of Example I.8.25, we consider the case where $X = \mathrm{U}(1)$ with local coordinate θ and $E = X \times \mathbb{C}^2$ is the trivial vector bundle of rank 2. Consider the second-order linear partial differential operator

$$P := \begin{pmatrix} \dfrac{d^2}{d\theta^2} & 0 \\ 0 & -\dfrac{d^2}{d\theta^2} \end{pmatrix}$$

acting on $C^\infty(X,E) \cong C^\infty(X) \oplus C^\infty(X)$ componentwise. Identifying T^*X with $X \times \mathbb{R}$, the principal symbol of P is given by

$$\sigma_P(\theta, \xi\, d\theta) = \begin{pmatrix} \xi^2 & 0 \\ 0 & -\xi^2 \end{pmatrix}.$$

□

Ellipticity

A linear partial differential operator P of order ℓ on E is said to be *elliptic* if, for every point $x \in X$ and every non-zero covector $\xi \in T^*_x X \setminus \{0\}$, the linear map

$$\sigma_P(\xi)\colon E_x \longrightarrow E_x$$

is an isomorphism. Note that, by definition, ellipticity is a purely local condition, determined entirely by the principal part of P.

For instance, in Example I.8.27, the operator P is elliptic whenever F has no zeros on X, or $F \equiv 0$ and G has no zeros on X. In Example I.8.28, P is elliptic.

We now state one of the most fundamental analytic properties of elliptic linear partial differential operators.

THEOREM I.8.29 (L^2 a priori estimate). *Let X be an orientable compact C^∞ manifold and $E \to X$ be a C^∞ vector bundle. Let $P\colon C^\infty(X,E) \to C^\infty(X,E)$ be an elliptic linear partial differential operator of order $\ell \in \mathbb{Z}_{\geq 1}$. Then, for any integer $\mu \in \mathbb{Z}_{\geq 0}$, there exists a constant $C_{P,\mu} > 0$ such that*

$$\|s\|_{W^{\mu+\ell,2}(X,E)} \leq C_{P,\mu}\Big(\|Ps\|_{W^{\mu,2}(X,E)} + \|s\|_{W^{\mu,2}(X,E)}\Big)$$

holds for all $s \in C^\infty(X,E)$.

PROOF. See [Kod, Appendix, Theorem 4.1]. □

The L^2 a priori estimate in Theorem I.8.29 has several important consequences for elliptic linear partial differential operators. We collect below some fundamental properties.

First, the L^2 a priori estimate implies that an elliptic linear partial differential operator defines a closed operator on Sobolev spaces when X is compact. More precisely, for a C^∞ vector bundle E on an orientable compact C^∞ manifold X and an elliptic linear partial differential operator $P\colon C^\infty(X,E) \to C^\infty(X,E)$ of order $\ell \in \mathbb{Z}_{\geq 1}$, fix $\mu \in \mathbb{Z}_{\geq 0}$ and regard P as an operator

$$P\colon W^{\mu,2}(X,E) \xrightarrow{\text{op}} W^{\mu,2}(X,E), \qquad \mathrm{Dom}(P) := W^{\mu+\ell,2}(X,E).$$

We claim that this operator is closed. To see this, consider the graph

$$\mathrm{Graph}(P) := \big\{(s, Ps) \in W^{\mu,2}(X,E) \times W^{\mu,2}(X,E) \;\big|\; s \in W^{\mu+\ell,2}(X,E)\big\}$$

of P. Let $\{s_\nu\}_{\nu\geq 1} \subset W^{\mu+\ell,2}(X,E)$ be a sequence such that

$$s_\nu \longrightarrow s \quad \text{and} \quad Ps_\nu \longrightarrow t \qquad \text{in } W^{\mu,2}(X,E).$$

By the L^2 a priori estimate in Theorem I.8.29, we have

$$\|s_\nu - s_\kappa\|_{W^{\mu+\ell,2}} \leq C_{P,\mu}\Big(\|P(s_\nu - s_\kappa)\|_{W^{\mu,2}} + \|s_\nu - s_\kappa\|_{W^{\mu,2}}\Big),$$

which shows that $\{s_\nu\}$ is a Cauchy sequence in $W^{\mu+\ell,2}(X,E)$. Hence $s \in W^{\mu+\ell,2}(X,E)$. On the other hand, by the basic estimate (32), the convergence $s_\nu \to s$ in $W^{\mu+\ell,2}(X,E)$ implies $Ps_\nu \to Ps$ in $W^{\mu,2}(X,E)$. Since $Ps_\nu \to t$ in $W^{\mu,2}(X,E)$ by assumption, we conclude that $Ps = t$. Therefore the graph of P is closed, and $P\colon W^{\mu,2}(X,E) \xrightarrow{\text{op}} W^{\mu,2}(X,E)$ is a closed operator.

The second fundamental consequence of ellipticity is the regularity of solutions.

THEOREM I.8.30 (Elliptic regularity). *Let X be an orientable compact C^∞ manifold and $E \to X$ be a C^∞ vector bundle. Let $P\colon C^\infty(X,E) \longrightarrow C^\infty(X,E)$ be an elliptic linear partial differential operator of order $\ell \in \mathbb{Z}_{\geq 1}$. Take $\mu \in \mathbb{Z}_{\geq 0}$ and $s \in \mathrm{Dom}(P\colon W^{\mu,2}(X,E) \xrightarrow{\text{op}} W^{\mu,2}(X,E))$. Assume*

$$Ps \in C^\infty(X,E).$$

Then

$$s \in C^\infty(X,E).$$

PROOF. We explain the one-derivative step used in the proof of [Kod, Appendix, § 6, Theorem 6.1] (See also [Dy, Theorem 15.1 and Remark 15.2]). Suppose that $s \in W^{k,2}(X,E)$ and $Ps \in W^{k-\ell+1,2}(X,E)$, where $k \geq \ell$. Choose a smooth cut-off function χ with compact support in a coordinate neighborhood, and write $v = \chi s$ in its local frame. For a coordinate direction e_j and sufficiently small real $h \neq 0$, set

$$\delta_{h,j} v(x) := \frac{v(x + he_j) - v(x)}{h}.$$

Choose $h_0 > 0$ so that all these translates, for $|h| < h_0$, are supported in a fixed compact subset of the same coordinate neighborhood. We regard them as global sections by extension by zero; local coordinate Sobolev norms on these sections are equivalent to the indicated norms on X.

The identity $P(\chi s) = \chi Ps + [P,\chi]s$, where $[P,\chi]$ has order at most $\ell - 1$, gives $Pv \in W^{k-\ell+1,2}$. The difference-quotient equation for v contains products of $\delta_{h,j} a_\alpha$ with translates of $D^\alpha v$, where a_α are the local coefficients of P. Their $W^{k-\ell,2}$ norms are uniformly bounded: the required derivatives of $\delta_{h,j} a_\alpha$ are bounded on the fixed compact set, independently of h, and $|\alpha| \leq \ell$. Also, the $W^{k-\ell,2}$ norm of $\delta_{h,j}(Pv)$ is bounded by a constant times the $W^{k-\ell+1,2}$ norm of Pv. For each fixed h, we already know that $\delta_{h,j} v \in W^{k,2}$, so density in this known space allows us to apply Theorem I.8.29 with exponent $k - \ell$. It follows that

$$\|\delta_{h,j} v\|_{W^{k,2}(X,E)} \leq C\big(\|P\delta_{h,j} v\|_{W^{k-\ell,2}(X,E)} + \|\delta_{h,j} v\|_{W^{k-\ell,2}(X,E)}\big) \leq C' \qquad (0 < |h| < h_0),$$

where the last term is controlled by $v \in W^{k,2}$ and $\ell \geq 1$. The constants do not depend on h. This uniform bound and $\delta_{h,j} v \to \partial v/\partial x^j$ in the sense of distributions imply that $\partial v/\partial x^j \in W^{k,2}$ (for example, apply Fatou's lemma to the local Fourier coefficients, as in [Kod, Appendix, Theorem 1.13]). Doing this in every coordinate direction and using a finite partition of unity gives $s \in W^{k+1,2}(X, E)$.

In the present setting, $s \in W^{\mu+\ell,2}(X, E)$ by the definition of $\mathrm{Dom}(P)$ and Ps is smooth. Repeating the preceding step gives $s \in W^{k,2}(X, E)$ for every non-negative integer k. The Sobolev embedding theorem then implies $s \in C^\infty(X, E)$. □

The third fundamental consequence of ellipticity concerns the structure of the kernel of P. In particular, ellipticity implies that the kernel does not depend on the Sobolev exponent and is finite-dimensional.

THEOREM I.8.31. *Let X be an orientable compact C^∞ manifold and $E \to X$ be a C^∞ vector bundle. Let $P\colon C^\infty(X, E) \longrightarrow C^\infty(X, E)$ be an elliptic linear partial differential operator of order $\ell \in \mathbb{Z}_{\geq 1}$. Then, for every $\mu \in \mathbb{Z}_{\geq 0}$, one has*

$$\mathrm{Ker}\big(P\colon W^{\mu,2}(X, E) \xrightarrow{\mathrm{op}} W^{\mu,2}(X, E)\big) = \mathrm{Ker}\big(P\colon C^\infty(X, E) \xrightarrow{\mathrm{op}} C^\infty(X, E)\big),$$

and this common kernel is a finite-dimensional vector space.

PROOF. Since $C^\infty(X, E) \subset W^{\mu,2}(X, E)$, the inclusion

$$\mathrm{Ker}\big(P\colon C^\infty(X, E) \xrightarrow{\mathrm{op}} C^\infty(X, E)\big) \subset \mathrm{Ker}\big(P\colon W^{\mu,2}(X, E) \xrightarrow{\mathrm{op}} W^{\mu,2}(X, E)\big)$$

is obvious. Conversely, let $s \in \mathrm{Ker}\big(P\colon W^{\mu,2}(X, E) \xrightarrow{\mathrm{op}} W^{\mu,2}(X, E)\big)$. Then $Ps = 0$, hence $Ps \in C^\infty(X, E)$. By Theorem I.8.30, we conclude that $s \in C^\infty(X, E)$, which proves the equality of the two kernels. From now on, we denote this common kernel simply by

$$\mathrm{Ker}\, P := \mathrm{Ker}\big(P\colon W^{0,2}(X, E) \xrightarrow{\mathrm{op}} W^{0,2}(X, E)\big),$$

and regard $\mathrm{Ker}\, P$ as a topological subspace of $W^{\ell,2}(X, E)$ endowed with the induced topology. Note that, as $\mathrm{Dom}(P\colon W^{0,2}(X, E) \xrightarrow{\mathrm{op}} W^{0,2}(X, E)) = W^{\ell,2}(X, E)$,

$$\mathrm{Ker}\, P = \mathrm{Ker}\big(P\colon W^{\ell,2}(X, E) \to W^{0,2}(X, E)\big).$$

Note also that $\mathrm{Ker}\, P$ is a closed subspace of $W^{\ell,2}(X, E)$, since $P\colon W^{\ell,2}(X, E) \to W^{0,2}(X, E)$ is continuous by the basic estimate (32). Hence $\mathrm{Ker}\, P$ is also a Hilbert space.

To show $\dim(\mathrm{Ker}\, P) < \infty$, by Theorem I.6.4 (*iii*), it suffices to show that the closed unit ball $B \subset \mathrm{Ker}\, P$ (with respect to the norm of $W^{\ell,2}(X, E)$) is compact in the topology induced from $W^{\ell,2}(X, E)$. By the L^2 a priori estimate (Theorem I.8.29 with $\mu = \ell$), there exists a constant $C > 0$ such that

$$\|s\|_{W^{2\ell,2}(X,E)} \leq C\, \|s\|_{W^{\ell,2}(X,E)} \qquad \text{for all } s \in \mathrm{Ker}\, P.$$

Since $\|s\|_{W^{\ell,2}(X,E)} \leq 1$ for any $s \in B$, it follows that B is bounded in $W^{2\ell,2}(X, E)$. On the other hand, by Rellich's theorem, the inclusion $W^{2\ell,2}(X, E) \hookrightarrow W^{\ell,2}(X, E)$ is compact. Therefore, B, being a bounded subset of $W^{2\ell,2}(X, E)$, is relatively compact in $W^{\ell,2}(X, E)$. Since B is closed in $W^{\ell,2}(X, E)$, it is compact. This shows that $\mathrm{Ker}\, P$ is locally compact, and hence finite-dimensional. □

Finally, the L^2 a priori estimate also yields a global decomposition theorem for elliptic operators. Here we describe such a theorem when P is *self-adjoint* for simplicity: i.e., when

$$(Ps,t)_{W^{0,2}(X,E)} = (s,Pt)_{W^{0,2}(X,E)} \qquad \text{for all } s,t \in C^\infty(X,E).$$

THEOREM I.8.32. *Let X be an orientable compact C^∞ manifold and $E \to X$ be a C^∞ vector bundle. Let $P\colon C^\infty(X,E) \longrightarrow C^\infty(X,E)$ be an elliptic linear partial differential operator of order $\ell \in \mathbb{Z}_{\geq 1}$. Assume moreover that P is self-adjoint. Then* $\mathrm{Image}(P\colon C^\infty(X,E) \longrightarrow C^\infty(X,E))$ *is a closed subspace of $C^\infty(X,E)$. Moreover,*

$$\mathrm{Image}(P\colon C^\infty(X,E) \longrightarrow C^\infty(X,E)) = C^\infty(X,E) \cap \mathrm{Ran}\left(P\colon W^{0,2}(X,E) \xrightarrow{\mathrm{op}} W^{0,2}(X,E)\right)$$

holds as vector spaces, and hence one has the decomposition

$$C^\infty(X,E) = \mathrm{Ker}(P\colon C^\infty(X,E) \to C^\infty(X,E)) \oplus \mathrm{Image}(P\colon C^\infty(X,E) \to C^\infty(X,E))$$

as an orthogonal direct sum in $W^{0,2}(X,E)$.

In the proof of Theorem I.8.32, we use the following:

LEMMA I.8.33. *Let X, E, P, and ℓ be as in Theorem I.8.32. Then there exists a constant $C > 0$ such that*

$$\|u\|_{W^{\ell,2}(X,E)} \leq C\,\|Pu\|_{W^{0,2}(X,E)}$$

holds for all $u \in \mathrm{Dom}(P\colon W^{0,2}(X,E) \xrightarrow{\mathrm{op}} W^{0,2}(X,E)) \cap (\mathrm{Ker}\,P)^\perp \subset W^{0,2}(X,E)$.

PROOF. Suppose, to the contrary, that no such constant exists. Then there exists a sequence $\{u_j\}_{j\in\mathbb{Z}_{\geq 0}} \subset \mathrm{Dom}\,P \cap (\mathrm{Ker}\,P)^\perp$ such that

$$\|u_j\|_{W^{\ell,2}(X,E)} = 1 \text{ (for all } j \in \mathbb{Z}_{\geq 0}), \qquad \|Pu_j\|_{W^{0,2}(X,E)} \longrightarrow 0.$$

By the Rellich theorem, the embedding $W^{\ell,2}(X,E) \hookrightarrow W^{0,2}(X,E)$ is compact. Hence, after passing to a subsequence, we may assume that $\{u_j\}_{j\in\mathbb{Z}_{\geq 0}}$ is a convergent sequence in $W^{0,2}(X,E)$. Denote by $u \in W^{0,2}(X,E)$ the limit: i.e., $u_j \to u$ in $W^{0,2}(X,E)$. Since P is closed and $Pu_j \to 0$ in $W^{0,2}(X,E)$, it follows that $u \in \mathrm{Dom}\,P$ and $Pu = 0$, that is, $u \in \mathrm{Ker}\,P$. On the other hand, since each $u_j \in (\mathrm{Ker}\,P)^\perp$ and $(\mathrm{Ker}\,P)^\perp$ is closed in $W^{0,2}(X,E)$, we have $u \in (\mathrm{Ker}\,P)^\perp$. Therefore $u = 0$.

On the other side, the L^2 a priori estimate (Theorem I.8.29) yields

$$\|u_j\|_{W^{\ell,2}(X,E)} \leq C\big(\|Pu_j\|_{W^{0,2}(X,E)} + \|u_j\|_{W^{0,2}(X,E)}\big) \qquad (\text{for all } j \in \mathbb{Z}_{\geq 0})$$

for some $C > 0$. Since $\|u_j\|_{W^{\ell,2}(X,E)} = 1$ and $\|Pu_j\|_{W^{0,2}} \to 0$, this implies that $\|u_j\|_{W^{0,2}(X,E)}$ is bounded from below by a positive constant, contradicting $u_j \to 0$ in $W^{0,2}(X,E)$. □

Proof of Theorem I.8.32. We work throughout with the $W^{0,2}(X,E)$ inner product. As mentioned above, since P is elliptic, the L^2 a priori estimate (Theorem I.8.29) implies that $P\colon W^{0,2}(X,E) \xrightarrow{\text{op}} W^{0,2}(X,E)$ is a densely defined closed operator. Symmetry on $C^\infty(X,E)$ and density in $W^{\ell,2}(X,E)$ give $P \subset P^\dagger$. For the reverse inclusion, take $u \in \operatorname{Dom} P^\dagger$. Testing against smooth sections shows that $Pu = P^\dagger u \in W^{0,2}(X,E)$ in the sense of distributions. The weak-solution form of the same elliptic regularity result gives

$$u \in W^{0,2}(X,E), \quad Pu \in W^{0,2}(X,E) \quad \Longrightarrow \quad u \in W^{\ell,2}(X,E) = \operatorname{Dom} P$$

(See [Kod, Appendix, Theorem 6.1 and Theorems 7.1–7.2]). Here the L^2 starting case is part of that weak-solution result; it does not presuppose membership in $\operatorname{Dom} P$. Thus the von Neumann adjoint $P^\dagger$ has the same domain and action as P.

First we apply Theorem I.6.19 with $H_1 = H_2 = W^{0,2}(X,E)$, $H_3 = 0$, $T = P$, and $S = 0$. Note that T and S are densely defined closed operators satisfying $\operatorname{Ran} T \subset \operatorname{Ker} S$. Let C be a constant as in Lemma I.8.33. Then

$$\|u\|^2_{W^{0,2}(X,E)} \le \|u\|^2_{W^{\ell,2}(X,E)} \le C^2 \, \|Pu\|^2_{W^{0,2}(X,E)} = C^2 \, \left(\|Pu\|^2_{W^{0,2}(X,E)} + |0|^2 \right)$$

holds for all

$$u \in \operatorname{Dom} T^\dagger \cap \operatorname{Dom} S \cap (\operatorname{Ker} S \cap \operatorname{Ker} T^\dagger)^\perp = \operatorname{Dom} P \cap (\operatorname{Ker} P)^\perp \subset W^{0,2}(X,E).$$

Therefore, by this theorem, one has the orthogonal decomposition

$$W^{0,2}(X,E) = \operatorname{Ran}(P\colon W^{0,2}(X,E) \xrightarrow{\text{op}} W^{0,2}(X,E)) \oplus \operatorname{Ker}(P\colon W^{0,2}(X,E) \xrightarrow{\text{op}} W^{0,2}(X,E)).$$

We now turn to the C^∞ setting. Let $f \in C^\infty(X,E)$. If $f \in \operatorname{Ran}(P\colon W^{0,2}(X,E) \xrightarrow{\text{op}} W^{0,2}(X,E))$, then $f = Pu$ for some $u \in W^{\ell,2}(X,E)$. By elliptic regularity (Theorem I.8.30), it follows that $u \in C^\infty(X,E)$, hence $f \in \operatorname{Image}(P\colon C^\infty(X,E) \to C^\infty(X,E))$. This proves the equality

$$\operatorname{Image}(P\colon C^\infty(X,E) \to C^\infty(X,E)) = C^\infty(X,E) \cap \operatorname{Ran}(P\colon W^{0,2}(X,E) \xrightarrow{\text{op}} W^{0,2}(X,E)).$$

By the orthogonal decomposition of $W^{0,2}(X,E)$ and the identification of kernels given by Theorem I.8.31, this implies that

$$C^\infty(X,E) = \operatorname{Ker}(P\colon C^\infty(X,E) \to C^\infty(X,E)) \;\oplus\; \operatorname{Image}(P\colon C^\infty(X,E) \to C^\infty(X,E))$$

as an orthogonal direct sum in $W^{0,2}(X,E)$. In particular, $\operatorname{Image}(P\colon C^\infty(X,E) \to C^\infty(X,E))$ is closed in $C^\infty(X,E)$. □

Remark I.8.34. Consider the setting of Theorem I.8.32, and assume in addition that P is *non-negative*, i.e.,

$$(Ps, s)_{W^{0,2}(X,E)} \ge 0 \qquad \text{for all } s \in C^\infty(X,E).$$

Then, by combining Theorem I.8.32 with Lemma I.8.33, one obtains the following: The L^2-orthogonal complement of $\mathrm{Ker}(P\colon C^\infty(X,E) \to C^\infty(X,E))$ intersects $C^\infty(X,E)$ in a closed subspace. On this orthogonal complement in $W^{0,2}(X,E)$, the self-adjoint realization of P, with domain $W^{\ell,2}(X,E) \cap (\mathrm{Ker}\, P)^\perp$, has spectrum bounded from below by a positive constant, which may be taken to be $1/C$ for the estimate constant $C > 0$ in Lemma I.8.33. In this sense, the positive spectrum of P admits a uniform lower bound, often referred to as a *spectral gap*. □

Remark I.8.35. In the proof of Theorem I.8.32, we only used the following analytic input on the underlying Hilbert space:
(*i*) $C^\infty(X,E)$ is dense;
(*ii*) P defines a densely defined closed operator;
(*iii*) the L^2 a priori estimate yields the estimate on $\mathrm{Dom}\, P \cap (\mathrm{Ker}\, P)^\perp$ (Lemma I.8.33);
(*iv*) P is self-adjoint so that its von Neumann adjoint coincides with P.
When X is compact, after fixing a Riemannian metric g on X and a C^∞ Hermitian metric h on E, we may regard $W^{0,2}(X,E)$ as the Hilbert space $L^2(X,E;g,h) := L^0_{(2)}(X,E;g,h)$ equipped with the inner product

$$(s,t)_{L^2(X,E;g,h)} = \int_X \langle s,\, t\rangle_h \, dV_g.$$

The induced norm is equivalent to any Sobolev $W^{0,2}$-norm on compact X. Therefore the same proof applies verbatim, and yields the following variant: if P is elliptic and self-adjoint with respect to $(\,\cdot\,,\,\cdot\,)_{L^2(X,E;g,h)}$, then $\mathrm{Image}(P\colon C^\infty(X,E) \to C^\infty(X,E))$ is closed in $C^\infty(X,E)$ and

$$C^\infty(X,E) = \mathrm{Ker}(P\colon C^\infty(X,E) \to C^\infty(X,E)) \oplus \mathrm{Image}(P\colon C^\infty(X,E) \to C^\infty(X,E))$$

as an orthogonal direct sum in $L^2(X,E;g,h)$. □

Strong ellipticity

In addition to ellipticity, a stronger positivity condition on the principal symbol plays an important role in analysis. From now on, we fix a Riemannian metric g on X and an Hermitian metric h on E.

Let P be a linear partial differential operator of order $\ell \in \mathbb{Z}_{\geq 2}$ on E. We say that P is *strongly elliptic* if ℓ is even and there exists a constant $c > 0$ such that

$$(-1)^{\ell/2}\, \mathrm{Re}\, \big\langle \sigma_P(x,\xi)\, e,\, e\big\rangle_h \;\geq\; c\, |\xi|_g^\ell\, |e|_h^2$$

holds for all $x \in X$, all $\xi \in T^*_x X \setminus \{0\}$, and all $e \in E_x \setminus \{0\}$.

By definition, strong ellipticity implies ellipticity. However, the converse does not hold in general. For instance, in Example I.8.28, the operator P is elliptic but not strongly elliptic.

While ellipticity already yields the L^2 a priori estimate stated in Theorem I.8.29, strong ellipticity leads to a sharper inequality of coercive type, known as *Gårding's inequality*.

THEOREM I.8.36 (Gårding's inequality). *Let X be an orientable compact C^∞ manifold and $E \to X$ be a C^∞ vector bundle. Fix a Riemannian metric g on X and a C^∞ Hermitian metric h on E. Let $P\colon C^\infty(X,E) \to C^\infty(X,E)$ be a linear partial differential operator that is strongly elliptic with respect to (g,h) and is of order $2m$ ($m \in \mathbb{Z}_{\geq 1}$). Then there exist constants $c_P > 0$ and $C_P > 0$ such that*

$$c_P \, \|s\|^2_{W^{m,2}(X,E)} \leq \mathrm{Re}\,(Ps, s)_{L^2(X,E;g,h)} + C_P \, \|s\|^2_{L^2(X,E;g,h)}$$

holds for all $s \in C^\infty(X,E)$.

PROOF. See [Kod, Appendix, Theorem 4.2]. □

We now present an application of Gårding's inequality in a concrete setting. As a preparatory step, let us recall the notation for *Wirtinger derivatives* on the complex plane. Let $z = x + \sqrt{-1}y$ be the standard complex coordinate on $\mathbb{C}$, where $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$. We define the Wirtinger derivatives by

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\,\frac{\partial}{\partial y}\right). \tag{35}$$

Using this notation, we prove the following regularity result.

THEOREM I.8.37. *Let N be a positive integer, let $\Omega \subset \mathbb{C}^N$ be an open set, and let $f \in L^2_{\mathrm{loc}}(\Omega) := L^2_{\mathrm{loc}}(\Omega, \Omega \times \mathbb{C})$ be a locally L^2 function on Ω. With respect to the standard complex Euclidean coordinates $(z^1, z^2, \ldots, z^N)$ on $\mathbb{C}^N$, assume that*

$$\frac{\partial f}{\partial \overline{z^j}} \in C^\infty(\Omega) \qquad \text{for all } j \in \{1, 2, \ldots, N\}.$$

Then $f \in C^\infty(\Omega)$.

Note that, in contrast to the elliptic regularity results discussed so far, which were formulated for elliptic operators on orientable compact C^∞ manifolds, the statement of Theorem I.8.37 concerns an open subset of $\mathbb{C}^N$.

From now on, we write

$$z^j = x^j + \sqrt{-1}\, y^j \qquad (j = 1, 2, \ldots, N),$$

where $x^j = \mathrm{Re}\, z^j$ and $y^j = \mathrm{Im}\, z^j$. A direct computation shows that, for each j,

$$\frac{\partial}{\partial z^j}\frac{\partial}{\partial \overline{z^j}} = \frac{1}{4}\left(\frac{\partial}{\partial x^j} - \sqrt{-1}\,\frac{\partial}{\partial y^j}\right)\left(\frac{\partial}{\partial x^j} + \sqrt{-1}\,\frac{\partial}{\partial y^j}\right) = \frac{1}{4}\left(\left(\frac{\partial}{\partial x^j}\right)^2 + \left(\frac{\partial}{\partial y^j}\right)^2\right).$$

Consequently, for the *Laplace operator*

$$\Delta := -\sum_{j=1}^{N}\left(\frac{\partial}{\partial x^j}\right)^2 - \sum_{j=1}^{N}\left(\frac{\partial}{\partial y^j}\right)^2,$$

we obtain the identity

$$\Delta = -4 \sum_{j=1}^{N} \frac{\partial}{\partial z^j} \frac{\partial}{\partial \overline{z^j}}. \tag{36}$$

Let

$$\xi = \sum_{j=1}^{N} \xi_j \, dx^j + \sum_{j=1}^{N} \eta_j \, dy^j \in T_z^* \mathbb{C}^N$$

be a cotangent vector at $z \in \mathbb{C}^N$ ($\xi_j, \eta_j \in \mathbb{R}$). With respect to the real coordinates $(x^1, y^1, \ldots, x^N, y^N)$, the principal symbol of the Laplace operator Δ is given by

$$\sigma_\Delta(z, \xi) = -\sum_{j=1}^{N} \xi_j^2 - \sum_{j=1}^{N} \eta_j^2.$$

Therefore Δ is strongly elliptic of order 2 (Here we equip $\mathbb{C}^N$ with the standard Euclidean metric, and the trivial complex line bundle over $\mathbb{C}^N$ with its standard Hermitian metric).

Let f be a function as in Theorem I.8.37. For each $j \in \{1, 2, \ldots, N\}$, we set

$$g_j := \frac{\partial f}{\partial \overline{z^j}} \in C^\infty(\Omega).$$

Our goal is to show that f is of class C^∞ in a neighborhood of each point $p \in \Omega$. Therefore, by translation invariance, we may assume that $p = 0 \in \Omega$ and work in a neighborhood of the origin: Indeed, the differential operators used in this argument have constant coefficients and are therefore invariant under translations. Moreover, by a suitable scaling of coordinates, we may further assume that the closed ball B centered at the origin with radius $1/4$ satisfies $B \subset \Omega$.

We fix a C^∞ cut-off function $\rho \colon \mathbb{C}^N \to \mathbb{R}$ such that

$$0 \leq \rho \leq 1, \qquad \operatorname{Supp} \rho \subset B, \qquad \rho \equiv 1 \text{ in a neighborhood of } 0.$$

Extend the compactly supported function ρf by zero to $\mathbb{C}^N$, and define its periodization by

$$\widetilde{f}(x) := \sum_{v \in (\mathbb{Z} + \sqrt{-1}\mathbb{Z})^N} \rho(x + v) \, f(x + v).$$

As is clear from the support condition on ρ, for each $x \in \mathbb{C}^N$ only finitely many terms in this sum are non-zero. By construction, $\widetilde{f}$ is invariant under translations by $(\mathbb{Z} + \sqrt{-1}\mathbb{Z})^N$ and belongs to $L^2_{\mathrm{loc}}(\mathbb{C}^N)$. Its L^2 norm is taken on one fundamental domain, or equivalently on the following torus. We now pass to the torus

$$X := \mathbb{C}^N / (\mathbb{Z} + \sqrt{-1}\mathbb{Z})^N,$$

which is an orientable compact C^∞ manifold of real dimension $n = 2N$. The differential operators $\partial/\partial z^j$, $\partial/\partial \overline{z^j}$, and Δ descend to well-defined differential operators on X.

Let $\pi\colon \mathbb{C}^N \to X$ be the quotient map. By construction, there exists an L^2 function F on X such that

$$F \circ \pi = \widetilde{f}.$$

On the fundamental domain $\{z \in \mathbb{C}^N \mid -1/2 \le \operatorname{Re} z^j, \operatorname{Im} z^j < 1/2 \text{ for } j = 1, 2, \ldots, N\}$ of π, we have

$$\frac{\partial \widetilde{f}}{\partial \overline{z^j}} = f\, \frac{\partial \rho}{\partial \overline{z^j}} + \rho\, g_j.$$

It follows that, for each $j \in \{1, 2, \ldots, N\}$,

$$\frac{\partial F}{\partial \overline{z^j}} \in L^2(X) := L^2(X, X \times \mathbb{C}). \tag{37}$$

In what follows, we use sequences $\{\widetilde{f}_\nu\}_{\nu \in \mathbb{Z}_{\ge 1}} \subset C^\infty(\mathbb{C}^N)$ and $\{F_\nu\}_{\nu \in \mathbb{Z}_{\ge 1}} \subset C^\infty(X)$ approximating $\widetilde{f}$ and F, respectively, constructed by a standard smoothing procedure as follows. Fix a function $\theta \in C^\infty_c(\mathbb{C}^N)$ such that $\operatorname{Image} \theta \subset \mathbb{R}_{\ge 0}$, $\operatorname{Supp} \theta$ is contained in the unit open ball, and

$$\int_{\mathbb{C}^N} \theta\, dV_{\mathrm{Euc}} = 1.$$

For each $\nu \in \mathbb{Z}_{\ge 1}$, define

$$\theta_\nu(x) := \nu^{2N}\, \theta(\nu x).$$

Then $\operatorname{Supp} \theta_\nu$ is contained in the open ball centered at the origin with radius $1/\nu$. Regard $\widetilde{f}$ as a function in $L^2_{\mathrm{loc}}(\mathbb{C}^N)$, and set

$$\widetilde{f}_\nu(x) := \int_{\mathbb{C}^N} \widetilde{f}(x - y)\, \theta_\nu(y)\, dV_{\mathrm{Euc}}(y),$$

where dV_{Euc} denotes the Euclidean volume form. Since $\widetilde{f}$ is locally integrable and θ_ν is smooth with compact support, $\widetilde{f}_\nu \in C^\infty(\mathbb{C}^N)$. Moreover, $\widetilde{f}$ is $(\mathbb{Z} + \sqrt{-1}\mathbb{Z})^N$-periodic, and hence so is $\widetilde{f}_\nu$. Therefore, $\widetilde{f}_\nu$ induces a C^∞ function F_ν on X defined by

$$F_\nu \circ \pi = \widetilde{f}_\nu.$$

Mini-Exercise I.8.38. In the above setting, let $\{\widetilde{f}_\nu\}_{\nu \in \mathbb{Z}_{\ge 1}} \subset C^\infty(\mathbb{C}^N)$ and $\{F_\nu\}_{\nu \in \mathbb{Z}_{\ge 1}} \subset C^\infty(X)$ be sequences constructed as above.
(*i*) Show that, as $\nu \to \infty$, $\widetilde{f}_\nu \to \widetilde{f}$ in $L^2_{\mathrm{loc}}(\mathbb{C}^N)$ and $F_\nu \to F$ in $L^2(X)$, with respect to the Lebesgue measures.
(*ii*) Show that

$$\frac{\partial \widetilde{f}_\nu}{\partial \overline{z^j}} \longrightarrow \frac{\partial \widetilde{f}}{\partial \overline{z^j}}, \qquad \frac{\partial F_\nu}{\partial \overline{z^j}} \longrightarrow \frac{\partial F}{\partial \overline{z^j}}$$

as $\nu \to \infty$, in $L^2_{\mathrm{loc}}(\mathbb{C}^N)$ and $L^2(X)$, respectively, with respect to the Lebesgue measures.
(*iii*) Equip X with the Euclidean metric g induced from $\mathbb{C}^N$ via the quotient map π, and

equip the trivial bundle $E := X \times \mathbb{C}$ with its standard Hermitian metric h. Show that, for each $u \in C^\infty(X, E)$,

$$(\Delta u,\, u)_{L^2(X,E;g,h)} = 4 \sum_{j=1}^{N} \left\| \frac{\partial u}{\partial \overline{z^j}} \right\|^2_{L^2(X,E;g,h)}.$$

With this preparation, and using the strong ellipticity of Δ, we first show the following lemma, which is the key step in the proof of Theorem I.8.37.

LEMMA I.8.39. *In the setting of Theorem I.8.37, assume that $0 \in \Omega$. Let U and V be open neighborhoods of 0 in Ω such that $\overline{U} \subset V$ and $\overline{V}$ is contained in the interior* $\operatorname{Int} B$ *of the ball B. Assume that*

$$\frac{\partial f}{\partial \overline{z^j}} \in C^\infty(V) \qquad \text{for all } j \in \{1, 2, \dots, N\}.$$

Then, for each $j \in \{1, 2, \dots, N\}$, we have

$$\frac{\partial f}{\partial z^j} \in L^2(U)$$

with respect to the Lebesgue measure.

PROOF. We continue to use the notation introduced above. In particular, we work with $F \in L^2(X)$ on the compact $n = 2N$ dimensional torus X; its derivatives $\partial F/\partial \overline{z^j}$ belong to $L^2(X)$ by (37). The cut-off function ρ is chosen so that $\operatorname{Supp} \rho \subset V$ and $\rho|_U \equiv 1$.

By construction, it suffices to show that

$$F \in W^{1,2}(X) := W^{1,2}(X, X \times \mathbb{C}),$$

since $F \circ \pi = f$ on U and hence the first-order derivatives of f belong to $L^2(U)$.

Equip X with the Euclidean metric g induced from $\mathbb{C}^N$ and the trivial line bundle $E := X \times \mathbb{C}$ with its standard Hermitian metric h. Applying Gårding's inequality (Theorem I.8.36) to Δ with these metrics gives

$$c_\Delta \|u\|^2_{W^{1,2}(X,E)} \le \operatorname{Re}(\Delta u, u)_{L^2(X,E;g,h)} + C_\Delta \|u\|^2_{L^2(X,E;g,h)} \qquad (u \in C^\infty(X, E))$$

for constants $c_\Delta, C_\Delta > 0$. Taking into account Mini-Exercise I.8.38 (*iii*), we obtain

$$c_\Delta \|u\|^2_{W^{1,2}(X,E)} \le 4 \sum_{j=1}^{N} \left\| \frac{\partial u}{\partial \overline{z^j}} \right\|^2_{L^2(X,E;g,h)} + C_\Delta \|u\|^2_{L^2(X,E;g,h)} \qquad (u \in C^\infty(X, E)).$$

Therefore we have

$$c_\Delta \|F_\nu - F_\mu\|^2_{W^{1,2}(X,E)} \le 4 \sum_{j=1}^{N} \left\| \frac{\partial F_\nu}{\partial \overline{z^j}} - \frac{\partial F_\mu}{\partial \overline{z^j}} \right\|^2_{L^2(X,E;g,h)} + C_\Delta \|F_\nu - F_\mu\|^2_{L^2(X,E;g,h)} \tag{38}$$

for any $\nu, \mu \in \mathbb{Z}_{\ge 1}$. For any $\varepsilon > 0$, by (37) and Mini-Exercise I.8.38 (*i*) and (*ii*), one can take $\nu_0 \in \mathbb{Z}_{\ge 0}$ such that the right-hand side of (38) is less than ε if $\nu, \mu > \nu_0$. Therefore one can construct a subsequence $\{F_{\nu_k}\}$ which is Cauchy in $W^{1,2}(X, E)$. Hence its limit F belongs to $W^{1,2}(X, E)$, which proves the lemma. □

Proof of Theorem I.8.37. As explained above, it suffices to prove the statement in a neighborhood of the origin, by assuming that $0 \in \Omega$. Fix a sufficiently small open ball $W \subset \Omega$ centered at 0. We argue by induction on $\mu \in \mathbb{Z}_{\geq 1}$ and show that there exists an open neighborhood $U_\mu \subset \Omega$ of $\overline{W}$ such that, for all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_N) \in (\mathbb{Z}_{\geq 0})^N$ with $|\alpha| + |\beta| \leq \mu$ ($|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_N$, $|\beta| := \beta_1 + \beta_2 + \cdots + \beta_N$), the derivative

$$\frac{\partial^{|\alpha|+|\beta|} f}{(\partial x^1)^{\alpha_1} \cdots (\partial x^N)^{\alpha_N} (\partial y^1)^{\beta_1} \cdots (\partial y^N)^{\beta_N}}$$

belongs to $L^2(U_\mu)$. Once this statement has been established for all $\mu \in \mathbb{Z}_{\geq 1}$, the desired regularity of f follows by choosing, once and for all, a C^∞ cut-off function ρ supported in W and equal to 1 on a neighborhood of 0. The construction above gives the same function F on the compact torus $X = \mathbb{C}^N/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})^N$ for all μ, and $F \in W^{\mu,2}(X, X \times \mathbb{C})$ for every $\mu \in \mathbb{Z}_{\geq 1}$. For each $\ell \in \mathbb{Z}_{\geq 0}$, choose $\mu > \ell + N$ and apply the Sobolev embedding theorem. Thus F is smooth. Since the pull-back $F \circ \pi$ agrees with f near 0 in W, f is smooth there as well.

The case $\mu = 1$ follows from Lemma I.8.39. Assume that the claim holds for $\mu = m$, and choose U_{m+1} such that $\overline{U_{m+1}} \subset U_m$ and $\overline{W} \subset U_{m+1}$. It suffices to prove that, for all j and all multi-indices α, β with $|\alpha| + |\beta| = m$,

$$\frac{\partial}{\partial z^j} \frac{\partial^{|\alpha|+|\beta|} f}{(\partial x^1)^{\alpha_1} \cdots (\partial x^N)^{\alpha_N} (\partial y^1)^{\beta_1} \cdots (\partial y^N)^{\beta_N}} \in L^2(U_{m+1}) \tag{39}$$

and

$$\frac{\partial}{\partial \overline{z^j}} \frac{\partial^{|\alpha|+|\beta|} f}{(\partial x^1)^{\alpha_1} \cdots (\partial x^N)^{\alpha_N} (\partial y^1)^{\beta_1} \cdots (\partial y^N)^{\beta_N}} \in L^2(U_m) \tag{40}$$

hold, since both $\partial/\partial x^j$ and $\partial/\partial y^j$ are linear combinations of $\partial/\partial z^j$ and $\partial/\partial \overline{z^j}$.

The validity of (40) follows by exchanging the order of differentiation and applying the assumption of the theorem. This exchange is valid in the sense of distributions, since partial derivatives of smooth test functions commute.

Finally, (39) follows from Lemma I.8.39 applied with $U = U_{m+1}$ and $V = U_m$. This completes the proof. □

Remark I.8.40. The regularity result of Theorem I.8.37 has a more general counterpart for sections with distribution coefficients. Let X be a C^∞ manifold, let $E \to X$ be a C^∞ vector bundle of finite rank, and let P be an elliptic linear partial differential operator on its sections with C^∞ coefficients. Let s be a *distributional section* of E, with distributions as its coefficients in local smooth frames (See §8.6 for distributions). If Ps is smooth, then s is smooth.

Localization and control of weak derivatives lead to regularity. Variable coefficients and arbitrary distributions require local elliptic estimates beyond the constant-coefficient L^2 argument above. These estimates follow from a local inverse modulo operators with smooth kernels, and improve regularity until Sobolev embedding applies (For a complete proof, see [Dy, Theorem 14.23; Theorem 15.1 and Remark 15.2]). □

8.8 De Rham cohomology

Throughout this subsection, unless otherwise stated, we fix the following setting: Let X be a C^∞ manifold of real dimension $n \in \mathbb{Z}_{\geq 0}$, and let $E \to X$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle of rank $r \in \mathbb{Z}_{\geq 1}$, where $\mathrm{GL}_r(\mathbb{C})$ denotes the group of all invertible complex $r \times r$ matrices.

Since E is $\mathrm{GL}_r(\mathbb{C})$-flat, there exists an open covering $\{U_j\}_{j\in J}$ of X and local trivializations $\Phi_{U_j} \colon \pi^{-1}(U_j) \to U_j \times \mathbb{C}^r$ such that all transition functions are $\mathrm{GL}_r(\mathbb{C})$-valued locally constant maps. For each $j \in J$, let $e^\lambda \colon U_j \to E|_{U_j}$ denote the section corresponding via Φ_{U_j} to the constant map valued at the λ-th standard basis vector of $\mathbb{C}^r$. Then $(e^1, e^2, \ldots, e^r)$ forms a local frame of E over U_j. Define a linear operator

$$D_{U_j} \colon \mathcal{A}^q(U_j, E|_{U_j}) \to \mathcal{A}^{q+1}(U_j, E|_{U_j})$$

by

$$D_{U_j}\left(\sum_{\lambda=1}^{r} \omega_\lambda \otimes e^\lambda\right) := \sum_{\lambda=1}^{r} d\omega_\lambda \otimes e^\lambda, \tag{41}$$

where $\omega_\lambda \in \mathcal{A}^q(U_j)$. Since the transition functions are locally constant, the operators $\{D_{U_j}\}$ glue to define a global operator (the *flat connection*)

$$D \colon \mathcal{A}^q(X, E) \to \mathcal{A}^{q+1}(X, E).$$

By construction, D is a continuous linear map and satisfies

$$D \circ D \equiv 0 \qquad \text{on } \mathcal{A}^q(X, E) \text{ for all } q \in \mathbb{Z}_{\geq 0}.$$

Hence we obtain a cochain complex

$$(\mathcal{A}^\bullet(X, E), D).$$

By restriction, we also obtain a cochain complex

$$(\mathcal{A}_c^\bullet(X, E), D).$$

Furthermore, interpreting the exterior derivative d in (41) in the distribution sense, we obtain cochain complexes

$$(\mathcal{K}^\bullet(X, E), D) \quad \text{and} \quad (\mathcal{K}_c^\bullet(X, E), D).$$

In this subsection, we investigate the cohomology groups

$$H^q(\mathcal{A}^\bullet(X, E), D),\ H^q(\mathcal{A}_c^\bullet(X, E), D),\ H^q(\mathcal{K}^\bullet(X, E), D),\ \text{and } H^q(\mathcal{K}_c^\bullet(X, E), D).$$

For each of these complexes, we endow $\operatorname{Ker} D$ and $\operatorname{Image} D$ with the subspace topology inherited from the corresponding TVSs $\mathcal{A}^\bullet(X, E)$, $\mathcal{A}_c^\bullet(X, E)$, $\mathcal{K}^\bullet(X, E)$, or $\mathcal{K}_c^\bullet(X, E)$, and we equip the cohomology groups with the quotient topology. Thus all cohomology groups considered here are regarded as TVSs.

$H^q(\mathcal{A}^\bullet(X,E),D)$ and $H^q(\mathcal{A}^\bullet_c(X,E),D)$

We first consider $H^q(\mathcal{A}^\bullet(X,E),D)$ and $H^q(\mathcal{A}^\bullet_c(X,E),D)$. These are called the *$q$-th de Rham cohomology of X with coefficients in E* and the *q-th de Rham cohomology of X with compact supports with coefficients in E*, respectively. In this note we write

$$H^q_{\mathrm{DR}}(X,E) := H^q(\mathcal{A}^\bullet(X,E),D),$$
$$H^q_{\mathrm{DR},c}(X,E) := H^q(\mathcal{A}^\bullet_c(X,E),D).$$

If E is the trivial line bundle $X\times\mathbb{C}$ with its trivial flat structure, then D coincides with the exterior derivative d. In this case these groups are simply called the *q-th de Rham cohomology of X* and the *q-th de Rham cohomology of X with compact supports*, and we write

$$H^q_{\mathrm{DR}}(X) := H^q(\mathcal{A}^\bullet(X,X\times\mathbb{C}),d),$$
$$H^q_{\mathrm{DR},c}(X) := H^q(\mathcal{A}^\bullet_c(X,X\times\mathbb{C}),d).$$

By restricting the exterior derivative to real C^∞ forms on X, one similarly defines real vector spaces, which we denote by $H^q_{\mathrm{DR}}(X,\mathbb{R})$ and $H^q_{\mathrm{DR},c}(X,\mathbb{R})$. When it is necessary to emphasize that we are working with complex coefficients, we may write $H^q_{\mathrm{DR}}(X,\mathbb{C})$ instead of $H^q_{\mathrm{DR}}(X)$, and similarly $H^q_{\mathrm{DR},c}(X,\mathbb{C})$ instead of $H^q_{\mathrm{DR},c}(X)$.

Remark I.8.41. Let $q\in\mathbb{Z}_{\geq 0}$. Consider a C^∞ singular q-chain $\sigma=\sum_\nu c_\nu\sigma_\nu\in C^{\mathrm{smooth}}_q(X)$ with $\partial\sigma=0$ ($c_\nu\in\mathbb{Z}$, $\sigma_\nu\colon(\mathrm{Simplex})_q\to X$). Then the linear map

$$\mathcal{A}^q(X)\ni\omega\mapsto\int_\sigma\omega:=\sum_\nu c_\nu\int_{\sigma_\nu}\omega\in\mathbb{C}$$

vanishes on $\mathrm{Image}(d\colon\mathcal{A}^{q-1}(X)\to\mathcal{A}^q(X))$ by Stokes' theorem. Hence the map

$$\int_\sigma\bullet\colon H^q_{\mathrm{DR}}(X)\to\mathbb{C}$$

is induced. For this reason, for a class $\alpha\in H^q_{\mathrm{DR}}(X)$, we often write, by abuse of notation,

$$\int_\sigma\alpha$$

for the integral of any representative of α along σ.

Similarly, for a fixed d-closed q-form $\omega\in\mathcal{A}^q(X)$, the map

$$C^{\mathrm{smooth}}_q(X)\ni\sigma\mapsto\int_\sigma\omega\in\mathbb{C}$$

vanishes on $\mathrm{Image}(\partial\colon C^{\mathrm{smooth}}_{q+1}(X)\to C^{\mathrm{smooth}}_q(X))$ again by Stokes' theorem. Thus for a class $\theta\in H_q\big(C^{\mathrm{smooth}}_\bullet(X)\big)$ we may write

$$\int_\theta\omega.$$

Based on the isomorphism (27), we may further write

$$\int_\theta \alpha$$

for $\alpha \in H^q_{\mathrm{DR}}(X)$ and $\theta \in H_q(X)$, since the value is independent of the choice of representatives.

If $\sigma \in \mathrm{Ker}(\partial\colon C_q^{\mathrm{sing}}(X) \to C_{q-1}^{\mathrm{sing}}(X))$ is a (not necessarily C^∞) singular q-cycle, we define

$$\int_\sigma \omega := \int_{\{\sigma\}} \omega,$$

where $\{\sigma\}$ denotes the homology class represented by σ (We emphasize that, if σ is a singular q-simplex which is not of class C^∞, then the pull-back $\sigma^*\omega$ of a C^∞ q-form ω need not be integrable over the standard simplex. Thus the integral $\int_\sigma \omega$ is not defined in general by directly pulling back ω via σ. The definition above is justified by replacing σ by a C^∞-approximation within the same homology class, as in the proof of the isomorphism (27)). □

Note that, for a C^∞ manifold Y and a C^∞ map $f\colon Y \to X$, the pull-back

$$f^*\colon \mathcal{A}^\bullet(X, E) \to \mathcal{A}^\bullet(Y, f^*E)$$

satisfies $Df^* = f^*D$. Hence we obtain an induced map

$$f^*\colon H^\bullet_{\mathrm{DR}}(X, E) \to H^\bullet_{\mathrm{DR}}(Y, f^*E).$$

If moreover f is proper, we similarly obtain

$$f^*\colon H^\bullet_{\mathrm{DR},c}(X, E) \to H^\bullet_{\mathrm{DR},c}(Y, f^*E).$$

Among the basic properties of de Rham cohomology, the following homotopy invariance property is fundamental.

Lemma I.8.42. *Let X and Y be C^∞ manifolds, and let $E \to X$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle. Let $f, g\colon Y \to X$ be C^∞ maps that are homotopic. Choose a smooth homotopy h from f to g, and let $\tau_h\colon f^*E \to g^*E$ be the flat bundle isomorphism obtained by transport along the paths $t \mapsto h(t, y)$, $y \in Y$. Then*

$$f^* = (\tau_h^{-1})_* g^*\colon H^q_{\mathrm{DR}}(X, E) \to H^q_{\mathrm{DR}}(Y, f^*E)$$

for any $q \in \mathbb{Z}_{\geq 0}$.

Proof. It suffices to construct a cochain homotopy. Such a smooth homotopy exists by Theorem I.8.13. Write $h\colon U \to X$, where U is an open neighborhood of $[0, 1] \times Y$ in $\mathbb{R} \times Y$, with $h(0, -) = f$ and $h(1, -) = g$. Flat transport in the t-direction identifies h^*E

over $[0,1]\times Y$ with the pull-back of f^*E under the projection to Y. We use this identification below; at $t=1$ it identifies g^*E with f^*E by τ_h^{-1}. We define

$$H\colon \mathcal{A}^{q+1}(X,E)\to\mathcal{A}^q(Y,f^*E)$$

as follows: For $\omega\in\mathcal{A}^{q+1}(X,E)$, write

$$h^*\omega=\alpha(t)+dt\wedge\beta(t),$$

where $\alpha(t)$ is a $(q+1)$-form on Y valued in f^*E depending smoothly on $t\in[0,1]$, and $\beta(t)$ is a q-form on Y valued in f^*E depending smoothly on t. Define

$$H(\omega):=\int_0^1\beta(\tau)\,d\tau.$$

Write D_X for the flat connection on E, D_Y for that on f^*E, and D_h for the pull-back connection on h^*E. In flat local frames the following computation is the usual exterior derivative calculation. Then

$$D_h(h^*\omega)=dt\wedge\frac{\partial\alpha(t)}{\partial t}+D_Y\alpha(t)-dt\wedge D_Y\beta(t).$$

Hence

$$\begin{aligned}H(D_X\omega)&=\int_0^1\left(\frac{\partial\alpha(\tau)}{\partial\tau}-D_Y\beta(\tau)\right)d\tau\\&=\alpha(1)-\alpha(0)-\int_0^1 D_Y\beta(\tau)\,d\tau,\end{aligned}$$

while

$$D_YH(\omega)=D_Y\int_0^1\beta(\tau)\,d\tau=\int_0^1 D_Y\beta(\tau)\,d\tau.$$

Therefore

$$D_YH+HD_X=(\tau_h^{-1})_*g^*-f^*.$$

Thus H defines a cochain homotopy, and the assertion follows. □

Corollary I.8.43. *Let X be a C^∞ manifold of real dimension $n\in\mathbb{Z}_{\geq 0}$, and let $E\to X$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle. If X is contractible, then*

$$H^q_{\mathrm{DR}}(X,E)\cong\begin{cases}\mathbb{C}^r & \text{if } q=0,\\ 0 & \text{otherwise.}\end{cases}$$

In particular,

$$H^q_{\mathrm{DR}}(X)\cong\begin{cases}\mathbb{C} & \text{if } q=0,\\ 0 & \text{otherwise.}\end{cases}$$

PROOF. By Lemma I.8.42, it suffices to compute the cohomology when X is a single point (a connected 0-dimensional manifold). In that case the computation follows directly from the definition. □

REMARK I.8.44. In the setting of Corollary I.8.43, the bundle E is trivial as a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle. Indeed, choose $x_0 \in X$ and a basis of E_{x_0}, and extend it by path lifting in the covering $E_{\text{discrete}} \to X$ of Lemma I.7.28. Since X is simply connected, homotopy lifting shows that the resulting basis at each point is independent of the chosen path from x_0. The resulting sections have locally constant coefficients in flat local trivializations, and hence form a global smooth flat frame of E. □

REMARK I.8.45. Consider the case where $X \subset \mathbb{R}^n$ is a star-shaped open subset. Assume that $0 \in X$ and, for any $x \in X$, the line segment $[0, x]$ is contained in X, for simplicity. In this case, the homotopy

$$h(t, x) := tx$$

shows that X is contractible. Hence, by Corollary I.8.43,

$$H^q_{\mathrm{DR}}(X) \cong \begin{cases} \mathbb{C} & \text{if } q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is known as the *Poincaré lemma*. Let us now describe explicitly the construction of a d-primitive for a d-closed form on X. Let $q > 0$ and let $\omega \in \mathrm{Ker}(d\colon \mathcal{A}^q(X) \to \mathcal{A}^{q+1}(X))$. In the proof of Lemma I.8.42, the cochain homotopy H was constructed, and $H(\omega)$ gives a d-primitive of ω. We now describe this more concretely.

Write

$$h^*\omega = \alpha(t) + dt \wedge \beta(t),$$

where $\alpha(t)$ is a q-form on X depending smoothly on $t \in [0, 1]$, and $\beta(t)$ is a $(q-1)$-form on X depending smoothly on t. Then

$$H(\omega) = \int_0^1 \beta(\tau)\, d\tau$$

(coefficientwise integration in t). Since $h(t, x) = tx$, if we write

$$\omega = \sum_{I \in \{1,2,\dots,n\}^q} a_I(x)\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q}$$

(where $I = (i_1, i_2, \dots, i_q)$), then

$$h^*\omega = \sum_{I \in \{1,2,\dots,n\}^q} a_I(tx)\, d(tx^{i_1}) \wedge \cdots \wedge d(tx^{i_q}).$$

From $d(tx^i) = t\, dx^i + x^i dt$, we obtain

$$\alpha(t) = \sum_{I \in \{1,2,\dots,n\}^q} t^q a_I(tx)\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_q},$$

and

$$\beta(t) = \sum_{j=1}^{q} \sum_{I \in \{1,2,\dots,n\}^q} (-1)^{j+1} t^{q-1} a_I(tx) \cdot x^{i_j}\, dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_q}.$$

Hence

$$H(\omega) = \sum_{j=1}^{q} \sum_{I \in \{1,2,\dots,n\}^q} (-1)^{j+1} \left(\int_0^1 t^{q-1} a_I(tx)\, dt \right) x^{i_j}\, dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_q},$$

where the notation $\widehat{dx^{i_j}}$ indicates that the factor dx^{i_j} is removed from the wedge product.

The extraction of $\beta(t)$ from $h^*\omega$ can be described using interior multiplication with respect to the vector field $\partial/\partial t$ on $(0,1) \times X$:

$$\beta(t) = \frac{\partial}{\partial t} \lrcorner\, h^*\omega.$$

This observation also yields another proof that $H(\omega)$ is a d-primitive of ω, via *Cartan's magic formula*

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} (\varphi_\varepsilon)^* h^*\omega = \frac{\partial}{\partial t} \lrcorner\, (dh^*\omega) + d\left(\frac{\partial}{\partial t} \lrcorner\, h^*\omega \right),$$

where φ_ε denotes the flow of $\partial/\partial t$. In the present case,

$$\varphi_\varepsilon(t,x) = (t+\varepsilon, x),$$

and writing $h_t := h(t, \bullet)$, we have

$$\int_0^1 \left(\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} (\varphi_\varepsilon)^* h^*\omega \right) dt = \int_0^1 \frac{d}{dt} (h_t^*\omega)\, dt = h_1^*\omega - h_0^*\omega = \omega.$$

Since $dh^*\omega = h^* d\omega = 0$, the right-hand side equals $dH(\omega)$, and the assertion follows. See also [Lee-Jo2, Lemma 17.9]. □

In the setting of Lemma I.8.42, we next treat the special case $Y = (0,1) \times X$.

LEMMA I.8.46. *Let X be a C^∞ manifold of real dimension $n \in \mathbb{Z}_{\geq 0}$, and let $E \to X$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle. Denote by I the interval $(0,1)$, and by $f\colon I \times X \to X$ the second projection. Then*

$$H^q_{\mathrm{DR}}(I \times X, f^*E) \cong H^q_{\mathrm{DR}}(X, E) \qquad (\textit{for any } q \in \mathbb{Z}_{\geq 0}),$$

and

$$H^{q+1}_{\mathrm{DR},c}(I \times X, f^*E) \cong H^q_{\mathrm{DR},c}(X, E) \qquad (\textit{for any } q \in \mathbb{Z}_{\geq 0}).$$

PROOF. As the first assertion follows immediately from Lemma I.8.42, in what follows we prove the second assertion. Define a map

$$F\colon \mathcal{A}_c^{q+1}(I \times X, f^*E) \to \mathcal{A}_c^q(X, E)$$

as follows: For $\omega = \alpha(t) + dt \wedge \beta(t)$, where $\alpha(t)$ is a $(q+1)$-form on X depending smoothly on t and $\beta(t)$ is a q-form on X depending smoothly on t, set

$$F(\omega) := \int_0^1 \beta(t)\, dt.$$

Fix $\theta \in C_c^\infty(I)$ such that $\int_0^1 \theta(t)\, dt = 1$, and define

$$G\colon \mathcal{A}_c^q(X, E) \to \mathcal{A}_c^{q+1}(I \times X, f^*E)$$

by

$$G(\eta) := \theta(t)\, dt \wedge \eta.$$

By construction, $FD = -DF$ and $DG = -GD$. Thus F and G preserve closed and exact forms and induce maps on cohomology in the indicated degrees. Since

$$F \circ G(\eta) = \left(\int_0^1 \theta(t)\, dt \right) \cdot \eta = \eta,$$

the induced map on cohomology is the identity on $H^\bullet_{\mathrm{DR},c}(X, E)$. It remains to show that $G \circ F$ induces the identity on $H^{q+1}_{\mathrm{DR},c}(I \times X, f^*E)$.

To show this, define

$$H\colon \mathcal{A}_c^{q+1}(I \times X, f^*E) \to \mathcal{A}_c^q(I \times X, f^*E)$$

by

$$H(\omega) := \int_0^t \beta(\tau)\, d\tau - \left(\int_0^t \theta(\tau)\, d\tau \right) \int_0^1 \beta(\tau)\, d\tau.$$

Since this vanishes on neighborhoods of $t = 0$ and $t = 1$, and since ω has compact support in $I \times X$, $H(\omega)$ also has compact support. Thus H is well-defined. Writing d_X for the exterior derivative on X, we compute

$$\begin{aligned} dH(\omega) = {} & dt \wedge \beta(t) + \int_0^t d_X \beta(\tau)\, d\tau \\ & - \theta(t)\, dt \wedge \int_0^1 \beta(\tau)\, d\tau - \left(\int_0^t \theta(\tau)\, d\tau \right) \int_0^1 d_X \beta(\tau)\, d\tau, \end{aligned}$$

and

$$\begin{aligned} H(d\omega) &= H\left(dt \wedge \frac{\partial \alpha(t)}{\partial t} + d_X \alpha(t) - dt \wedge d_X \beta(t)\right) \\ &= \int_0^t \left(\frac{\partial \alpha(\tau)}{\partial \tau} - d_X \beta(\tau)\right) d\tau \\ &\quad - \left(\int_0^t \theta(\tau)\, d\tau\right) \int_0^1 \left(\frac{\partial \alpha(\tau)}{\partial \tau} - d_X \beta(\tau)\right) d\tau \\ &= \alpha(t) - \int_0^t d_X \beta(\tau)\, d\tau + \left(\int_0^t \theta(\tau)\, d\tau\right) \int_0^1 d_X \beta(\tau)\, d\tau, \end{aligned}$$

where we use that $\alpha(0) = \alpha(1) = 0$ since ω has compact support in $I \times X$. Combining the above computations, we obtain

$$dH(\omega) + H(d\omega) = \omega - G \circ F(\omega).$$

Thus H is a cochain homotopy between the identity and $G \circ F$, and the proof is complete. □

Remark I.8.47. By applying Lemma I.8.46 inductively on the real dimension, we obtain the following for $I = (0, 1)$ and $n \in \mathbb{Z}_{\geq 0}$:

$$H^q_{\mathrm{DR}}(X) \cong \begin{cases} \mathbb{C} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (\text{if } X \text{ is } C^\infty \text{ homeomorphic to } I^n), \tag{42}$$

and

$$H^q_{\mathrm{DR},c}(X) \cong \begin{cases} \mathbb{C} & \text{if } q = n \\ 0 & \text{otherwise} \end{cases} \qquad (\text{if } X \text{ is } C^\infty \text{ homeomorphic to } I^n). \tag{43}$$

The isomorphism (42) is already contained in Corollary I.8.43, and clearly $H^0_{\mathrm{DR}}(X)$ consists of constant functions in this case. Let us record a concrete description of a generator of $H^n_{\mathrm{DR},c}(X) \cong \mathbb{C}$ in (43). Consider the linear map

$$F \colon \mathcal{A}^n_c(X) \to \mathbb{C}$$

defined by

$$F(\omega) := \int_X \omega.$$

Then F is a surjective linear map (linearity is clear, and surjectivity follows by choosing a non-negative bump function in $\mathcal{A}^n_c(X)$ with non-zero integral). Moreover, Stokes' theorem yields

$$\mathrm{Image}(d \colon \mathcal{A}^{n-1}_c(X) \to \mathcal{A}^n_c(X)) \subset \mathrm{Ker}\, F.$$

Consequently F induces a well-defined linear map $\widetilde{F} \colon H^n_{\mathrm{DR},c}(X) \to \mathbb{C}$. Since $H^n_{\mathrm{DR},c}(X) \cong \mathbb{C}$ by (43) and $\widetilde{F} \neq 0$, the map $\widetilde{F}$ is an isomorphism. In particular, the cohomology class of any $\omega \in \mathcal{A}^n_c(X)$ with $\int_X \omega = 1$ gives a generator of $H^n_{\mathrm{DR},c}(X)$. □

MINI-EXERCISE I.8.48. Let $X := \mathbb{R}^2 \setminus \{(0,0)\}$. Using the Euclidean coordinates (x, y) on $\mathbb{R}^2$, consider the 1-form

$$\omega := \frac{-y\,dx + x\,dy}{x^2 + y^2} \in \mathcal{A}^1(X).$$

(i) Show that ω is d-closed.
(ii) Show that ω is not d-exact.
(iii) Give explicitly a basis of $H^1_{\mathrm{DR}}(X)$.

Mayer–Vietoris sequence

Let X be a C^∞ manifold of real dimension n, and let $E \to X$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle. Take open subsets $U, V \subset X$ such that $X = U \cup V$.

(1) The case without compact supports. Consider the sequence of cochain complexes

$$0 \longrightarrow \mathcal{A}^\bullet(X, E) \longrightarrow \mathcal{A}^\bullet(U, E) \oplus \mathcal{A}^\bullet(V, E) \longrightarrow \mathcal{A}^\bullet(U \cap V, E) \longrightarrow 0,$$

where the first map is the restriction

$$\mathcal{A}^\bullet(X, E) \ni \omega \longmapsto (\omega|_U,\ \omega|_V) \in \mathcal{A}^\bullet(U, E) \oplus \mathcal{A}^\bullet(V, E),$$

and the second map is

$$\mathcal{A}^\bullet(U, E) \oplus \mathcal{A}^\bullet(V, E) \ni (\xi, \eta) \longmapsto \xi|_{U\cap V} - \eta|_{U\cap V} \in \mathcal{A}^\bullet(U \cap V, E).$$

All maps commute with the flat connection D, hence this is a sequence of cochain complexes. Exactness is proved by the standard partition-of-unity argument associated with the open covering $\{U, V\}$ (see, for example, [BT, Chapter I §2]). Passing to cohomology, we obtain a long exact sequence

$$0 \longrightarrow H^0_{\mathrm{DR}}(X, E) \longrightarrow H^0_{\mathrm{DR}}(U, E) \oplus H^0_{\mathrm{DR}}(V, E) \longrightarrow H^0_{\mathrm{DR}}(U \cap V, E)$$

$$\longrightarrow H^1_{\mathrm{DR}}(X, E) \longrightarrow H^1_{\mathrm{DR}}(U, E) \oplus H^1_{\mathrm{DR}}(V, E) \longrightarrow H^1_{\mathrm{DR}}(U \cap V, E)$$

$$\longrightarrow H^2_{\mathrm{DR}}(X, E) \longrightarrow H^2_{\mathrm{DR}}(U, E) \oplus H^2_{\mathrm{DR}}(V, E) \longrightarrow H^2_{\mathrm{DR}}(U \cap V, E) \longrightarrow \dots$$

(2) The case with compact supports. Consider the sequence of cochain complexes

$$0 \longrightarrow \mathcal{A}^\bullet_c(U \cap V, E) \longrightarrow \mathcal{A}^\bullet_c(U, E) \oplus \mathcal{A}^\bullet_c(V, E) \longrightarrow \mathcal{A}^\bullet_c(X, E) \longrightarrow 0,$$

where the first map is the diagonal embedding

$$\mathcal{A}^\bullet_c(U \cap V, E) \ni \omega \longmapsto (\omega,\ \omega) \in \mathcal{A}^\bullet_c(U, E) \oplus \mathcal{A}^\bullet_c(V, E),$$

and the second map is

$$\mathcal{A}^\bullet_c(U,E)\oplus\mathcal{A}^\bullet_c(V,E)\ni(\xi,\eta)\longmapsto\xi-\eta\in\mathcal{A}^\bullet_c(X,E),$$

where we view ξ and η as elements of $\mathcal{A}^\bullet_c(X,E)$ by extending them by 0 outside U and V, respectively. Again the maps commute with D, and exactness follows from the same partition-of-unity argument. Passing to cohomology, we obtain a long exact sequence

$$\begin{aligned}
&0\longrightarrow H^0_{\mathrm{DR},c}(U\cap V,E)\longrightarrow H^0_{\mathrm{DR},c}(U,E)\oplus H^0_{\mathrm{DR},c}(V,E)\longrightarrow H^0_{\mathrm{DR},c}(X,E)\\
&\longrightarrow H^1_{\mathrm{DR},c}(U\cap V,E)\longrightarrow H^1_{\mathrm{DR},c}(U,E)\oplus H^1_{\mathrm{DR},c}(V,E)\longrightarrow H^1_{\mathrm{DR},c}(X,E)\\
&\longrightarrow H^2_{\mathrm{DR},c}(U\cap V,E)\longrightarrow H^2_{\mathrm{DR},c}(U,E)\oplus H^2_{\mathrm{DR},c}(V,E)\longrightarrow H^2_{\mathrm{DR},c}(X,E)\longrightarrow\dots
\end{aligned}$$

The above long exact sequences are called the *Mayer–Vietoris sequence* (for de Rham cohomology, and for de Rham cohomology with compact supports, respectively).

Besides being useful for explicit computations, Mayer–Vietoris sequences are also fundamental in the theory. For instance, they enter the proof of various forms of *Poincaré duality* and the finite-dimensionality of (compactly supported) de Rham cohomology under suitable finiteness assumptions on X when it is oriented. In particular, for each $q\in\{0,1,\dots,n\}$, one considers the pairing

$$H^{n-q}_{\mathrm{DR},c}(X,E^*)\times H^q_{\mathrm{DR}}(X,E)\ni(\alpha,\beta)\longmapsto\int_X\alpha\wedge\beta\in\mathbb{C},\tag{44}$$

where $\alpha\wedge\beta$ is regarded as a compactly supported $\mathbb{C}$-valued n-form via the natural contraction $E^*\otimes E\to\mathbb{C}$. We show the following version.

Proposition I.8.49. *Let X be an oriented C^∞ manifold of real dimension n, and let $E\to X$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle. Assume one of the following:*
(i) X is compact, or
(ii) there exists a C^∞ exhaustion function $f\colon X\to\mathbb{R}$ (i.e., $\{x\in X\mid f(x)\le c\}$ is compact for every $c\in\mathbb{R}$) such that $f(X)$ is unbounded from above and the set of all critical values of f is bounded from above.
Then for each $q\in\{0,1,\dots,n\}$ we have

$$\dim H^{n-q}_{\mathrm{DR},c}(X,E^*)=\dim H^q_{\mathrm{DR}}(X,E)<\infty,$$

and the pairing (44) induces a map

$$H^q_{\mathrm{DR}}(X,E)\longrightarrow\bigl(H^{n-q}_{\mathrm{DR},c}(X,E^*)\bigr)^\vee$$

which is a TVS isomorphism.

PROOF. We first explain a reduction for the last assertion. It suffices to forget the topological structures and replace topological duals by algebraic duals, and to show that the induced map

$$H^q_{\mathrm{DR}}(X,E) \longrightarrow \left(H^{n-q}_{\mathrm{DR},c}(X,E^*)\right)^*$$

is bijective. Indeed, assuming the first assertion, put

$$m_q := \dim H^{n-q}_{\mathrm{DR},c}(X,E^*) = \dim H^q_{\mathrm{DR}}(X,E) < \infty.$$

Once the algebraic pairing is non-degenerate, its pairings with fixed elements separate points of both cohomology groups. These are continuous linear functionals: for a fixed compactly supported form, integration is continuous on the space of smooth forms, and for a fixed smooth form it is continuous on each fixed-support space of test forms, hence on their locally convex inductive limit. By Stokes' theorem they descend to the quotient topologies on cohomology. Thus $H^q_{\mathrm{DR}}(X,E)$ and $H^{n-q}_{\mathrm{DR},c}(X,E^*)$ are Hausdorff. Hence, by Theorem I.6.4 (i), both are TVS isomorphic to $\mathbb{C}^{m_q}$, and for finite-dimensional Hausdorff TVSs the algebraic and topological duals coincide canonically. This justifies the reduction.

For the proof, we use a good covering in the sense of [BT]. An open covering $\{U_j\}_{j\in J}$ of X is said to be a *good covering in the sense of [BT]* if every non-empty finite intersection $U_{j_0}\cap U_{j_1}\cap\cdots\cap U_{j_\ell}$ is C^∞ homeomorphic to $\mathbb{R}^n$. It is known that any C^∞ manifold admits a locally finite good covering in this sense, and if X is compact then one may take J to be finite. One can construct such coverings by choosing a locally finite covering by sufficiently small coordinate neighborhoods that are "convex" in a suitable sense, so that their non-empty finite intersections retain the required property; see [BT, Theorem 5.1] and [Weil, p. 120–122].

Assume that X is compact. In this case, by the above, it suffices to prove the proposition under the assumption that X admits a good covering $\{U_j\}_{j\in J}$ with J finite. We argue by induction on $\#J$. If $\#J = 1$, then X is C^∞ homeomorphic to $\mathbb{R}^n$, and the assertions follow from (42), (43), together with the description of representatives discussed in Remark I.8.47. Let $m \in \mathbb{Z}_{\geq 1}$ and assume that the proposition holds for any oriented C^∞ manifold of real dimension n which admits a good covering by at most m open sets. Now suppose that X admits a good covering $\{U_j\}_{j\in J}$ with $\#J = m+1$. Choose $j_0 \in J$ and put

$$U := U_{j_0}, \qquad V := \bigcup_{j\in J\setminus\{j_0\}} U_j.$$

Then $\{U,V\}$ is an open covering of X, and V admits a good covering by m open sets. Moreover, $U\cap V$ admits a good covering by at most m open sets as well (by intersecting the covering of V with U), since $U\cap U_j$ is again C^∞ homeomorphic to $\mathbb{R}^n$ whenever it is non-empty. Consider the Mayer–Vietoris sequence

$$\cdots \longrightarrow H^{q-1}_{\mathrm{DR}}(U,E)\oplus H^{q-1}_{\mathrm{DR}}(V,E) \longrightarrow H^{q-1}_{\mathrm{DR}}(U\cap V,E)$$

$$\longrightarrow H^q_{\mathrm{DR}}(X,E) \longrightarrow H^q_{\mathrm{DR}}(U,E)\oplus H^q_{\mathrm{DR}}(V,E) \longrightarrow H^q_{\mathrm{DR}}(U\cap V,E)$$

and the corresponding Mayer–Vietoris sequence for compact supports

$$H^{n-q}_{\mathrm{DR},c}(U\cap V,E^*)\longrightarrow H^{n-q}_{\mathrm{DR},c}(U,E^*)\oplus H^{n-q}_{\mathrm{DR},c}(V,E^*)\longrightarrow H^{n-q}_{\mathrm{DR},c}(X,E^*)$$

$$\longrightarrow H^{n-q+1}_{\mathrm{DR},c}(U\cap V,E^*)\longrightarrow H^{n-q+1}_{\mathrm{DR},c}(U,E^*)\oplus H^{n-q+1}_{\mathrm{DR},c}(V,E^*)\longrightarrow\dots$$

We also consider the algebraic dual sequence of the latter. (For vector spaces, the algebraic dual functor is exact; c.f., Corollary I.5.9). By the induction hypothesis, all groups associated with U, V, and $U\cap V$ appearing above are finite-dimensional, and the Poincaré-type statement (bijectivity of the induced algebraic-dual map) holds for these pieces. The pairing (44) is compatible with the restrictions, extensions by zero, and connecting morphisms up to signs. With the difference-map convention in the above sequences, use the pairing on U and its negative on V in the direct-sum term, and absorb the signs of the connecting morphisms into the remaining vertical maps. This gives a commutative diagram with exact rows and with vertical maps induced by the pairing up to signs (compare [BT, Lemma 5.6]). Then the five lemma yields the desired bijectivity for X. Finite-dimensionality follows from the same long exact sequences.

Next, assume that there exists a C^∞ exhaustion function $f\colon X\to\mathbb{R}$ such that $f(X)$ is unbounded from above and the set of all critical values of f is bounded from above. Replacing f by $f-C$ for a suitable constant C, we may assume that every critical value of f is less than -1. Let X^+ and X^- be two copies of the sublevel set $\{x\in X\mid f(x)<1\}$, and put

$$Y:=\{x\in X\mid f(x)=0\},\qquad E^\pm:=E|_{X^\pm}.$$

Since 0 is a regular value, Y is a C^∞ submanifold of real dimension $n-1$. Since f is an exhaustion function, Y is compact. By Ehresmann's fibration theorem (Theorem I.8.7), the open set $\{x\in X\mid f(x)>-1\}$ is C^∞ diffeomorphic to $Y\times(-1,\infty)$. In particular, $X^\pm$ is C^∞ diffeomorphic to X, hence it suffices to prove the assertion for $(X^\pm,E^\pm)$. Put

$$W:=\{x\in X\mid -1<f(x)<1\},$$

and identify W with $Y\times(-1,1)$ using Ehresmann's theorem. Let $i\colon W\to W$ be the C^∞ diffeomorphism corresponding to

$$Y\times(-1,1)\ni(y,t)\longmapsto(y,-t)\in Y\times(-1,1).$$

Gluing X^+ and X^- along the common open subset W via the identification induced by i, we obtain an oriented compact C^∞ manifold Z (the *double* of X^+). On the collar $W\cong Y\times(-1,1)$, flat transport in the t-direction identifies both $E^+|_W$ and $E^-|_W$ with the pull-back of $E|_Y$ under the projection $W\to Y$. Since $i(y,t)=(y,-t)$ preserves this projection, these identifications give a flat bundle isomorphism covering i, along which E^+ and E^- glue to a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle $\widetilde{E}$ on Z.

Consider the Mayer–Vietoris sequence for the open covering $\{X^+, X^-\}$ of Z. Using the identifications $X^+ \approx X^-$, we obtain, for each $q \in \{0, 1, \dots, n\}$, exact sequences of the form

$$H^{q-1}_{\mathrm{DR}}(W, E|_W) \longrightarrow H^q_{\mathrm{DR}}(Z, \widetilde{E}) \longrightarrow H^q_{\mathrm{DR}}(X^+, E^+)^{\oplus 2} \longrightarrow H^q_{\mathrm{DR}}(W, E|_W) \longrightarrow H^{q+1}_{\mathrm{DR}}(Z, \widetilde{E})$$

and

$$H^{n-q-1}_{\mathrm{DR},c}(Z, \widetilde{E}^*) \longrightarrow H^{n-q}_{\mathrm{DR},c}(W, E|_W^*) \longrightarrow H^{n-q}_{\mathrm{DR},c}(X^+, (E^+)^*)^{\oplus 2} \longrightarrow H^{n-q}_{\mathrm{DR},c}(Z, \widetilde{E}^*) \longrightarrow H^{n-q+1}_{\mathrm{DR},c}(W, E|_W^*).$$

Now Z is compact, so the assertion for (i) applies to $(Z, \widetilde{E})$. Moreover, W is C^∞ diffeomorphic to the product of the compact manifold Y and the interval $(-1, 1)$, hence W admits a good covering by finitely many open sets; thus the assertion also holds for $(W, E|_W)$. Using the above exact sequences, we deduce finite-dimensionality for $H^q_{\mathrm{DR}}(X^+, E^+)$ and $H^{n-q}_{\mathrm{DR},c}(X^+, (E^+)^*)$. Furthermore, combining these exact sequences with the sign-adjusted commutative diagrams induced by the pairing (44) and applying the five lemma as above, we obtain the desired bijectivity for (X^+, E^+). Finally, since X^+ is C^∞ diffeomorphic to X, the conclusion follows for (X, E). □

$H^q(\mathcal{K}^\bullet(X, E), D)$ and $H^q(\mathcal{K}^\bullet_c(X, E), D)$

We next discuss the cohomology groups of the complexes of currents. We begin with $H^q(\mathcal{K}^\bullet_c(X, E), D)$. By applying Lemma I.6.52 and Lemma I.6.55, we obtain the following.

Lemma I.8.50. *Let X be a C^∞ manifold of real dimension n, and let $E \to X$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle. Let $q \in \mathbb{Z}_{\geq 0}$. If $H^{q+1}_{\mathrm{DR}}(X, E^*)$ is Hausdorff, then there exists a natural vector space isomorphism*

$$H^{n-q}(\mathcal{K}^\bullet_c(X, E), D) \cong (H^q_{\mathrm{DR}}(X, E^*))^\vee.$$

In particular, if $\dim H^q_{\mathrm{DR}}(X, E^*) < \infty$ and $\dim H^{q+1}_{\mathrm{DR}}(X, E^*) < \infty$, then

$$\dim H^{n-q}(\mathcal{K}^\bullet_c(X, E), D) = \dim H^q_{\mathrm{DR}}(X, E^*).$$

Proof. Consider the sequence of Fréchet spaces

$$\mathcal{A}^{q-1}(X, E^*) \xrightarrow{D} \mathcal{A}^q(X, E^*) \xrightarrow{D} \mathcal{A}^{q+1}(X, E^*),$$

where both arrows are the flat connection D. Applying Lemma I.6.52 (i), we obtain a natural linear surjection

$$\iota \colon H^{n-q}(\mathcal{K}^\bullet_c(X, E), D) \longrightarrow (H^q_{\mathrm{DR}}(X, E^*))^\vee.$$

Assume that $H^{q+1}_{\mathrm{DR}}(X, E^*)$ is Hausdorff. Then $\mathrm{Image}\big(D \colon \mathcal{A}^q(X, E^*) \to \mathcal{A}^{q+1}(X, E^*)\big)$ is closed in $\mathrm{Ker}\big(D \colon \mathcal{A}^{q+1}(X, E^*) \to \mathcal{A}^{q+2}(X, E^*)\big)$. Moreover, $\mathrm{Ker}\big(D \colon \mathcal{A}^{q+1}(X, E^*) \to \mathcal{A}^{q+2}(X, E^*)\big)$ is a closed subspace of $\mathcal{A}^{q+1}(X, E^*)$. Therefore $\mathrm{Image}\big(D \colon \mathcal{A}^q(X, E^*) \to \mathcal{A}^{q+1}(X, E^*)\big)$ is a closed subspace of $\mathcal{A}^{q+1}(X, E^*)$. By Lemma I.6.52 (ii), the map ι is bijective. This proves the first assertion.

If $\dim H^{q+1}_{\mathrm{DR}}(X, E^*) < \infty$, then Lemma I.6.55 implies that $H^{q+1}_{\mathrm{DR}}(X, E^*)$ is Hausdorff, so the above isomorphism holds. If, in addition, $m := \dim H^q_{\mathrm{DR}}(X, E^*) < \infty$, then the same lemma shows that $H^q_{\mathrm{DR}}(X, E^*)$ is Hausdorff. By Theorem I.6.4 (i), it is TVS isomorphic to $\mathbb{C}^m$, hence its topological dual is an m-dimensional vector space. This yields the last assertion. □

Remark I.8.51. Let $\sigma \in C^{\mathrm{smooth}}_q(X)$ be a C^∞ singular q-chain such that $\partial\sigma = 0$. As we have already seen in Example I.8.20, one can associate to σ a compactly supported $(n-q)$-current $T_\sigma \in \mathcal{K}^{n-q}_c(X)$ in a natural way. Moreover, as observed in Remark I.8.22, we have $dT_\sigma = 0$. Hence we obtain a class $\{T_\sigma\} \in H^{n-q}(\mathcal{K}^\bullet_c(X), d)$. By construction, this class corresponds, via Lemma I.8.50, to the functional

$$\int_\sigma \in (H^q_{\mathrm{DR}}(X))^\vee$$

considered in Remark I.8.41. □

As in the proof of Lemma I.8.50, one can use Lemma I.6.52 and Lemma I.6.55 to relate $H^{n-q}(\mathcal{K}^\bullet(X, E), D)$ to $(H^q_{\mathrm{DR},c}(X, E^*))^\vee$. However, a more careful discussion is needed in general: Indeed, in Lemma I.6.52 (ii), for a sequence of strict countable LF-spaces such as

$$\mathcal{A}^{q-1}_c(X, E^*) \to \mathcal{A}^q_c(X, E^*) \to \mathcal{A}^{q+1}_c(X, E^*),$$

the additional hypotheses required are stronger.

On the other hand, after developing the general theory of sheaves in the next chapter, one can show in complete generality (for any oriented C^∞ manifold X of real dimension n and any $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle $E \to X$) that there is a vector space isomorphism

$$H^q(\mathcal{K}^\bullet(X, E), D) \cong H^q_{\mathrm{DR}}(X, E). \tag{45}$$

We postpone the proof of this fact until Chapter II (see Example II.3.65), and here we show only the following special version.

Lemma I.8.52 (Poincaré lemma for currents). *Assume that $X \subset \mathbb{R}^n$ is an open neighborhood of the origin and that X is C^∞ homeomorphic to $(0,1)^n$. Then for any $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle $E \to X$, the isomorphism (45) holds. In particular,*

$$H^q(\mathcal{K}^\bullet(X, E), D) \cong \begin{cases} \mathbb{C}^r & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since X is contractible, Remark I.8.44 shows that E is a trivial vector bundle on X. Thus, by considering each direct summand, it suffices to prove the statement for $E = X \times \mathbb{C}$ and $D = d$.

We may work on the cube $(0,1)^n$ with its standard orientation. For $n = 0$ the assertion is immediate, so assume $n \geq 1$. The compact-support homotopy in Lemma I.8.46 can be

iterated in the coordinate directions. It gives a degree -1 operator K on $\mathcal{A}_c^\bullet((0,1)^n)$ such that

$$dK + Kd = \mathrm{id} - \Pi, \qquad \Pi(\omega) = \begin{cases} \left(\int_{(0,1)^n} \omega\right)\Theta & \text{if } \deg\omega = n, \\ 0 & \text{otherwise,} \end{cases}$$

where Θ is a fixed compactly supported n-form of integral 1. Indeed, if K' and Π' are the corresponding operators for the remaining coordinates, the interval operators F, G, H give $K = H - GK'F$ and $\Pi = G\Pi'F$; the identity follows from $FD = -DF$, $DG = -GD$, and $DH + HD = \mathrm{id} - GF$.

These operators are continuous on the test-form spaces. To see this, fix a compact set containing the support of the input. In the interval formula the support of the output is contained in the product of the projection of this set to the remaining coordinates and the convex hull of its interval projection with $\operatorname{Supp}\theta$. This is again a compact subset of the cube. Differentiation under the integrals bounds every fixed-order seminorm of the output by a constant times a finite-order seminorm of the input. The same properties persist under the finite iteration. The universal property of the locally convex inductive limit therefore gives the asserted continuity.

We can now transpose the homotopy. For a q-current T and a test form φ of degree $n - q + 1$, set

$$(\mathscr{H}_q T)(\varphi) := (-1)^q T(K\varphi).$$

The sign agrees with the convention $dT(\varphi) = (-1)^{q+1}T(d\varphi)$, and gives

$$d\mathscr{H} + \mathscr{H}d = \mathrm{id} - {}^t\Pi.$$

The transpose ${}^t\Pi$ is zero in positive current degrees; in degree 0 it sends T to the constant current whose value on an n-form φ is $T(\Theta)\int\varphi$. Hence the current cohomology is $\mathbb{C}$ in degree 0 and zero in other degrees. The integration map from smooth forms sends constants to these constant currents, so this computation also gives the stated comparison with de Rham cohomology. □

Using Lemma I.8.52, under the assumptions of Proposition I.8.49 we can already summarize the relation between de Rham cohomology and its current version as follows.

Proposition I.8.53. *Let X be an oriented C^∞ manifold of real dimension n, and let $E \to X$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle. Assume that X satisfies (i) or (ii) in Proposition I.8.49. Then for each $q \in \{0, 1, \ldots, n\}$ there are natural isomorphisms*

$$H^q_{\mathrm{DR}}(X, E) \cong H^q(\mathcal{K}^\bullet(X, E), D)$$

and

$$H^{n-q}_{\mathrm{DR},c}(X, E^*) \cong H^{n-q}(\mathcal{K}^\bullet_c(X, E^*), D),$$

and all these groups are finite-dimensional vector spaces of the same dimension.

PROOF. We consider the natural maps induced by integration pairings:

$$\mathcal{A}^q(X,E) \ni \alpha \longmapsto \left(\varphi \longmapsto \int_X \alpha \wedge \varphi\right) \in \mathcal{K}^q(X,E),$$

where $\varphi \in \mathcal{A}_c^{n-q}(X,E^*)$, and

$$\mathcal{A}_c^{n-q}(X,E^*) \ni \beta \longmapsto \left(\psi \longmapsto \int_X \beta \wedge \psi\right) \in \mathcal{K}_c^{n-q}(X,E^*),$$

where $\psi \in \mathcal{A}^q(X,E)$.

By Proposition I.8.49 and Lemma I.8.50, it remains to show that the induced map

$$H^q_{\mathrm{DR}}(X,E) \longrightarrow H^q(\mathcal{K}^\bullet(X,E),D)$$

is an isomorphism.

Let $U, V \subset X$ be open subsets with $X = U \cup V$. As in the construction of the Mayer–Vietoris sequence, one has a short exact sequence of cochain complexes

$$0 \to \mathcal{K}^\bullet(X,E) \to \mathcal{K}^\bullet(U,E) \oplus \mathcal{K}^\bullet(V,E) \to \mathcal{K}^\bullet(U \cap V,E) \to 0,$$

where the maps are given by restriction and $(\xi,\eta) \longmapsto \xi|_{U\cap V} - \eta|_{U\cap V}$. Hence we obtain a long exact sequence on cohomology. Using the naturality of the integration pairings with respect to restrictions and connecting morphisms, we can compare this long exact sequence with the Mayer–Vietoris sequence for $H^\bullet_{\mathrm{DR}}(-,E)$, and we obtain a commutative diagram of long exact sequences. Using Lemma I.8.52 as the base case, one proves the desired isomorphism by induction on the number of sets in a finite good covering in the sense of [BT], together with the five lemma, exactly as in the proof of Proposition I.8.49 (Note that, when X is compact, the argument is simpler, since by definition $H^q_{\mathrm{DR}}(X,E) = H^q_{\mathrm{DR},c}(X,E)$ and $H^q(\mathcal{K}^\bullet(X,E),D) = H^q(\mathcal{K}_c^\bullet(X,E),D)$, so the conclusion follows directly from Proposition I.8.49 together with Lemma I.8.50). Under assumption (ii), one applies the same argument as in the proof of Proposition I.8.49 by considering the double. □

Singular–de Rham correspondence

Here we prove the following theorem in the case where the coefficient bundle is the trivial line bundle.

THEOREM I.8.54. *Let X be a C^∞ manifold of real dimension n. Then*

$$H^q_{\mathrm{DR}}(X) \;\cong\; H^q(X,\mathbb{C}) \qquad (\textit{for all } q \in \mathbb{Z}_{\geq 0})$$

holds.

PROOF. Let $\{U_v\}_{v\in X^0}$ be the locally finite open covering as in Lemma I.7.20, and take a partition of unity $\{\rho_v\}_{v\in X^0}$ subordinate to this covering.

Consider the double complex defined as follows. For integers $p, q \in \mathbb{Z}_{\geq 0}$, set

$$C^{p,q} := \prod_{\substack{(v_0,\dots,v_q)\in (X^0)^{q+1} \\ U_{v_0}\cap\cdots\cap U_{v_q}\neq\emptyset}} \mathcal{A}^p(U_{v_0}\cap U_{v_1}\cap\cdots\cap U_{v_q}).$$

Define

$$\check{\delta}\colon C^{p,q} \longrightarrow C^{p,q+1}$$

by letting

$$(\check{\delta}c)_{v_0,\dots,v_{q+1}} := \sum_{\ell=0}^{q+1}(-1)^\ell\, c_{v_0,\dots,\widehat{v_\ell},\dots,v_{q+1}} \qquad \big(U_{v_0}\cap U_{v_1}\cap\cdots\cap U_{v_{q+1}}\neq\emptyset\big),$$

where we interpret $c_{v_0,\dots,\widehat{v_\ell},\dots,v_{q+1}}$ as 0 if $U_{v_0}\cap\cdots\cap\widehat{U_{v_\ell}}\cap\cdots\cap U_{v_{q+1}}=\emptyset$ (c.f., the definition of the homomorphism in Proposition I.7.24). Then $\check{\delta}\circ\check{\delta}=0$. On the other hand, define

$$d\colon C^{p,q} \longrightarrow C^{p+1,q}$$

by the exterior derivative on each component:

$$(dc)_{v_0,\dots,v_q} := d\,c_{v_0,\dots,v_q}.$$

Clearly we have $d\circ d=0$ and $d\check{\delta}=\check{\delta}d$.

By the isomorphism (26), it suffices to show that

$$H^q_{\mathrm{DR}}(X) \;\cong\; H^q\big(\check{C}^\bullet(\{U_v\},\mathbb{C})\big).$$

By Theorem A.1.1 in the appendix of this note (or the general theory of spectral sequences for double complexes), it is enough to verify that the following two sequences are exact:

$$C^{p,q}\longrightarrow C^{p+1,q}\longrightarrow C^{p+2,q} \qquad\text{and}\qquad C^{p,q}\longrightarrow C^{p,q+1}\longrightarrow C^{p,q+2}$$

for all $p, q \in \mathbb{Z}_{\geq 0}$. Here p is the form degree and q is the covering degree. After interchanging the two indices to match the appendix, the total differential is $\check{\delta}+(-1)^q d$; the two unsigned differentials commute, as required there. We use the comparison induced by the inclusions of the two edge complexes into this total complex, without an additional sign in any degree; see Remarks A.1.4 and A.1.5 for translations between sign conventions.

First, every nonempty finite intersection $U_{v_0}\cap U_{v_1}\cap\cdots\cap U_{v_q}$ is a contractible open submanifold. Its identity map is smoothly homotopic to a constant map by Theorem I.8.13. Hence Corollary I.8.43 applies, and its de Rham cohomology vanishes in positive degrees. Therefore each sequence $C^{p,q}\longrightarrow C^{p+1,q}\longrightarrow C^{p+2,q}$ is exact.

Next we prove that each sequence $C^{p,q}\longrightarrow C^{p,q+1}\longrightarrow C^{p,q+2}$ is exact. Let $q>0$ and let $c=(c_{v_0,\dots,v_q})\in C^{p,q}$ satisfy $\check{\delta}c=0$. Define $b\in C^{p,q-1}$ by

$$b_{v_0,\dots,v_{q-1}} := \sum_{v\in X^0}\rho_v\, c_{v,v_0,\dots,v_{q-1}}.$$

Since the covering is locally finite and each ρ_v has support contained in U_v, the above sum is locally finite, hence defines a well-defined element of $\mathcal{A}^p(U_{v_0} \cap \cdots \cap U_{v_{q-1}})$. We claim that $\check{\delta} b = c$. Indeed, for $U_{v_0} \cap \cdots \cap U_{v_q} \neq \emptyset$ we have

$$\begin{aligned}(\check{\delta} b)_{v_0,\ldots,v_q} &= \sum_{\ell=0}^{q} (-1)^\ell \, b_{v_0,\ldots,\widehat{v_\ell},\ldots,v_q} \\ &= \sum_{\ell=0}^{q} (-1)^\ell \sum_{v \in X^0} \rho_v \, c_{v,v_0,\ldots,\widehat{v_\ell},\ldots,v_q} \\ &= \sum_{v \in X^0} \rho_v \left(\sum_{\ell=0}^{q} (-1)^\ell \, c_{v,v_0,\ldots,\widehat{v_\ell},\ldots,v_q} \right).\end{aligned}$$

Since $\check{\delta} c = 0$, we have

$$0 = (\check{\delta} c)_{v,v_0,\ldots,v_q} = c_{v_0,\ldots,v_q} - \sum_{\ell=0}^{q} (-1)^\ell \, c_{v,v_0,\ldots,\widehat{v_\ell},\ldots,v_q},$$

hence

$$\sum_{\ell=0}^{q} (-1)^\ell \, c_{v,v_0,\ldots,\widehat{v_\ell},\ldots,v_q} = c_{v_0,\ldots,v_q}.$$

Substituting this into the previous expression yields

$$(\check{\delta} b)_{v_0,\ldots,v_q} = \sum_{v \in X^0} \rho_v \, c_{v_0,\ldots,v_q}.$$

Finally, since $\{\rho_v\}$ is a partition of unity, we have $\sum_{v \in X^0} \rho_v = 1$, and therefore $(\check{\delta} b)_{v_0,\ldots,v_q} = c_{v_0,\ldots,v_q}$. Thus $\check{\delta} b = c$, proving exactness in the $\check{\delta}$-direction. This completes the proof. The comparison with singular cohomology uses the ordered-simplex orientations of Proposition I.7.24. With these conventions it agrees with integration of forms over oriented smooth singular simplices; Stokes' theorem identifies the differentials. See also [BT, §§8–9].

□

Remark I.8.55. Assume here that X is oriented. In fact, for essentially the same reason as the argument described in the proof of Theorem I.8.54, one also has

$$H^q(\mathcal{K}^\bullet(X), d) \;\cong\; H^q(X, \mathbb{C}) \qquad (q \in \mathbb{Z}_{\geq 0}).$$

One may then deduce the vector space isomorphism (45). However, for the open covering $\{U_v\}_{v \in X^0}$ as in Lemma I.7.20, which is used in the proof of Theorem I.8.54, although each finite intersection is contractible, it is not clear whether one can arrange it to be C^∞ homeomorphic to $(0,1)^n$. Therefore, in order to apply Lemma I.8.52 directly, one would need additional work (for instance, by taking a sufficiently fine barycentric subdivision of the triangulation used to construct the covering). Alternatively, one can also pursue an argument

based on a good covering in the sense of [BT], as in the proof of Proposition I.8.49, at least under suitable finiteness assumptions on the geometry of X. In this note we do not go further in this direction. Instead, in Chapter II, after introducing sheaf cohomology, this issue will be replaced by the standard refinement argument built into the definition of Čech cohomology. □

We have seen in Proposition I.8.49 that, for a C^∞ manifold X satisfying (i) or (ii) there, the de Rham cohomology groups are finite-dimensional. In particular, for each $q \in \{0, 1, \dots, n\}$,

$$\dim H^q_{\mathrm{DR}}(X) = \dim H^{n-q}_{\mathrm{DR},c}(X) < \infty.$$

By the singular–de Rham correspondence (Theorem I.8.54) together with the universal coefficient theorem for cohomology (in particular, the isomorphism (23)), this common value is nothing but the Betti number $b_q(X)$. In particular, if X is compact, then we obtain $b_q(X) = b_{n-q}(X)$.

As another topological application of de Rham cohomology, we record the following.

Lemma I.8.56. *Let X be a connected C^∞ manifold of real dimension n, and let $Y \subset X$ be a connected oriented C^∞ submanifold of real dimension $n-1$ whose real normal line bundle in X is trivial. Then $X \setminus Y$ is connected or has exactly two connected components. Moreover, if the first Betti number of X is 0, then $X \setminus Y$ has exactly two connected components.*

Proof. By the tubular neighborhood theorem (Theorem I.8.4) and the triviality of the real normal line bundle, Y admits an open neighborhood $V \subset X$ that is C^∞ diffeomorphic to $Y \times (-1, 1)$. Put $M := X \setminus Y$, so that $X = V \cup M$. The Mayer–Vietoris sequence for de Rham cohomology gives an exact sequence

$$0 \to H^0_{\mathrm{DR}}(X) \to H^0_{\mathrm{DR}}(M) \oplus H^0_{\mathrm{DR}}(V) \to H^0_{\mathrm{DR}}(V \setminus Y) \to H^1_{\mathrm{DR}}(X).$$

Since X is connected, we have $H^0_{\mathrm{DR}}(X) \cong \mathbb{C}$. Also V is connected, hence $H^0_{\mathrm{DR}}(V) \cong \mathbb{C}$. On the other hand, $V \setminus Y$ is C^∞ diffeomorphic to $Y \times \big((-1, 0) \sqcup (0, 1)\big)$, which has exactly two connected components because Y is connected. Thus $H^0_{\mathrm{DR}}(V \setminus Y) \cong \mathbb{C}^2$. Hence we obtain an exact sequence

$$0 \to \mathbb{C} \to H^0_{\mathrm{DR}}(M) \oplus \mathbb{C} \to \mathbb{C}^2.$$

By construction, the map $\mathbb{C} \to \mathbb{C}^2$ coming from the V-component is given by $a \mapsto (-a, -a)$, so the map $H^0_{\mathrm{DR}}(M) \oplus \mathbb{C} \to \mathbb{C}^2$ has rank 1 or 2.

If its rank is 1, then we obtain an exact sequence

$$0 \to \mathbb{C} \to H^0_{\mathrm{DR}}(M) \oplus \mathbb{C} \to \mathbb{C} \to 0,$$

hence $H^0_{\mathrm{DR}}(M) \cong \mathbb{C}$, i.e., M is connected. If its rank is 2, then we have an exact sequence

$$0 \to \mathbb{C} \to H^0_{\mathrm{DR}}(M) \oplus \mathbb{C} \to \mathbb{C}^2 \to 0,$$

hence $H^0_{\mathrm{DR}}(M) \cong \mathbb{C}^2$, i.e., M has exactly two connected components. This proves the first assertion.

For the second assertion, assume that $b_1(X) = 0$. Then by Theorem I.8.54 we have $H^1_{\mathrm{DR}}(X) = 0$. Therefore the map $H^0_{\mathrm{DR}}(M) \oplus H^0_{\mathrm{DR}}(V) \to H^0_{\mathrm{DR}}(V \setminus Y)$ is surjective, so its rank is 2. Hence $M = X \setminus Y$ has exactly two connected components. □

Remark I.8.57. If $X = \mathrm{U}(1)$ and Y is a one-point subset, then $X \setminus Y$ is connected. Thus, in the situation of Lemma I.8.56, both possibilities (one component or two components) can indeed occur. □

Remark I.8.58. When $X = \mathbb{R}^2$, let $Y \subset X$ be a connected compact C^∞ submanifold of real dimension 1. Then Lemma I.8.56 implies that $X \setminus Y$ has two connected components. The bounded one is called the *inside* of Y, and the unbounded one is called the *outside* of Y. This can be viewed as a special case of the *Jordan curve theorem* in the case where Y is the image of a C^∞ embedding $\gamma\colon \mathrm{U}(1) \to X = \mathbb{R}^2$. In fact, the Jordan curve theorem asserts that the same conclusion holds even when $\gamma\colon \mathrm{U}(1) \to \mathbb{R}^2$ is merely injective and continuous (a *Jordan curve*), and its proof is not elementary; see [Mu2, §61]. □

8.9 Harmonic forms and the Hodge Laplacian

Throughout this subsection, unless otherwise stated, we fix the following setting. Let X be an oriented C^∞ manifold of real dimension $n \in \mathbb{Z}_{\geq 0}$, g a Riemannian metric on X, and $E \to X$ be a $\mathrm{U}(r)$-flat vector bundle of rank $r \in \mathbb{Z}_{\geq 1}$. We emphasize that, unlike the previous subsubsections, the notions introduced below depend not only on the C^∞ structure of X but also on the choice of g. Moreover, since E is $\mathrm{U}(r)$-flat, it admits a flat metric; we fix such a metric h on E once and for all. In what follows, the operators and inner products will depend on the pair (g, h).

For each $p \in \{0, 1, \dots, n\}$, we agree to write, more simply, $(\bullet, \bullet)_{g,h}$ for the L^2-inner product induced by the pair (g, h) that was introduced earlier (when X is compact):

$$(u, v)_{g,h} := (u, v)_{L^p_{(2)}(X,E;g,h)} \left(= \int_X \langle u,\, v\rangle_{g,h}\, dV_g \right), \qquad u, v \in \mathcal{A}^p(X, E).$$

Note that the integral makes sense, for example, when at least one of u and v has compact support.

For each p, the *Hodge star operator* $* = *_{g,h}$ is the C^∞ fiberwise isomorphism

$$*\colon E \otimes \bigwedge^p T^*X \longrightarrow E \otimes \bigwedge^{n-p} T^*X$$

characterized by the following property: the induced continuous linear map

$$*\colon \mathcal{A}^p(X, E) \longrightarrow \mathcal{A}^{n-p}(X, E)$$

satisfies the identity

$$*u \wedge \overline{v} = \langle u, v\rangle_{g,h}\, dV_g \qquad \big(u, v \in \mathcal{A}^p(X, E)\big),$$

where the left-hand side is understood by contracting the E-coefficients using the fixed Hermitian metric h, so that it becomes an ordinary complex-valued n-form. Note that $*$ satisfies the involution formula

$$** u = (-1)^{p(n-p)}\, u \qquad \text{for all } u \in \mathcal{A}^p(X,E).$$

Let $D\colon \mathcal{A}^p(X,E) \to \mathcal{A}^{p+1}(X,E)$ be the flat connection on E. Assume that $u \in \mathcal{A}^{p+1}(X,E)$ and $v \in \mathcal{A}^p(X,E)$, and that at least one of u and v has compact support. Then, by the Leibniz rule and Stokes' theorem, we compute

$$\begin{aligned}
(u, Dv)_{g,h} &= \int_X *u \wedge \overline{Dv} \\
&= \int_X *u \wedge D\overline{v} \\
&= (-1)^{n-p-1} \int_X \Big(d(*u \wedge \overline{v}) - D(*u) \wedge \overline{v} \Big) \\
&= (-1)^{n-p} \int_X D(*u) \wedge \overline{v} \\
&= (-1)^{n-p} \cdot (-1)^{(n-p)p} \int_X ** D(*u) \wedge \overline{v} \\
&= (-1)^{(n-p)(p+1)} \int_X \langle * D * u, v \rangle_{g,h}\, dV_g \\
&= (-1)^{(n-p)(p+1)}\, (* D * u, v)_{g,h}
\end{aligned}$$

Motivated by this, we define the *formal adjoint* of D (with respect to (g,h)) by

$$D^\dagger\ (= D^\dagger_{g,h})\colon \mathcal{A}^{p+1}(X,E) \longrightarrow \mathcal{A}^p(X,E), \qquad D^\dagger u := (-1)^{(n-p)(p+1)}\ * D * u.$$

With this definition, the above computation shows that

$$(u, Dv)_{g,h} = (D^\dagger u, v)_{g,h}$$

whenever at least one of u and v has compact support. We now define the *Hodge Laplacian* (or Laplace–Beltrami operator) associated with (g,h) by

$$\Delta\ (= \Delta_{g,h}) := DD^\dagger + D^\dagger D.$$

For each p, this gives a linear differential operator

$$\Delta\colon \mathcal{A}^p(X,E) \longrightarrow \mathcal{A}^p(X,E).$$

If $n \geq 1$, Δ is a linear partial differential operator of order 2; when $n = 0$, one has $D = \Delta = 0$.

Mini-Exercise I.8.59. Let $X = \mathbb{R}^n$ be endowed with the Euclidean metric g. Let $E := X \times \mathbb{C}$ be the trivial complex line bundle with the flat metric $h \equiv 1$. Note that, in this case, the Hodge Laplacian is $\Delta := dd^\dagger + d^\dagger d$. Show that, on $\mathcal{A}^0(X,E) = C^\infty(\mathbb{R}^n)$, one has

$$\Delta f = -\sum_{j=1}^{n} \left(\frac{\partial}{\partial x^j}\right)^2 f \qquad \big(f \in C^\infty(\mathbb{R}^n)\big),$$

where $(x^1, x^2, \ldots, x^n)$ denotes the standard Euclidean coordinates on $\mathbb{R}^n$.

Lemma I.8.60. *We keep the notation and the setting above, and fix $p \in \{0, 1, \ldots, n\}$. Then for each $x \in X$ and each $\xi \in T^*_x X$, the principal symbol of the Hodge Laplacian $\Delta\colon \mathcal{A}^p(X,E) \to \mathcal{A}^p(X,E)$ satisfies*

$$\sigma_\Delta(x,\xi) = -|\xi|_g^2 \cdot \mathrm{id}_{\bigwedge^p T^*_x X \otimes E_x},$$

where

$$\mathrm{id}_{\bigwedge^p T^*_x X \otimes E_x}\colon \bigwedge^p T^*_x X \otimes E_x \longrightarrow \bigwedge^p T^*_x X \otimes E_x$$

is the identity map. In particular, Δ is strongly elliptic.

Proof. Fix $x \in X$ and $\xi \in T^*_x X$. By the definition of the principal symbol and the definition $\Delta = DD^\dagger + D^\dagger D$, we have

$$\sigma_\Delta(x,\xi) = \sigma_D(x,\xi)\,\sigma_{D^\dagger}(x,\xi) + \sigma_{D^\dagger}(x,\xi)\,\sigma_D(x,\xi).$$

By Lemma I.8.26, we may compute these symbols using any choice of local coordinates and local trivializations.

Choose local coordinates $(x^1, x^2, \ldots, x^n)$ centered at x which are normal for g at x. Thus the vectors

$$e_j := \left.\frac{\partial}{\partial x^j}\right|_x \in T_x X \qquad (1 \le j \le n)$$

form an orthonormal basis of $T_x X$, and we denote the dual basis of $T^*_x X$ by

$$e^j := dx^j\big|_x \in T^*_x X.$$

Since E is $\mathrm{U}(r)$-flat, we may also choose a local h-unitary frame of E near x which is parallel at x with respect to D. In particular, in this frame the connection matrix of D vanishes at x, so the first-order part of D at x coincides with the exterior derivative on each component. Therefore the principal symbol of D is the same as that of d on scalar-valued forms, tensored with the identity map on E_x. Hence it is enough to treat the case $E = X \times \mathbb{C}$ and $D = d$.

For operators between different vector bundles, we use the same definition of the principal symbol by the highest-order coefficients, now taking values in the fiberwise linear maps between the two bundles. First we claim that the principal symbol of $d\colon \mathcal{A}^p(X) \longrightarrow \mathcal{A}^{p+1}(X)$ at (x,ξ) is the wedge product with ξ:

$$\sigma_d(x,\xi)\colon \bigwedge^p T^*_x X \longrightarrow \bigwedge^{p+1} T^*_x X, \qquad \sigma_d(x,\xi)(\eta) = \xi \wedge \eta.$$

Indeed, choose local coordinates $(x^1, x^2, \dots, x^n)$ on a neighborhood U of x. For a strictly increasing multi-index $I = (i_1, i_2, \dots, i_p)$ with $1 \le i_1 < \cdots < i_p \le n$, put $dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. Then any $u \in \mathcal{A}^p(U)$ can be written uniquely as

$$u = \sum_I \phi_I \, dx^I \qquad (\phi_I \in C^\infty(U, \mathbb{C})).$$

By the definition of the exterior derivative, we have

$$du = \sum_I \sum_{j=1}^n \frac{\partial \phi_I}{\partial x^j} \, dx^j \wedge dx^I.$$

Thus, viewing d as a linear partial differential operator of order 1 acting on the trivial bundle $\bigwedge^p T^*U$ with frame $\{dx^I\}_I$, we may write

$$d = \sum_{j=1}^n \big(dx^j \wedge\big) \circ \frac{\partial}{\partial x^j},$$

where $\big(dx^j \wedge\big)$ denotes the $\mathbb{C}$-linear map $\bigwedge^p T^*U \to \bigwedge^{p+1} T^*U$ given by wedge product with dx^j.

Now write

$$\xi = \sum_{j=1}^n \xi_j \, dx^j\big|_x \in T^*_x X.$$

Since d is of order 1, the definition (33) yields

$$\sigma_d(x, \xi) = \sum_{j=1}^n \xi_j \cdot \big(dx^j \wedge\big) = \xi \wedge,$$

i.e., $\sigma_d(x, \xi)(\eta) = \xi \wedge \eta$ for $\eta \in \bigwedge^p T^*_x X$. This proves the claim.

Next we determine the principal symbol of $d^\dagger$. Recall that on p-forms we defined

$$d^\dagger \colon \mathcal{A}^{p+1}(X) \longrightarrow \mathcal{A}^p(X), \qquad d^\dagger u := (-1)^{(n-p)(p+1)} \ast d \ast u.$$

Fix $x \in X$ and $\xi \in T^*_x X$. Since $\ast \colon \mathcal{A}^k(X) \to \mathcal{A}^{n-k}(X)$ is fiberwise (i.e., a linear differential operator of order 0), its principal symbol at (x, ξ) is simply the induced linear map on fibers, which we denote by

$$\ast_x \colon \bigwedge^k T^*_x X \longrightarrow \bigwedge^{n-k} T^*_x X.$$

In particular, the composition $\ast \circ d \circ \ast$ is a linear differential operator of order 1, and its principal symbol is obtained by composing the symbols of the factors. Using the previous computation, we obtain, for $\eta \in \bigwedge^{p+1} T^*_x X$,

$$\sigma_{d^\dagger}(x, \xi)(\eta) = (-1)^{(n-p)(p+1)} \ast_x \Big(\xi \wedge \ast_x \eta\Big).$$

Equivalently,

$$\sigma_{d^\dagger}(x,\xi) = (-1)^{(n-p)(p+1)} \; *_x \circ (\xi\wedge) \circ *_x \colon \bigwedge^{p+1} T^*_x X \longrightarrow \bigwedge^{p} T^*_x X.$$

Write

$$\xi = \sum_{j=1}^{n} \xi_j \, e^j, \qquad |\xi|_g^2 := \sum_{j=1}^{n} \xi_j^2.$$

For a strictly increasing multi-index $I = (i_1, i_2, \ldots, i_{p+1})$ with $1 \le i_1 < i_2 < \cdots < i_{p+1} \le n$, we write

$$e^I := e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_{p+1}},$$

and let $I^c = (j_1, j_2, \ldots, j_{n-p-1})$ be the unique strictly increasing multi-index such that

$$\{i_1, i_2, \ldots, i_{p+1}\} \sqcup \{j_1, j_2, \ldots, j_{n-p-1}\} = \{1, 2, \ldots, n\}.$$

We fix the sign convention for $*_x$ by requiring that

$$e^I \wedge e^{I^c} = \epsilon(I)\, dV_{g,x} \qquad \text{with } \epsilon(I) \in \{\pm 1\}.$$

Note that we have

$$*_x e^I = (-1)^{(p+1)(n-p-1)} \, \epsilon(I) \, e^{I^c}.$$

Indeed, since $\langle e^I, e^I \rangle_{g,h} = 1$, by definition we have $*_x e^I \wedge e^I = dV_{g,x}$, and rewriting $e^{I^c} \wedge e^I = (-1)^{(p+1)(n-p-1)} \, e^I \wedge e^{I^c}$ yields the equation above.

To compute the action of $*_x \circ (e^j\wedge) \circ *_x$ on the basis $\{e^I\}$, fix $I = (i_1, i_2, \ldots, i_{p+1})$ with $1 \le i_1 < i_2 < \cdots < i_{p+1} \le n$ and $j \in \{1, 2, \ldots, n\}$. If $j \notin \{i_1, i_2, \ldots, i_{p+1}\}$, then $j \in I^c$, so e^j appears as one of the factors of e^{I^c}; hence $e^j \wedge e^{I^c} = 0$, and therefore

$$\big(*_x \circ (e^j\wedge) \circ *_x\big)(e^I) = 0.$$

If $j \in \{i_1, i_2, \ldots, i_{p+1}\}$, let $\ell \in \{1, 2, \ldots, p+1\}$ be the unique integer with $j = i_\ell$. Write

$$I \setminus \{j\} = (i_1, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_{p+1}),$$

which is strictly increasing. In this case $j \notin I^c$, so $e^j \wedge e^{I^c} \neq 0$. We claim that

$$e^j \wedge e^{I^c} = (-1)^{j-\ell} \, e^{(I\setminus\{j\})^c}. \tag{46}$$

Indeed, since $\{i_1, \ldots, i_{p+1}\} \sqcup \{j_1, \ldots, j_{n-p-1}\} = \{1, \ldots, n\}$ and $j = i_\ell$, the set $\{1, 2, \ldots, j-1\}$ contains exactly $(\ell - 1)$ elements of $\{i_1, \ldots, i_{p+1}\}$ and hence exactly $(j-1) - (\ell-1) = j - \ell$ elements of $\{j_1, \ldots, j_{n-p-1}\}$, i.e., of I^c. Therefore, when we rewrite $e^j \wedge e^{I^c}$ into the increasing wedge $e^{(I\setminus\{j\})^c}$, the factor e^j has to be moved past precisely $j - \ell$ factors, which gives the sign $(-1)^{j-\ell}$ and proves (46).

Using the explicit action of $*_x$ on an orthonormal frame, we obtain, for $I = (i_1, i_2, \ldots, i_{p+1})$ and $j = i_\ell$,

$$*_x e^I = (-1)^{(p+1)(n-p-1)} \, \epsilon(I) \, e^{I^c}, \qquad *_x e^{(I\setminus\{j\})^c} = (-1)^{p(n-p)} \, \epsilon((I \setminus \{j\})^c) \, e^{I\setminus\{j\}}.$$

Therefore,

$$\big(*_x \circ (e^j\wedge) \circ *_x\big)(e^I) = (-1)^{(p+1)(n-p-1)+p(n-p)}\,\epsilon(I)\,(-1)^{j-\ell}\,\epsilon((I\setminus\{j\})^c)\,e^{I\setminus\{j\}}.$$

Consequently,

$$\sigma_{d^\dagger}(x,\xi)(e^I) = (-1)^{np+1}\cdot\epsilon(I)\sum_{\ell=1}^{p+1}\epsilon((I\setminus\{i_\ell\})^c)\,(-1)^{i_\ell-\ell}\,\xi_{i_\ell}\,e^{I\setminus\{i_\ell\}}. \tag{47}$$

Similarly, for $I=(i_1,i_2,\dots,i_p)$ with $1\le i_1<i_2<\cdots<i_p\le n$ and $j\notin\{i_1,i_2,\dots,i_p\}$, let

$$N_I(j) := \#\{\,\nu\in\{1,2,\dots,p\}\mid i_\nu<j\,\},$$

so that inserting j into the strictly increasing list I produces the strictly increasing multi-index $\{j\}\cup I$ and

$$e^j\wedge e^I = (-1)^{N_I(j)}\,e^{\{j\}\cup I}.$$

Hence the formula $\sigma_d(x,\xi)=\xi\wedge$ gives

$$\sigma_d(x,\xi)(e^I) = \sum_{j\notin\{i_1,\dots,i_p\}}\xi_j\,(-1)^{N_I(j)}\,e^{\{j\}\cup I}. \tag{48}$$

We are ready to compute $\sigma_\Delta(x,\xi)=\sigma_d(x,\xi)\sigma_{d^\dagger}(x,\xi)+\sigma_{d^\dagger}(x,\xi)\sigma_d(x,\xi)$ on e^I. We begin with the term $\sigma_{d^\dagger}(x,\xi)\sigma_d(x,\xi)$. Let $I=(i_1,i_2,\dots,i_p)$ be a strictly increasing multi-index with $1\le i_1<\cdots<i_p\le n$.

In (48), for each $j\in\{1,2,\dots,n\}\setminus\{i_1,i_2,\dots,i_p\}$, let $\ell_0\in\{1,2,\dots,p+1\}$ be the unique integer such that j is the ℓ_0-th entry of the strictly increasing tuple $\{j\}\cup I$. Then $\ell_0=N_I(j)+1$. We apply (47) to the $(p+1)$-tuple $\{j\}\cup I$. In the summation in (47), the only term that can contribute to the coefficient of e^I is the term where one removes the entry j, i.e., the ℓ_0-th entry. Since $(\{j\}\cup I)\setminus\{j\}=I$ and $((\{j\}\cup I)\setminus\{j\})^c=I^c$, the coefficient of e^I in $\sigma_{d^\dagger}(x,\xi)\sigma_d(x,\xi)(e^I)$ equals

$$(-1)^{np+1}\sum_{j\notin\{i_1,\dots,i_p\}}\xi_j^2\,(-1)^{N_I(j)}\,\epsilon(\{j\}\cup I)\,\epsilon(I^c)\,(-1)^{j-\ell_0}.$$

Using $\ell_0=N_I(j)+1$, we rewrite the j-dependent sign as

$$(-1)^{N_I(j)}\,(-1)^{j-\ell_0} = (-1)^{j-1}.$$

We relate $\epsilon(\{j\}\cup I)$ and $\epsilon(I^c)$ to $\epsilon(I)$. First, by comparing the identities

$$e^I\wedge e^{I^c} = \epsilon(I)\,dV_{g,x} \qquad\text{and}\qquad e^{\{j\}\cup I}\wedge e^{(\{j\}\cup I)^c} = \epsilon(\{j\}\cup I)\,dV_{g,x},$$

and observing that $(\{j\}\cup I)^c = I^c\setminus\{j\}$ and that $e^{\{j\}\cup I}$ differs from $e^j\wedge e^I$ by the sign $(-1)^{N_I(j)}$, one checks that

$$\epsilon(\{j\}\cup I) = (-1)^{p+j-1}\,\epsilon(I) \qquad \big(j\notin\{i_1,i_2,\dots,i_p\}\big) \tag{49}$$

(Indeed, the same computation as above shows that the total exponent is $p+j-1$ modulo 2). Secondly, by applying the defining identity of $\epsilon(\,\cdot\,)$ to the complementary multi-index I^c and using $e^{I^c}\wedge e^I=(-1)^{p(n-p)}\,e^I\wedge e^{I^c}$, we obtain

$$\epsilon(I^c)=(-1)^{p(n-p)}\,\epsilon(I). \tag{50}$$

Combining (49) and (50), we compute

$$\epsilon(\{j\}\cup I)\,\epsilon(I^c)\,(-1)^{j-1}=(-1)^{p+j-1}\,\epsilon(I)\cdot(-1)^{p(n-p)}\,\epsilon(I)\cdot(-1)^{j-1}=(-1)^{p(n-p)+p},$$

because $\epsilon(I)^2=1$ and $(-1)^{p+j-1}(-1)^{j-1}=(-1)^p$.

Consequently, when we compute $\sigma_{d^\dagger}(x,\xi)\sigma_d(x,\xi)(e^I)$, the coefficient of e^I is

$$(-1)^{np+1}\,(-1)^{p(n-p)+p}\sum_{j\notin\{i_1,\dots,i_p\}}\xi_j^2=-\sum_{j\notin\{i_1,\dots,i_p\}}\xi_j^2.$$

Next we treat the term $\sigma_d(x,\xi)\sigma_{d^\dagger}(x,\xi)$. Here one must be careful about the degree: the operator $d^\dagger$ that appears in $dd^\dagger$ is

$$d^\dagger\colon\mathcal{A}^p(X)\longrightarrow\mathcal{A}^{p-1}(X),\qquad d^\dagger u:=(-1)^{(n-(p-1))p}\,*\,d\,*\,u,$$

so in applying (47) we have to replace p there by $(p-1)$. With this understood, let $I=(i_1,i_2,\dots,i_p)$ be a strictly increasing multi-index with $1\le i_1<\cdots<i_p\le n$ and write $e^I:=e^{i_1}\wedge\cdots\wedge e^{i_p}$. Applying (47) with p replaced by $p-1$ to the p-form e^I, we obtain

$$\sigma_{d^\dagger}(x,\xi)(e^I)=(-1)^{n(p-1)+1}\cdot\epsilon(I)\sum_{\ell=1}^{p}\epsilon\big((I\setminus\{i_\ell\})^c\big)\,(-1)^{i_\ell-\ell}\,\xi_{i_\ell}\,e^{I\setminus\{i_\ell\}}, \tag{51}$$

where $I\setminus\{i_\ell\}=(i_1,\dots,i_{\ell-1},i_{\ell+1},\dots,i_p)$.

We now apply $\sigma_d(x,\xi)=\xi\wedge$ to (51). For each ℓ, the term $e^{I\setminus\{i_\ell\}}$ is sent by $\xi\wedge$ to

$$\xi_{i_\ell}\,e^{i_\ell}\wedge e^{I\setminus\{i_\ell\}}\;+\sum_{j\notin\{i_1,\dots,i_p\}}\xi_j\,e^j\wedge e^{I\setminus\{i_\ell\}},$$

and only the first summand can contribute to the coefficient of e^I. Since

$$e^{i_\ell}\wedge e^{I\setminus\{i_\ell\}}=(-1)^{\ell-1}\,e^I,$$

the contribution of the ℓ-th summand in (51) to the coefficient of e^I in $\sigma_d(x,\xi)\sigma_{d^\dagger}(x,\xi)(e^I)$ equals

$$(-1)^{n(p-1)+1}\,\xi_{i_\ell}^2\,\epsilon(I)\,\epsilon\big((I\setminus\{i_\ell\})^c\big)\,(-1)^{i_\ell-\ell}\,(-1)^{\ell-1}=(-1)^{n(p-1)+1}\,\xi_{i_\ell}^2\,\epsilon(I)\,\epsilon\big((I\setminus\{i_\ell\})^c\big)\,(-1)^{i_\ell-1}.$$

Thus, to determine the diagonal coefficient it remains to relate $\epsilon\big((I\setminus\{i_\ell\})^c\big)$ to $\epsilon(I)$. Write $I^c=(k_1,k_2,\dots,k_{n-p})$ so that

$$\{i_1,\dots,i_p\}\sqcup\{k_1,\dots,k_{n-p}\}=\{1,2,\dots,n\},\qquad e^I\wedge e^{I^c}=\epsilon(I)\,dV_{g,x}.$$

Fix $\ell \in \{1, 2, \dots, p\}$ and put $J := I \setminus \{i_\ell\}$. Then J^c is obtained from I^c by inserting the element i_ℓ. Applying (49) with I replaced by I^c (whose length is $n-p$) and $j = i_\ell$, we obtain

$$\epsilon(J^c) = \epsilon\big((I \setminus \{i_\ell\})^c\big) = (-1)^{(n-p)+i_\ell-1}\,\epsilon(I^c). \tag{52}$$

On the other hand, by applying the defining identity of $\epsilon(\,\cdot\,)$ to the complementary multi-index I^c and using $e^{I^c} \wedge e^I = (-1)^{p(n-p)}\, e^I \wedge e^{I^c}$, we have

$$\epsilon(I^c) = (-1)^{p(n-p)}\,\epsilon(I). \tag{53}$$

Combining (52) and (53), we compute

$$\begin{aligned}\epsilon(I)\,\epsilon\big((I \setminus \{i_\ell\})^c\big)\,(-1)^{i_\ell-1} &= \epsilon(I)\cdot(-1)^{(n-p)+i_\ell-1}\,\epsilon(I^c)\cdot(-1)^{i_\ell-1}\\ &= (-1)^{n-p}\,\epsilon(I)\,\epsilon(I^c) = (-1)^{n-p}\,(-1)^{p(n-p)},\end{aligned}$$

which is independent of ℓ. Therefore the coefficient of e^I in $\sigma_d(x,\xi)\sigma_{d^\dagger}(x,\xi)(e^I)$ equals

$$(-1)^{n(p-1)+1}\,(-1)^{n-p+p(n-p)}\sum_{\ell=1}^{p}\xi_{i_\ell}^2 = -\sum_{\ell=1}^{p}\xi_{i_\ell}^2.$$

Combining this with the computation for $\sigma_{d^\dagger}(x,\xi)\sigma_d(x,\xi)(e^I)$ obtained above, we conclude that the coefficient of e^I in $\sigma_\Delta(x,\xi)(e^I)$ is

$$-\sum_{\ell=1}^{p}\xi_{i_\ell}^2 \;-\sum_{j\notin\{i_1,\dots,i_p\}}\xi_j^2 = -\sum_{j=1}^{n}\xi_j^2 = -|\xi|_g^2.$$

Moreover, all remaining terms in $\sigma_d\sigma_{d^\dagger}$ and $\sigma_{d^\dagger}\sigma_d$ are off-diagonal (they change the index set by removing one element and inserting a different one), and they occur with opposite signs in the two compositions, hence cancel in the sum. Consequently,

$$\sigma_\Delta(x,\xi)(e^I) = -|\xi|_g^2\, e^I.$$

Since the elements e^I (with I ranging over strictly increasing p-tuples) form a basis of $\bigwedge^p T_x^*X$, this proves that

$$\sigma_\Delta(x,\xi) = -|\xi|_g^2 \cdot \mathrm{id}_{\bigwedge^p T_x^*X}.$$

In particular, Δ is elliptic. □

We call an element of $\mathrm{Ker}\big(\Delta\colon \mathcal{A}^0(X,E) \to \mathcal{A}^0(X,E)\big)$ a *harmonic section* of E (or a *harmonic function* if E is the trivial line bundle), and an element of $\mathrm{Ker}\big(\Delta\colon \mathcal{A}^p(X,E) \to \mathcal{A}^p(X,E)\big)$ a *harmonic p-form* with values in E.

Lemma I.8.61. *In the setting fixed above, with the orientation of X, the metrics g, h, and the flat connection D, fix $p \in \{0,1,\dots,n\}$. Then for every $u \in \mathcal{A}_c^p(X,E)$, one has*

$$(\Delta u, u)_{g,h} = \|Du\|_{g,h}^2 + \|D^\dagger u\|_{g,h}^2.$$

In particular, for $u \in \mathcal{A}_c^p(X,E)$, u is harmonic if and only if $Du = 0$ and $D^\dagger u = 0$.

PROOF. By definition $\Delta = DD^\dagger + D^\dagger D$, hence

$$(\Delta u, u)_{g,h} = (DD^\dagger u, u)_{g,h} + (D^\dagger D u, u)_{g,h}.$$

Since u has compact support, integration by parts yields the formal adjointness

$$(Dv, w)_{g,h} = (v, D^\dagger w)_{g,h} \qquad \big(v \in \mathcal{A}_c^{p-1}(X,E),\ w \in \mathcal{A}_c^p(X,E)\big),$$

and similarly

$$(D^\dagger v, w)_{g,h} = (v, Dw)_{g,h} \qquad \big(v \in \mathcal{A}_c^{p+1}(X,E),\ w \in \mathcal{A}_c^p(X,E)\big).$$

Applying these identities with $v = D^\dagger u$ and $w = u$ for the first term, and with $v = Du$ and $w = u$ for the second term, we obtain

$$(DD^\dagger u, u)_{g,h} = (D^\dagger u, D^\dagger u)_{g,h} = \|D^\dagger u\|^2_{g,h}, \qquad (D^\dagger D u, u)_{g,h} = (Du, Du)_{g,h} = \|Du\|^2_{g,h},$$

which proves the desired identity. The last assertion follows immediately, since the right-hand side is a sum of two non-negative terms. □

Hodge theory on compact manifolds

From now on, in addition to the setting fixed above, we assume that X is compact and orientable. In the previous subsections we have already proved the finiteness of Betti numbers and Poincaré duality for X. Our proof there was based on the existence of a finite good covering in the sense of [BT] together with the Mayer–Vietoris sequence and an induction on the number of open sets. There is also a different approach based on Morse theory: for a C^∞ function $f\colon X \to \mathbb{R}$ with only non-degenerate critical points, one obtains a handle decomposition of X from f, and comparing the handle decompositions arising from f and $-f$ leads to Poincaré duality (Recall that we have already discussed the finiteness of Betti numbers in this note from this Morse-theoretic viewpoint; we will not pursue the further details of the Morse-theoretic proof of Poincaré duality here). Here let us present yet another method, based on *harmonic integral theory*. If $n = 0$, the compact manifold X is a finite set, $D = \Delta = 0$, and the cohomology and harmonic spaces are simply the spaces of sections in degree 0 and zero in other degrees; the decomposition and duality follow directly. For the elliptic argument below we therefore assume $n \geq 1$. The starting point is that the Hodge Laplacian Δ is an elliptic operator (Lemma I.8.60) and is self-adjoint with respect to $(\bullet, \bullet)_{g,h}$. Therefore we may apply the general elliptic theory.

Fix $p \in \{0, 1, \ldots, n\}$. By Theorem I.8.32 (see also Remark I.8.35) applied to $P = \Delta\colon \mathcal{A}^p(X,E) \to \mathcal{A}^p(X,E)$, we obtain an orthogonal direct sum decomposition in the inner product $(\bullet, \bullet)_{g,h}$:

$$\mathcal{A}^p(X,E) = \operatorname{Ker}\big(\Delta\colon \mathcal{A}^p(X,E) \to \mathcal{A}^p(X,E)\big) \ \oplus\ \operatorname{Image}\big(\Delta\colon \mathcal{A}^p(X,E) \to \mathcal{A}^p(X,E)\big).$$

One has also a sharper decomposition in terms of D and $D^\dagger$. Indeed, it follows from Lemma I.8.61 that

$$\operatorname{Ker}\Delta = \operatorname{Ker} D \cap \operatorname{Ker} D^\dagger \qquad \text{on } \mathcal{A}^p(X,E).$$

Moreover, by orthogonality relations

$$\operatorname{Image} D \ \perp\ \operatorname{Image} D^\dagger, \qquad \operatorname{Ker}\Delta \ \perp\ \operatorname{Image} D, \qquad \operatorname{Ker}\Delta \ \perp\ \operatorname{Image} D^\dagger$$

(with respect to $(-,-)_{g,h}$), one obtains the (Hodge) orthogonal decomposition

$$\mathcal{A}^p(X,E) = \operatorname{Ker}\Delta \ \oplus\ \operatorname{Image} D \ \oplus\ \operatorname{Image} D^\dagger$$

and

$$\operatorname{Ker}(D\colon \mathcal{A}^p(X,E) \to \mathcal{A}^{p+1}(X,E)) = \operatorname{Ker}\Delta \ \oplus\ \operatorname{Image} D$$

as an orthogonal direct sum in the inner product $(\bullet,\bullet)_{g,h}$. Therefore we have

$$H^p_{\mathrm{DR}}(X,E) \cong \mathbb{H}^p(X,E) := \{u \in \mathcal{A}^p(X,E) \mid \Delta u = 0\},$$

which is finite-dimensional by virtue of Theorem I.8.31 and the ellipticity of Δ.

We next explain how Poincaré duality can be seen naturally from harmonic theory. By definition of $D^\dagger$ and the identity $\Delta = DD^\dagger + D^\dagger D$, one checks that $*$ commutes with the Hodge Laplacian in the following sense: for every $u \in \mathcal{A}^p(X,E)$ one has

$$\begin{aligned}
\Delta(*u) &= DD^\dagger(*u) + D^\dagger D(*u) \\
&= D\Big((-1)^{(p-1)(n-p)} * D * (*u)\Big) \ +\ (-1)^{(p-2)(n-p+1)} * D * \big(D(*u)\big) \\
&= (-1)^{(p-1)(n-p)} D * D * *u \ +\ (-1)^{(p-2)(n-p+1)} * D * D * u \\
&= (-1)^{(p-1)(n-p)} * * D * Du \ +\ (-1)^{(p-2)(n-p+1)} * D * D * u \\
&= (-1)^{(p-1)(n-p)+(n-p)(p+1)} * D^\dagger Du \ +\ (-1)^{(p-2)(n-p+1)+(n-p-1)p} * DD^\dagger u \\
&= *D^\dagger Du \ +\ *DD^\dagger u = *(\Delta u).
\end{aligned}$$

Consequently, $*$ preserves harmonicity. Since $*\circ *$ is a pointwise scalar multiplication by ± 1 on $\mathcal{A}^p(X,E)$, the restriction

$$*\colon \mathbb{H}^p(X,E) \longrightarrow \mathbb{H}^{n-p}(X,E)$$

is a linear bijection. In particular, we obtain a canonical identification

$$\mathbb{H}^p(X,E) \cong \mathbb{H}^{n-p}(X,E), \qquad u \longmapsto *u.$$

To relate this to Poincaré duality in our formulation, we also note that the Hermitian metric h on E induces a fiberwise conjugate-linear identification between E and its dual bundle E^*. Extending this pointwise identification to E-valued differential forms yields a conjugate-linear bijection

$$\mathbb{H}^p(X,E) \ \cong\ \mathbb{H}^p(X,E^*).$$

Combining these identifications with the Hodge isomorphisms

$$H^p_{\mathrm{DR}}(X,E) \cong \mathbb{H}^p(X,E), \qquad H^{n-p}_{\mathrm{DR},c}(X,E^*) \cong \mathbb{H}^{n-p}(X,E^*),$$

we may represent every cohomology class by a unique harmonic form, and the Hodge star identifies harmonic representatives in complementary degrees (after passing from E to E^* via the Hermitian pairing), which yields Poincaré duality in the present setting.

Remarks on non-compact manifolds

We keep the setting of the previous discussion: Let (X, g) be an oriented C^∞ Riemannian manifold of real dimension n, $E \to X$ be a $\mathrm{U}(r)$-flat bundle, and let h be the flat metric on E. Unlike §8.9, we do *not* assume that X is compact in what follows, and we briefly discuss which parts of the Hodge theory remain valid (and what additional issues arise) on the Hilbert space $L^q_{(2)}(X, E; g, h)$ in the non-compact case (see also Remark I.8.24).

For the flat connection

$$D\colon \mathcal{A}^p(X, E) \to \mathcal{A}^{p+1}(X, E) \qquad (p \in \mathbb{Z}_{\geq 0}),$$

up to this point, for $u \in \mathcal{A}^{p+1}(X, E)$ we have written $D^\dagger u := (-1)^{(n-p)(p+1)} * D * u$. Strictly speaking, this operator is the *formal adjoint* of D: It behaves as an adjoint of D only at the level of compactly supported smooth forms, in the sense that the usual integration-by-parts identity holds with no boundary term. On the other hand, once we regard D as a densely defined operator on the Hilbert spaces $L^p_{(2)}(X, E; g, h)$, there is another notion of adjoint, namely the *von Neumann adjoint* of D defined by the Riesz representation theorem (Definition I.6.17). Thus, the symbol $D^\dagger$ is potentially ambiguous (Note that, in the compact case, these two adjoints agree on the natural domains; this is the reason why we intentionally used the same symbol † up to this point). In the non-compact case, however, they need not coincide unless one imposes extra conditions. For this reason, from now on we shall write $D^\dagger_{\mathrm{formal}}$ for the operator defined by the Hodge star: for $u \in \mathcal{A}^{p+1}(X, E)$,

$$D^\dagger_{\mathrm{formal}} u := (-1)^{(n-p)(p+1)} * D * u.$$

Example I.8.62. Here we describe a concrete warning based on [De4, Chapter VIII, (3.1)]. Let $X = (0, 1) \subset \mathbb{R}$ with the Euclidean metric g, and let $E = X \times \mathbb{C}$ with the flat metric $h \equiv 1$. We identify $L^0_{(2)}(X, E; g, h)$ with the usual Hilbert space $L^2((0, 1))$ with respect to the Lebesgue measure, and identify $L^1_{(2)}(X, E; g, h)$ with $L^2((0, 1))\, dx$, where x denotes the Euclidean coordinate on $\mathbb{R}$. Consider the exterior derivative

$$d\colon \mathcal{A}^0(X, E) \to \mathcal{A}^1(X, E).$$

Under these identifications, we may write

$$d\colon C^\infty((0, 1)) \ni f \longmapsto f'(x)\, dx \in C^\infty((0, 1))\, dx,$$

and the formal adjoint $d^\dagger_{\mathrm{formal}}$ is given by

$$d^\dagger_{\mathrm{formal}}\big(v(x)\, dx\big) = -v'(x) \qquad \big(v \in C^\infty((0, 1))\big).$$

By construction, for $f, v \in C^\infty_c((0, 1)) = \mathcal{A}^0_c(X, E)$, the identity

$$(df,\, v\, dx)_{L^1_{(2)}(X,E;g,h)} = (f,\, d^\dagger_{\mathrm{formal}}(v\, dx))_{L^0_{(2)}(X,E;g,h)} \tag{54}$$

holds. However, once one tries to interpret d and $d^\dagger_{\mathrm{formal}}$ as densely defined operators

$$d\colon L^0_{(2)}(X, E; g, h) \xrightarrow{\mathrm{op}} L^1_{(2)}(X, E; g, h), \qquad d^\dagger_{\mathrm{formal}}\colon L^1_{(2)}(X, E; g, h) \xrightarrow{\mathrm{op}} L^0_{(2)}(X, E; g, h),$$

the adjointness relation (54) need not persist for larger domains. Indeed, suppose that we choose operator domains so that

$$C^\infty(\mathbb{R})\big|_{(0,1)} \subset \operatorname{Dom} d \qquad \text{and} \qquad C^\infty(\mathbb{R})\big|_{(0,1)}\, dx \subset \operatorname{Dom} d^\dagger_{\text{formal}}.$$

Then for $f, v \in C^\infty(\mathbb{R})\big|_{(0,1)}$ we have

$$\int_0^1 f'(x)\,\overline{v(x)}\,dx = -\int_0^1 f(x)\,\overline{v'(x)}\,dx + \Big(f(1)\,\overline{v(1)} - f(0)\,\overline{v(0)}\Big),$$

so, for instance, when $f(x) = v(x) = x$,

$$(df,\, v\,dx)_{L^1_{(2)}(X,E;g,h)} \neq (f,\, d^\dagger_{\text{formal}}(v\,dx))_{L^0_{(2)}(X,E;g,h)}.$$

In other words, in the non-compact case, the formal integration-by-parts computation does not by itself determine the Hilbert space adjoint: boundary terms may reappear as soon as one enlarges the class of test forms. □

Example I.8.62 shows that, in order to make the formal adjoint $D^\dagger_{\text{formal}}$ behave as the von Neumann adjoint of D on $L^p_{(2)}(X, E; g, h)$, one must impose extra conditions that allow one to control boundary behaviour "at infinity". As we will see in Theorem I.8.63, a convenient hypothesis for this is the completeness of (X, g).

THEOREM I.8.63 ([De4, Chapter VIII, (3.2)]). *Let (X, g) be an oriented C^∞ Riemannian manifold of real dimension n, let $E \to X$ be a $\mathrm{U}(r)$-flat bundle, and let h be the flat metric on E. For each $p \in \{0, 1, \dots, n\}$, let $D\colon \mathcal{A}^p(X, E) \to \mathcal{A}^{p+1}(X, E)$ be the flat connection. Assume that (X, g) is complete (i.e., $(X, \operatorname{dist}_g)$ is a complete metric space). Define an operator*

$$D\colon L^p_{(2)}(X, E; g, h) \xrightarrow{\text{op}} L^{p+1}_{(2)}(X, E; g, h)$$

by setting

$$\operatorname{Dom} D := \big\{u \in L^p_{(2)}(X, E; g, h) \mid Du \in L^{p+1}_{(2)}(X, E; g, h)\big\},$$

where Du is understood in the sense of distributions, and by defining $Du := Du$ on this domain. Similarly, define

$$D^\dagger_{\text{formal}}\colon L^{p+1}_{(2)}(X, E; g, h) \xrightarrow{\text{op}} L^p_{(2)}(X, E; g, h)$$

by setting

$$\operatorname{Dom} D^\dagger_{\text{formal}} := \big\{v \in L^{p+1}_{(2)}(X, E; g, h) \mid D^\dagger_{\text{formal}} v \in L^p_{(2)}(X, E; g, h)\big\},$$

where $D^\dagger_{\text{formal}} v$ is understood in the sense of distributions. Then the following hold for $p \in \{0, 1, \dots, n\}$.
(i) $\mathcal{A}^p_c(X, E)$ is dense in

$$\operatorname{Dom}\big(D\colon L^p_{(2)}(X, E; g, h) \xrightarrow{\text{op}} L^{p+1}_{(2)}(X, E; g, h)\big)$$

for the graph norm $u \mapsto \|u\|_{L^p_{(2)}(X,E;g,h)} + \|Du\|_{L^{p+1}_{(2)}(X,E;g,h)}$.
(ii) $\mathcal{A}^p_c(X,E)$ *is dense in*

$$\mathrm{Dom}(D^\dagger_{\mathrm{formal}}\colon L^p_{(2)}(X,E;g,h) \xrightarrow{\mathrm{op}} L^{p-1}_{(2)}(X,E;g,h))$$

for the graph norm $v \mapsto \|v\|_{L^p_{(2)}(X,E;g,h)} + \|D^\dagger_{\mathrm{formal}} v\|_{L^{p-1}_{(2)}(X,E;g,h)}$.
(iii) $\mathcal{A}^p_c(X,E)$ *is dense in*

$$\mathrm{Dom}(D\colon L^p_{(2)}(X,E;g,h) \xrightarrow{\mathrm{op}} L^{p+1}_{(2)}(X,E;g,h)) \cap \mathrm{Dom}(D^\dagger_{\mathrm{formal}}\colon L^p_{(2)}(X,E;g,h) \xrightarrow{\mathrm{op}} L^{p-1}_{(2)}(X,E;g,h))$$

for the norm

$$u \longmapsto \|u\|_{L^p_{(2)}(X,E;g,h)} + \|Du\|_{L^{p+1}_{(2)}(X,E;g,h)} + \|D^\dagger_{\mathrm{formal}} u\|_{L^{p-1}_{(2)}(X,E;g,h)}.$$

(iv) $D^\dagger_{\mathrm{formal}}$ *is the von Neumann adjoint of* D*, and* D *is the von Neumann adjoint of* $D^\dagger_{\mathrm{formal}}$. *In particular, for all* $u \in \mathrm{Dom}\, D$ *and* $v \in \mathrm{Dom}\, D^\dagger_{\mathrm{formal}}$,

$$(Du,\, v)_{L^{p+1}_{(2)}(X,E;g,h)} = (u,\, D^\dagger_{\mathrm{formal}} v)_{L^p_{(2)}(X,E;g,h)}.$$

(v) Let $u \in L^p_{(2)}(X,E;g,h)$*. If* $\Delta u \in L^p_{(2)}(X,E;g,h)$*, then* $u \in \mathrm{Dom}\, D \cap \mathrm{Dom}\, D^\dagger_{\mathrm{formal}}$ *and*

$$(u,\, \Delta u)_{L^p_{(2)}(X,E;g,h)} = \|Du\|^2_{L^{p+1}_{(2)}(X,E;g,h)} + \|D^\dagger_{\mathrm{formal}} u\|^2_{L^{p-1}_{(2)}(X,E;g,h)}.$$

Proof. We refer to [De4, Chapter VIII, (3.2)] for the proof. Roughly speaking, the key point is that one needs approximation of L^2-forms by smooth forms not only in L^2, but also in the graph norms appearing in *(i)*–*(iii)*, hence one must approximate while keeping track of the L^2-norms of D and $D^\dagger_{\mathrm{formal}}$ as well. A smoothing argument (based on convolution in local charts) handles regularization, but the essential extra step in the non-compact case is a cut-off procedure producing compact support while controlling derivatives: This is where the completeness of the metric space (X, dist_g) is used. Arguing on each connected component separately, we may assume that X is connected. Fix a point $x_0 \in X$ and consider the function

$$\psi_0(x) := \frac{1}{2}\,\mathrm{dist}_g(x_0, x).$$

By completeness, ψ_0 is an exhaustion function (i.e., $\{x \in X \mid \psi_0(x) \leq c\}$ is compact for every $c \in \mathbb{R}$). By considering the smoothing of this function, one can then construct a smooth exhaustion function $\psi\colon X \to \mathbb{R}$ with a uniform bound on its differential, namely $|d\psi|_g \leq 1$ everywhere (see [De4, Chapter VIII, (2.4)]). Such a function allows one to choose cut-off functions with uniformly controlled derivatives, which in turn makes it possible to approximate in the above graph norms. □

Remark I.8.64. Even when (X, g) is complete and Theorem I.8.63 holds, general Hilbert space theory alone does *not* guarantee that $\operatorname{Ran} D$ or $\operatorname{Ran} D^{\dagger}_{\text{formal}}$ is closed. Therefore, for L^2-Hodge-type decompositions on non-compact manifolds, one cannot in general remove closures from orthogonal decompositions without additional analytic input. For further discussion and refined statements, we refer to [De4, Chapter VIII, (3.2)]. □

We conclude this subsection by recording two fundamental results on harmonic forms that will play an important role in several applications in complex analytic geometry later in this note.

Theorem I.8.65 (Aronszajn's unique continuation). *Let X be a connected and oriented C^∞ manifold, and let $E \to X$ be a C^∞ vector bundle. Fix $p \in \{0, 1, \dots, \dim_{\mathbb{R}} X\}$. Let $P\colon \mathcal{A}^p(X, E) \to \mathcal{A}^p(X, E)$ be a linear partial differential operator of order 2. Assume that its principal symbol has the form*

$$\sigma_P(x, \xi) = -a_x(\xi, \xi)\mathbf{I},$$

where a_x is a real positive definite quadratic form on $T^*_x X$ depending smoothly on x, $\xi \in T^*_x X$ is real, and $\mathbf{I}$ is the identity on $\bigwedge^p T^*_x X \otimes E_x$. Terms of order at most one may couple the components. Let $u \in \mathcal{A}^p(X, E)$ satisfy $Pu \equiv 0$ on X. If $u \equiv 0$ on some non-empty open subset of X, then $u \equiv 0$ on X. In particular, assume that $E \to X$ is $\mathrm{U}(r)$-flat, let h be the flat metric on E, and fix a C^∞ Riemannian metric g on X. If $u, v \in \mathcal{A}^p(X, E)$ are harmonic forms, then $u = v$ on a non-empty open subset of X implies $u = v$ on X.

Proof. We use the system version of Aronszajn's unique continuation theorem [Ar, pp. 235–236 and Remark 3, p. 248], also recalled in [Sa, §2, p. 212]. In local coordinates and a local frame, all components have the same real scalar elliptic principal part. Since the remaining coefficients are smooth, its value on each component of a solution is locally bounded by a constant times the sum of the absolute values of all components and their first derivatives. These are precisely the inequalities required for that system form (Apply it to real and imaginary parts).

Let $u \in \mathcal{A}^p(X, E)$ satisfy $Pu \equiv 0$ on X, and assume that $u \equiv 0$ on some non-empty open subset of X. Put

$$U := \{\, x \in X \mid u \equiv 0 \text{ on some open neighborhood of } x \,\}.$$

Then U is an open subset of X by definition, and $U \neq \emptyset$ by hypothesis. We claim that U is also closed. Let $x \in \overline{U}$. Choose a coordinate open ball $V \subset X$ centered at x, and fix a local trivialization of E over V. Since $x \in \overline{U}$, we have $V \cap U \neq \emptyset$. Then it follows from Aronszajn's theorem that $u|_V \equiv 0$. Therefore $x \in U$, which implies that U is closed.

Since X is connected and $U \subset X$ is non-empty, open and closed, we conclude that $U = X$. Hence $u \equiv 0$ on X. For the last assertion, put $w := u - v \in \mathcal{A}^p(X, E)$. Then $\Delta w = 0$ on X. The principal-symbol calculation for the Hodge Laplacian (Lemma I.8.60) gives $\sigma_\Delta(x, \xi) = -|\xi|^2_g \mathbf{I}$, so $P := \Delta$ satisfies the required scalar-principal-part hypothesis. If $u = v$ on a non-empty open subset, then $w \equiv 0$ there, hence $w \equiv 0$ on X by the first part. Thus $u = v$ on X. □

THEOREM I.8.66 (Greene–Wu's theorem). *Let X be a connected non-compact C^∞ manifold of real dimension $n \geq 2$, and let g be a C^∞ Riemannian metric on X. Then there exist $2n+1$ real-valued harmonic functions $h_1, h_2, \ldots, h_{2n+1} \in C^\infty(X)$ (i.e., each h_j belongs to $\operatorname{Ker}\Delta$ on $\mathcal{A}^0(X,E)$, where $E := X \times \mathbb{C}$ is the trivial bundle endowed with the flat metric $h \equiv 1$) such that the map*

$$X \ni x \mapsto \big(h_1(x), h_2(x), \ldots, h_{2n+1}(x)\big) \in \mathbb{R}^{2n+1}$$

is a proper C^∞ embedding.

PROOF. Refer to [GW, Theorem, p. 231]. □

8.10 Foliation

Let $n, m \in \mathbb{Z}_{\geq 0}$ with $m \leq n$, and let X be a C^∞ manifold of real dimension n. A C^∞ *foliation* $\mathcal{F}$ on X (of real dimension m) means a family $\{\mathcal{L}_\alpha\}_{\alpha \in A}$ of connected C^∞ immersed submanifolds of X such that $X = \coprod_{\alpha \in A} \mathcal{L}_\alpha$ and the following condition holds: There exist an open covering $\{U_j\}_{j \in J}$ of X and, for each $j \in J$, a local coordinate system $(x_j^1, x_j^2, \ldots, x_j^m,\ y_j^1, y_j^2, \ldots, y_j^{n-m})$ on U_j such that, for every $j, k \in J$ with $U_j \cap U_k \neq \emptyset$, the coordinate change on $U_j \cap U_k$ can be written as

$$\begin{cases} x_k^\mu = \Phi_{kj}^{(\mu)}(x_j^1, x_j^2, \ldots, x_j^m,\ y_j^1, y_j^2, \ldots, y_j^{n-m}) & (\mu \in \{1, 2, \ldots, m\}), \\ y_k^\nu = \Psi_{kj}^{(\nu)}(x_j^1, x_j^2, \ldots, x_j^m,\ y_j^1, y_j^2, \ldots, y_j^{n-m}) & (\nu \in \{1, 2, \ldots, n-m\}), \end{cases}$$

and moreover:

- For every $j, k \in J$, the functions $\Psi_{kj}^{(\nu)}$ depend only on the y-coordinates, i.e., for each $\nu \in \{1, 2, \ldots, n-m\}$ one can write

$$y_k^\nu = \Psi_{kj}^{(\nu)}(y_j^1, y_j^2, \ldots, y_j^{n-m}) \qquad \text{on } U_j \cap U_k. \tag{55}$$

- For every $j \in J$ and every $t = (t^1, t^2, \ldots, t^{n-m}) \in \mathbb{R}^{n-m}$, each connected component of the subset

$$\{\, x \in U_j \mid y_j^\nu(x) = t^\nu \ (\nu \in \{1, 2, \ldots, n-m\}) \,\}$$

  is an open subset of some $\mathcal{L}_\alpha$ (called a *plaque*), and $(x_j^1, x_j^2, \ldots, x_j^m)$ gives a C^∞ local coordinate system on that open subset of $\mathcal{L}_\alpha$.

Each $\mathcal{L}_\alpha$ is called a *leaf* of $\mathcal{F}$. We often write $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha \in A}$ or $\mathcal{L}_\alpha \in \mathcal{F}$. A chart $\{(U_j, (x_j^1, x_j^2, \ldots, x_j^m, y_j^1, y_j^2, \ldots, y_j^{n-m}))\}_{j \in J}$ satisfying the condition above is called a *foliation chart*. The coordinates $(y_j^1, y_j^2, \ldots, y_j^{n-m})$ are called *transversal coordinates*. Once a system of local functions $\{(U_j, (y_j^1, y_j^2, \ldots, y_j^{n-m}))\}$ which satisfies the transition rule (55) is given, one may say that $\mathcal{F}$ is the foliation locally defined by the equations "$y_j^\bullet = (\text{constant})$". The *real codimension* of $\mathcal{F}$ is defined to be $n - m$.

Note that a leaf $\mathcal{L}_\alpha$ of a foliation need not be a submanifold of X in our sense (Recall that in this note we use the term "submanifold" in the embedded sense); in general, it is only an immersed submanifold.

Associated with a C^∞ foliation $\mathcal{F}$ on X, one has a natural C^∞ real subbundle (a *distribution*) $T\mathcal{F} \subset TX$, called the *tangent bundle* of $\mathcal{F}$. By definition, for each $x \in X$, the fiber $(T\mathcal{F})_x \subset T_xX$ is the tangent space at x to the leaf of $\mathcal{F}$ passing through x. In terms of a foliation chart $\{(U_j, (x_j^1, \dots, x_j^m, y_j^1, \dots, y_j^{n-m}))\}_{j\in J}$, the restriction $T\mathcal{F}|_{U_j}$ is the real subbundle of TU_j which admits the local frame

$$\frac{\partial}{\partial x_j^1}, \ \frac{\partial}{\partial x_j^2}, \ \cdots, \ \frac{\partial}{\partial x_j^m}.$$

In particular, $T\mathcal{F}$ is an *involutive distribution*: for every open set $U \subset X$, the Lie bracket of any two C^∞ sections of $T\mathcal{F}|_U$ is again a C^∞ section of $T\mathcal{F}|_U$. Conversely, given a C^∞ involutive distribution $\mathcal{D} \subset TX$ of constant real rank m, it is known that there exists a C^∞ foliation $\mathcal{F}$ on X such that $T\mathcal{F} = \mathcal{D}$ (the *Frobenius theorem*, see [Lee-Jo2, Theorem 19.12, Theorem 19.21]). In view of this correspondence, some references use the term "foliation" to mean an involutive distribution of constant rank.

Let $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha\in A}$ be a C^∞ foliation on X of real dimension m. Let $Y \in \mathcal{F}$ be a *compact leaf*, meaning that $Y = \mathcal{L}_\alpha$ for some $\alpha \in A$ and that Y is compact. Note that Y is an embedded C^∞ submanifold of X. For such a leaf Y (or more generally, for a leaf which is an embedded submanifold of X), we will often study the dynamical behavior of $\mathcal{F}$ near Y via the *holonomy of $\mathcal{F}$ along Y*, a group homomorphism

$$\mathrm{Hol}_{\mathcal{F},Y} \colon \pi_1(Y, *)^{\mathrm{op}} \longrightarrow \mathrm{Diff}^\infty(\mathbb{R}^{n-m}, 0),$$

where $\mathrm{Diff}^\infty(\mathbb{R}^{n-m}, 0)$ denotes the group of *germs* at 0 of C^∞ local diffeomorphisms fixing 0:

$$\begin{aligned}&\mathrm{Diff}^\infty(\mathbb{R}^{n-m}, 0)\\ &:= \{(U, \varphi) \mid U \subset \mathbb{R}^{n-m} \colon \text{open}, 0 \in U, \ \varphi \colon U \to \mathbb{R}^{n-m} \colon C^\infty, \ \varphi(0) = 0, \ (D\varphi)(0) \colon \text{invertible}\}/\sim .\end{aligned}$$

Here $D\varphi$ denotes the Jacobian matrix, and the relation "$\sim$" is defined by $(U, \varphi) \sim (V, \psi)$ if and only if there exists a neighborhood W of 0 with $W \subset U \cap V$ such that $\varphi|_W = \psi|_W$. Note that, by the inverse function theorem, every element of $\mathrm{Diff}^\infty(\mathbb{R}^{n-m}, 0)$ is represented by a local diffeomorphism near 0, and the composition of local maps induces a well-defined group structure on $\mathrm{Diff}^\infty(\mathbb{R}^{n-m}, 0)$.

We now recall a concrete construction of $\mathrm{Hol}_{\mathcal{F},Y}$ using a foliation atlas $\{(U_j, (x_j^1, x_j^2, \dots, x_j^m, y_j^1, y_j^2, \dots, y_j^{n-m}))\}_{j\in J}$ as in the definition of foliations. Fix a base point $* \in Y$, and choose $j_0 \in J$ such that $* \in U_{j_0}$. By taking a refinement of $\{U_j\}_{j\in J}$ and by translating the transversal coordinates if necessary, we may (and shall) assume that

$$Y \cap U_{j_0} \subset \{y_{j_0}^1 = y_{j_0}^2 = \cdots = y_{j_0}^{n-m} = 0\}.$$

Let $\gamma \colon [0, 1] \to Y$ be a loop based at $*$. Choose a subdivision $0 = t_0 < t_1 < \cdots < t_N = 1$ and indices $j_0, j_1, \dots, j_N \in J$ with $j_N = j_0$ such that

$$\gamma([t_\nu, t_{\nu+1}]) \subset U_{j_\nu} \qquad (\nu \in \{0, 1, \dots, N-1\}).$$

For each ν, after translating the transversal coordinates on U_{j_ν} if necessary, we may also assume that

$$Y \cap U_{j_\nu} \subset \{y^1_{j_\nu} = y^2_{j_\nu} = \cdots = y^{n-m}_{j_\nu} = 0\},$$

so that the transversal coordinate change maps

$$\Psi_{j_{\nu+1}j_\nu} := \left(\Psi^1_{j_{\nu+1}j_\nu},\ \Psi^2_{j_{\nu+1}j_\nu},\ \ldots,\ \Psi^{n-m}_{j_{\nu+1}j_\nu}\right)$$

satisfy $\Psi_{j_{\nu+1}j_\nu}(0) = 0$ on $U_{j_\nu} \cap U_{j_{\nu+1}}$. We then define

$$\mathrm{Hol}_{\mathcal{F},Y}([\gamma]) := \Psi_{j_0 j_{N-1}} \circ \Psi_{j_{N-1}j_{N-2}} \circ \cdots \circ \Psi_{j_2 j_1} \circ \Psi_{j_1 j_0}. \tag{56}$$

Roughly speaking, $\mathrm{Hol}_{\mathcal{F},Y}([\gamma])$ measures how transversal coordinates are changed after one travels along γ and returns to the base point: equivalently, it tracks nearby plaques along the same path γ. The inverse path $\mathrm{Inv}_{\mathrm{path}}(\gamma)$ gives the inverse germ. Since our product traverses the first path first, the displayed composition satisfies $\mathrm{Hol}_{\mathcal{F},Y}([\gamma_1][\gamma_2]) = \mathrm{Hol}_{\mathcal{F},Y}([\gamma_2]) \circ \mathrm{Hol}_{\mathcal{F},Y}([\gamma_1])$. This explains the opposite group in its domain, exactly as for monodromy in §7.6. If one changes the initial choice of transversal coordinates near $*$ (equivalently, the choice of j_0 and the normalization $y^\bullet_{j_0} = 0$ along $Y \cap U_{j_0}$), the resulting representation is conjugated in $\mathrm{Diff}^\infty(\mathbb{R}^{n-m}, 0)$. On the other hand, once the transversal coordinates near $*$ are fixed, the above construction of $\mathrm{Hol}_{\mathcal{F},Y}([\gamma])$ is independent of the choice of subdivision and the indices $j_1, j_2, \ldots, j_{N-1}$, and depends only on the homotopy class $[\gamma] \in \pi_1(Y, *)$ (see [CC, Proposition 2.3.2]). In this sense, $\mathrm{Hol}_{\mathcal{F},Y} \colon \pi_1(Y,*)^{\mathrm{op}} \to \mathrm{Diff}^\infty(\mathbb{R}^{n-m}, 0)$ is well-defined once the transversal coordinates near $*$ are fixed, and is canonical up to conjugation when only $(X, \mathcal{F}, Y)$ are fixed.

Example I.8.67. Let $n, m \in \mathbb{Z}_{\geq 0}$ with $m \leq n$, X a C^∞ manifold of real dimension n, Z a C^∞ manifold of real dimension $n-m$, and let $f \colon X \to Z$ be a C^∞ submersion. For each $z \in Z$, the fiber $f^{-1}(z)$ is a C^∞ submanifold of X of real dimension m, and we obtain a C^∞ foliation $\mathcal{F}$ on X by declaring that the leaves of $\mathcal{F}$ are the connected components of the fibers of f. In particular, every leaf of $\mathcal{F}$ is a C^∞ submanifold of X in this example.

By the local normal form of a submersion (Mini-Exercise I.8.1 (i)), local coordinates of Z provide transversal coordinates for each leaf of this foliation. In this situation the holonomy along any leaf is trivial: Indeed, if Y is a leaf of $\mathcal{F}$ and $* \in Y$ is a base point, then for a loop $\gamma \colon [0,1] \to Y$ based at $*$ we have $f \circ \gamma \equiv f(*)$. Using the foliation charts coming from f, the associated changes of transversal coordinates are induced solely from coordinate changes on Z, and hence the germ obtained by transporting transversal coordinates along γ is the identity element of $\mathrm{Diff}^\infty(\mathbb{R}^{n-m}, 0)$. Therefore $\mathrm{Hol}_{\mathcal{F},Y}$ is the trivial homomorphism. From a dynamical point of view, foliations arising from submersions are among the simplest examples. □

Example I.8.68. Let $n, m \in \mathbb{Z}_{\geq 0}$ with $n \geq m$. Let X be a C^∞ manifold of real dimension m, and let F be a C^∞ manifold of real dimension $n-m$. Let G be a subgroup of the group of all C^∞ diffeomorphisms of F. Let $\pi \colon E \to X$ be a G-flat fiber bundle with fiber F. Then we have an open covering $\{U_j\}_{j \in J}$ of X and a system of local trivializations $\{\Phi_{U_j} \colon \pi^{-1}(U_j) \to$

$U_j \times F\}_{j\in J}$ such that each transition can be written by using G-valued locally constant maps. Since the changes of trivializations are C^∞ diffeomorphisms, the bundle charts endow the total space E with the structure of a C^∞ manifold of real dimension n.

The total space E carries two natural C^∞ foliations which are *transversal* to each other in the sense that, whenever two plaques (one from each foliation) meet, they meet transversally. One of them is induced by the projection map: since $\pi\colon E \to X$ is a C^∞ submersion, it defines a C^∞ foliation of real dimension $n-m$ as in Example I.8.67. In what follows, we are mainly interested in the other foliation $\mathcal{F}$ of real dimension m.

Fix $j \in J$. For each $\xi \in F$, consider the subset

$$P_{j,\xi} := \Phi_{U_j}^{-1}(U_j \times \{\xi\}) \subset \pi^{-1}(U_j).$$

Then $P_{j,\xi}$ is a C^∞ embedded submanifold of $\pi^{-1}(U_j)$, and $\pi|_{P_{j,\xi}}\colon P_{j,\xi} \to U_j$ is a C^∞ diffeomorphism. If $U_j \cap U_k \neq \emptyset$, then on $\pi^{-1}(U_j \cap U_k)$ we have

$$\Phi_{U_j} \circ \Phi_{U_k}^{-1}(x,\xi) = (x, \varphi_{jk}(x)(\xi))$$

for a G-valued locally constant function φ_{jk}. Therefore, whenever two plaques $P_{j,\xi}$ and $P_{k,\xi'}$ meet, they coincide in a neighborhood of each intersection point (after possibly restricting to a connected component of $U_j \cap U_k$). Take the connected components of the sets $P_{j,\xi}$ as connected plaques. Join two such plaques when they can be connected by a finite chain of plaques with consecutive members intersecting. The union of each resulting class is a leaf; these maximal unions define a C^∞ foliation $\mathcal{F}$ on E. In the bundle chart Φ_{U_j}, the transversal coordinates for $\mathcal{F}$ are given by local coordinates on F.

In the rest of this example, assume that X is connected. Fix a base point $x_0 \in X$ and identify its fiber with F by a local trivialization. Composing transition maps along paths gives the endpoint monodromy

$$\rho_E\colon \pi_1(X,x_0)^{\mathrm{op}} \to G.$$

This description does not require the action on F to be free. Suppose that $\xi_0 \in F$ is fixed by $\operatorname{Image}\rho_E$. Transporting ξ_0 from x_0 to x is then independent of the chosen path and defines a global C^∞ section $s_{\xi_0}\colon X \to E$. After refining to connected trivializing neighborhoods, its local fiber coordinates are constants ξ_j satisfying $\xi_j = \varphi_{jk}\xi_k$ on overlaps. They need not all equal ξ_0 in the originally chosen trivializations. Let $Y := s_{\xi_0}(X) \subset E$ be its image. Then Y is a C^∞ embedded submanifold of E, the restriction $\pi|_Y\colon Y \to X$ is a C^∞ diffeomorphism, and Y is a leaf of the foliation $\mathcal{F}$. The holonomy representation of $\mathcal{F}$ along Y (with respect to a choice of transversal coordinates near a base point in Y) is naturally identified with the germ at ξ_0 of the action of the monodromy representation ρ_E via $\pi|_Y\colon Y \to X$, since the expression of ρ_E in terms of transition functions (Mini-Exercise I.7.30 (ii)) coincides with the right-hand side of (56). □

9 From complex analysis in one variable

In this section we review a few basic facts from complex analysis in one variable. We assume that the reader is already familiar with the standard material from textbooks such as

[Ah], [Lan1]. In particular, we assume that the reader has studied the following topics: the definition of *holomorphicity* of a function on a *domain* in the complex plane (i.e., a connected open subset), the *Cauchy–Riemann equations*, *Cauchy's integral theorem*, power series and their radii of convergence (including standard convergence tests such as the *Weierstraß M-test*, the *Cauchy–Hadamard formula*, and the *ratio test*), *Morera's theorem*, *Montel's theorem*, the *identity theorem*, *maximum principles* (both for $|f|$ of a holomorphic function (*maximum modulus principle*) and for *subharmonic functions*), *rational functions* and *fractional linear transformations* (*Möbius transformations*), *meromorphic functions*, and the *residue theorem*.

In what follows, the term *domain* always means a connected open subset. Unless explicitly stated otherwise, throughout this section we write

$$z = x + \sqrt{-1}\, y$$

for the standard complex coordinate on $\mathbb{C}$, where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. In this note the *Riemann sphere* is identified with the complex projective line $\mathbb{P}^1$: For the *homogeneous coordinates* $[z : w]$ on $\mathbb{P}^1$, the open set $\{[z : w] \in \mathbb{P}^1 \mid w \neq 0\}$ is regarded as the copy of $\mathbb{C}$ inside $\mathbb{P}^1$ via the map

$$\mathbb{C} \ni z \longmapsto [z : 1] \in \mathbb{P}^1.$$

The *point at infinity* $z = \infty$ is identified with $[1 : 0] \in \mathbb{P}^1$. In this sense, a meromorphic function f on an open set $U \subset \mathbb{C}$ may be viewed as a map $f \colon U \to \mathbb{P}^1$.

We begin with a brief remark on the regularity of holomorphic functions on open subsets of $\mathbb{C}$. Let $U \subset \mathbb{C}$ be open, and let $f \colon U \to \mathbb{C}$ be a function. In many textbooks, holomorphicity of f is defined by the condition that, for every $z_0 \in U$, the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This implies that f is real-differentiable (and hence continuous) on U, and satisfies the Cauchy–Riemann equations pointwise. In particular, using the Wirtinger derivatives (see (35)), one has

$$\frac{\partial f}{\partial \overline{z}} \equiv 0, \qquad \frac{\partial f}{\partial z} = f'$$

pointwise on U (see [Ah, Chapter 2 §1] or [Lan1, Chapter I §5, 6]). A priori, this definition only asserts the existence of the complex derivative in the sense of the difference quotient along the complex direction, and does not by itself provide higher regularity. However, in reality, a holomorphic function is (complex) analytic, and in particular is C^∞. A classical method to show the C^∞-regularity is to use Cauchy's integral formula and its consequences; however, if one wishes to derive Cauchy's integral theorem and Cauchy's integral formula directly from the definition of holomorphicity without circular reasoning, some care is required. One standard approach is to adopt Goursat's proof of Cauchy's theorem (see [Ah, Chapter 4 §1.4] or [Lan1, Chapter III §3]). One may also connect this argument with the regularity result already proved in Theorem I.8.37. On a sufficiently small disc, Goursat's theorem gives a primitive F with $F' = f$. Since f is continuous, F is of class C^1; hence its classical equation $\partial F/\partial \overline{z} = 0$ also holds in the sense of distributions, by integration by parts. Applying

Theorem I.8.37 to F shows that F, and therefore f, is C^∞ (See also *Weyl's lemma*, [Forste, Chapter 3 §24]).

Next, we recall the decomposition of the exterior derivative

$$d = \partial + \overline{\partial}$$

on open subsets of $\mathbb{C}$. For a C^∞ function $f\colon U \to \mathbb{C}$, we define

$$\partial f := \frac{\partial f}{\partial z}\, dz, \qquad \overline{\partial} f := \frac{\partial f}{\partial \overline{z}}\, d\overline{z}.$$

For a 1-form $\omega = f\, dz + g\, d\overline{z}$ on U with C^∞ functions $f, g\colon U \to \mathbb{C}$, we set

$$\partial \omega := \frac{\partial g}{\partial z}\, dz \wedge d\overline{z}, \qquad \overline{\partial} \omega := \frac{\partial f}{\partial \overline{z}}\, d\overline{z} \wedge dz.$$

If f is holomorphic on U, then $\overline{\partial} f \equiv 0$, hence the 1-form $f(z)\, dz$ is d-closed. In particular, for a holomorphic function f on U, the line integral

$$\int_\gamma f(z)\, dz$$

is well-defined for any singular 1-cycle γ in the sense of Remark I.8.41, and it induces a homomorphism

$$\int_\bullet f(z)\, dz\colon H_1(U) \to \mathbb{C}.$$

If U is simply connected and f is holomorphic on U, then $H_1(U) = 0$ (by the relation between $\pi_1(U, *)$ and $H_1(U)$ discussed in Example I.7.12), and hence for every loop γ in U one has

$$\int_\gamma f(z)\, dz = 0.$$

This viewpoint is equivalent to the classical proof of Cauchy's theorem via Stokes' theorem (in real dimension 2, one may also call this Green's theorem).

Once one has this form of Cauchy's theorem, many standard consequences follow quickly. For instance, every holomorphic function on a simply connected domain admits a holomorphic primitive (Compare also the related discussion around Morera's theorem). Moreover, one can derive Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

and Cauchy's differentiation formula

$$\frac{d^n f}{dz^n}(z_0) = \frac{n!}{2\pi\sqrt{-1}} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad \big(n \in \mathbb{Z}_{\geq 0}\big),$$

where $z_0 \in U$ and γ is the positively oriented circle centered at z_0 of radius r strictly smaller than the Euclidean distance from z_0 to ∂U. From these one obtains Cauchy's estimate

$$\left|\frac{d^n f}{dz^n}(z_0)\right| \leq \frac{n!}{r^n} \sup_{|z-z_0|=r} |f(z)| \qquad \big(n \in \mathbb{Z}_{\geq 0}\big),$$

and hence, for any subset $K \subset U$ such that the Euclidean distance from K to ∂U is larger than $r > 0$,

$$\sup_{z \in K} \left|\frac{d^n f}{dz^n}(z)\right| \leq \frac{n!}{r^n} \sup_{z \in U} |f(z)| \qquad \big(n \in \mathbb{Z}_{\geq 0}\big).$$

Although these are basic results, their scope of applications is wide. For example, one deduces Liouville's theorem (bounded entire holomorphic functions are constant), and from it the Fundamental Theorem of Algebra (Theorem I.1.1): Indeed, if a polynomial $P(z) \in \mathbb{C}[z]$ has no zero in $\mathbb{C}$, then $f(z) := 1/P(z)$ is an entire holomorphic function, and it is bounded; thus f is constant by Liouville's theorem, so P must be constant. Refer to [Ah, Chapter 4] or [Lan1, Chapter V] for further details. Note that Cauchy's estimate also yields uniform bounds on derivatives on relatively compact subsets for uniformly bounded families of holomorphic functions, which leads to Montel's theorem (on a fixed open subset of $\mathbb{C}$, any family of holomorphic functions is normal if it is uniformly bounded on every compact set, see [Ah, Chapter 5 §5.4] or [Lan1, Chapter XIV §2]. One may also note here that Morera's theorem is often useful in verifying the holomorphicity of the limit of a sequence of holomorphic functions converging uniformly on compact subsets).

At this point, let us also briefly recall the analyticity of holomorphic functions. If f is holomorphic on an open set $U \subset \mathbb{C}$, then for every point $z_0 \in U$ it admits a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n \, (z - z_0)^n$$

in a neighborhood of z_0, and each coefficient a_n can be expressed by Cauchy's differentiation formula (see [Ah, Chapter 4 §3.3] or [Lan1, Chapter II Theorem 5.2] for example). Moreover, Cauchy's estimates show that this power series converges uniformly on every closed disc centered at z_0 which is contained in U. In this sense, a holomorphic function is locally represented by a convergent power series, and hence is not only of class C^∞ but, more strongly, analytic. Conversely, a convergent power series defines a holomorphic function (at least) on its disc of convergence, since it is a locally uniform limit of polynomials of z. For this reason, holomorphic functions are often also called *(complex) analytic functions*; however, in order to avoid confusion, we shall consistently use the term "holomorphic" in this note. Note that, by applying Taylor expansion, one can soon deduce the *open mapping theorem* for non-constant holomorphic functions ([Ah, Chapter 4 §1.2] or [Lan1, Chapter V Theorem 1.3]), the maximum principle for (the modulus of) holomorphic functions (the *maximum modulus principle*, [Ah, Chapter 4 §3.4] or [Lan1, Chapter II §6, Chapter III Corollary 1.4]), and the *identity theorem* (or the *uniqueness theorem*, see [Ah, Chapter 4 §3.2] for example).

For an open set $U \subset \mathbb{C}$, we write $\mathcal{O}_{\mathbb{C}}(U)$ for the $\mathbb{C}$-algebra of holomorphic functions on U. In view of the discussion so far, the reader will hopefully find it natural to endow $\mathcal{O}_{\mathbb{C}}(U)$ with

the topology of locally uniform convergence. For each non-empty compact subset $K \subset U$, consider the seminorm

$$\mathcal{O}_{\mathbb{C}}(U) \ni f \longmapsto \max_K |f|.$$

These seminorms define a TVS topology on $\mathcal{O}_{\mathbb{C}}(U)$. When $U = \emptyset$, we use the unique topology on $\mathcal{O}_{\mathbb{C}}(U) = \{0\}$. For non-empty U, choose a compact exhaustion $\{K_\nu\}_{\nu\geq 0}$ of U (i.e., non-empty compact subsets with $K_\nu \subset \operatorname{Int} K_{\nu+1}$ and $U = \bigcup_{\nu\geq 0} K_\nu$). The topology may then be described by a countable family of seminorms (compare the exhaustion $\{K_\nu\}$ used in Example I.6.41). Moreover, one can easily show the completeness (use Montel's theorem for example), hence $\mathcal{O}_{\mathbb{C}}(U)$ is a Fréchet space. Combining Montel's theorem with the characterization of TVS boundedness in Mini-Exercise I.6.36, one further sees that every closed and TVS bounded subset of $\mathcal{O}_{\mathbb{C}}(U)$ is compact (the *Heine–Borel property*, or the *Montel property*). The natural linear bijection

$$\operatorname{Ker}\bigl(\overline{\partial}\colon \mathcal{A}^0(U) \to \mathcal{A}^1(U)\bigr) \longrightarrow \mathcal{O}_{\mathbb{C}}(U)$$

is continuous, where the domain is viewed as a closed subspace of the Fréchet space $\mathcal{A}^0(U) = C^\infty(U)$ (it is closed since $\overline{\partial}$ is continuous. To see that this bijection is well-defined, recall that the equation $\overline{\partial} f \equiv 0$ is precisely the Cauchy–Riemann equation, and hence characterizes holomorphicity of f, refer to [Lan1, Chapter I §6] for example). Since the topology on $C^\infty(U)$ is defined by seminorms involving derivatives, it is a priori stronger than the topology on $\mathcal{O}_{\mathbb{C}}(U)$; hence the continuity is immediate. Conversely, Cauchy's estimates show that the inverse map is also continuous. In this sense, the Fréchet space $\mathcal{O}_{\mathbb{C}}(U)$ is naturally identified (including its topology) with the closed subspace $\operatorname{Ker}(\overline{\partial}\colon \mathcal{A}^0(U) \to \mathcal{A}^1(U))$ of $C^\infty(U)$.

For instance, when U is the unit disc

$$\Delta := \{z \in \mathbb{C} \mid |z| < 1\},$$

Taylor expansions show that polynomials are dense in $\mathcal{O}_{\mathbb{C}}(\Delta)$. Similarly, for $r \in \mathbb{R}_{\geq 0}$ and $\ell \in \mathbb{R}_{>r} \cup \{\infty\}$, if

$$U = \{z \in \mathbb{C} \mid r < |z| < \ell\},$$

then Laurent expansions show that rational functions are dense in $\mathcal{O}_{\mathbb{C}}(U)$. The reader is also encouraged to recall the classification of isolated singularities via Laurent series: given an open set $U \subset \mathbb{C}$, a point $z_0 \in U$, and a holomorphic function $f\colon U \setminus \{z_0\} \to \mathbb{C}$, one classifies z_0 as a removable singularity, a pole, or an essential singularity (see [Ah, Chapter 4 §3.1] or [Lan1, Chapter V §3] for example).

Remark I.9.1. Let $r \in \mathbb{R}_{>0}$ and $\ell \in \mathbb{R}_{>r}$, and set $U := \{z \in \mathbb{C} \mid r < |z| < \ell\}$ as above. At this point, the reader may also wish to recall why every holomorphic function $f \in \mathcal{O}_{\mathbb{C}}(U)$ admits a Laurent expansion. Fix a real number ρ with $r < \rho < \ell$, and let

$$\gamma_\rho\colon [0, 2\pi] \to U, \qquad \gamma_\rho(\theta) := \rho\, e^{\sqrt{-1}\theta},$$

be the positively oriented circle of radius ρ. Then, for each point $z_0 \in \mathbb{C}$ with $|z_0| < \rho$, one defines

$$F(z_0) := \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta.$$

This gives a holomorphic function F on the disc $\{z \in \mathbb{C} \mid |z| < \rho\}$. Similarly, on the disc

$$\{z \in \mathbb{C} \mid |z| > \rho\} \cup \{\infty\} = \{[z:1] \in \mathbb{P}^1 \mid |z| > \rho\} \cup \{[1:0]\} = \{[1:w] \in \mathbb{P}^1 \mid |w| < 1/\rho\},$$

define

$$G(z) := -\frac{1}{2\pi\sqrt{-1}} \int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad (|z| > \rho), \qquad G(\infty) := 0.$$

In the coordinate $w := 1/z$, this is holomorphic also at $w = 0$, since $(\zeta - z)^{-1} = w/(w\zeta - 1)$. Now, by varying ρ in the interval (r, ℓ), one sees that the functions F fit together to define a holomorphic function on the interior $\{[z:1] \in \mathbb{P}^1 \mid |z| < \ell\}$ of $\{z \in \mathbb{C} \mid |z| = \ell\}$, while the functions G fit together to define a holomorphic function on the exterior $\{[1:w] \in \mathbb{P}^1 \mid |w| < 1/r\}$ of $\{z \in \mathbb{C} \mid |z| = r\}$. For $z \in U$, Cauchy's formula on an annulus bounded by circles of radii $r < \rho_1 < |z| < \rho_2 < \ell$ gives the decomposition $f(z) = F(z) + G(z)$. Taking the Taylor expansions of these two functions, one recovers the Laurent expansion of f: the non-negative powers come from F, and the negative powers come from G (refer to [Ah, Chapter 5 §1.3] or [Lan1, Chapter V §2]). This proof already shows two points which are worth keeping in mind.

- From a suitable function defined on (a neighborhood of) the circle $\{z \in \mathbb{C} \mid |z| = \rho\}$, one can construct holomorphic functions on the interior $\{[z:1] \in \mathbb{P}^1 \mid |z| < \rho\}$, and on the exterior $\{[1:w] \in \mathbb{P}^1 \mid |w| < 1/\rho\}$ by means of Cauchy-type integrals (see also [Lan1, Chapter V Theorem 1.2] for example. The reader may also compare this with the construction of solutions to the Dirichlet problem on a disc by means of the Poisson kernel, see [Ah, Chapter 4 §6], [Forste, Chapter 3 §22], or [Lan1, Chapter VIII §4–5]).
- Even when the given function on $\{z \in \mathbb{C} \mid |z| = \rho\}$ arises as the restriction of a holomorphic function defined on a neighborhood of that circle, the holomorphic functions obtained in this way need not be analytic continuations of the original function across the circle.

□

When f is not identically zero near z_0 and z_0 is either a removable singularity (or f was holomorphic on U to begin with) or a pole, one can write the Laurent expansion

$$f(z) = \sum_{n=N}^{\infty} a_n\, (z - z_0)^n, \qquad a_N \neq 0,$$

for some integer N. We denote this integer by $\mathrm{ord}_{z_0}(f)$:

$$\mathrm{ord}_{z_0}(f) := N.$$

If z_0 is a zero of f, then $\mathrm{ord}_{z_0}(f)$ is the multiplicity of the zero. If z_0 is a pole of f, then $-\mathrm{ord}_{z_0}(f)$ is the (non-negative integer) order of the pole. If f is identically zero near z_0, we set $\mathrm{ord}_{z_0}(f) := \infty$ by convention.

MINI-EXERCISE I.9.2. Let $U \subset \mathbb{C}$ be an open set and $z_0 \in U$. We define

$$\mathcal{O}_{\mathbb{C},z_0} := \{(V, f) \mid V \subset U \text{ is open, } z_0 \in V, \text{ and } f \in \mathcal{O}_{\mathbb{C}}(V)\}/\sim,$$

where $(V, f) \sim (W, g)$ if and only if there exists an open neighborhood $\Omega \subset V \cap W$ of z_0 such that $f|_\Omega = g|_\Omega$.
(i) Show that $\mathcal{O}_{\mathbb{C},z_0}$ carries a natural ring structure, and that it is canonically isomorphic, as a ring, to the ring $\mathbb{C}\{w\}$ of convergent power series in one variable w (by sending w to the germ of $z - z_0$).
(ii) Determine all ideals of the ring $\mathcal{O}_{\mathbb{C},z_0}$.

Up to this point, one may say that we have mostly emphasized consequences of Cauchy's integral formula obtained by applying it locally, typically by integrating along small circles. In fact, it is not an exaggeration to say that many of the statements above can be derived essentially from the properties of the kernel function

$$(z_0, z) \longmapsto \frac{1}{2\pi\sqrt{-1}} \cdot \frac{1}{z - z_0}.$$

On the other hand, if one reinterprets such line integrals along more general loops, then one can also extract global topological information of the domain. For instance, fix a point $z_0 \in \mathbb{C}$. The 1-form

$$\omega_{z_0} := \frac{1}{2\pi\sqrt{-1}} \cdot \frac{dz}{z - z_0}$$

is d-closed on $\mathbb{C} \setminus \{z_0\}$, and therefore the integral

$$\frac{1}{2\pi\sqrt{-1}} \int_\bullet \frac{dz}{z - z_0}$$

induces a group homomorphism

$$\pi_1(\mathbb{C} \setminus \{z_0\}, *) \longrightarrow \mathbb{Z}, \qquad [\gamma] \longmapsto \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{dz}{z - z_0}.$$

Via the orientation-preserving homotopy equivalence $\mathbb{C} \setminus \{z_0\} \simeq \mathrm{U}(1)$, this homomorphism is nothing but the winding number: See also [Lan1, Chapter IV §1].

Motivated by this observation, let $U \subset \mathbb{C}$ be an open set and let $f \colon U \to \mathbb{P}^1$ be a meromorphic function. Let $\gamma \colon [0, 1] \to U$ be a loop such that γ avoids the zeros and poles of f. Then $f \circ \gamma$ is a loop in $\mathbb{C}^* \subset \mathbb{P}^1$, hence the integral

$$\frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{df}{f} = \frac{1}{2\pi\sqrt{-1}} \int_\gamma f^* \left(\frac{d\zeta}{\zeta}\right) = \frac{1}{2\pi\sqrt{-1}} \int_{f\circ\gamma} \frac{d\zeta}{\zeta}$$

is well-defined, where ζ denotes the standard inhomogeneous coordinate on the chart $\{[\,\zeta : 1\,]\} \subset \mathbb{P}^1$. The right-hand side shows that this integral computes the winding number of the loop $f \circ \gamma$ around 0. In this sense, the left-hand side (the *argument integral*) may be viewed

as expressing the topological content underlying the argument principle ([Ah, Chapter 4 §3.3, 5.2], [Lan1, Chapter VI Theorem 1.5]).

Related to this, the reader is encouraged to review Rouché's theorem ([Ah, Chapter 4 §5.2], [Lan1, Chapter VI Theorem 1.6]) and its application to the Fundamental Theorem of Algebra (one may apply Rouché's theorem using an estimate such as (1)). It is also instructive to compare such an analytic proof with the topological proof given in [Ha1, Chapter I, §1.1]. The reader is also encouraged to review the generalized variant of the argument principle described in [Ah, Chapter 4 §5.2].

At this point, it is also useful to keep in mind the *Riemann mapping theorem* (see, for example, [Ah, Chapter 6 §1.1] or [Lan1, Chapter XIV] for details; more generally, the reader is encouraged to know at least the statement of *Koebe's uniformization theorem* at this stage. Its proof is highly nontrivial, and we shall not discuss it in this section). By using these results (together with universal coverings of domains), many questions concerning the complex geometry of domains in $\mathbb{C}$, or more generally in $\mathbb{P}^1$, may be reduced to the three basic model domains $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$, $\mathbb{C}$, and $\mathbb{P}^1$. It is classically known that the holomorphic automorphism groups $\mathrm{Aut}(\Delta)$, $\mathrm{Aut}(\mathbb{C})$, and $\mathrm{Aut}(\mathbb{P}^1)$ of these three domains admit explicit descriptions in terms of *fractional linear transformations* (or *Möbius transformations*. See, for example, [Ah, Chapter 3, §3] or [Lan1, Chapter VII, §5]). Let us briefly review the groups $\mathrm{Aut}(\Delta)$, $\mathrm{Aut}(\mathbb{C})$, and $\mathrm{Aut}(\mathbb{P}^1)$.

First let us show that $\mathrm{Aut}(\mathbb{C})$ coincides with the affine transformation group:

$$\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b \mid a, b \in \mathbb{C},\ a \neq 0\}.$$

Indeed, the inclusion "$\supset$" is obvious, so it suffices to show that every $f \in \mathrm{Aut}(\mathbb{C})$ is affine. The point at infinity is an isolated singularity of f, and it cannot be an essential singularity: this follows from the fact that a holomorphic function comes arbitrarily close to every complex value in every neighborhood of an essential singularity (*Casorati–Weierstraß' theorem*, see [Ah, Chapter 4 Theorem 9] or [Lan1, Chapter V, Theorem 3.2]). On the other hand, if it were a removable singularity of f, then f would be a bounded holomorphic function on $\mathbb{C}$, and hence would be constant by Liouville's theorem, contrary to the assumption that $f \in \mathrm{Aut}(\mathbb{C})$. Therefore ∞ must be a pole of f, that is, f is a polynomial. Since $f \in \mathrm{Aut}(\mathbb{C})$, for each value $w \in \mathbb{C}$ the equation $f(z) = w$ has exactly one solution counting multiplicity. Hence, by the Fundamental Theorem of Algebra (Theorem I.1.1), f must be of degree one. This proves the claim.

Next, let us show that

$$\mathrm{Aut}(\mathbb{P}^1) = \left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; a, b, c, d \in \mathbb{C},\ ad - bc \neq 0 \right\} \cong \mathrm{PGL}_2(\mathbb{C}),$$

where $\mathrm{PGL}_2(\mathbb{C}) := \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^*$ denotes the projective general linear group. Again, the inclusion "$\supset$" is obvious, so it suffices to show that every $f \in \mathrm{Aut}(\mathbb{P}^1)$ is a fractional linear transformation. As one can send any three distinct points of $\mathbb{P}^1$ to 0, 1, and ∞ by a suitable fractional linear transformation, after composing with suitable fractional linear transformations if necessary, we may reduce to the case where f fixes 0, 1, and ∞. Restricting f to

$\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$, we may regard it as an element of $\mathrm{Aut}(\mathbb{C})$. Therefore the claim follows from the result proved above.

Finally, as for the determination of $\mathrm{Aut}(\Delta)$, we do not write out the details here, since they are explained carefully in [Lan1, Chapter VII, §2]. The conclusion is that

$$\mathrm{Aut}(\Delta) = \left\{ z \mapsto t\,\frac{z-a}{1-\overline{a}z} \;\middle|\; a \in \Delta,\ t \in \mathrm{U}(1) \right\}.$$

The proof is based on Schwarz's lemma (see [Ah, Chapter 4, §3.3, Theorem 13], [J, Theorem 2.3.1], or [Lan1, Chapter VII, §1]). Here we also note that the following invariant form of Schwarz's lemma, called the *Schwarz–Pick theorem*, is known (see, for example, [J, Theorem 2.3.2]): For every holomorphic function $f\colon \Delta \to \Delta$,

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \leq \frac{|z_1 - z_2|}{|1 - \overline{z_1} z_2|}$$

holds for all $z_1, z_2 \in \Delta$, and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

holds for all $z \in \Delta$.

Finally, the reader is also encouraged at this stage to review *harmonic* and *subharmonic* functions. For the relation between the real and imaginary parts of holomorphic functions and harmonic functions, as well as for the basic properties of harmonic functions (the *mean-value property*, *Poisson's formula*, the *maximum principle*, *Harnack's principle*, *Harnack's inequality*, and so on), the reader may consult the clear exposition in [Ah, Chapter 4, §6; Chapter 6, §3]. As for subharmonic functions, their basic properties (such as the *maximum principle* and the *mean-value inequality*) are discussed in [Ah, Chapter 6, §4.1].

Mini-Exercise I.9.3. Let $h\colon \mathbb{C} \to \mathbb{R}_{>0}$ be a harmonic function. Show that h is constant.

At this point, let us make a brief remark on our definition of subharmonic functions. For a function $u\colon U \to \mathbb{R} \cup \{-\infty\}$ on an open set $U \subset \mathbb{C}$, we say that u is subharmonic if it is upper semicontinuous and satisfies the mean-value inequality. In some references, one further requires that u be not identically equal to $-\infty$ on each connected component of U, but we shall *not* impose this additional condition in this note.

For each point $z_0 \in U$ and each sufficiently small positive number $r > 0$, the mean-value inequality is written in the form

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r e^{\sqrt{-1}\theta})\, d\theta.$$

Hence this inequality holds for every $z_0 \in U$ and every $r > 0$ such that $\overline{\Delta_r(z_0)} \subset U$, where

$$\Delta_r(z_0) := \{ z \in \mathbb{C} \mid |z - z_0| < r \}.$$

In particular, for every $t \in (0, r)$, we obtain

$$t\, u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + te^{\sqrt{-1}\theta})\, t\, d\theta.$$

Integrating both sides with respect to t, we get

$$\begin{aligned} \frac{r^2}{2} u(z_0) &= \int_0^r t\, u(z_0)\, dt \\ &\leq \int_0^r \left(\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + te^{\sqrt{-1}\theta})\, t\, d\theta \right) dt \\ &= \frac{1}{2\pi} \int_{\Delta_r(z_0)} u(z)\, d\lambda, \end{aligned}$$

where $d\lambda$ denotes the Lebesgue measure on $\mathbb{C}$. Thus one also obtains the area version of the mean-value inequality:

$$u(z_0) \leq \frac{1}{\pi r^2} \int_{\Delta_r(z_0)} u\, d\lambda, \tag{57}$$

at least whenever the right-hand side is well-defined.

As the following lemma shows, a subharmonic function on a domain which is not identically equal to $-\infty$ is locally L^1, and hence may be regarded as a distribution. Accordingly, whenever we write $\partial\overline{\partial} u$ or similar expressions for such a function u, they are to be understood in the sense of currents (Recall the arguments in Example I.8.19).

Lemma I.9.4. *Let $U \subset \mathbb{C}$ be a domain and $u \colon U \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function which is not identically equal to $-\infty$. Then $u \in L^1_{\mathrm{loc}}(U)$ and $\{z \in U \mid u(z) = -\infty\}$ has no interior point.*

Proof. Take a point $z_0 \in U$ such that $u(z_0) > -\infty$, and take $r > 0$ such that $\overline{\Delta_r(z_0)} \subset U$. Since u is upper semicontinuous, it is bounded above on $\overline{\Delta_r(z_0)}$. Subtracting a sufficiently large constant if necessary, we may assume that $u < 0$ on $\Delta_r(z_0)$. Then (57) gives

$$\frac{1}{\pi r^2} \int_{\Delta_r(z_0)} |u|\, d\lambda = \frac{-1}{\pi r^2} \int_{\Delta_r(z_0)} u\, d\lambda \leq -u(z_0) < \infty,$$

from which $u|_{\Delta_r(z_0)} \in L^1(\Delta_r(z_0))$ follows. Note that from this we also have that $\{z \in \Delta_r(z_0) \mid u(z) = -\infty\}$ has no interior point, since any non-empty open subset is of positive Lebesgue measure.

Denote by A the set of all points $a \in U$ that satisfy the following property: for any positive constant $r > 0$ which is smaller than the Euclidean distance from a to ∂U, $u|_{\Delta_r(a)} \in L^1(\Delta_r(a))$ holds. Note that $\{z \in A \mid u(z) = -\infty\}$ has no interior point for the same reason as above. The preceding mean-value estimate shows that every point where u is finite belongs to A. In particular, $z_0 \in A$, so A is non-empty. We first show that A is closed in U.

Take a point $z_1 \in \overline{A} \cap U$ and a positive number r smaller than the distance from z_1 to ∂U. Take also a positive number $\varepsilon > 0$ such that $r + 2\varepsilon$ is also smaller than the distance

from z_1 to ∂U. As $z_1 \in \overline{A}$, one can take a point $z_2 \in \Delta_\varepsilon(z_1) \cap A$. As $\overline{\Delta_r(z_1)} \subset \overline{\Delta_{r+\varepsilon}(z_2)} \subset U$ and $z_2 \in A$, we have that $u|_{\Delta_r(z_1)} \in L^1(\Delta_r(z_1))$. Therefore $z_1 \in A$. Thus A is closed in U. For $a \in A$, choose a sufficiently small disc centered at a on which u is integrable. The points where u is finite are dense in this disc and all belong to A. Since A is closed in U, the whole disc belongs to A. Thus A is also open, and connectedness gives $A = U$, proving the assertion. □

Mini-Exercise I.9.5. Let $U \subset \mathbb{C}$ be a domain and $f\colon U \to \mathbb{C}$ be a holomorphic function such that $f \not\equiv 0$. Show that $z \mapsto \log|f(z)|^2$ is subharmonic on U. Show also the following *Lelong–Poincaré formula*:

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2 = \sum_{a\in U} \mathrm{ord}_a(f)\,\delta_a \in \mathcal{K}^2(U)\ (= (C_c^\infty(U))^\vee), \tag{58}$$

where δ_a denotes the Dirac measure supported at $a \in U$ (i.e., $\delta_a(\varphi) = \varphi(a)$ for $\varphi \in C_c^\infty(U)$).

Mini-Exercise I.9.6. Let $U \subset \mathbb{C}$ be a domain and let $u \in L^1_{\mathrm{loc}}(U)$ be real-valued almost everywhere. Show that the following are equivalent:
(i) The L^1_{loc}-equivalence class of u admits a subharmonic representative.
(ii) $\sqrt{-1}\partial\overline{\partial}u \in \mathcal{K}^2(U)$ $(= (C_c^\infty(U))^\vee)$ is semi-positive; that is, for every $\varphi \in \mathcal{A}_c^0(U)$ $(= C_c^\infty(U))$, if $\varphi(x) \geq 0$ for every $x \in U$, then $(\sqrt{-1}\partial\overline{\partial}u)(\varphi) \geq 0$ holds.
Show also that the subharmonic representative in (i) is unique.

Mini-Exercise I.9.7. Let $r > 0$, and fix positive numbers ρ and λ such that $0 < \rho < \lambda < r$. For each positive number ℓ, set $\Delta_\ell := \{z \in \mathbb{C} \mid |z| < \ell\}$. Prove the following statements concerning subharmonic functions on Δ_r.
(i) Let u be a subharmonic function on Δ_r. Show that, for every $z_0 \in \Delta_\rho$,

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\lambda^2 - |z_0|^2}{|\lambda e^{\sqrt{-1}\theta} - z_0|^2}\, u(\lambda e^{\sqrt{-1}\theta})\, d\theta.$$

(ii) Let $A \in \mathbb{R}$, and let $\{u_k\}_{k\in\mathbb{Z}_{\geq 1}}$ be a sequence of subharmonic functions on Δ_r satisfying the following two conditions:
(a) The sequence $\{\sup_{\Delta_\lambda} u_k\}_{k\in\mathbb{Z}_{\geq 1}}$ is bounded above.
(b) For every $z_0 \in \Delta_r$,

$$\limsup_{k\to\infty} u_k(z_0) \leq A.$$

Show that, for every $\alpha \in \mathbb{R}$ with $\alpha > A$, there exists a positive integer N, depending only on r, ρ, λ, $\{u_k\}_{k\in\mathbb{Z}_{\geq 1}}$, A, and α, such that

$$\sup_{\Delta_\rho} u_k \leq \alpha$$

for every $k \in \mathbb{Z}_{>N}$.

In this section, we have organized the material in complex analysis that may reasonably be regarded as standard undergraduate background. Since this note will deal with several complex variables from Chapter III onward, we have intentionally avoided discussing here topics that will be treated there in a more unified manner. For example, at this stage the reader should already be able to understand the definition of a Riemann surface without difficulty, and many facts may already be formulated and proved in that setting; however, we postpone such discussions until Chapter III, where complex manifolds in arbitrary dimension will be introduced. Likewise, we have not discussed cohomology groups with coefficients in analytic sheaves, nor related facts from the theory of the $\overline{\partial}$-equation. If the reader has enough time, it is highly recommended to study some basic facts on Riemann surfaces at this point, for example from textbooks such as [Forste] or [J]. Moreover, although we have not gone into details here, it is also helpful to learn the theory of *doubly periodic functions* (*elliptic functions*), the *Weierstraß $\wp$-function* and related special functions (see [Ah, Chapter 7]), as well as the basic theory of analytic continuation for holomorphic functions of one variable and its relation to covering maps and homotopy (see [Ah, Chapter 8, §1]), because such preparation is expected to make the discussions in the following chapters easier to follow.

Exercises for Chapter I

Exercise I.1. Prove Lemma I.1.3 in the following two ways.
(i) Using Cauchy's estimate for the roots of a polynomial (Lemma I.1.2), show that the map $\mathrm{Sym}^n(\mathbb{C}) \longrightarrow \mathbb{C}^n$ defined by the usual relation between the roots and the coefficients is a homeomorphism.
(ii) Apply Rouché's theorem to show that the set of roots varies continuously with t.

Exercise I.2. In this exercise, for an Hermitian matrix H and a positive definite matrix ω of the same order, the *eigenvalues of H with respect to ω* mean the eigenvalues of $\sqrt{\omega^{-1}}H\sqrt{\omega^{-1}}$. Let n, N be positive integers, $U \subset \mathbb{R}^N$ be an open subset, and $t \mapsto H(t)$ and $t \mapsto \omega(t)$ be continuous functions defined on U such that, for any $t \in U$, $H(t)$ is an Hermitian matrix of order n and $\omega(t)$ is a positive definite matrix of order n. For each $t \in U$, denote by $\gamma_1(t) \geq \gamma_2(t) \geq \cdots \geq \gamma_n(t)$ the eigenvalues of $H(t)$ with respect to $\omega(t)$ counted with their algebraic multiplicity. Show that $\gamma_j \colon U \to \mathbb{R}$ is continuous for each $j \in \{1, 2, \ldots, n\}$.

Exercise I.3. Let (X, d) be a metric space, $Y \subset X$ a compact subset, and $\{U_j\}_{j\in J}$ an open covering of Y by open subsets of X. For a positive number $r > 0$ and each $j \in J$, define

$$U_j^{(r)} := \bigl\{x \in Y \,\bigm|\, d(x, X \setminus U_j) > r\bigr\}.$$

If $U_j = X$, we set $U_j^{(r)} := Y$. Show that there exists a positive number $r_0 > 0$ such that, for every $r \in (0, r_0)$, the family $\{U_j^{(r)}\}_{j\in J}$ is an open covering of Y. Moreover, examine whether the above statement remains true if the compactness assumption on Y is dropped.

Exercise I.4. For $\tau \in \mathbb{R}_{>1}$, define

$$\mathrm{Dio}(\tau) := \Big\{ \alpha \in \mathbb{R} \setminus \mathbb{Q} \;\Big|\; \exists\, C > 0 \text{ such that } d(q\alpha, \mathbb{Z}) \geq C\, q^{-\tau} \text{ for all } q \in \mathbb{Z}_{>0} \Big\},$$

where d is the Euclidean distance.
(i) Show that the set $\mathrm{Dio}(\tau)$ is meagre in $\mathbb{R}$.
(ii) Show that the complement of $\mathrm{Dio}(\tau)$ has Lebesgue measure zero.
(iii) Let

$$\mathrm{Dio}_{>1} := \bigcup_{\tau > 1} \mathrm{Dio}(\tau).$$

Determine whether $\mathrm{Dio}_{>1}$ is meagre or comeagre, and whether it has full Lebesgue measure in $\mathbb{R}$.

Exercise I.5. Let $X := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the sphere. For each point $p \in X$, let δ_p denote the Dirac measure supported at p, i.e.

$$\delta_p(\varphi) := \varphi(p) \qquad (\varphi \in C_c^\infty(X)).$$

Let $N \in \mathbb{Z}_{>0}$, let $p_1, p_2, \ldots, p_N \in X$ be distinct points, and let $n_1, n_2, \ldots, n_N \in \mathbb{Z}$. Determine the necessary and sufficient condition for the existence of $T \in \mathcal{K}^1(X)$ such that

$$dT = \sum_{j=1}^{N} n_j\, \delta_{p_j} \in \mathcal{K}^2(X).$$

When this condition is satisfied, construct such a current T explicitly.

Exercise I.6. Let $r \in \mathbb{R}_{>0}$ and $\ell \in \mathbb{R}_{>r}$, and set $U_0 := \{z \in \mathbb{C} \mid |z| < \ell\}$ and

$$U_1 := \{z \in \mathbb{C} \mid |z| > r\} \cup \{\infty\} = \{[z : 1] \in \mathbb{P}^1 \mid |z| > r\} \cup \{[1 : 0]\} = \{[1 : w] \in \mathbb{P}^1 \mid |w| < 1/r\}.$$

Denote by W_j the Fréchet space of holomorphic functions on the disc U_j for $j = 0, 1$. Determine the kernel and the cokernel of the map

$$\delta \colon W_0 \oplus W_1 \to \mathcal{O}_{\mathbb{C}}(U_0 \cap U_1)$$

defined by letting $\delta(f_0, f_1) := -f_0 + f_1$ for $f_j \in W_j$ $(j = 0, 1)$. One may also ask how the kernel and the cokernel change if the regularity condition is replaced by another one. For example, what happens if one considers the function

$$\delta \colon C^\infty(U_0) \oplus C^\infty(U_1) \to C^\infty(U_0 \cap U_1)$$

defined by the same equation $\delta(f_0, f_1) := -f_0 + f_1$? What happens if one replaces holomorphic functions by locally constant functions?

Exercise I.7. Let $U \subset \mathbb{C}$ be a simply connected domain. Show that the ring $\mathcal{O}_{\mathbb{C}}(U)$ is not Noetherian. Give also an explicit example of such a domain U and an ideal $I \subset \mathcal{O}_{\mathbb{C}}(U)$ which is not finitely generated.

Chapter II

Preliminaries in some general frameworks

1 A Siegel–type majorant theorem

In this section, we show the following:

THEOREM II.1.1 (Siegel–majorant theorem [Sie]). *Let $\{e_n\}_{n\in\mathbb{Z}_{\geq 1}}$ be a sequence of positive real numbers. Assume the following:*
(i) There exists a positive constant $\alpha > 0$ such that, for any positive integer n, $e_n \geq (2n)^{-\alpha}$ holds.
(ii) For any positive integers p and q with $q < p$, $e_{p-q} \leq e_p + e_q$ holds.
Let

$$A(u) = u + \sum_{n=2}^{\infty} a_n u^n$$

be the formal power series determined by the functional equation

$$\sum_{n=2}^{\infty} e_{n-1}\, a_n\, u^n = R \cdot \frac{(A(u))^2}{1 - MA(u)}, \tag{59}$$

where $R, M \in \mathbb{R}_{>0}$ are positive constants (see Remark II.1.2 below). Then $A(u)$ has a positive radius of convergence.

REMARK II.1.2. The functional equation (59) determines the coefficients $\{a_n\}_{n\in\mathbb{Z}_{\geq 2}}$ recursively: for each $n \geq 2$, the coefficient a_n is uniquely determined by the lower-order coefficients $a_2, \ldots, a_{n-1}$. Indeed, expanding

$$\frac{A(u)^2}{1 - MA(u)} = A(u)^2 \sum_{k=0}^{\infty} (MA(u))^k = \sum_{k=0}^{\infty} M^k A(u)^{k+2},$$

and comparing coefficients in (59), one obtains explicit formulas. For example, to see how the first few coefficients are determined, let us examine the expansion up to order 4. Writing

$A(u) = u + a_2u^2 + a_3u^3 + a_4u^4 + O(u^5)$, we have

$$\begin{aligned} A(u)^2 &= u^2 + 2a_2u^3 + (2a_3 + a_2^2)u^4 + O(u^5), \\ A(u)^3 &= u^3 + 3a_2u^4 + O(u^5), \\ A(u)^4 &= u^4 + O(u^5). \end{aligned}$$

Hence

$$\begin{aligned} \frac{A(u)^2}{1 - MA(u)} &= A(u)^2 + MA(u)^3 + M^2A(u)^4 + O(u^5) \\ &= u^2 + (2a_2 + M)u^3 + \big(2a_3 + a_2^2 + 3Ma_2 + M^2\big)u^4 + O(u^5). \end{aligned}$$

On the other hand, the left-hand side of (59) is

$$\sum_{n=2}^{\infty} e_{n-1}a_nu^n = e_1a_2u^2 + e_2a_3u^3 + e_3a_4u^4 + O(u^5).$$

Comparing coefficients of u^2, u^3, u^4 in (59), we obtain

$$a_2 = Re_1^{-1},$$

$$a_3 = Re_2^{-1}\,(2a_2 + M),$$

and

$$a_4 = Re_3^{-1}\,\big(2a_3 + a_2^2 + 3Ma_2 + M^2\big).$$

To describe the general case, it is convenient to set

$$a_1 := 1$$

so that $A(u) = \sum_{n\geq 1} a_nu^n$. With this convention, the coefficient comparison in (59) yields, for each $k \geq 2$,

$$e_{k-1}\,a_k = R\sum_{r\geq 2} M^{r-2} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_r=k \\ \ell_1\geq 1,\ \ell_2\geq 1,\ \ldots,\ \ell_r\geq 1}} a_{\ell_1}\cdot a_{\ell_2}\cdots a_{\ell_r}. \tag{60}$$

Here the sum runs over all integers $r \geq 2$ and all ordered r-tuples $(\ell_1, \ldots, \ell_r)$ of positive integers with total sum k, that is, all compositions of k into r positive parts (with order taken into account, different orderings of the same integers are counted separately).

In what follows, we keep the convention $a_1 = 1$ and always interpret such sums in the sense of (60). □

Note that assumption (i) is equivalent to

$$e_n^{-1} \leq (2n)^{\alpha} \quad (n \geq 1). \tag{61}$$

A typical source of sequences $\{e_n\}$ satisfying assumptions (i) and (ii) comes from Diophantine approximation. For example, let $\theta \in \mathbb{R}$ be a Diophantine number, and set

$$e_n := d(n\theta, \mathbb{Z}) := \min_{m\in\mathbb{Z}} |n\theta - m| \qquad (n \geq 1).$$

Then the Diophantine condition implies a lower bound of the form $e_n \geq Cn^{-\tau}$ for suitable constants $C > 0$ and $\tau > 1$, so that assumption (i) is satisfied after a harmless renormalization of constants: More precisely, by letting

$$\alpha := \max\left\{\tau,\ \log_2 \frac{1}{C}\right\},$$

assumption (i) is satisfied. Assumption (ii) follows from the elementary inequality

$$d((p-q)\theta, \mathbb{Z}) \leq d(p\theta, \mathbb{Z}) + d(q\theta, \mathbb{Z}),$$

which is simply the triangle inequality on $\mathbb{R}/\mathbb{Z}$.

From a historical point of view, Theorem II.1.1 is due to Siegel [Sie], where the corresponding majorant argument appears in the study of the analytic linearization problem in complex dynamics. The abstract formulation presented here is not original; it can be found in [U1, Lemma 5].

Remark II.1.3. It is in fact sufficient to prove Theorem II.1.1 in the special case $M = R = 1$. Indeed, as observed in Remark II.1.2, the functional equation (59) determines the coefficients recursively: for each $n \geq 2$, the coefficient a_n is uniquely determined by the lower-order coefficients $a_2, \ldots, a_{n-1}$. Inspecting the explicit recursion, one sees that increasing the constants M and R only enlarges the right-hand side of the recursion. Consequently, if the convergence of $A(u)$ is established for a larger pair of constants, it follows *a fortiori* for any smaller pair. Therefore, setting $K := \max\{M, R\}$, it is enough to prove the convergence of the formal power series defined by (59) with both constants M and R replaced by K.

Let $A(u) = u + \sum_{n=2}^{\infty} a_n u^n$ be a formal power series satisfying (59) with $M = R = K$. Define a rescaled formal power series

$$B(u) := K\, A\Big(\frac{u}{K}\Big) = u + \sum_{n=2}^{\infty} b_n u^n, \qquad b_n := K^{1-n} a_n \quad (n \geq 2).$$

Then we compute

$$\sum_{n=2}^{\infty} e_{n-1} b_n u^n = \sum_{n=2}^{\infty} e_{n-1} K^{1-n} a_n u^n = K \sum_{n=2}^{\infty} e_{n-1} a_n \left(\frac{u}{K}\right)^n = K \cdot \frac{KA(u/K)^2}{1 - KA(u/K)}$$

Hence $B(u)$ satisfies the functional equation

$$\sum_{n=2}^{\infty} e_{n-1}\, b_n\, u^n = \frac{(B(u))^2}{1 - B(u)},$$

which is exactly (59) in the special case $M = R = 1$. Since linear rescalings of the variable preserve the positivity of the radius of convergence, the convergence of $B(u)$ implies that of $A(u)$. Thus the proof of Theorem II.1.1 reduces to the case $M = R = 1$. □

In the proof of Theorem II.1.1, we apply the following:

LEMMA II.1.4 (=[Sie, Lemma 1]). *Let $t \geq 2$ be an integer. Let $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$ be positive integers with $r \geq 0$, $s \geq 2$, and $r+s = t$. Set $k := \sum_{p=1}^{r} x_p + \sum_{q=1}^{s} y_q$, and assume that*

$$\sum_{q=1}^{s} y_q > \frac{k}{2} \qquad \text{and} \qquad y_q \leq \frac{k}{2} \ (\text{for all } q = 1, 2, \ldots, s)$$

hold. Then

$$\left(\prod_{p=1}^{r} x_p\right)\left(\prod_{q=1}^{s} y_q^2\right) \geq k^3\, 8^{1-t}. \tag{62}$$

Lemma II.1.4 is a purely combinatorial inequality, proved by Siegel through a careful and rather intricate combination of elementary inequalities. While the argument is completely elementary, reproducing it here would require a substantial amount of space and might distract the reader from the main line of the proof of Theorem II.1.1. Therefore, we defer an explanation of Siegel's proof to Appendix §A.2.

Outline of the proof of Theorem II.1.1. Here we sketch the structure of the proof, which is in accordance with Siegel's original proof in [Sie]; the detailed arguments and estimates are carried out below. In the proof, by Remark II.1.3, we may (and do) assume that $M = R = 1$. Then the functional equation (59) takes the form

$$\sum_{n=2}^{\infty} e_{n-1}\, a_n\, u^n = \frac{A(u)^2}{1 - A(u)} \qquad a_1 := 1. \tag{63}$$

To understand the effect of the coefficients e_{n-1}, we first consider, as a comparison, the following *unweighted* functional equation. Let

$$B(u) = \sum_{n \geq 1} b_n u^n, \qquad b_1 := 1,$$

be the formal power series defined by

$$B(u) - u = \frac{B(u)^2}{1 - B(u)}. \tag{64}$$

Equivalently,

$$\sum_{n=2}^{\infty} b_n u^n = \frac{B(u)^2}{1 - B(u)}.$$

This equation is algebraic and can be solved explicitly; in particular, $B(u)$ has a positive radius of convergence (Indeed, from (64) we have

$$(B(u) - u)(1 - B(u)) = B(u)^2,$$

that is,

$$2B(u)^2 - (1+u)B(u) + u = 0.$$

Solving this quadratic equation in $B(u)$, we obtain

$$B(u) = \frac{(1+u) \pm \sqrt{(1+u)^2 - 8u}}{4} = \frac{(1+u) \pm \sqrt{1-6u+u^2}}{4}.$$

Since $B(0) = 0$ must hold, we choose the minus sign and get

$$B(u) = \frac{(1+u) - \sqrt{1-6u+u^2}}{4}. \tag{65}$$

In particular, $B(u)$ is analytic in a neighborhood of $u = 0$). Moreover, comparing coefficients yields a purely combinatorial recursion: for each $k \geq 2$,

$$b_k = \sum_{r \geq 2} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_r=k \\ \ell_i \geq 1}} b_{\ell_1} \cdot b_{\ell_2} \cdots b_{\ell_r}. \tag{66}$$

Returning to (63), taking absolute values and comparing coefficients shows that the coefficients of $A(u)$ satisfy

$$e_{k-1}\,|a_k| \leq \sum_{r \geq 2} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_r=k \\ \ell_i \geq 1}} |a_{\ell_1}| \cdot |a_{\ell_2}| \cdots |a_{\ell_r}| \qquad (k \geq 2). \tag{67}$$

Thus, up to the factor e_{k-1} on the left-hand side, the recursion for $|a_k|$ has exactly the same combinatorial structure as the recursion (66) for b_k.

From the above observation, now we can say that the essential difficulty is to control the accumulation of the inverse factors e_{k-1}^{-1} that arise when one solves (67) recursively for $|a_k|$. To capture the maximal possible growth produced by iterations of such recursions, we introduce a positive sequence $\{\delta_k\}_{k \in \mathbb{Z}_{\geq 1}}$, defined recursively as a worst-case majorant. The core of the proof consists in showing that δ_k grows at most exponentially, since δ_k will be constructed so that $|a_k| \leq b_k\,\delta_k$ holds for any positive integer k (see Lemma II.1.5 below).

Definition of the sequence $\{\delta_k\}_{k \geq 1}$ Define a sequence $\{\delta_k\}_{k \geq 1}$ of positive numbers recursively as follows:

$$\delta_1 := 1,$$

and for each integer $k \geq 2$ let ω_k denote the maximum of all products

$$\delta_{\ell_1}\delta_{\ell_2} \cdots \delta_{\ell_s}$$

taken over all integers s and all s-tuples $(\ell_1, \ell_2, \ldots, \ell_s) \in \mathbb{Z}_+^s$ satisfying

$$\ell_1 + \ell_2 + \cdots + \ell_s = k, \qquad k \geq \ell_1 \geq \ell_2 \geq \cdots \geq \ell_s \geq 1, \qquad 2 \leq s \leq k,$$

and put

$$\delta_k := \frac{\omega_k}{e_{k-1}}.$$

LEMMA II.1.5. *Keep the setting of the outline above, and assume $M = R = 1$. Let $A(u) = \sum_{n\geq 1} a_n u^n$ be the formal solution of (63) with $a_1 = 1$, and let $B(u) = \sum_{n\geq 1} b_n u^n$ be the solution of (64) with $b_1 = 1$. Then, for every integer $k \geq 1$, one has*

$$|a_k| \leq b_k\, \delta_k, \tag{68}$$

where δ_k is as above.

PROOF. We prove (68) by induction on k. The case $k = 1$ is clear since $|a_1| = 1 = b_1\delta_1$.

Assume $k \geq 2$ and that (68) holds for all integers $1 \leq j \leq k-1$. From the inductive assumption and (67), one has

$$e_{k-1}\,|a_k| \leq \sum_{r\geq 2} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_r=k \\ \ell_i\geq 1}} (b_{\ell_1}\delta_{\ell_1})\cdot(b_{\ell_2}\delta_{\ell_2})\cdots(b_{\ell_r}\delta_{\ell_r}).$$

Now use the definition of ω_k: for each $r \geq 2$ and each $(\ell_1, \ldots, \ell_r)$ with $\ell_1 + \cdots + \ell_r = k$, we have $\delta_{\ell_1}\cdot\delta_{\ell_2}\cdots\delta_{\ell_r} \leq \omega_k$, because ω_k is the maximum of such products (reordering the ℓ_i if needed does not change the product). Hence

$$e_{k-1}\,|a_k| \leq \omega_k \sum_{r\geq 2} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_r=k \\ \ell_i\geq 1}} b_{\ell_1}\cdot b_{\ell_2}\cdots b_{\ell_r}.$$

Finally, the coefficient recursion (66) for b_k gives

$$\sum_{r\geq 2} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_r=k \\ \ell_i\geq 1}} b_{\ell_1}\cdot b_{\ell_2}\cdots b_{\ell_r} = b_k.$$

Therefore $e_{k-1}\,|a_k| \leq \omega_k\, b_k$, which completes the induction. □

To estimate the growth of δ_k, as a preparation, we first give a uniform upper bound for $\prod e_{m_j}^{-1}$ in terms of the successive gaps $m_{j-1} - m_j$ in Lemma II.1.6. By combining this with the combinatorial lemma (Lemma II.1.4), we prove δ_k is at most exponential in k in Lemma II.1.7.

We now begin the detailed proof following the outline above. Define the constants

$$N_1 := 2^{2\alpha+1}, \qquad N_2 := 8^{\alpha} N_1 = 2^{5\alpha+1},$$

and set

$$\Lambda_k := k^{-2\alpha} N_2^{k-1}$$

for each $k \in \mathbb{Z}_{\geq 1}$.

LEMMA II.1.6. *Assume (i) and (ii), and that $M = R = 1$. Let $r \geq 0$ and let $m_0, m_1, \ldots, m_r$ be positive integers such that $m_0 > m_1 > \cdots > m_r > 0$. Then*

$$\prod_{j=0}^{r} e_{m_j}^{-1} \leq N_1^{r+1}\, m_0^{\alpha} \prod_{j=1}^{r} (m_{j-1} - m_j)^{\alpha}. \tag{69}$$

PROOF. First let us note that, for integers $p > q \geq 1$,

$$\min\{e_p^{-1}, e_q^{-1}\} \leq 2\, e_{p-q}^{-1} \tag{70}$$

holds. Indeed, (ii) gives $e_{p-q} \leq e_p + e_q$, hence

$$e_{p-q}^{-1} \geq \frac{1}{e_p + e_q} \geq \frac{1}{2\max\{e_p, e_q\}} = \frac{1}{2}\min\{e_p^{-1}, e_q^{-1}\},$$

which is equivalent to (70). By using this inequality, we show the assertion by induction on $r \geq 1$.

When $r = 0$, the assertion directly follows from (61). Let us show the case $r = 1$. Let $m_0 > m_1 \geq 1$. By (70), we have

$$\min\{e_{m_0}^{-1}, e_{m_1}^{-1}\} \leq 2\, e_{m_0-m_1}^{-1}.$$

Therefore

$$e_{m_0}^{-1} e_{m_1}^{-1} \leq 2\, e_{m_0-m_1}^{-1} \max\{e_{m_0}^{-1}, e_{m_1}^{-1}\}.$$

Using (61) and $m_1 \leq m_0$, we obtain

$$e_{m_0}^{-1} e_{m_1}^{-1} \leq 2\,(2(m_0 - m_1))^{\alpha}\,(2m_0)^{\alpha} \leq N_1^2\, m_0^{\alpha}(m_0 - m_1)^{\alpha},$$

which proves (69) for $r = 1$.

Assume $r \geq 2$ and that (69) holds for all smaller values of r. Let $h \in \{0, 1, \ldots, r\}$ be an index such that $e_{m_h}^{-1}$ is minimal among $e_{m_0}^{-1}, \ldots, e_{m_r}^{-1}$. Applying (70) to the neighboring pairs, we obtain

$$e_{m_h}^{-1} \leq 2\, e_{m_{h-1}-m_h}^{-1} \quad (\text{if } h \geq 1), \qquad e_{m_h}^{-1} \leq 2\, e_{m_h-m_{h+1}}^{-1} \quad (\text{if } h \leq r-1).$$

Combining these inequalities with (61), we obtain

$$e_{m_h}^{-1} \leq 2^{\alpha+1}\,(m_{h-1} - m_h)^{\alpha} \qquad \text{if } h \geq 1,$$

and

$$e_{m_h}^{-1} \leq 2^{\alpha+1}\,(m_h - m_{h+1})^{\alpha} \qquad \text{if } h \leq r-1.$$

Apply the induction hypothesis to the single decreasing sequence obtained by omitting m_h from $m_0, m_1, \ldots, m_r$. If $1 \leq h \leq r-1$, put $a := m_{h-1} - m_h$ and $b := m_h - m_{h+1}$. The two gaps a and b are replaced by the single gap $a + b$, so

$$\prod_{j=0}^{r} e_{m_j}^{-1} \leq N_1^r m_0^{\alpha}\, e_{m_h}^{-1}(a+b)^{\alpha} \prod_{\substack{1 \leq j \leq r \\ j \neq h, h+1}} (m_{j-1} - m_j)^{\alpha}.$$

The preceding estimates give $e_{m_h}^{-1} \leq 2^{\alpha+1}\min\{a, b\}^{\alpha}$. Since $(a+b)\min\{a, b\} \leq 2ab$, we obtain

$$e_{m_h}^{-1}(a+b)^{\alpha} \leq 2^{2\alpha+1} a^{\alpha} b^{\alpha} = N_1 a^{\alpha} b^{\alpha},$$

which proves (69) in this case. If $h = 0$, the induction hypothesis for $m_1 > \cdots > m_r$, together with $m_1^{\alpha} \leq m_0^{\alpha}$ and $e_{m_0}^{-1} \leq 2^{\alpha+1}(m_0 - m_1)^{\alpha}$, gives the same conclusion because $2^{\alpha+1} \leq N_1$. If $h = r$, apply the induction hypothesis to $m_0 > \cdots > m_{r-1}$ and use $e_{m_r}^{-1} \leq 2^{\alpha+1}(m_{r-1} - m_r)^{\alpha}$. This completes the proof. □

LEMMA II.1.7. *With the notation above,*

$$\delta_k \leq \Lambda_k \tag{71}$$

holds for any $k \in \mathbb{Z}_{\geq 1}$. In particular, δ_k grows at most exponentially.

PROOF. The case $k = 1$ is clear. Assume $k \geq 2$ and that (71) holds for all positive integers less than k.

First note that for any $j, \ell \geq 1$, one has

$$\frac{\Lambda_j \Lambda_\ell}{\Lambda_{j+\ell}} = \frac{(j+\ell)^{2\alpha}}{j^{2\alpha}\ell^{2\alpha}} \cdot \frac{1}{N_2} = \left(\frac{1}{\ell} + \frac{1}{j}\right)^{2\alpha} \cdot \frac{1}{N_2} \leq \frac{2^{2\alpha}}{N_2} = \frac{2^{2\alpha}}{2^{5\alpha+1}} \leq 1.$$

Therefore we have $\Lambda_j \Lambda_\ell \leq \Lambda_{j+\ell}$ for any $j, \ell \geq 1$. Consequently, by induction on the number of factors, we have the following: for any $\lambda \in \mathbb{Z}_{\geq 1}$ and $j, j_1, j_2, \ldots, j_\lambda \in \mathbb{Z}_{\geq 1}$,

$$j_1 + j_2 + \cdots + j_\lambda = j \implies \Lambda_{j_1} \cdot \Lambda_{j_2} \cdots \Lambda_{j_\lambda} \leq \Lambda_j \tag{72}$$

holds.

Note that, just by directly applying (72) for an admissible $(\ell_1, \ell_2, \ldots, \ell_s)$, we only have

$$\delta_{\ell_1} \cdots \delta_{\ell_s} \leq \Lambda_{\ell_1} \cdots \Lambda_{\ell_s} \leq \Lambda_k \tag{73}$$

and hence $\omega_k \leq \Lambda_k$, which is not sufficient to show the assertion.

Take $s \in \{2, 3, \ldots, k\}$ and s-tuples $(\ell_1, \ell_2, \ldots, \ell_s) \in \mathbb{Z}_+^s$ satisfying

$$\ell_1 + \ell_2 + \cdots + \ell_s = k, \qquad k \geq \ell_1 \geq \ell_2 \geq \cdots \geq \ell_s \geq 1$$

such that

$$\delta_k = \frac{1}{e_{k-1}} \cdot \delta_{\ell_1} \cdot \delta_{\ell_2} \cdots \delta_{\ell_s}. \tag{74}$$

Step 1. Let us set

$$k_0 := k, \qquad k_1 := \ell_1, \qquad \Delta_0 := \delta_{\ell_2} \delta_{\ell_3} \cdots \delta_{\ell_s}.$$

With this notation, (74) can be rewritten as

$$\delta_k = e_{k_0-1}^{-1} \, \Delta_0 \cdot \delta_{k_1}. \tag{75}$$

Here Δ_0 is a product of δ_j's whose indices are all $\leq k/2$. We call Δ_0 the (only) *block* in this case (and we say that the *number of blocks* is 1 $(= r + 1$, by letting $r := 0))$. Moreover, the sum of the indices appearing in Δ_0 equals

$$\sum_{\ell_\bullet \text{ appearing in } \Delta_0} \ell_\bullet = \ell_2 + \ell_3 + \cdots + \ell_s = k - k_1,$$

which is less than $k/2$ if $k_1 > k/2$. If $k_1 \leq k/2$, there is no further decomposition to perform after replacing Δ_0 with $\Delta_0 \cdot \delta_{k_1}$.

Step 2 (Only when $k_1 > k/2$). Assume next that $k_1 = \ell_1 > k/2$. Apply the defining decomposition (74) to δ_{ℓ_1}: choose an integer $s_1 \in \{2, 3, \dots, \ell_1\}$ and an s_1-tuple $(\ell_{1,1}, \ell_{1,2}, \dots, \ell_{1,s_1}) \in \mathbb{Z}_+^{s_1}$ with

$$\ell_{1,1} + \ell_{1,2} + \cdots + \ell_{1,s_1} = \ell_1, \qquad \ell_{1,1} \geq \ell_{1,2} \geq \cdots \geq \ell_{1,s_1} \geq 1,$$

such that

$$\delta_{\ell_1} = \frac{1}{e_{\ell_1 - 1}} \, \delta_{\ell_{1,1}} \cdot \delta_{\ell_{1,2}} \cdots \delta_{\ell_{1,s_1}}. \tag{76}$$

Substituting (76) into (74), we obtain

$$\delta_k = e_{k-1}^{-1} \, e_{\ell_1 - 1}^{-1} \left(\delta_{\ell_2} \cdot \delta_{\ell_3} \cdots \delta_{\ell_s} \right) \left(\delta_{\ell_{1,2}} \cdot \delta_{\ell_{1,3}} \cdots \delta_{\ell_{1,s_1}} \right) \cdot \delta_{\ell_{1,1}}.$$

Now set

$$k_0 := k, \qquad k_1 := \ell_1, \qquad k_2 := \ell_{1,1},$$

and define blocks by

$$\Delta_0 := \delta_{\ell_2} \cdot \delta_{\ell_3} \cdots \delta_{\ell_s}, \qquad \Delta_1 := \delta_{\ell_{1,2}} \cdot \delta_{\ell_{1,3}} \cdots \delta_{\ell_{1,s_1}}.$$

Then we can rewrite the above equality as

$$\delta_k = \left(e_{k_0 - 1}^{-1} \Delta_0 \right) \left(e_{k_1 - 1}^{-1} \Delta_1 \right) \delta_{k_2}.$$

Note that all indices appearing in Δ_0 satisfy $\leq \ell_2 \leq \ell_1 = k_1$; moreover, since $k_1 > k/2$, we have

$$\sum_{\ell_\bullet \text{ appearing in } \Delta_0} \ell_\bullet = \ell_2 + \ell_3 + \cdots + \ell_s = k - \ell_1 = k_0 - k_1 < \frac{k}{2}$$

and

$$\sum_{\ell_\bullet \text{ appearing in } \Delta_1} \ell_\bullet = \ell_{1,2} + \ell_{1,3} + \cdots + \ell_{1,s_1} = \ell_1 - \ell_{1,1} = k_1 - k_2,$$

which is less than $k/2$ if $k_2 > k/2$. In particular, every index appearing in Δ_0 is $\leq k/2$. Thus Δ_0 is already a product of δ_j's with indices $\leq k/2$. For Δ_1, the indices are bounded by $\ell_{1,1} \leq k_1$; if additionally $\ell_{1,1} \leq k/2$, then the procedure stops here by replacing Δ_1 with $\Delta_1 \cdot \delta_{k_2}$. We call Δ_0 and Δ_1 the two *blocks* in this case, so that the *number of blocks* is 2 ($= r + 1$ by letting $r := 1$).

Iteration. Starting from the expression in Step 2, we continue as follows. Assume that we have obtained integers

$$k_0 := k > k_1 > \cdots > k_p$$

and blocks $\Delta_0, \Delta_1, \dots, \Delta_{p-1}$ such that

$$\delta_k = \left(e_{k_0 - 1}^{-1} \Delta_0 \right) \cdot \left(e_{k_1 - 1}^{-1} \Delta_1 \right) \cdots \left(e_{k_{p-1} - 1}^{-1} \Delta_{p-1} \right) \cdot \delta_{k_p}, \tag{77}$$

where every index appearing in each Δ_ν is $\leq k/2$.

Case 1: $k_p \le k/2$. In this case we stop. We absorb the remaining factor δ_{k_p} into the last block by replacing Δ_{p-1} with $\Delta_{p-1} \cdot \delta_{k_p}$. Then (77) becomes

$$\delta_k = \left(e_{k_0-1}^{-1}\Delta_0\right) \cdot \left(e_{k_1-1}^{-1}\Delta_1\right) \cdots \left(e_{k_{p-1}-1}^{-1}\Delta_{p-1}\right).$$

We set $r := p-1$. Then $k_0 := k > k_1 > \cdots > k_r > k/2$ holds, and the last block $\Delta_r = \Delta_{p-1}$ is a product of finitely many δ_j's with $j \le k/2$.

Case 2: $k_p > k/2$. Apply the defining decomposition to δ_{k_p}: choose an integer $s_p \ge 2$ and an s_p-tuple $(\mu_{p,1}, \mu_{p,2}, \ldots, \mu_{p,s_p}) \in \mathbb{Z}_+^{s_p}$ such that

$$\mu_{p,1} + \mu_{p,2} + \cdots + \mu_{p,s_p} = k_p, \qquad \mu_{p,1} \ge \mu_{p,2} \ge \cdots \ge \mu_{p,s_p} \ge 1,$$

and

$$\delta_{k_p} = \frac{1}{e_{k_p-1}}\, \delta_{\mu_{p,1}} \cdot \delta_{\mu_{p,2}} \cdots \delta_{\mu_{p,s_p}}.$$

Set

$$k_{p+1} := \mu_{p,1}, \qquad \Delta_p := \delta_{\mu_{p,2}} \cdot \delta_{\mu_{p,3}} \cdots \delta_{\mu_{p,s_p}}.$$

Substituting this into (77), we obtain

$$\delta_k = \left(e_{k_0-1}^{-1}\Delta_0\right) \cdot \left(e_{k_1-1}^{-1}\Delta_1\right) \cdots \left(e_{k_p-1}^{-1}\Delta_p\right) \cdot \delta_{k_{p+1}}.$$

Since $k_{p+1} < k_p$, repeating Case 2 yields a strictly decreasing chain

$$k_0 := k > k_1 > \cdots > k_r > \frac{k}{2}, \qquad k_{r+1} \le \frac{k}{2},$$

and after stopping at Case 1 and absorbing the remaining factor into the last block, we obtain a factorization

$$\delta_k = \left(e_{k_0-1}^{-1}\Delta_0\right) \cdot \left(e_{k_1-1}^{-1}\Delta_1\right) \cdots \left(e_{k_r-1}^{-1}\Delta_r\right), \tag{78}$$

where every index appearing in each Δ_ν is $\le k/2$.

By construction, every index appearing in each Δ_ν is $\le k/2$. We call each Δ_ν a *block*. Hence the number of blocks is $r+1$. Moreover, the indices in Δ_ν add up to

$$\sum_{\ell_\bullet \text{ appearing in } \Delta_\nu} \ell_\bullet = k_\nu - k_{\nu+1} \qquad (\nu = 0, 1, \ldots, r-1).$$

For the last block Δ_r (after the absorption performed in Case 1), the indices add up to

$$\sum_{\ell_\bullet \text{ appearing in } \Delta_r} \ell_\bullet = (k_r - k_{r+1}) + k_{r+1} = k_r.$$

Consequently,

$$\sum_{\nu=0}^{r} \sum_{\ell_\bullet \text{ appearing in } \Delta_\nu} \ell_\bullet = k_0 = k.$$

Let s be the number of factors in Δ_r, and set $t := r + s$.

Since every index appearing in Δ_p is less than or equal to $k/2 < k$, we may apply (73) to each Δ_p. Writing

$$x_p := k_{p-1} - k_p \quad (p = 1, 2, \dots, r), \qquad \Delta_r = \delta_{y_1} \cdot \delta_{y_2} \cdots \delta_{y_s} \quad \Big(1 \le y_q \le \frac{k}{2},\ \sum_{q=1}^{s} y_q = k_r\Big),$$

we obtain

$$\prod_{p=0}^{r} \Delta_p \le \Lambda_{x_1} \cdots \Lambda_{x_r} \cdot \Lambda_{y_1} \cdots \Lambda_{y_s}. \tag{79}$$

Applying Lemma II.1.6 to the decreasing chain $k_0 - 1 > k_1 - 1 > \cdots > k_r - 1 > 0$, we obtain

$$\prod_{p=0}^{r} e_{k_p-1}^{-1} \le N_1^{r+1}\,(k-1)^{\alpha} \prod_{p=1}^{r} (k_{p-1} - k_p)^{\alpha} = N_1^{r+1}\,(k-1)^{\alpha} \prod_{p=1}^{r} x_p^{\alpha}. \tag{80}$$

Combining (78), (79), and (80), and using $\Lambda_m = m^{-2\alpha} N_2^{m-1}$, we get

$$\begin{aligned}
\delta_k &\le \left(\prod_{p=0}^{r} e_{k_p-1}^{-1}\right) \cdot \left(\prod_{p=0}^{r} \Delta_p\right) \\
&\le N_1^{r+1} (k-1)^{\alpha} \left(\prod_{p=1}^{r} x_p^{\alpha}\right) \cdot \left(\prod_{p=1}^{r} x_p^{-2\alpha} N_2^{x_p - 1}\right) \left(\prod_{q=1}^{s} y_q^{-2\alpha} N_2^{y_q - 1}\right) \\
&= N_1^{r+1} N_2^{k-r-s}\,(k-1)^{\alpha} \cdot \left(\prod_{p=1}^{r} x_p^{-\alpha}\right) \left(\prod_{q=1}^{s} y_q^{-2\alpha}\right).
\end{aligned}$$

Equivalently,

$$\delta_k \le (k-1)^{-2\alpha} N_2^{k} \cdot \left(N_1^{r+1} N_2^{-(r+s)} \cdot \frac{(k-1)^{3\alpha}}{\left(\prod_{p=1}^{r} x_p\right)^{\alpha} \left(\prod_{q=1}^{s} y_q^2\right)^{\alpha}} \right). \tag{81}$$

Now note that $\sum_{q=1}^{s} y_q = k_r > k/2$ and $y_q \le k/2$ by construction, and

$$\sum_{p=1}^{r} x_p + \sum_{q=1}^{s} y_q = k.$$

Hence we can apply Lemma II.1.4 (with $t = r + s$) to obtain

$$\Big(\prod_{p=1}^{r} x_p\Big)\Big(\prod_{q=1}^{s} y_q^2\Big) \ge k^3\, 8^{\,1-(r+s)}.$$

Therefore,

$$\frac{(k-1)^{3\alpha}}{\left(\prod_{p=1}^{r} x_p\right)^{\alpha}\left(\prod_{q=1}^{s} y_q^2\right)^{\alpha}} \le \left(\frac{(k-1)^3}{k^3}\right)^{\alpha} \cdot 8^{\alpha(r+s-1)} \le 8^{\alpha(r+s-1)}. \tag{82}$$

Using $N_2 = 8^{\alpha} N_1$, we have

$$N_1^{r+1} N_2^{-(r+s)} \cdot 8^{\alpha(r+s-1)} \le N_1^{r+s-1} N_2^{-(r+s)} \cdot 8^{\alpha(r+s-1)} = \frac{1}{N_2}\left(\frac{8^{\alpha} N_1}{N_2}\right)^{r+s-1} = \frac{1}{N_2}.$$

Thus, retaining the factor $((k-1)/k)^{3\alpha}$ in (82) and using (81), we obtain

$$\begin{aligned} \delta_k &\le (k-1)^{-2\alpha} N_2^{k-1} \left(\frac{k-1}{k}\right)^{3\alpha} \\ &= \left(\frac{k-1}{k}\right)^{\alpha} k^{-2\alpha} N_2^{k-1} \le \Lambda_k, \end{aligned}$$

which completes the induction step. □

Proof of Theorem II.1.1. By Remark II.1.3, it suffices to consider the case $M = R = 1$. Let

$$B(u) = \sum_{n \ge 1} b_n u^n, \qquad b_1 := 1,$$

be the power series defined by (64). By the explicit expression (65), the series $B(u)$ has a positive radius of convergence. Hence there exists a positive constant L_0 such that

$$b_n \le L_0^n \qquad (n \ge 1).$$

On the other hand, by Lemma II.1.7, there exists a positive constant L_1 such that

$$\delta_n \le L_1^n \qquad (n \ge 1).$$

By Lemma II.1.5, it follows that

$$|a_n| \le b_n \, \delta_n \le (L_0 L_1)^n \qquad (n \ge 1).$$

Therefore the power series $A(u) = \sum_{n \ge 1} a_n u^n$ has a positive radius of convergence, which completes the proof. □

2 On matrices arising from certain graphs

In this section, we collect some elementary linear-algebraic facts on matrices arising from certain graphs, which will be used later as formal tools in complex geometry. Although we will follow the presentation in [KUed2], the main ingredients are essentially contained in classical results of Bourbaki [Bour, Chap. 5, §3.5, Lemme 4]; see also [BHPV, Lemma 2.10].

Let A be a square matrix of order $N \ge 1$ satisfying the following three properties.

($\mathbf{P}_1$) A is a symmetric matrix with real entries.

($\mathbf{P}_2$) Any non-diagonal entry of A is non-negative.

($\mathbf{P}_3$) For any partition (I, J) of $\{1, 2, \ldots, N\}$, there exist elements $i \in I$ and $j \in J$ such that the (i, j) entry of A is positive.

Here we say that (I, J) is a *partition* of $\{1, 2, \ldots, N\}$ if I and J are non-empty subsets of $\{1, 2, \ldots, N\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{1, 2, \ldots, N\}$ hold.

Let us briefly comment on the meaning of **($\mathbf{P}_3$)**. Denote by $\Gamma(A)$ the finite simple graph with N vertices $v_1, v_2, \ldots, v_N$, where for each pair of distinct indices $i \neq j$ in $\{1, 2, \ldots, N\}$, the vertices v_i and v_j are joined by an edge if and only if the (i, j)-entry of A is positive. By **($\mathbf{P}_1$)**, this graph is well-defined as an undirected graph. Then condition **($\mathbf{P}_3$)** is equivalent to the connectedness of the graph $\Gamma(A)$ (Indeed, the negation of **($\mathbf{P}_3$)** means that there exists a partition (I, J) of $\{1, 2, \ldots, N\}$ such that no vertex v_i with $i \in I$ is joined by an edge to a vertex v_j with $j \in J$. Hence $\Gamma(A)$ is the disjoint union of the two non-empty subgraphs spanned by $\{v_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$, so $\Gamma(A)$ is not connected. Conversely, if $\Gamma(A)$ is not connected, then the vertex set I of one connected component and the set J of all remaining vertices give a partition (I, J) for which every (i, j)-entry with $i \in I$ and $j \in J$ is zero, so **($\mathbf{P}_3$)** fails).

It is known that this connectedness condition is, in turn, equivalent to the matrix A being *irreducible* in the sense of non-negative matrix theory (see [HJ, Definitions 6.2.21–6.2.22] and [HJ, Theorem 6.2.24]).

Under the assumptions **($\mathbf{P}_1$)**–**($\mathbf{P}_3$)**, we record the following two results:

Lemma II.2.1 (=[KUed2, Lemma 3.3]). *Let N be a positive integer, A a square matrix of order N which satisfies* **($\mathbf{P}_1$)**, **($\mathbf{P}_2$)**, *and* **($\mathbf{P}_3$)**, *and $\mathfrak{m} \in \mathbb{R}^N$ be an eigenvector associated to the largest eigenvalue of A. Then either $\mathfrak{m}$ or $-\mathfrak{m}$ is an element of $\mathbb{R}^N_+$.*

Proposition II.2.2 (=[KUed2, Proposition 3.2]). *Let N be a positive integer and A be a square matrix of order N which satisfies* **($\mathbf{P}_1$)**, **($\mathbf{P}_2$)**, *and* **($\mathbf{P}_3$)**. *Then the following holds for the largest eigenvalue $\lambda(A)$ of A.*
(i) $\lambda(A) > 0$ if and only if there exists an element $\mathfrak{m} \in \mathbb{R}^N_+$ such that $A\mathfrak{m} \in \mathbb{R}^N_+$.
(ii) $\lambda(A) = 0$ if and only if there exists an element $\mathfrak{m} \in \mathbb{R}^N_+$ such that $A\mathfrak{m} = 0$.
(iii) $\lambda(A) < 0$ if and only if there exists an element $\mathfrak{m} \in \mathbb{R}^N_+$ such that $-A\mathfrak{m} \in \mathbb{R}^N_+$.
In each of the assertions (i), (ii), and (iii), "$\mathfrak{m} \in \mathbb{R}^N_+$" can be replaced with "$\mathfrak{m} \in \mathbb{Z}^N_+$" when all entries of A are integers.

Although elementary proofs of these results are given in [KUed2], one can also obtain alternative proofs by using the Perron–Frobenius theorem.

Mini-Exercise II.2.3. Following [KUed2], give a self-contained exposition of the proofs of Lemma II.2.1 and Proposition II.2.2.

Mini-Exercise II.2.4. By adding a sufficiently large positive constant c times the identity matrix of order N to A, derive Lemma II.2.1 and Proposition II.2.2 from Theorem II.2.5 below.

THEOREM II.2.5 (Perron–Frobenius). *Let N be a positive integer, and let A be a square matrix of order N satisfying* $\mathbf{(P_1)}$, $\mathbf{(P_2)}$, *and* $\mathbf{(P_3)}$. *Assume furthermore that all diagonal entries of A are non-negative. Then the following statements hold.*
(i) All eigenvalues of A are real numbers. In what follows, the largest eigenvalue of A is denoted by $\lambda(A)$.
(ii) The largest eigenvalue $\lambda(A)$ is non-negative and is an algebraically simple root of the characteristic polynomial of A.
(iii) There exists a unique element $\mathfrak{m} \in \mathbb{R}^N_+$ such that

$$A\mathfrak{m} = \lambda(A)\mathfrak{m} \qquad \text{and} \qquad \|\mathfrak{m}\|_{L^\infty} = 1,$$

where $\|\mathfrak{m}\|_{L^\infty} := \max\{m_1, m_2, \ldots, m_N\}$ for $\mathfrak{m} = {}^t(m_1, m_2, \ldots, m_N) \in \mathbb{R}^N_+$.
(iv) Assume that there exist an element $\mathfrak{m} \in \mathbb{R}^N_+$ and a non-negative real number ℓ such that $A\mathfrak{m} = \ell\,\mathfrak{m}$. Then $\ell = \lambda(A)$.

PROOF. The case $N = 1$ is obvious, so we assume $N > 1$. By assumption $\mathbf{(P_1)}$, the matrix A is real symmetric, hence all its eigenvalues are real. This proves (i). Statements (ii) and (iii) are classical consequences of the Perron–Frobenius theorem for irreducible non-negative matrices; see [HJ, Theorem 8.4.4]. Finally, statement (iv) follows from the characterization of the largest eigenvalue in terms of non-negative eigenvectors; see [HJ, Theorem 8.3.4]. □

3 Categories and sheaves

3.1 Categories and related notions

A *category* $\mathfrak{C}$ consists of the following data: *objects* and *morphisms* (also called *arrows* in some literature), satisfying the axioms recalled below. In this note, the collection of objects of $\mathfrak{C}$ is denoted by $\mathrm{ob}(\mathfrak{C})$. For each pair of objects $X, Y \in \mathrm{ob}(\mathfrak{C})$, the collection of morphisms from X to Y is denoted by

$$\mathrm{Hom}_{\mathfrak{C}}(X, Y).$$

The notation

$$\mathfrak{C}(X, Y) := \mathrm{Hom}_{\mathfrak{C}}(X, Y)$$

is also standard. In these notes, we use the notation $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$ whenever possible, reserving $\mathfrak{C}(X, Y)$ for cases in which the subscript makes the notation difficult to read. An element $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$, a morphism from X to Y, is often displayed as

$$f\colon X \to Y,$$

since, in many concrete categories appearing in this note, objects can be regarded as sets endowed with additional structures, and morphisms as structure-preserving maps between them; in such situations this notation is quite natural. Even in general, however, we shall follow this standard convention and write $f\colon X \to Y$, even when f is not literally a map between sets in the usual sense.

Here we deliberately use the word "collection" rather than "set", since these are not assumed to be sets in the sense of axiomatic set theory; they may be proper classes. For example, for the category (Set) of sets and maps, the collection $\mathrm{ob}((\mathsf{Set}))$ is the collection of all sets, and hence is not a set in our ZFC framework: indeed, if it were a set V, then one would have $V \in V$, contradicting the axiom of foundation (one may also keep in mind the classical Russell paradox here). We shall not pursue foundational issues any further in this note. Basically, such set-theoretic difficulties will be avoided in the complex analytic geometry discussed in this note by the following conventions:

- whenever a category $\mathfrak{C}$ is used as an indexing category, we shall always assume that $\mathfrak{C}$ is *small*, namely that $\mathrm{ob}(\mathfrak{C})$ and all $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$ $(X, Y \in \mathrm{ob}(\mathfrak{C}))$ are sets;
- in general, we shall mostly deal with *locally small* categories $\mathfrak{C}$, namely categories such that $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$ is a set for every $X, Y \in \mathrm{ob}(\mathfrak{C})$, and all objects that arise essentially in our arguments will be understood to remain within a set-theoretically harmless range.

The axioms required for $\mathfrak{C}$ to be a category are as follows:

- each morphism f belongs to $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$ for a unique pair of objects $X, Y \in \mathrm{ob}(\mathfrak{C})$ (in other words, for each morphism f, there are uniquely determined objects $X, Y \in \mathrm{ob}(\mathfrak{C})$ such that $f\colon X \to Y$);
- for each triple of objects $X, Y, Z \in \mathrm{ob}(\mathfrak{C})$, there is a composition law

$$\mathrm{Hom}_{\mathfrak{C}}(Y, Z) \times \mathrm{Hom}_{\mathfrak{C}}(X, Y) \to \mathrm{Hom}_{\mathfrak{C}}(X, Z), \qquad (g, f) \mapsto g \circ f;$$

- composition is associative: if

$$f\colon W \to X, \qquad g\colon X \to Y, \qquad h\colon Y \to Z$$

  are morphisms, then

$$(h \circ g) \circ f = h \circ (g \circ f);$$

- for each object $X \in \mathrm{ob}(\mathfrak{C})$, there exists an identity morphism

$$\mathrm{id}_X \in \mathrm{Hom}_{\mathfrak{C}}(X, X)$$

  such that

$$f \circ \mathrm{id}_X = f \qquad \text{for every} \qquad f\colon X \to Y,$$

  and

$$\mathrm{id}_X \circ g = g \qquad \text{for every} \qquad g\colon Z \to X$$

  (It is immediate from the axioms that such an identity morphism is necessarily unique. Throughout this note, if X is an object of a category, the symbol id_X always denotes the identity morphism of X, unless otherwise specified).

The first and most basic way of using the language of categories is to make explicit what the objects are and what the morphisms between them are. In many examples, the objects are sets endowed with some additional structure, and the morphisms are maps preserving that structure. Let us recall some standard examples.

- The category (Set) is the category whose objects are sets and whose morphisms are maps between sets.
- The category (Top) is the category whose objects are topological spaces and whose morphisms are continuous maps.
- The category (Group) is the category whose objects are groups and whose morphisms are group homomorphisms.
- The category (Ab) is the category whose objects are abelian groups and whose morphisms are group homomorphisms between them.

Similarly, in algebra and geometry we shall use the following categories frequently.

- The category (Ring) is the category whose objects are rings and whose morphisms are ring homomorphisms (in the sense fixed in Chapter I).
- For a ring R, the category (Mod(R)) is the category whose objects are R-modules and whose morphisms are R-module homomorphisms. Under this convention, (Ab) is nothing but (Mod($\mathbb{Z}$)). Also, (Mod($\mathbb{C}$)) is the category of complex vector spaces and complex-linear maps; this category is often denoted by (Vect).
- For a ring R, the category (CxMod(R)) is the category whose objects are complexes of R-modules and whose morphisms are morphisms of complexes, in the sense recalled in Chapter I. Chain complexes and cochain complexes are formally analogous. In this note, however, unless otherwise specified, an object of (CxMod(R)) will be understood to be a cochain complex.
- The category (C^∞Mfd) is the category whose objects are C^∞ manifolds and whose morphisms are C^∞ maps between them.

These examples explain the most familiar point of view on categories: a category provides a framework in which one specifies a class of mathematical objects together with the appropriate structure-preserving maps between them (See also [Lee-Jo1, p. 210] for further geometric examples). However, the notion of a category is broader than this concrete picture.

For example, a *monoid* can be regarded as a category with exactly one object. More precisely, if M is a monoid (i.e., a set endowed with an associative binary operation and an identity element), we define a category with one object $*$ by setting

$$\mathrm{Hom}(*,*) := M,$$

and by defining composition by the multiplication of M. A group is the special case where every element of M admits an inverse with respect to the multiplication.

Partially ordered sets (i.e., sets endowed with a reflexive, antisymmetric, and transitive relation) give another important class of examples. Let $(I, \leq)$ be a partially ordered set. We may regard I as a category by taking the elements of I as objects and by declaring that there is a unique morphism from i to j if and only if $i \leq j$. In other words,

$$\mathrm{Hom}(i,j) = \begin{cases} \{i \to j\} & \text{if } i \leq j, \\ \emptyset & \text{otherwise.} \end{cases}$$

The reflexivity and transitivity of $\leq$ correspond precisely to the existence of identity morphisms and the composition law. More generally, the same construction works for a *preorder*, namely a set endowed with a reflexive and transitive relation. Conversely, a category in which there is at most one morphism between any two objects may be viewed as a preorder.

Directed graphs are also closely related to categories. A small category may be viewed as a directed graph equipped with identity morphisms and a composition law satisfying associativity. Conversely, a directed graph gives rise to a category by taking paths in the graph as morphisms; this is called the *path category* of the directed graph.

Thus the notion of a category simultaneously generalizes several different kinds of structures: mathematical objects with structure-preserving maps, monoids and groups, partially ordered sets and preorders, and directed graphs with composable paths. In particular, in an abstract category it is usually meaningless to speak about an "element" of an object (Arguments based on elements are natural in categories such as (Set), (Group), or $(\mathsf{Mod}(R))$ for example, but they do not make sense in an arbitrary category). In this note, partially ordered sets, preorders, and directed graphs will mainly appear as indexing devices for diagrams and related constructions.

Before proceeding further, let us introduce one more general construction that produces a new category from a given category.

Example II.3.1. Let $\mathfrak{C}$ be a category. The *opposite category* (also called the *dual category*) of $\mathfrak{C}$, denoted by

$$\mathfrak{C}^{\mathrm{op}},$$

is obtained by reversing the direction of every morphism in $\mathfrak{C}$. More precisely, the objects of $\mathfrak{C}^{\mathrm{op}}$ are the same as those of $\mathfrak{C}$:

$$\mathrm{ob}(\mathfrak{C}^{\mathrm{op}}) = \mathrm{ob}(\mathfrak{C}),$$

and, for objects $X, Y \in \mathrm{ob}(\mathfrak{C}^{\mathrm{op}})$, we set

$$\mathrm{Hom}_{\mathfrak{C}^{\mathrm{op}}}(X,Y) := \mathrm{Hom}_{\mathfrak{C}}(Y,X).$$

□

Mini-Exercise II.3.2. Let $\mathfrak{C}$ be a category. Define identity morphisms and a composition law on $\mathfrak{C}^{\mathrm{op}}$ in a natural way, and show that $\mathfrak{C}^{\mathrm{op}}$ is indeed a category. Also show that

$$(\mathfrak{C}^{\mathrm{op}})^{\mathrm{op}} = \mathfrak{C}.$$

By reversing all morphisms in a categorical definition or statement, one obtains its *dual*. Since $(\mathfrak{C}^{\mathrm{op}})^{\mathrm{op}} = \mathfrak{C}$, the dual of the dual is the original definition or statement. This observation is often called the *duality principle*: every assertion formulated purely in terms of objects, morphisms, identity morphisms, and composition has a formally dual assertion obtained by reversing all morphisms.

Definition II.3.3. Let $\mathfrak{C}$ be a category. A morphism

$$f\colon X \to Y$$

in $\mathfrak{C}$ is called an *isomorphism* if there exists a morphism

$$g\colon Y \to X$$

such that

$$g \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ g = \mathrm{id}_Y.$$

In this case, the morphism g is uniquely determined by f and is called the *inverse* of f; it is denoted by

$$f^{-1}\colon Y \to X.$$

Two objects $X, Y \in \mathrm{ob}(\mathfrak{C})$ are said to be *isomorphic* if there exists an isomorphism from X to Y. In this case, we write

$$X \cong Y.$$

□

Remark II.3.4. In familiar categories such as (Ab) and $(\mathsf{Mod}(R))$, a morphism is an isomorphism if and only if it is both injective and surjective as a map of sets. This familiar characterization should not, however, be taken as the definition of an isomorphism in an arbitrary category. Indeed, in a general category, an object need not have elements in any meaningful sense. Consequently, expressions such as "for every element of Y" or "two elements of X have the same image" may not make sense at all. Before using words such as "injective" or "surjective" in an abstract categorical setting, one must therefore ask how such notions can be formulated purely in terms of objects and morphisms (refer to the explanation on *monomorphisms* and *epimorphisms* in [Mac, §I.3, I.4] and [St, Tag 003A] for example). □

Relationships between categories are described by *functors*, and relationships between functors are further described by *natural transformations* (for the definitions, see [Mac, §I.5] and [St, Tag 0013], for example). Here we briefly recall only the compatibility conditions contained in these definitions by drawing the following commutative diagrams. For a functor F from a category $\mathfrak{C}$ to another category $\mathfrak{D}$, it is required that F send the composition diagram on the left below to the commutative diagram on the right for all composable morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathfrak{C}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g\circ f}\searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array} \qquad \overset{F}{\rightsquigarrow} \qquad \begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ & {\scriptstyle F(g\circ f)}\searrow & \downarrow{\scriptstyle F(g)} \\ & & F(Z). \end{array}$$

For functors F and G from $\mathfrak{C}$ to $\mathfrak{D}$, a natural transformation τ from F to G consists of a family of morphisms

$$\tau_W \colon F(W) \to G(W) \qquad \big(W \in \mathrm{ob}(\mathfrak{C})\big)$$

such that the following diagram commutes for every morphism $f \colon X \to Y$ in $\mathfrak{C}$:

$$\begin{array}{ccc} X & \overset{\tau}{\rightsquigarrow} & F(X) \xrightarrow{\tau_X} G(X) \\ {\scriptstyle f}\downarrow & & {\scriptstyle F(f)}\downarrow \qquad\qquad \downarrow{\scriptstyle G(f)} \\ Y & & F(Y) \xrightarrow[\tau_Y]{} G(Y) \end{array}$$

(One should also keep in mind that a functor is required to satisfy $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object $X \in \mathrm{ob}(\mathfrak{C})$).

In this note, we use the following notation: For categories $\mathfrak{C}$ and $\mathfrak{D}$, a functor F from $\mathfrak{C}$ to $\mathfrak{D}$ is written as

$$F \colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{D}.$$

For functors $F, G \colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{D}$, a natural transformation τ from F to G is written as

$$\tau \colon F \xrightarrow{\text{nat}} G.$$

The composition of functors will be denoted by the same symbol $\circ$ as the composition of morphisms. Thus, if

$$F \colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{D}, \qquad G \colon \mathfrak{D} \xrightarrow{\text{functor}} \mathfrak{E}$$

are functors, then their composition is denoted by

$$G \circ F \colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{E}.$$

We shall also use $\circ$ for the composition of natural transformations whenever no confusion can arise.

The symbols $\to$, $\xrightarrow{\text{functor}}$, and $\xrightarrow{\text{nat}}$ are used only to make the level of the objects under consideration visually clear: morphisms in a category, functors between categories, and natural transformations between functors. From a more categorical point of view, this distinction is not intrinsic. For example, if one considers a category of categories, then functors themselves become morphisms in that category. In this note, however, we shall not make systematic use of such higher-level categorical language, and the above notation will be convenient.

For categories $\mathfrak{C}$ and $\mathfrak{D}$, we denote by

$$\mathrm{Fun}(\mathfrak{C}, \mathfrak{D})$$

the collection of functors from $\mathfrak{C}$ to $\mathfrak{D}$. Whenever no set-theoretic difficulty arises—for example, when $\mathfrak{C}$ is small and $\mathfrak{D}$ is locally small—we also regard

$$\mathrm{Fun}(\mathfrak{C}, \mathfrak{D})$$

as the category whose objects are functors from $\mathfrak{C}$ to $\mathfrak{D}$ and whose morphisms are natural transformations. If

$$F, G \colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{D}$$

are functors, we denote by

$$\mathrm{Nat}(F, G)$$

the collection of natural transformations from F to G. Thus,

$$\mathrm{Hom}_{\mathrm{Fun}(\mathfrak{C},\mathfrak{D})}(F, G) = \mathrm{Nat}(F, G)$$

whenever the functor category under consideration is set-theoretically harmless. In this case, an isomorphism in the category $\mathrm{Fun}(\mathfrak{C}, \mathfrak{D})$ is called a *natural isomorphism*. More generally, for $F, G \in \mathrm{Fun}(\mathfrak{C}, \mathfrak{D})$, a natural transformation

$$\tau \colon F \xrightarrow{\text{nat}} G$$

is called a natural isomorphism if and only if each component

$$\tau_X \colon F(X) \to G(X)$$

is an isomorphism.

Example II.3.5. For every category $\mathfrak{C}$, the functor

$$\mathrm{id}_{\mathfrak{C}} \colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{C}$$

which is the identity on both objects and morphisms is called the *identity functor* of $\mathfrak{C}$. □

Example II.3.6. *Forgetful functors* (functors obtained by forgetting part of the structure) give some typical examples of functors. For instance, a group can be regarded simply as a set by forgetting the group structure. This gives a functor

$$(\mathsf{Group}) \xrightarrow{\text{functor}} (\mathsf{Set}).$$

Similarly, an R-module can be regarded as an abelian group by forgetting the scalar multiplication by R, and hence we have a functor

$$(\mathsf{Mod}(R)) \xrightarrow{\text{functor}} (\mathsf{Ab}).$$

□

Example II.3.7. For each integer $n \in \mathbb{Z}$, the n-th cohomology construction gives a functor

$$H^n \colon (\mathsf{CxMod}(R)) \xrightarrow{\text{functor}} (\mathsf{Mod}(R)).$$

Indeed, a morphism of cochain complexes $f^\bullet \colon C^\bullet \to D^\bullet$ induces an R-module homomorphism

$$H^n(C^\bullet) \longrightarrow H^n(D^\bullet).$$

□

Example II.3.8. Diagrams also fit naturally into this language. If $\mathfrak{I}$ is a small category, then a *diagram in* $\mathfrak{C}$ *indexed by* $\mathfrak{I}$ is simply a functor

$$F\colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{C}.$$

Thus a diagram includes not only objects $F(i)$ of $\mathfrak{C}$ indexed by objects $i \in \mathrm{ob}(\mathfrak{I})$, but also morphisms $F(a)\colon F(i) \to F(j)$ corresponding to morphisms $a\colon i \to j$ in $\mathfrak{I}$, with the compatibility required by functoriality. For example, if $\mathfrak{I}$ is the category associated with a partially ordered set $(I, \leq)$, then such a functor is a system of objects together with compatible morphisms corresponding to the order relations. Such systems will be discussed in more detail in the subsection on projective and inductive systems below. □

Example II.3.9. Let $\mathfrak{C}$ be a locally small category, and let $X \in \mathrm{ob}(\mathfrak{C})$. The assignment

$$Y \longmapsto \mathrm{Hom}_{\mathfrak{C}}(X, Y)$$

defines a functor

$$\mathrm{Hom}_{\mathfrak{C}}(X, -)\colon \mathfrak{C} \xrightarrow{\text{functor}} (\mathsf{Set}).$$

Indeed, a morphism

$$h\colon Y \to Z$$

in $\mathfrak{C}$ induces a map

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{C}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathfrak{C}}(X, Z), \\ f &\longmapsto h \circ f. \end{aligned}$$

This functor is called the *covariant Hom functor* associated with X.

On the other hand, the assignment

$$Y \longmapsto \mathrm{Hom}_{\mathfrak{C}}(Y, X)$$

defines a functor

$$\mathrm{Hom}_{\mathfrak{C}}(-, X)\colon \mathfrak{C}^{\mathrm{op}} \xrightarrow{\text{functor}} (\mathsf{Set}).$$

Equivalently, a morphism

$$h\colon Y \to Z$$

in $\mathfrak{C}$ induces a map in the reverse direction

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{C}}(Z, X) &\longrightarrow \mathrm{Hom}_{\mathfrak{C}}(Y, X), \\ f &\longmapsto f \circ h. \end{aligned}$$

This functor is called the *contravariant Hom functor* associated with X. □

More generally, a functor

$$F\colon \mathfrak{C}^{\mathrm{op}} \xrightarrow{\text{functor}} \mathfrak{D}$$

is also called a *contravariant functor from* $\mathfrak{C}$ *to* $\mathfrak{D}$. When it is useful to emphasize the distinction, an ordinary functor

$$F\colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{D}$$

is called a *covariant functor*.

Example II.3.10. Consider the functors

$$T^n, T^{n+1} \colon (\mathsf{CxMod}(R)) \xrightarrow{\text{functor}} (\mathsf{Mod}(R))$$

which send a cochain complex $C^\bullet$ to its degree n and degree $n+1$ terms:

$$T^n(C^\bullet) := C^n, \qquad T^{n+1}(C^\bullet) := C^{n+1}.$$

The differential of a cochain complex gives, for each $C^\bullet$, an R-module homomorphism

$$d^n_C \colon C^n \to C^{n+1}.$$

The defining condition for a morphism of cochain complexes says exactly that these homomorphisms are compatible with morphisms of complexes. In other words, the differentials define a natural transformation

$$d^n \colon T^n \xrightarrow{\text{nat}} T^{n+1}.$$

□

Universal morphisms, the Yoneda lemma, and adjoint functors

In elementary finite-dimensional linear algebra, choosing bases allows one to describe linear maps by (finite) matrices. More fundamentally, even without choosing a basis of the target space, a linear map is uniquely determined by the images of the elements of a basis of the source space. We shall now formulate, in categorical terms, the principle that makes this familiar simplification in the description and treatment of morphisms possible.

Let

$$U \colon (\mathsf{Vect}) \xrightarrow{\text{functor}} (\mathsf{Set})$$

be the forgetful functor. Let V be a vector space, let $B \subset V$ be a basis, and denote by

$$\iota_B \colon B \to U(V)$$

the inclusion map. For every vector space W, composition with ι_B gives a bijection

$$\begin{aligned} \mathrm{Hom}_{(\mathsf{Vect})}(V, W) &\longrightarrow \mathrm{Hom}_{(\mathsf{Set})}(B, U(W)), \\ f &\longmapsto U(f) \circ \iota_B. \end{aligned}$$

Indeed, a map from B to $U(W)$ extends uniquely to a linear map from V to W. Moreover, these bijections are compatible with linear maps between the vector spaces W. Thus, choosing a basis permits us to describe all morphisms from V in terms of maps of sets from the simpler object B.

This observation may be formulated without starting from a previously given vector space. For a set S, let

$$\mathrm{Span}(S) := \bigoplus_{s \in S} \mathbb{C} \cdot s = \left\{ \sum_{s \in S} a_s s \;\middle|\; a_s \in \mathbb{C}, \quad a_s = 0 \text{ for all but finitely many } s \right\}$$

(Here each element $s \in S$ is regarded as a formal vector, and no non-trivial finite linear relation among the elements of S is imposed. Addition and scalar multiplication are defined coefficientwise). Then S is a basis of $\mathrm{Span}(S)$. Let

$$\iota_S \colon S \to U(\mathrm{Span}(S)), \qquad s \longmapsto s := 1 \cdot s$$

be the canonical map. For every vector space W and every map

$$f \colon S \to U(W),$$

there exists a unique linear map

$$\overline{f} \colon \mathrm{Span}(S) \to W$$

such that

$$U(\overline{f}) \circ \iota_S = f,$$

which is given explicitly by

$$\overline{f}\left(\sum_{s \in S} a_s s\right) = \sum_{s \in S} a_s f(s).$$

Equivalently, for every vector space W, we have a bijection

$$\mathrm{Hom}_{(\mathsf{Vect})}(\mathrm{Span}(S), W) \cong \mathrm{Hom}_{(\mathsf{Set})}(S, U(W))$$

which is *natural* in W.

Remark II.3.11. The word "natural" is used here in the precise categorical sense. For the fixed set S, consider the functors

$$H_1, H_2 \colon (\mathsf{Vect}) \xrightarrow{\text{functor}} (\mathsf{Set})$$

defined by

$$H_1(W) := \mathrm{Hom}_{(\mathsf{Vect})}(\mathrm{Span}(S), W)$$

and

$$H_2(W) := \mathrm{Hom}_{(\mathsf{Set})}(S, U(W)).$$

For a linear map

$$h \colon W \to W',$$

the corresponding maps are given by

$$H_1(h)(f) := h \circ f$$

and

$$H_2(h)(g) := U(h) \circ g.$$

If

$$\Phi_W \colon \mathrm{Hom}_{(\mathsf{Vect})}(\mathrm{Span}(S), W) \longrightarrow \mathrm{Hom}_{(\mathsf{Set})}(S, U(W))$$

denotes the bijection

$$\Phi_W(f) := U(f) \circ \iota_S,$$

then the family $\{\Phi_W\}_W$ defines a natural isomorphism

$$\Phi \colon H_1 \xrightarrow{\text{nat}} H_2.$$

Explicitly, this means that, for every linear map

$$h \colon W \to W'$$

and every linear map

$$f \colon \mathrm{Span}(S) \to W,$$

one has

$$\Phi_{W'}(h \circ f) = U(h) \circ \Phi_W(f).$$

The word "canonical", by contrast, is not usually a formally defined categorical term in the same sense. It generally indicates that an object or a morphism is distinguished without making arbitrary auxiliary choices. Thus, "natural" and "canonical" are related, but they are not synonymous: naturality is a precise compatibility condition with morphisms, whereas canonicity concerns the absence of arbitrary choices or the distinguished character of a construction.

A natural isomorphism need not be unique, while a construction described as canonical is generally expected to be natural whenever an appropriate functorial formulation is available. In the construction above, the map

$$\iota_S \colon S \to U(\mathrm{Span}(S))$$

is canonical. On the other hand, an isomorphism between a given vector space V and a vector space of the form $\mathrm{Span}(S)$ generally requires the choice of a basis of V, and hence is not canonical (at least before such a basis is chosen). □

The following definition extracts the categorical content of the construction above.

Definition II.3.12. Let $\mathfrak{C}$ and $\mathfrak{D}$ be locally small categories, let

$$G \colon \mathfrak{D} \xrightarrow{\text{functor}} \mathfrak{C}$$

be a functor, and let $X \in \mathrm{ob}(\mathfrak{C})$. A *universal morphism from X to G* is a pair (R, u) consisting of an object $R \in \mathrm{ob}(\mathfrak{D})$ and a morphism

$$u \colon X \to G(R)$$

such that, for every object $Y \in \mathrm{ob}(\mathfrak{D})$ and every morphism

$$f \colon X \to G(Y),$$

there exists a unique morphism

$$\overline{f}\colon R \to Y$$

satisfying

$$G(\overline{f}) \circ u = f.$$

Dually, a *universal morphism from G to X* is a pair (R, v) consisting of an object $R \in \mathrm{ob}(\mathfrak{D})$ and a morphism

$$v\colon G(R) \to X$$

such that, for every object $Y \in \mathrm{ob}(\mathfrak{D})$ and every morphism

$$f\colon G(Y) \to X,$$

there exists a unique morphism

$$\overline{f}\colon Y \to R$$

satisfying

$$v \circ G(\overline{f}) = f.$$

□

In other words, a universal morphism from X to G is "initial" among morphisms of the form

$$X \to G(Y),$$

whereas a universal morphism from G to X is "terminal" among morphisms of the form

$$G(Y) \to X,$$

where morphisms between such morphisms are understood to be those making the evident triangles commute (Equivalently, for readers familiar with comma categories, these are respectively an initial object of $(X \downarrow G)$ and a terminal object of $(G \downarrow X)$).

In the vector-space construction above, the notation in Definition II.3.12 corresponds to

$$\mathfrak{C} = (\mathsf{Set}), \qquad \mathfrak{D} = (\mathsf{Vect}), \qquad G = U, \qquad X = S, \qquad R = \mathrm{Span}(S), \qquad u = \iota_S.$$

Thus $(\mathrm{Span}(S), \iota_S)$ is a universal morphism from S to U (If B is a basis of a vector space V, then (V, ι_B) is a universal morphism from B to U).

Proposition II.3.13. *A universal morphism from an object to a functor is unique up to a unique isomorphism compatible with its universal morphism. More precisely, by using the notation in Definition II.3.12, if (R, u) and (R', u') are universal morphisms from X to G, then there exists a unique isomorphism*

$$\varphi\colon R \to R'$$

such that

$$G(\varphi) \circ u = u'.$$

The analogous assertion holds for universal morphisms from G to X.

PROOF. By the universal property of (R, u) applied to $u' \colon X \to G(R')$, there exists a unique morphism

$$\varphi \colon R \to R'$$

such that $G(\varphi) \circ u = u'$. Similarly, there exists a unique morphism

$$\psi \colon R' \to R$$

such that $G(\psi) \circ u' = u$. We then have

$$G(\psi \circ \varphi) \circ u = u = G(\mathrm{id}_R) \circ u.$$

The uniqueness in the universal property of (R, u) implies that

$$\psi \circ \varphi = \mathrm{id}_R.$$

Similarly, $\varphi \circ \psi = \mathrm{id}_{R'}$. Hence φ is an isomorphism. Its uniqueness is again part of the universal property. □

Universal morphisms can be reformulated in terms of representable functors. Let $\mathfrak{D}$ be a locally small category. A functor

$$H \colon \mathfrak{D} \xrightarrow{\text{functor}} (\mathsf{Set})$$

is said to be *representable* if there exist an object $R \in \mathrm{ob}(\mathfrak{D})$ and a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{D}}(R, -) \cong H.$$

In this case, we say that R represents H. The basic mechanism behind such representations is given by the Yoneda lemma.

THEOREM II.3.14 (Yoneda lemma). *Let $\mathfrak{D}$ be a locally small category, let $R \in \mathrm{ob}(\mathfrak{D})$, and let*

$$H \colon \mathfrak{D} \xrightarrow{\text{functor}} (\mathsf{Set})$$

be a functor. Then the map

$$\begin{aligned} \mathrm{Nat}(\mathrm{Hom}_{\mathfrak{D}}(R, -), H) &\longrightarrow H(R), \\ \tau &\longmapsto \tau_R(\mathrm{id}_R) \end{aligned}$$

is a bijection. Its inverse sends $u \in H(R)$ to the natural transformation

$$\tau^u \colon \mathrm{Hom}_{\mathfrak{D}}(R, -) \xrightarrow{\text{nat}} H$$

whose component at $Y \in \mathrm{ob}(\mathfrak{D})$ is given by

$$\tau^u_Y(f) := H(f)(u) \qquad \text{for } f \in \mathrm{Hom}_{\mathfrak{D}}(R, Y).$$

Dually, if

$$K \colon \mathfrak{D}^{\mathrm{op}} \xrightarrow{\text{functor}} (\mathsf{Set})$$

is a functor, then there is a bijection

$$\mathrm{Nat}(\mathrm{Hom}_{\mathfrak{D}}(-, R), K) \cong K(R).$$

PROOF. Let $u \in H(R)$. If $f\colon R \to Y$ and $g\colon Y \to Z$ are morphisms in $\mathfrak{D}$, then

$$H(g)(\tau_Y^u(f)) = H(g)(H(f)(u)) = H(g \circ f)(u) = \tau_Z^u(g \circ f).$$

Hence τ^u is a natural transformation. Moreover,

$$\tau_R^u(\mathrm{id}_R) = H(\mathrm{id}_R)(u) = u.$$

Conversely, let

$$\tau\colon \mathrm{Hom}_{\mathfrak{D}}(R, -) \xrightarrow{\text{nat}} H$$

be a natural transformation, and set

$$u := \tau_R(\mathrm{id}_R).$$

For every morphism $f\colon R \to Y$, naturality of τ gives

$$\tau_Y(f) = H(f)(\tau_R(\mathrm{id}_R)) = H(f)(u) = \tau_Y^u(f).$$

Thus $\tau = \tau^u$, which proves the first assertion. The contravariant assertion follows by applying the first assertion to the opposite category. □

We now return to the setting of Definition II.3.12. Fix $X \in \mathrm{ob}(\mathfrak{C})$, and define a functor

$$H_X\colon \mathfrak{D} \xrightarrow{\text{functor}} (\mathsf{Set})$$

by

$$H_X(Y) := \mathrm{Hom}_{\mathfrak{C}}(X, G(Y))$$

and

$$H_X(h)(f) := G(h) \circ f$$

for $h\colon Y \to Y'$ and $f\colon X \to G(Y)$. For an object $R \in \mathrm{ob}(\mathfrak{D})$ and a morphism

$$u\colon X \to G(R),$$

the Yoneda lemma associates to $u \in \mathrm{Hom}_{\mathfrak{C}}(X, G(R)) = H_X(R)$ the natural transformation

$$\Phi^u\colon \mathrm{Hom}_{\mathfrak{D}}(R, -) \xrightarrow{\text{nat}} H_X,$$
$$\Phi_Y^u(h) := G(h) \circ u.$$

The pair (R, u) is a universal morphism from X to G if and only if Φ^u is a natural isomorphism. Equivalently, (R, u) consists of a representation of the functor

$$Y \longmapsto \mathrm{Hom}_{\mathfrak{C}}(X, G(Y))$$

together with the element corresponding to id_R. Thus the Yoneda lemma explains how the single morphism $u\colon X \to G(R)$ encodes, simultaneously and naturally in Y, all the bijections

$$\mathrm{Hom}_{\mathfrak{D}}(R, Y) \cong \mathrm{Hom}_{\mathfrak{C}}(X, G(Y)).$$

Dually, a universal morphism (R, v) from G to X is equivalent to a representation

$$\mathrm{Hom}_{\mathfrak{D}}(-, R) \cong \mathrm{Hom}_{\mathfrak{C}}(G(-), X)$$

of functors from $\mathfrak{D}^{\mathrm{op}}$ to (Set).

So far, the object X has been fixed. Suppose now that a universal morphism from X to G is chosen for every object $X \in \mathrm{ob}(\mathfrak{C})$.

Proposition II.3.15. *Let*

$$G\colon \mathfrak{D} \xrightarrow{\text{functor}} \mathfrak{C}$$

be a functor between locally small categories. Assume that, for every object $X \in \mathrm{ob}(\mathfrak{C})$, a universal morphism

$$\eta_X\colon X \to G(F(X))$$

from X to G has been chosen, where $F(X) \in \mathrm{ob}(\mathfrak{D})$. Then the assignment $X \mapsto F(X)$ extends uniquely to a functor

$$F\colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{D}$$

such that

$$\eta\colon \mathrm{id}_{\mathfrak{C}} \xrightarrow{\text{nat}} G \circ F, \qquad X \longmapsto \eta_X,$$

is a natural transformation. Moreover, the universal properties give bijections

$$\begin{aligned}\Phi_{X,Y}\colon \mathrm{Hom}_{\mathfrak{D}}(F(X), Y) &\longrightarrow \mathrm{Hom}_{\mathfrak{C}}(X, G(Y)),\\ f &\longmapsto G(f) \circ \eta_X,\end{aligned}$$

which are natural in both $X \in \mathrm{ob}(\mathfrak{C})$ and $Y \in \mathrm{ob}(\mathfrak{D})$.

Proof. For the following proof, also refer to [Lei, Corollary 2.3.7] or [St, Tag 0A8B]. Let $a\colon X \to X'$ be a morphism in $\mathfrak{C}$. Applying the universal property of $\eta_X\colon X \to G(F(X))$ to the morphism

$$\eta_{X'} \circ a\colon X \to G(F(X')),$$

we obtain a unique morphism

$$F(a)\colon F(X) \to F(X')$$

such that

$$G(F(a)) \circ \eta_X = \eta_{X'} \circ a.$$

The uniqueness in the universal property implies $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(b \circ a) = F(b) \circ F(a)$ for composable morphisms a and b. Thus F is a functor, and the defining equality says precisely that η is natural.

For fixed X and Y, the map $\Phi_{X,Y}$ is a bijection by the universal property of η_X. If

$$a\colon X' \to X, \qquad b\colon Y \to Y', \qquad f\colon F(X) \to Y,$$

then

$$\begin{aligned}\Phi_{X',Y'}(b \circ f \circ F(a)) &= G(b) \circ G(f) \circ G(F(a)) \circ \eta_{X'} \\ &= G(b) \circ G(f) \circ \eta_X \circ a \\ &= G(b) \circ \Phi_{X,Y}(f) \circ a.\end{aligned}$$

Hence the bijections are natural in both variables. □

The preceding proposition motivates the following definition.

Definition II.3.16. Let $\mathfrak{C}$ and $\mathfrak{D}$ be locally small categories, and let

$$F\colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{D}, \qquad G\colon \mathfrak{D} \xrightarrow{\text{functor}} \mathfrak{C}$$

be functors. We say that F is *left adjoint* to G, or that G is *right adjoint* to F, if there is a family of bijections

$$\Phi_{X,Y}\colon \mathrm{Hom}_{\mathfrak{D}}(F(X), Y) \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(X, G(Y))$$

which is natural in $X \in \mathrm{ob}(\mathfrak{C})$ and $Y \in \mathrm{ob}(\mathfrak{D})$. Explicitly, naturality means that, for morphisms

$$a\colon X' \to X, \qquad b\colon Y \to Y', \qquad f\colon F(X) \to Y,$$

one has

$$\Phi_{X',Y'}(b \circ f \circ F(a)) = G(b) \circ \Phi_{X,Y}(f) \circ a.$$

In this case, we write

$$F \dashv G.$$

□

Let $\mathfrak{C}$ and $\mathfrak{D}$ be locally small categories, and let

$$F\colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{D}, \qquad G\colon \mathfrak{D} \xrightarrow{\text{functor}} \mathfrak{C}$$

be functors. Assume $F \dashv G$, and let $\Phi_{X,Y}$ denote the corresponding natural bijections. The morphisms

$$\eta_X := \Phi_{X,F(X)}(\mathrm{id}_{F(X)})\colon X \to G(F(X))$$

form a natural transformation

$$\eta\colon \mathrm{id}_{\mathfrak{C}} \xrightarrow{\text{nat}} G \circ F,$$

called the *unit* of the adjunction. Similarly, the morphisms

$$\varepsilon_Y := \Phi^{-1}_{G(Y),Y}(\mathrm{id}_{G(Y)})\colon F(G(Y)) \to Y$$

form a natural transformation

$$\varepsilon\colon F \circ G \xrightarrow{\text{nat}} \mathrm{id}_{\mathfrak{D}},$$

called the *counit* of the adjunction. The adjunction bijections and their inverses are given by

$$\Phi_{X,Y}(f) = G(f) \circ \eta_X$$

and

$$\Phi^{-1}_{X,Y}(g) = \varepsilon_Y \circ F(g),$$

respectively. In particular, for every $X \in \mathrm{ob}(\mathfrak{C})$, the pair $(F(X), \eta_X)$ is a universal morphism from X to G, and, for every $Y \in \mathrm{ob}(\mathfrak{D})$, the pair $(G(Y), \varepsilon_Y)$ is a universal morphism from F to Y. Thus an adjunction may be regarded as a family of universal morphisms, one for every object, organized functorially.

Example II.3.17. The construction above defines a functor

$$\mathrm{Span}\colon (\mathsf{Set}) \xrightarrow{\text{functor}} (\mathsf{Vect}).$$

For a map

$$a\colon S \to T,$$

the linear map

$$\mathrm{Span}(a)\colon \mathrm{Span}(S) \to \mathrm{Span}(T)$$

is defined by

$$\mathrm{Span}(a)\left(\sum_{s\in S} c_s s\right) := \sum_{s\in S} c_s a(s),$$

where each $a(s) \in T$ is regarded as a formal basis vector of $\mathrm{Span}(T)$.

For every set S and every vector space V, let

$$\Phi_{S,V}\colon \mathrm{Hom}_{(\mathsf{Vect})}(\mathrm{Span}(S), V) \longrightarrow \mathrm{Hom}_{(\mathsf{Set})}(S, U(V))$$

be the bijection defined by

$$\Phi_{S,V}(f) := U(f) \circ \iota_S.$$

These bijections are natural in both S and V. More explicitly, if

$$a\colon S' \to S$$

is a map of sets,

$$h\colon V \to V'$$

is a linear map, and

$$f\colon \mathrm{Span}(S) \to V$$

is a linear map, then

$$\Phi_{S',V'}\big(h \circ f \circ \mathrm{Span}(a)\big) = U(h) \circ \Phi_{S,V}(f) \circ a.$$

Hence

$$\mathrm{Span} \dashv U.$$

Let B be a basis of a vector space V, and let

$$j_B \colon B \to U(V)$$

be the inclusion map. The universal property induces a unique isomorphism

$$\theta_B \colon \mathrm{Span}(B) \longrightarrow V$$

such that

$$U(\theta_B) \circ \iota_B = j_B,$$

where

$$\iota_B \colon B \to U(\mathrm{Span}(B))$$

is the canonical map. Equivalently, θ_B sends each formal basis vector $b \in B$ to the corresponding vector $b \in V$. Thus the usual description of a vector space in terms of a basis is recovered from the universal property and the uniqueness of universal morphisms.

More generally, for a ring R, the functor

$$\mathrm{Free}_R \colon (\mathsf{Set}) \xrightarrow{\text{functor}} (\mathsf{Mod}(R)), \qquad S \longmapsto R^{(S)} := \bigoplus_{s \in S} R \cdot s,$$

is left adjoint to the forgetful functor $U_R \colon (\mathsf{Mod}(R)) \xrightarrow{\text{functor}} (\mathsf{Set})$. □

See also [Lei, §2.1] and [Mac, §IV.2] for further examples.

Projective and inductive systems

Before introducing limits and colimits for diagrams indexed by arbitrary small categories, let us recall the more classical language of projective and inductive systems. The corresponding constructions for modules over a ring will be assumed to be familiar to the reader; see [Lan2, Chapter III, §10] and [AM, Chapter 2, Exercises 14–18].

Definition II.3.18. A non-empty partially ordered set $(I, \leq)$ is called a *directed set* if, for every $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$. A subset $J \subset I$ is said to be *cofinal* in I if, for every $i \in I$, there exists $j \in J$ such that $i \leq j$. □

Thus every finite collection of elements of a directed set admits a common upper bound. Also, every cofinal subset of a directed set is itself directed.

Definition II.3.19. Let $(I, \leq)$ be a directed set, regarded as a category in the manner explained above. A *projective system* in a category $\mathfrak{C}$ indexed by I consists of objects X_i $(i \in I)$ and morphisms

$$\rho_{ij} \colon X_j \to X_i \qquad (i \leq j)$$

such that

$$\rho_{ii} = \mathrm{id}_{X_i}$$

and

$$\rho_{ik} = \rho_{ij} \circ \rho_{jk} \qquad (i \leq j \leq k).$$

Equivalently, a projective system indexed by I is a functor

$$I^{\mathrm{op}} \xrightarrow{\text{functor}} \mathfrak{C}.$$

Dually, an *inductive system* in $\mathfrak{C}$ indexed by I consists of objects X_i $(i \in I)$ and morphisms

$$\alpha_{ij} \colon X_i \to X_j \qquad (i \leq j)$$

such that

$$\alpha_{ii} = \mathrm{id}_{X_i}$$

and

$$\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij} \qquad (i \leq j \leq k).$$

Equivalently, an inductive system indexed by I is a functor

$$I \xrightarrow{\text{functor}} \mathfrak{C}.$$

□

For example, for a ring R, the projective limit of a projective system of R-modules consists of compatible families of elements, whereas the inductive limit of an inductive system of R-modules is obtained by identifying elements which represent the same element at a later stage. More precisely, if $x \in M_i$ and $y \in M_j$, then their images in the inductive limit agree if and only if there exists $k \in I$ such that

$$i \leq k, \qquad j \leq k, \qquad \text{and} \qquad \alpha_{ik}(x) = \alpha_{jk}(y).$$

The existence of a common index k is precisely where directedness is used.

Cofinal subsets permit us to discard inessential indices. If $J \subset I$ is cofinal, then, whenever the relevant limits exist, restricting a projective or inductive system from I to J induces canonical isomorphisms

$$\varprojlim_{i \in I} X_i \cong \varprojlim_{j \in J} X_j$$

and

$$\varinjlim_{j \in J} X_j \cong \varinjlim_{i \in I} X_i.$$

Example II.3.20. Let X be a topological space, and let $S \subset X$. Denote by

$$\mathcal{N}_X(S) := \{U \subset X \mid U \text{ is open and } S \subset U\}$$

the collection of open neighborhoods of S, ordered by reverse inclusion:

$$U \leq V \quad \Longleftrightarrow \quad V \subset U.$$

Then $\mathcal{N}_X(S)$ is a directed set. Indeed, for $U, V \in \mathcal{N}_X(S)$, the open neighborhood $U \cap V$ satisfies

$$U \leq U \cap V \qquad \text{and} \qquad V \leq U \cap V.$$

Thus, when $\mathcal{N}_X(S)$ is regarded as a category, there is a unique morphism $U \to V$ precisely when $V \subset U$. This convention is convenient for constructions equipped with restriction morphisms from larger open sets to smaller ones.

A family $\mathcal{B} \subset \mathcal{N}_X(S)$ is a fundamental system of open neighborhoods of S if, for every $U \in \mathcal{N}_X(S)$, there exists $B \in \mathcal{B}$ such that

$$S \subset B \subset U.$$

With the reverse-inclusion ordering above, this says exactly that $\mathcal{B}$ is cofinal in $\mathcal{N}_X(S)$. □

Filtered and cofiltered indexing categories

The notion of a directed set has the following categorical generalization.

Definition II.3.21. A category $\mathfrak{I}$ is called *filtered* if it satisfies the following conditions:

- $\mathfrak{I}$ has at least one object;
- for every pair of objects $A, B \in \mathrm{ob}(\mathfrak{I})$, there exist an object $C \in \mathrm{ob}(\mathfrak{I})$ and morphisms

$$A \to C, \qquad B \to C;$$

- for every pair of morphisms

$$f, g \colon A \to B,$$

there exist an object $C \in \mathrm{ob}(\mathfrak{I})$ and a morphism

$$h \colon B \to C$$

such that

$$h \circ f = h \circ g.$$

A category $\mathfrak{I}$ is called *cofiltered* if $\mathfrak{I}^{\mathrm{op}}$ is filtered. □

If a partially ordered set is regarded as a category, there is at most one morphism between any two objects. Hence the third condition in Definition II.3.21 is automatic. It follows that a partially ordered set is filtered as a category if and only if it is directed in the sense of Definition II.3.18. Consequently, if I is a directed set, then I is filtered and I^{op} is cofiltered.

The relation between the traditional terminology and the general categorical terminology may be summarized as follows.

Traditional terminology	**Categorical terminology**
An inductive system indexed by a directed set I	A diagram $I \xrightarrow{\text{functor}} \mathfrak{C}$ indexed by the filtered category I
A projective system indexed by a directed set I	A diagram $I^{\text{op}} \xrightarrow{\text{functor}} \mathfrak{C}$ indexed by the cofiltered category I^{op}
An inductive limit, or direct limit	A filtered colimit
A projective limit, or inverse limit	A cofiltered limit

In this note, we shall mainly use the traditional terms *projective limit* and *inductive limit* for systems indexed by directed sets. The terms *inverse limit* and *direct limit*, respectively, are also common. We shall use the general categorical terms *limit* and *colimit* when the indexing category is arbitrary or when we wish to emphasize the universal property.

The directedness assumption has an important practical meaning. For a directed set, every finite collection of indices admits a common upper bound. Thus finitely many elements belonging to different stages of an inductive system can be transported to one sufficiently large stage and compared there. More generally, filteredness allows a finite amount of data, together with finitely many compatibility relations, to be treated simultaneously at a later stage. This is the source of many of the particularly good properties of filtered inductive limits (Proposition II.3.43 below is one of the typical examples). Dually, cofilteredness plays the corresponding role for projective systems.

The directedness, filteredness, or cofilteredness assumption is not, however, needed in order to define limits and colimits themselves. A limit or colimit can be considered for a diagram indexed by an arbitrary small category. In fact, many basic finite constructions, such as products, equalizers, and fiber products, arise from indexing categories which are not filtered. We now pass to this general setting.

Limits and colimits

Let $\mathfrak{I}$ be a small category, and let $\mathfrak{C}$ be a locally small category. For an object $X \in \mathrm{ob}(\mathfrak{C})$, the *constant diagram* with value X is the functor

$$\Delta_{\mathfrak{I}} X \colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{C}$$

which sends every object of $\mathfrak{I}$ to X and every morphism of $\mathfrak{I}$ to id_X. If

$$f \colon X \to Y$$

is a morphism in $\mathfrak{C}$, then the morphism

$$\Delta_{\mathfrak{I}} f \colon \Delta_{\mathfrak{I}} X \xrightarrow{\text{nat}} \Delta_{\mathfrak{I}} Y$$

is the natural transformation whose component at every object of $\mathfrak{I}$ is f. This defines the *diagonal functor*

$$\Delta_{\mathfrak{I}} \colon \mathfrak{C} \xrightarrow{\text{functor}} \mathrm{Fun}(\mathfrak{I}, \mathfrak{C}).$$

Remark II.3.22. Now we are assuming that $\mathfrak{I}$ is a small category and $\mathfrak{C}$ is a locally small category. In this case, the category $\mathrm{Fun}(\mathfrak{I}, \mathfrak{C})$ is locally small: indeed, for functors

$$F, G \colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{C},$$

the collection $\mathrm{Nat}(F, G)$ is a subset of

$$\prod_{i \in \mathrm{ob}(\mathfrak{I})} \mathrm{Hom}_{\mathfrak{C}}(F(i), G(i)),$$

which is a set because $\mathfrak{I}$ is small and $\mathfrak{C}$ is locally small. □

Let F be a diagram in $\mathfrak{C}$ indexed by $\mathfrak{I}$: i.e.,

$$F \colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{C}$$

(Recall Example II.3.8). A natural transformation

$$\lambda \colon \Delta_{\mathfrak{I}} X \xrightarrow{\text{nat}} F$$

consists of morphisms

$$\lambda_A \colon X \to F(A) \qquad (A \in \mathrm{ob}(\mathfrak{I}))$$

such that

$$F(a) \circ \lambda_A = \lambda_B$$

for every morphism $a \colon A \to B$ in $\mathfrak{I}$. Such a natural transformation is often called a *cone* from X to F. Dually, a natural transformation

$$\mu \colon F \xrightarrow{\text{nat}} \Delta_{\mathfrak{I}} X$$

consists of morphisms

$$\mu_A \colon F(A) \to X$$

such that

$$\mu_B \circ F(a) = \mu_A$$

for every morphism $a \colon A \to B$, and is often called a *cocone* from F to X. We shall mainly use the description in terms of natural transformations rather than the words "cone" and "cocone".

Definition II.3.23. A *limit* of F, also called a *projective limit* of F, is a universal morphism from the diagonal functor $\Delta_{\mathfrak{I}}$ to the object F of $\mathrm{Fun}(\mathfrak{I}, \mathfrak{C})$. Thus it consists of an object

$$\varprojlim_{\mathfrak{I}} F \in \mathrm{ob}(\mathfrak{C})$$

and a natural transformation

$$\pi \colon \Delta_{\mathfrak{I}} \big(\varprojlim_{\mathfrak{I}} F \big) \xrightarrow{\text{nat}} F$$

such that, for every object $X \in \mathrm{ob}(\mathfrak{C})$ and every natural transformation

$$\lambda\colon \Delta_{\mathfrak{I}} X \xrightarrow{\text{nat}} F,$$

there exists a unique morphism

$$\overline{\lambda}\colon X \to \varprojlim_{\mathfrak{I}} F$$

satisfying

$$\pi \circ \Delta_{\mathfrak{I}}(\overline{\lambda}) = \lambda.$$

Dually, a *colimit* of F, also called an *inductive limit* of F, is a universal morphism from the object F to the diagonal functor $\Delta_{\mathfrak{I}}$. Thus it consists of an object

$$\varinjlim_{\mathfrak{I}} F \in \mathrm{ob}(\mathfrak{C})$$

and a natural transformation

$$\iota\colon F \xrightarrow{\text{nat}} \Delta_{\mathfrak{I}}\big(\varinjlim_{\mathfrak{I}} F\big)$$

such that, for every object $X \in \mathrm{ob}(\mathfrak{C})$ and every natural transformation

$$\mu\colon F \xrightarrow{\text{nat}} \Delta_{\mathfrak{I}} X,$$

there exists a unique morphism

$$\overline{\mu}\colon \varinjlim_{\mathfrak{I}} F \to X$$

satisfying

$$\Delta_{\mathfrak{I}}(\overline{\mu}) \circ \iota = \mu.$$

□

Equivalently, the universal properties in Definition II.3.23 give natural bijections

$$\mathrm{Hom}_{\mathfrak{C}}\big(X, \varprojlim_{\mathfrak{I}} F\big) \cong \mathrm{Nat}(\Delta_{\mathfrak{I}} X, F)$$

and

$$\mathrm{Hom}_{\mathfrak{C}}\big(\varinjlim_{\mathfrak{I}} F, X\big) \cong \mathrm{Nat}(F, \Delta_{\mathfrak{I}} X).$$

Thus $\varprojlim_{\mathfrak{I}} F$ represents the functor

$$X \longmapsto \mathrm{Nat}(\Delta_{\mathfrak{I}} X, F),$$

whereas $\varinjlim_{\mathfrak{I}} F$ represents the functor

$$X \longmapsto \mathrm{Nat}(F, \Delta_{\mathfrak{I}} X).$$

Since limits and colimits are defined as universal morphisms, Proposition II.3.13 implies that, whenever they exist, the limit and the colimit of F are unique up to unique isomorphisms compatible with the corresponding natural transformations. When the indexing category is clear from the context, we simply write

$$\varprojlim F \qquad \text{and} \qquad \varinjlim F.$$

When the objects of the indexing category are denoted by i, we also write

$$\varprojlim_{i\in\mathrm{ob}(\mathfrak{I})} F(i) \qquad \text{and} \qquad \varinjlim_{i\in\mathrm{ob}(\mathfrak{I})} F(i),$$

usually abbreviated to

$$\varprojlim_{i} F(i) \qquad \text{and} \qquad \varinjlim_{i} F(i).$$

If the diagram is written as a system $\{X_i\}$, these are written as

$$\varprojlim_{i} X_i \qquad \text{and} \qquad \varinjlim_{i} X_i.$$

If I is a directed set and $F\colon I^{\mathrm{op}} \xrightarrow{\text{functor}} \mathfrak{C}$ is a projective system, the limit above is its projective limit in the traditional sense. Similarly, if $F\colon I \xrightarrow{\text{functor}} \mathfrak{C}$ is an inductive system, the colimit above is its inductive limit.

The preceding universal properties can also be expressed by adjunctions. If limits exist in $\mathfrak{C}$ for every diagram indexed by $\mathfrak{I}$, then the assignment

$$F \longmapsto \varprojlim_{\mathfrak{I}} F$$

defines a functor

$$\varprojlim_{\mathfrak{I}}\colon \mathrm{Fun}(\mathfrak{I},\mathfrak{C}) \xrightarrow{\text{functor}} \mathfrak{C}$$

which is right adjoint to the diagonal functor:

$$\Delta_{\mathfrak{I}} \dashv \varprojlim_{\mathfrak{I}}.$$

Similarly, if colimits exist in $\mathfrak{C}$ for every diagram indexed by $\mathfrak{I}$, then

$$\varinjlim_{\mathfrak{I}}\colon \mathrm{Fun}(\mathfrak{I},\mathfrak{C}) \xrightarrow{\text{functor}} \mathfrak{C}$$

is left adjoint to the diagonal functor:

$$\varinjlim_{\mathfrak{I}} \dashv \Delta_{\mathfrak{I}}.$$

Consequently, whenever both constructions exist for all diagrams indexed by $\mathfrak{I}$, one has

$$\varinjlim_{\mathfrak{I}} \dashv \Delta_{\mathfrak{I}} \dashv \varprojlim_{\mathfrak{I}}.$$

Basic finite examples

We next consider limits and colimits of diagrams indexed by some particularly simple finite categories. Before listing them, let us explain the diagrammatic notation used below.

A small category, especially a *finite category* (a category that has only finitely many objects and morphisms), may be described by specifying its objects and some generating morphisms, together with relations among their compositions. When the indexing category is generated by a directed graph, we often display a diagram by drawing only the values of a functor on the vertices and on the generating morphisms. Identity morphisms are not displayed, and morphisms obtained by composing the displayed morphisms are also usually omitted. Unless additional relations are stated, the indexing category is understood to be the path category generated by the displayed directed graph.

For example, the notation

$$X \longleftarrow Z \longrightarrow Y$$

denotes a diagram

$$F\colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{C},$$

where $\mathfrak{I}$ is the category generated by three objects i, j, k and two non-identity morphisms

$$k \to i, \qquad k \to j,$$

and where

$$F(i) = X, \qquad F(j) = Y, \qquad F(k) = Z.$$

The two displayed morphisms in $\mathfrak{C}$ are the images under F of the two generating morphisms of $\mathfrak{I}$.

Definition II.3.24. A small category $\mathfrak{I}$ is called *finitely presented* if it is isomorphic to a quotient of the path category of a directed graph with finitely many vertices and edges by finitely many equations $p_1 = q_1, p_2 = q_2, \ldots, p_s = q_s$ between paths with the same initial and terminal vertices, where $s \in \mathbb{Z}_{\geq 0}$. Here the quotient identifies paths by the smallest equivalence relation containing these equations and preserved by concatenation with paths on either side. A path of length zero represents the identity of its vertex. The graph and the equations together are a *finite presentation* of $\mathfrak{I}$. □

Example II.3.25. The category generated by $0 \longrightarrow 1$, with no additional relations, has two objects and just one non-identity morphism. In contrast, one vertex $*$ with two loops a, b and the relations $a \circ b = \mathrm{id}_*$ and $b \circ a = \mathrm{id}_*$ presents the one-object category associated with $\mathbb{Z}$: its morphisms are the integer powers of a, with $b = a^{-1}$. It is finitely presented but is not a finite category. One loop without these relations would instead give only the non-negative powers, namely the category associated with the monoid $\mathbb{Z}_{\geq 0}$. Every finite category is finitely presented: take its non-identity morphisms as edges and impose its finite composition table as relations. □

A category is called *discrete* if its only morphisms are identity morphisms. Thus a diagram indexed by a discrete category is simply a family of objects of $\mathfrak{C}$, with no additional morphisms between them. A limit or colimit indexed by a finite category is called a *finite limit* or a *finite colimit*, respectively. Several familiar constructions arise as limits and colimits of diagrams indexed by such simple categories (See also examples listed in [Lei, §5.1, 5.2]).

- The *empty category* $\emptyset$ is the category having no objects and no morphisms. A limit of the unique diagram
$$\emptyset \xrightarrow{\text{functor}} \mathfrak{C}$$
is a *terminal object* of $\mathfrak{C}$, namely it is an object T such that, for every object $X \in \mathrm{ob}(\mathfrak{C})$, there exists a unique morphism $X \to T$. Dually, a colimit of the empty diagram is an *initial object*: an object I such that, for every object $X \in \mathrm{ob}(\mathfrak{C})$, there exists a unique morphism $I \to X$. Note that the existence of an initial or terminal object is not automatic. For example, regard the partially ordered set $(\mathbb{Z}, \leq)$ as a category in the usual way. An initial object in this category would be a least integer, while a terminal object would be a greatest integer. Since neither exists, the empty diagram in this category has neither a colimit nor a limit.

- Let $\mathfrak{I}$ be the discrete category with two objects. A diagram
$$F\colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{C}$$
amounts simply to a pair of objects $X, Y \in \mathrm{ob}(\mathfrak{C})$. Its limit is a *product* of X and Y. Thus a product is an object $X \times Y$ together with morphisms
$$p_X\colon X \times Y \to X, \qquad p_Y\colon X \times Y \to Y$$
such that, for every object T and every pair of morphisms
$$f\colon T \to X, \qquad g\colon T \to Y,$$
there exists a unique morphism
$$(f, g)\colon T \to X \times Y$$
satisfying
$$p_X \circ (f, g) = f, \qquad p_Y \circ (f, g) = g.$$
Dually, the colimit of the same diagram is a *coproduct* of X and Y.

- Let $\mathfrak{I}$ be the category with two objects i, j and two distinct non-identity morphisms
$$a, b\colon i \to j.$$
A diagram
$$F\colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{C}$$

is usually displayed as

$$A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

where

$$A = F(i), \qquad B = F(j), \qquad f = F(a), \qquad g = F(b).$$

Its limit is the *equalizer* of f and g. Dually, its colimit is the *coequalizer* of f and g.

- Let $\mathfrak{I}$ be the category generated by three objects i, j, k and two non-identity morphisms

$$i \to k, \qquad j \to k.$$

A diagram indexed by $\mathfrak{I}$ is displayed as

$$X \longrightarrow Z \longleftarrow Y.$$

Its limit is the *fiber product*, or *pullback*, usually denoted by

$$X \times_Z Y.$$

Dually, the colimit of a diagram of the form

$$X \longleftarrow Z \longrightarrow Y$$

is the *pushout*.

Mini-Exercise II.3.26. Let $\mathfrak{I}$ be a small category and $\mathfrak{C}$ a locally small category, and let $F\colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{C}$ be a diagram.
(*i*) Suppose that $\mathfrak{I}$ has an initial object i_0. For $i \in \mathrm{ob}(\mathfrak{I})$, denote by $u_i\colon i_0 \longrightarrow i$ the unique morphism. Show that

$$\big(F(i_0), \{F(u_i)\colon F(i_0) \longrightarrow F(i)\}_{i\in\mathrm{ob}(\mathfrak{I})}\big)$$

is a limit of F: i.e.,

$$\varprojlim_{\mathfrak{I}} F \cong F(i_0).$$

(*ii*) Suppose that $\mathfrak{I}$ has a terminal object i_1. For $i \in \mathrm{ob}(\mathfrak{I})$, denote by $v_i\colon i \longrightarrow i_1$ the unique morphism. Show that

$$\big(F(i_1), \{F(v_i)\colon F(i) \longrightarrow F(i_1)\}_{i\in\mathrm{ob}(\mathfrak{I})}\big)$$

is a colimit of F: i.e.,

$$\varinjlim_{\mathfrak{I}} F \cong F(i_1).$$

(*iii*) Let I be a directed set having a greatest element m. Deduce that, for every inductive system

$$F\colon I \xrightarrow{\text{functor}} \mathfrak{C}$$

and every projective system

$$G\colon I^{\mathrm{op}} \xrightarrow{\text{functor}} \mathfrak{C},$$

one has

$$\varinjlim_{i\in I} F(i) \cong F(m)$$

and

$$\varprojlim_{i\in I} G(i) \cong G(m),$$

respectively.

Example II.3.27. Let R be a ring and $f\colon M \to N$ be an R-module homomorphism. In $(\mathsf{Mod}(R))$, the equalizer of

$$M \underset{0}{\overset{f}{\rightrightarrows}} N$$

is $\operatorname{Ker} f$ together with its inclusion into M, while the coequalizer is $\operatorname{Coker} f$ together with the quotient morphism from N. □

The preceding example shows that equalizers and coequalizers exist and have familiar descriptions in $(\mathsf{Mod}(R))$. In a general category, however, even the equalizer of two morphisms need not exist.

Example II.3.28. Consider in $(C^\infty\mathsf{Mfd})$ the two smooth maps

$$h, 0\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}$$

defined by

$$h(x,y) := xy \qquad \text{and} \qquad 0(x,y) := 0,$$

where (x,y) are the standard coordinates of $\mathbb{R}^2$. This diagram has no equalizer in $(C^\infty\mathsf{Mfd})$. Intuitively, its set-theoretic equalizer would be $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$, which has a singular crossing at the origin and is not a smooth manifold. A rigorous proof of the non-existence of an equalizer is left to the following Mini-Exercise. □

Mini-Exercise II.3.29. In the setting of Example II.3.28, prove that the diagram

$$\mathbb{R}^2 \underset{0}{\overset{h}{\rightrightarrows}} \mathbb{R}$$

has no equalizer in $(C^\infty\mathsf{Mfd})$.

Example II.3.30. Let R be a ring. Consider a diagram in $(\mathsf{Mod}(R))$ of the form

$$M_1 \xrightarrow{f_1} M_0 \xleftarrow{f_2} M_2.$$

Its limit is the fiber product

$$M_1 \times_{M_0} M_2 = \{(x_1, x_2) \in M_1 \times M_2 \mid f_1(x_1) = f_2(x_2)\}.$$

The universal property says that giving an R-module homomorphism

$$N \to M_1 \times_{M_0} M_2$$

is equivalent to giving R-module homomorphisms

$$g_1 \colon N \to M_1, \qquad g_2 \colon N \to M_2$$

such that

$$f_1 \circ g_1 = f_2 \circ g_2.$$

□

Example II.3.31. In the category (Set), the product of two sets A and B is their usual Cartesian product $A \times B$. For every set T, giving a map

$$T \to A \times B$$

is equivalent to giving a pair of maps

$$T \to A, \qquad T \to B.$$

Thus the usual Cartesian product is precisely the categorical product in (Set). The same pattern appears in (Top), (Group), and ($\mathsf{Mod}(R)$) (where R is a ring), with the appropriate structure imposed on the product. □

Example II.3.32. In the category (Set), the coproduct of two sets A and B is their disjoint union

$$A \sqcup B.$$

Let

$$\iota_A \colon A \to A \sqcup B, \qquad \iota_B \colon B \to A \sqcup B$$

be the canonical maps. For every set T, giving a map

$$f \colon A \sqcup B \to T$$

is equivalent to giving a pair of maps

$$f_A \colon A \to T, \qquad f_B \colon B \to T.$$

More precisely, the correspondence is given by

$$f \longmapsto \big(f \circ \iota_A, f \circ \iota_B\big).$$

Conversely, for every pair (f_A, f_B), the map $f \colon A \sqcup B \to T$ defined by letting

$$f(x) := \begin{cases} f_A(x) & \text{if } x \in A \\ f_B(x) & \text{if } x \in B \end{cases}$$

is the unique one such that $f \circ \iota_A = f_A$ and $f \circ \iota_B = f_B$ (Here, as in our convention for disjoint unions, we tacitly replace A and B, if necessary, by the disjoint copies $A \times \{0\}$ and $B \times \{1\}$, and assume that A and B have no common element). Thus the disjoint union is precisely the categorical coproduct in (Set). □

Example II.3.33. Let $\{M_i\}_{i\in I}$ be a family of R-modules. Its product in $(\mathsf{Mod}(R))$ is the usual direct product

$$\prod_{i\in I} M_i,$$

whereas its coproduct is the direct sum

$$\bigoplus_{i\in I} M_i = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} M_i \;\middle|\; x_i = 0 \text{ for all but finitely many } i \right\}.$$

There is a canonical injective homomorphism

$$\bigoplus_{i\in I} M_i \longrightarrow \prod_{i\in I} M_i.$$

If I is finite, this homomorphism is an isomorphism. Thus finite products and finite coproducts in $(\mathsf{Mod}(R))$ are canonically isomorphic. The same holds in (Ab). Such a finite product–coproduct is also called a *biproduct*.

For an infinite index set I, however, the direct sum and the direct product are generally different. For example,

$$\bigoplus_{i\in\mathbb{Z}_{\geq 0}} \mathbb{Z} \subsetneq \prod_{i\in\mathbb{Z}_{\geq 0}} \mathbb{Z},$$

since the element $(1, 1, 1, \ldots)$ belongs to the product but not to the direct sum (Moreover, these two groups are not even isomorphic: the direct sum is countable, whereas the direct product is uncountable, since it contains the set $\{0,1\}^{\mathbb{Z}_{\geq 0}}$).

This coincidence of finite products and finite coproducts is a special feature of categories such as (Ab) and $(\mathsf{Mod}(R))$; it does not hold in (Set) in general. □

Compatibility of limits, colimits, and adjoint functors

We first record two general compatibility results for limits and colimits.

Proposition II.3.34. *Let $\mathfrak{I}$ and $\mathfrak{J}$ be small categories, let $\mathfrak{C}$ be a locally small category, and let*

$$F\colon \mathfrak{I}\times\mathfrak{J} \xrightarrow{\text{functor}} \mathfrak{C}$$

be a diagram. Whenever all the limits appearing below exist, there are canonical isomorphisms

$$\varprojlim_{i\in\mathrm{ob}(\mathfrak{I})} \left(\varprojlim_{j\in\mathrm{ob}(\mathfrak{J})} F(i,j) \right) \cong \varprojlim_{(i,j)\in\mathrm{ob}(\mathfrak{I}\times\mathfrak{J})} F(i,j) \cong \varprojlim_{j\in\mathrm{ob}(\mathfrak{J})} \left(\varprojlim_{i\in\mathrm{ob}(\mathfrak{I})} F(i,j) \right).$$

Similarly, whenever all the colimits appearing below exist, there are canonical isomorphisms

$$\varinjlim_{i\in\mathrm{ob}(\mathfrak{I})} \left(\varinjlim_{j\in\mathrm{ob}(\mathfrak{J})} F(i,j) \right) \cong \varinjlim_{(i,j)\in\mathrm{ob}(\mathfrak{I}\times\mathfrak{J})} F(i,j) \cong \varinjlim_{j\in\mathrm{ob}(\mathfrak{J})} \left(\varinjlim_{i\in\mathrm{ob}(\mathfrak{I})} F(i,j) \right).$$

PROOF. See [Lei, Proposition 6.2.8], [Mac, §IX. 2, 8], and [St, Tag 002M]. Note that, for categories $\mathfrak{I}$ and $\mathfrak{J}$, their *product category*

$$\mathfrak{I} \times \mathfrak{J}$$

is defined as follows. Its objects are pairs

$$(i, j), \qquad i \in \mathrm{ob}(\mathfrak{I}), \quad j \in \mathrm{ob}(\mathfrak{J}),$$

and its morphisms are pairs

$$(a, b) \colon (i, j) \to (i', j'),$$

where $a \colon i \to i'$ is a morphism in $\mathfrak{I}$ and $b \colon j \to j'$ is a morphism in $\mathfrak{J}$. Identity morphisms and compositions are defined componentwise: $\mathrm{id}_{(i,j)} = (\mathrm{id}_i, \mathrm{id}_j)$ and $(a', b') \circ (a, b) = (a' \circ a, b' \circ b)$. If $\mathfrak{I}$ and $\mathfrak{J}$ are small, then $\mathfrak{I} \times \mathfrak{J}$ is also small. □

Thus limits commute with limits, and colimits commute with colimits. By contrast, limits and colimits do not commute with one another in general. The following important result gives one situation in which they do commute.

PROPOSITION II.3.35. *Let $\mathfrak{I}$ be a small filtered category, let $\mathfrak{J}$ be a finite category, and let*

$$F \colon \mathfrak{I} \times \mathfrak{J} \xrightarrow{\text{functor}} (\mathsf{Set})$$

be a diagram. Then the canonical map

$$\varinjlim_{i \in \mathrm{ob}(\mathfrak{I})} \left(\varprojlim_{j \in \mathrm{ob}(\mathfrak{J})} F(i, j) \right) \longrightarrow \varprojlim_{j \in \mathrm{ob}(\mathfrak{J})} \left(\varinjlim_{i \in \mathrm{ob}(\mathfrak{I})} F(i, j) \right)$$

is a bijection.

PROOF. See [Mac, §IX. 2 Theorem 1] and [St, Tag 002W]. □

We next explain how adjunctions interact with limits and colimits.

PROPOSITION II.3.36. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be locally small categories, and let*

$$F \colon \mathfrak{C} \xrightarrow{\text{functor}} \mathfrak{D}, \qquad G \colon \mathfrak{D} \xrightarrow{\text{functor}} \mathfrak{C}$$

be functors such that $F \dashv G$. Let $\mathfrak{I}$ be a small category. Then the following holds:
(i) Let $A \colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{C}$ be a diagram. If $\varinjlim_{\mathfrak{I}} A$ exists in $\mathfrak{C}$, then

$$F\left(\varinjlim_{\mathfrak{I}} A \right)$$

together with the morphisms obtained by applying F to the canonical morphisms

$$A(i) \to \varinjlim_{\mathfrak{I}} A$$

is a colimit of the diagram

$$F \circ A \colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{D}.$$

Equivalently, there is a canonical isomorphism

$$F\left(\varinjlim_{\mathfrak{I}} A\right) \cong \varinjlim_{\mathfrak{I}} (F \circ A).$$

(ii) Let $B \colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{D}$ *be a diagram. If* $\varprojlim_{\mathfrak{I}} B$ *exists in* $\mathfrak{D}$*, then*

$$G\left(\varprojlim_{\mathfrak{I}} B\right)$$

together with the morphisms obtained by applying G *to the canonical morphisms*

$$\varprojlim_{\mathfrak{I}} B \to B(i)$$

is a limit of the diagram

$$G \circ B \colon \mathfrak{I} \xrightarrow{\text{functor}} \mathfrak{C}.$$

Equivalently, there is a canonical isomorphism

$$G\left(\varprojlim_{\mathfrak{I}} B\right) \cong \varprojlim_{\mathfrak{I}} (G \circ B).$$

Proof. Put $L := \varinjlim_{\mathfrak{I}} A$. For every object $Y \in \mathrm{ob}(\mathfrak{D})$, the adjunction and the universal property of L give natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{D}}(F(L), Y) &\cong \mathrm{Hom}_{\mathfrak{C}}(L, G(Y)) \\ &\cong \mathrm{Nat}\big(A, \Delta_{\mathfrak{I}} G(Y)\big) \\ &\cong \mathrm{Nat}\big(F \circ A, \Delta_{\mathfrak{I}} Y\big). \end{aligned}$$

The last bijection is obtained by applying the adjunction bijection to every component of a natural transformation. Hence $F(L)$ satisfies the universal property of the colimit of $F \circ A$.

Dually, put $M := \varprojlim_{\mathfrak{I}} B$. For every object $X \in \mathrm{ob}(\mathfrak{C})$, we have natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{C}}(X, G(M)) &\cong \mathrm{Hom}_{\mathfrak{D}}(F(X), M) \\ &\cong \mathrm{Nat}\big(\Delta_{\mathfrak{I}} F(X), B\big) \\ &\cong \mathrm{Nat}\big(\Delta_{\mathfrak{I}} X, G \circ B\big). \end{aligned}$$

Therefore $G(M)$ satisfies the universal property of the limit of $G \circ B$. □

For comparison, see also [St, Tag 0038].

MINI-EXERCISE II.3.37. Let X and Y be sets, $f\colon X \to Y$ a map, and Λ be a non-empty index set. Let

$$A_\lambda \subset X, \qquad B_\lambda \subset Y \qquad (\lambda \in \Lambda)$$

be families of subsets. For each of the following pairs of subsets, determine whether the two subsets always coincide:

$$f\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) \quad \text{and} \quad \bigcup_{\lambda\in\Lambda} f(A_\lambda), \qquad f\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \quad \text{and} \quad \bigcap_{\lambda\in\Lambda} f(A_\lambda),$$

$$f^{-1}\left(\bigcup_{\lambda\in\Lambda} B_\lambda\right) \quad \text{and} \quad \bigcup_{\lambda\in\Lambda} f^{-1}(B_\lambda), \qquad f^{-1}\left(\bigcap_{\lambda\in\Lambda} B_\lambda\right) \quad \text{and} \quad \bigcap_{\lambda\in\Lambda} f^{-1}(B_\lambda).$$

Exactness in module categories and abelian categories

Let R be a ring. In the category $(\mathsf{Mod}(R))$, and likewise in the familiar special cases (Ab) and (Vect), kernels, cokernels, images, quotient modules, and (short) exact sequences play a fundamental role. The notion of an abelian category abstracts this familiar structure.

A category $\mathfrak{A}$ is called an *abelian category* if each set $\mathrm{Hom}_{\mathfrak{A}}(A, B)$ is an abelian group, composition is bilinear, finite products and finite coproducts exist and agree in the canonical manner, every morphism has a kernel and a cokernel, and the canonical morphism

$$\operatorname{Coim} f \longrightarrow \operatorname{Image} f$$

is an isomorphism for every morphism $f\colon A \to B$, where

$$\operatorname{Image} f := \operatorname{Ker}\big(B \to \operatorname{Coker} f\big)$$

and

$$\operatorname{Coim} f := \operatorname{Coker}\big(\operatorname{Ker} f \to A\big).$$

See [I, Chapter I §1–5], [Mac, §VIII. 1–3], and [St, Tag 00ZX] for the precise definitions and basic properties (We shall not pursue these notions in detail here; we only note that the zero morphism $0\colon A \to B$ is defined as the zero element of $\mathrm{Hom}_{\mathfrak{A}}(A, B)$, which is assumed to be an abelian group, and that $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are defined abstractly as the equalizer and coequalizer, respectively, of f and 0; compare Example II.3.27).

Although we have listed the defining conditions above, no familiarity with the general theory of abelian categories will be required in this note. Whenever a category $\mathfrak{A}$ is said to be abelian, the reader may, for the purposes of this note, think of $\mathfrak{A}$ as behaving formally like $(\mathsf{Mod}(R))$ for a ring R: the sets of morphisms are abelian groups, and zero morphisms, kernels, cokernels, images, quotient objects, and exact sequences are available and satisfy the familiar formal properties. We shall introduce any further facts that we need when they arise. Note that this point of view should not be interpreted as saying that every abelian

category is literally a category of the form $(\mathsf{Mod}(R))$. Abelian categories arise naturally in many other forms. For example, if $\mathfrak{I}$ is a small category, then

$$\mathrm{Fun}\big(\mathfrak{I}, (\mathsf{Mod}(R))\big)$$

is an abelian category, with kernels, cokernels, and exactness computed objectwise. Similarly,

$$(\mathsf{CxMod}(R))$$

is an abelian category, with these constructions computed degree by degree. Compare [Mac, §VIII. 3] and [St, Tag 03OM] for the pointwise argument for functor categories, and [St, Tag 010V] for cochain complexes.

For composable morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in an abelian category with $g \circ f = 0$, the sequence is said to be *exact at* B if the canonical morphism

$$\mathrm{Image}\, f \longrightarrow \mathrm{Ker}\, g$$

is an isomorphism. With this notion of exactness, the *five lemma*, the *snake lemma*, and the usual manipulations with *short exact sequences* are valid in an abelian category. See [Mac, §VIII. 3–4] and [St, Tag 010E, Tag 05QB, and Tag 010H] for example.

Definition II.3.38. Let $\mathfrak{A}$ and $\mathfrak{B}$ be abelian categories, and let

$$H : \mathfrak{A} \xrightarrow{\text{functor}} \mathfrak{B}$$

be a functor. The functor H is called *left exact* if it preserves finite limits, and it is called *right exact* if it preserves finite colimits. It is called *exact* if it is both left exact and right exact. □

For functors between abelian categories, these conditions can be recognized by means of short exact sequences. More precisely, H is left exact if and only if, for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

the sequence

$$0 \longrightarrow H(A) \longrightarrow H(B) \longrightarrow H(C)$$

is exact. Similarly, H is right exact if and only if

$$H(A) \longrightarrow H(B) \longrightarrow H(C) \longrightarrow 0$$

is exact for every short exact sequence as above. The functor H is exact if and only if it sends every short exact sequence to a short exact sequence.

Remark II.3.39. In many treatments, a left exact or right exact functor between abelian categories is assumed from the outset to be additive. Here, a functor

$$H\colon \mathfrak{A} \xrightarrow{\text{functor}} \mathfrak{B}$$

between abelian categories is called *additive* if, for every pair of objects $A, B \in \mathrm{ob}(\mathfrak{A})$, the induced map $\mathrm{Hom}_{\mathfrak{A}}(A,B) \longrightarrow \mathrm{Hom}_{\mathfrak{B}}(H(A),H(B))$ is a homomorphism of abelian groups.

We have not imposed additivity separately in Definition II.3.38, but this causes no difference. Indeed, if H preserves finite limits, then it preserves the zero object and finite products. Since finite products and finite coproducts coincide in an abelian category, it follows that H preserves finite direct sums. The addition of two morphisms $f, g\colon A \to B$ can be expressed in terms of direct sums as

$$f + g = \nabla_B \circ (f \oplus g) \circ \Delta_A,$$

where $\Delta_A\colon A \to A \oplus A$ and $\nabla_B\colon B \oplus B \to B$ are the diagonal and codiagonal morphisms (see [St, Tag 0DLP]). Since H preserves the zero object and finite direct sums, it preserves these morphisms and therefore satisfies $H(f+g) = H(f) + H(g)$. Thus H is additive. The same argument, with finite colimits in place of finite limits, shows that every right exact functor is additive.

Additivity also follows directly from either of the short-exact-sequence criteria above. Indeed, applying such a criterion to the split short exact sequence

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0$$

shows that the canonical comparison morphism between $H(A \oplus B)$ and $H(A) \oplus H(B)$ is an isomorphism. A functor between additive categories which preserves finite direct sums is additive.

Thus every left exact or right exact functor considered here is automatically additive. In particular, when we later construct derived functors of such functors, no additional additivity assumption will be necessary. See [St, Tag 010N]. □

The compatibility of adjoint functors with limits and colimits immediately gives the following important consequence.

Proposition II.3.40. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be locally small abelian categories, and let*

$$F\colon \mathfrak{A} \xrightarrow{\text{functor}} \mathfrak{B}, \qquad G\colon \mathfrak{B} \xrightarrow{\text{functor}} \mathfrak{A}$$

be functors such that $F \dashv G$. Then F is right exact and G is left exact. Consequently, a functor between abelian categories which is both a left adjoint and a right adjoint is exact.

Proof. By Proposition II.3.36, the left adjoint F preserves all colimits which exist, and the right adjoint G preserves all limits which exist. Since every abelian category has all finite limits and finite colimits, it follows that F preserves finite colimits and G preserves finite limits. Thus F is right exact and G is left exact. □

EXAMPLE II.3.41. Let R be a commutative ring, and let N be an R-module. There is an adjunction

$$- \otimes_R N \dashv \mathrm{Hom}_R(N, -),$$

expressed by the natural isomorphisms

$$\mathrm{Hom}_R(M \otimes_R N, L) \cong \mathrm{Hom}_R\big(M, \mathrm{Hom}_R(N, L)\big).$$

As we saw in Chapter I §5.2 in connection with Tor and Ext, the functor $-\otimes_R N$ is right exact, while $\mathrm{Hom}_R(N, -)$ is left exact. The adjunction above, together with Proposition II.3.40, gives another proof of these facts and explains them as instances of a general principle.

In addition, let $\mathfrak{I}$ be a small category, and let $M\colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{Mod}(R))$ be a diagram. Since left adjoints preserve arbitrary colimits and right adjoints preserve arbitrary limits, one has canonical isomorphisms

$$\left(\varinjlim_{\mathfrak{I}} M\right) \otimes_R N \cong \varinjlim_{\mathfrak{I}} \big(M(-) \otimes_R N\big) \tag{83}$$

and

$$\mathrm{Hom}_R\left(N,\ \varprojlim_{\mathfrak{I}} M\right) \cong \varprojlim_{\mathfrak{I}} \mathrm{Hom}_R\big(N, M(-)\big) \tag{84}$$

whenever the displayed limits and colimits exist (Note that the limits and colimits appearing here always exist in $(\mathsf{Mod}(R))$; see Remark II.3.45 below). □

We finally record the exactness properties of inductive and projective limits in module categories.

Let R be a ring, and let $\mathfrak{I}$ be a small category. As is mentioned above, the category $\mathrm{Fun}\big(\mathfrak{I}, (\mathsf{Mod}(R))\big)$ is an abelian category: kernels, cokernels, images, and exactness are computed object by object. Since

$$\varinjlim_{\mathfrak{I}} \dashv \Delta_{\mathfrak{I}} \dashv \varprojlim_{\mathfrak{I}},$$

Proposition II.3.40 gives the following.

COROLLARY II.3.42. *Let R be a ring, let $\mathfrak{I}$ be a small category, and let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a sequence of diagrams

$$A, B, C\colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{Mod}(R))$$

which is exact at every object of $\mathfrak{I}$.

Then

$$\varinjlim_{\mathfrak{I}} A \longrightarrow \varinjlim_{\mathfrak{I}} B \longrightarrow \varinjlim_{\mathfrak{I}} C \longrightarrow 0$$

is exact, and

$$0 \longrightarrow \varprojlim_{\mathfrak{I}} A \longrightarrow \varprojlim_{\mathfrak{I}} B \longrightarrow \varprojlim_{\mathfrak{I}} C$$

is exact.

Thus arbitrary colimits of R-modules are right exact, while arbitrary limits of R-modules are left exact. Additional assumptions are needed in order to obtain exactness at the remaining end of these sequences.

PROPOSITION II.3.43. *Let R be a ring, let $\mathfrak{I}$ be a small filtered category, and let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a sequence of diagrams

$$A, B, C \colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{Mod}(R))$$

which is exact at every object of $\mathfrak{I}$. Then the induced sequence

$$0 \longrightarrow \varinjlim_{\mathfrak{I}} A \longrightarrow \varinjlim_{\mathfrak{I}} B \longrightarrow \varinjlim_{\mathfrak{I}} C \longrightarrow 0$$

is exact.

PROOF. For the result in the language of directed systems, see [St, Tag 00DB]. The formulation for an arbitrary small filtered indexing category follows by replacing it with a cofinal directed system; see [St, Tag 0032]. □

One useful consequence concerns cohomology of complexes.

COROLLARY II.3.44. *Let R be a ring, $\mathfrak{I}$ a small filtered category, and*

$$K \colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{CxMod}(R))$$

be a diagram of cochain complexes of R-modules. Then, for every integer q, the canonical morphism

$$\varinjlim_{i \in \mathrm{ob}(\mathfrak{I})} H^q\big(K(i)\big) \longrightarrow H^q\left(\varinjlim_{i \in \mathrm{ob}(\mathfrak{I})} K(i)\right)$$

is an isomorphism.

PROOF. Colimits of complexes of R-modules are computed degree by degree. Since filtered colimits of R-modules are exact, they commute with the formation of kernels, images, and quotients. Hence they commute with $H^q(K^\bullet) = \operatorname{Ker} d^q / \operatorname{Image} d^{q-1}$. □

REMARK II.3.45. Let R be a ring. For every small category $\mathfrak{I}$, every diagram

$$F \colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{Mod}(R))$$

has both a limit and a colimit in $(\mathsf{Mod}(R))$. Indeed, arbitrary small products and coproducts of R-modules exist, and equalizers and coequalizers are given by kernels and cokernels, respectively. The general constructions of limits from products and equalizers, and of colimits from coproducts and coequalizers, are given in [St, Tag 002N and Tag 002P]. Similarly, the category $(\mathsf{CxMod}(R))$ has all small limits and colimits, computed degree by degree.

For a general abelian category, however, the axioms guarantee only the existence of finite limits and finite colimits; see [St, Tag 010D]. The existence of arbitrary filtered colimits or cofiltered limits does not follow from the axioms of an abelian category alone. Since equalizers and coequalizers in an abelian category $\mathfrak{A}$ are given by kernels and cokernels of differences of morphisms, respectively, the existence of all small products implies the existence of all small limits, while the existence of all small coproducts implies the existence of all small colimits. In particular, the conditions commonly denoted by AB3 and AB3* guarantee the existence of all small colimits and all small limits, respectively; see [St, Tag 079A]. Consequently, an abelian category having all small products and coproducts is both *complete* and *cocomplete*, that is, all small limits and colimits exist in it (Here a small limit or colimit means one indexed by a small category). □

The situation for projective limits is different. Let R be a ring, let I be a directed set, and suppose that

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0 \qquad (i \in I)$$

is a compatible projective system of short exact sequences of R-modules. The left exactness of the projective-limit functor gives an exact sequence

$$0 \longrightarrow \varprojlim_{i\in I} A_i \longrightarrow \varprojlim_{i\in I} B_i \longrightarrow \varprojlim_{i\in I} C_i.$$

The last morphism need not be surjective in general. There are useful additional hypotheses under which this surjectivity can be recovered. For example, for projective sequences of abelian groups or modules, suitable *Mittag–Leffler conditions* provide standard sufficient criteria (See [St, Tag 02MY] for example).

Right derived functors

Left exact functors will occur repeatedly in the rest of these notes. Although a left exact functor preserves the exactness at the first two terms of a short exact sequence, it need not preserve surjectivity at the last term. It is therefore useful to extract higher data which record, in a systematic way, this failure of exactness (The terminology used for this construction may at first sound more technical than the construction itself. In fact, it is precisely the same procedure already used to define Ext in Chapter I, § 5.2: one chooses an injective resolution, applies the given functor termwise, and then takes the cohomology of the resulting complex).

Let $\mathfrak{A}$ be an abelian category. An object $I \in \mathrm{ob}(\mathfrak{A})$ is called *injective* if, for every monomorphism

$$A \longrightarrow B$$

and every morphism

$$A \longrightarrow I,$$

there exists a morphism

$$B \longrightarrow I$$

which extends the given morphism from A. The category $\mathfrak{A}$ is said to have *enough injectives* if every object of $\mathfrak{A}$ admits a monomorphism into an injective object.

If $\mathfrak{A}$ has enough injectives, then every object $A \in \mathrm{ob}(\mathfrak{A})$ admits an *injective resolution*, namely, an exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

in which every I^q is injective.

Let $\mathfrak{B}$ be another abelian category, and let

$$F\colon \mathfrak{A} \xrightarrow{\text{functor}} \mathfrak{B}$$

be an *additive* left exact functor (here "additive" means that, for every pair of objects $A, B \in \mathrm{ob}(\mathfrak{A})$, the induced map

$$\mathrm{Hom}_{\mathfrak{A}}(A, B) \longrightarrow \mathrm{Hom}_{\mathfrak{B}}(F(A), F(B)), \qquad f \longmapsto F(f),$$

is a homomorphism of abelian groups). For an object $A \in \mathrm{ob}(\mathfrak{A})$, choose an injective resolution

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots .$$

Applying F termwise gives a cochain complex

$$F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow \cdots .$$

Definition II.3.46. With the notation and assumptions above, for every integer $q \geq 0$, the *q-th right derived functor* of F evaluated at A is defined by

$$R^qF(A) := H^q\big(F(I^\bullet)\big) := \frac{\mathrm{Ker}\big(F(d^q)\colon F(I^q) \longrightarrow F(I^{q+1})\big)}{\mathrm{Image}\big(F(d^{q-1})\colon F(I^{q-1}) \longrightarrow F(I^q)\big)},$$

where $d^q\colon I^q \longrightarrow I^{q+1}$ denotes the differential of the injective resolution, and the quotient is understood as a quotient object in the abelian category $\mathfrak{B}$ (the denominator is understood to be the zero subobject of $F(I^0)$ when $q = 0$). □

It is a standard fact that the resulting object is independent of the chosen injective resolution up to a canonical isomorphism. Moreover, the construction is functorial in A, and therefore defines a functor

$$R^qF\colon \mathfrak{A} \xrightarrow{\text{functor}} \mathfrak{B}$$

for every integer $q \geq 0$. See [I, Chapter I, § 7] and [St, Tag 013P and Tag 0156].

The functors R^qF with $q > 0$ are called the *higher right derived functors* of F.

The basic properties needed later are summarized in the following proposition.

Proposition II.3.47. *Let*

$$F\colon \mathfrak{A} \xrightarrow{\text{functor}} \mathfrak{B}$$

be an additive left exact functor between abelian categories, and assume that $\mathfrak{A}$ *has enough injectives. Then the following hold:*
(*i*) *One has a natural isomorphism* $R^0F \cong F$.
(*ii*) *If* I *is an injective object of* $\mathfrak{A}$*, then* $R^qF(I) = 0$ *for every integer* $q > 0$.
(*iii*) *Every short exact sequence*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathfrak{A}$ *induces a natural long exact sequence*

$$\begin{aligned} 0 &\longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'') \\ &\longrightarrow R^1F(A') \longrightarrow R^1F(A) \longrightarrow R^1F(A'') \\ &\longrightarrow R^2F(A') \longrightarrow R^2F(A) \longrightarrow R^2F(A'') \longrightarrow \cdots . \end{aligned}$$

Proof. Refer to [I, Chapter I, § 7] and [St, Tag 015B]. □

For example, for a fixed R-module M, the functor

$$\mathrm{Hom}_R(M, -)\colon (\mathsf{Mod}(R)) \xrightarrow{\text{functor}} (\mathsf{Mod}(R))$$

is left exact, and its right derived functors are

$$R^q\mathrm{Hom}_R(M, -) = \mathrm{Ext}^q_R(M, -)$$

by definition.

We shall also use resolutions by objects which are not necessarily injective but which behave like injective objects as far as the computation of the right derived functors of F is concerned.

Definition II.3.48. With $\mathfrak{A}$, $\mathfrak{B}$, and F as above, an object $A \in \mathrm{ob}(\mathfrak{A})$ is called *F-acyclic* if $R^qF(A) = 0$ for every integer $q > 0$. □

Every injective object is F-acyclic, but in applications there are often more concrete classes of F-acyclic objects.

Proposition II.3.49 (Acyclicity theorem). *Let* $\mathfrak{A}$*,* $\mathfrak{B}$*, and* F *be as above, and let*

$$0 \longrightarrow A \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots$$

be an exact resolution in $\mathfrak{A}$*, and assume that every* A^j *is* F*-acyclic. Then there are canonical isomorphisms*

$$R^qF(A) \cong H^q\big(F(A^\bullet)\big)$$

for every integer $q \geq 0$.

Proof. Refer to [I, Chapter I, Theorem 7.5] and [St, Tag 05TA]. Here, following [St, Tag 015J], we indicate the standard double-complex mechanism behind the result when $\mathfrak{B} = (\mathsf{Mod}(R))$ for a ring R. In this case, one can choose a suitable first-quadrant double injective resolution

$$A^\bullet \longrightarrow I^{\bullet,\bullet}$$

of the complex $A^\bullet$ in $\mathfrak{A}$ (Such a resolution is called a *Cartan–Eilenberg injective resolution*; see [St, Tag 015G and Tag 015I]. In the present situation, we choose such a resolution adapted to the exact augmented complex $0 \to A \to A^\bullet$ so that its two edge complexes will compute, after applying F, the cohomology of $F(A^\bullet)$ and the right derived functors of F evaluated at A, respectively). Set

$$C^{p,q} := F(I^{p,q}).$$

We use commuting unsigned differentials d and δ on $C^{p,q}$ and total differential $\delta + (-1)^p d$, as in Appendix 1.1. If the resolution is written with anticommuting differentials, first replace its vertical differential in bidegree (p, q) by $(-1)^p$ times that differential, as explained in Remark A.1.4. With the adapted choice above, the defining exactness properties of $I^{\bullet,\bullet}$, together with the F-acyclicity of the objects A^p, provide the row and column exactness required in Theorem A.1.1. The resulting comparison of the two edge complexes identifies $H^q\big(F(A^\bullet)\big)$ with $R^qF(A)$, from which the assertion follows.

For a general abelian category $\mathfrak{B}$, the same idea is implemented by the corresponding double-complex or spectral-sequence argument in an abelian category. □

For the general theory of right derived functors and injective resolutions, see [I, Chapter I, § 7] and [St, Tag 0156 and Tag 015B]. For acyclic objects and acyclic resolutions, see also [St, Tag 05T3].

3.2 Sheaves on a topological space

A geometric structure on a topological space determines which functions, sections, or other local objects are to be regarded as admissible on each open subset. Conversely, the assignment of such local objects to open subsets often encodes a substantial part of the geometric structure itself. The language of sheaves provides a systematic framework for describing this local-to-global structure. For example, if T is a topological space and $U \subset T$ is open, let

$$\mathcal{C}_T(U) := \{f \colon U \to \mathbb{C} \mid f \text{ is continuous}\}.$$

For open subsets $V \subset U$, restriction of functions gives a morphism

$$\mathcal{C}_T(U) \longrightarrow \mathcal{C}_T(V), \qquad f \longmapsto f|_V.$$

Likewise, if M is a C^∞ manifold, then

$$\mathcal{C}^\infty_M(U) := \{f \colon U \to \mathbb{C} \mid f \text{ is of class } C^\infty\}$$

comes with the same type of restriction morphisms. In the first case the assignment records the topology of T through its continuous functions, while in the second case it records the differentiable structure of M through its smooth functions.

Let X be a topological space. We denote by

$$(\mathsf{Open}(X))$$

the category whose objects are the open subsets of X and whose morphisms are inclusions. Thus, for open subsets $U, V \subset X$, there exists a unique morphism $U \to V$ if $U \subset V$, and no such morphism otherwise.

In the definition below, $\mathfrak{S}$ will be a category such as

$$(\mathsf{Set}), \quad (\mathsf{Group}), \quad (\mathsf{Ab}), \quad (\mathsf{Ring}), \quad (\mathsf{Mod}(R)), \quad \text{or} \quad (\mathsf{Vect}).$$

Unless otherwise stated, we shall mainly consider categories of rings, modules, abelian groups, or vector spaces, since the local objects occurring later usually carry natural algebraic operations.

Definition II.3.50. Let X be a topological space and let $\mathfrak{S}$ be as above. A *presheaf on X with values in $\mathfrak{S}$* is a functor

$$\mathcal{F}\colon (\mathsf{Open}(X))^{\mathrm{op}} \xrightarrow{\text{functor}} \mathfrak{S}.$$

For an inclusion $V \subset U$, the corresponding morphism

$$\rho_V^U\colon \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

is called the *restriction morphism*. We often write

$$s|_V := \rho_V^U(s)$$

for $s \in \mathcal{F}(U)$.

A presheaf $\mathcal{F}$ on X with values in $\mathfrak{S}$ is called a *sheaf on X with values in $\mathfrak{S}$* if the following two conditions hold for every open subset $U \subset X$ and every open covering

$$U = \bigcup_{i \in I} U_i.$$

- If $s, t \in \mathcal{F}(U)$ and
$$s|_{U_i} = t|_{U_i}$$
for every $i \in I$, then $s = t$.
- If sections $s_i \in \mathcal{F}(U_i)$ satisfy
$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$
for every $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$ such that
$$s|_{U_i} = s_i$$
for every $i \in I$.

The section s in the second condition is unique by the first condition.

When the value category $\mathfrak{S}$ is fixed or clear from the context, we simply speak of a presheaf or a sheaf on X. For specific value categories, we also use the usual terminology, such as a sheaf of sets, groups, abelian groups, rings, or R-modules (where R is a ring). □

The first condition says that equality of sections can be checked locally, while the second says that compatible local sections can be glued to a global section. For the general definitions and basic examples, see [I, Chapter II] and [St, Tag 006S and Tag 0071].

REMARK II.3.51. Note that the empty family is an open covering of $\emptyset$. Applying the first sheaf condition to this covering shows that any two sections in $\mathcal{F}(\emptyset)$ are equal. Applying the second sheaf condition to the empty family of local sections shows that a section in $\mathcal{F}(\emptyset)$ exists. In particular, for a sheaf of sets, $\mathcal{F}(\emptyset)$ is a one-point set. For a sheaf of groups it is the trivial group, for a sheaf of abelian groups or modules it is the zero object, and for a sheaf of rings it is the zero ring. □

REMARK II.3.52. Let $\mathcal{F}$ be a presheaf on a topological space X, $U \subset X$ an open subset, and $\{U_j\}_{j\in J}$ be an open covering of U. For the value categories considered in this note, the two sheaf conditions in Definition II.3.50 are equivalent to the statement that

$$\mathcal{F}(U) \longrightarrow \prod_{j\in J} \mathcal{F}(U_j) \rightrightarrows \prod_{(j,k)\in J\times J} \mathcal{F}(U_j \cap U_k)$$

is an equalizer diagram: That is, the restriction morphism

$$\mathcal{F}(U) \longrightarrow \prod_{j\in J} \mathcal{F}(U_j), \qquad s \longmapsto \big(s|_{U_j}\big)_{j\in J},$$

identifies $\mathcal{F}(U)$ with the collection of families $(s_j)_{j\in J}$ satisfying

$$s_j|_{U_j\cap U_k} = s_k|_{U_j\cap U_k}$$

for every $j, k \in J$. Here the two parallel morphisms, denoted by δ_0 and δ_1, are given by

$$\delta_0\big((s_j)_{j\in J}\big) = \big(s_j|_{U_j\cap U_k}\big)_{(j,k)\in J\times J}$$

and

$$\delta_1\big((s_j)_{j\in J}\big) = \big(s_k|_{U_j\cap U_k}\big)_{(j,k)\in J\times J}.$$

This categorical formulation can be generalized beyond the category $(\mathsf{Open}(X))$. More generally, one may equip a category with a suitable notion of covering, called a *Grothendieck topology* (a category equipped with such a topology is called a *site*) and one can define sheaves on it. We shall not develop this more general theory here and shall mainly consider sheaves on topological spaces as in Definition II.3.50. □

The presheaves on X with values in $\mathfrak{S}$, with natural transformations as morphisms, form a category denoted by

$$(\mathsf{PreSheaf}(X, \mathfrak{S})).$$

The sheaves form a *full subcategory*

$$(\mathsf{Sheaf}(X, \mathfrak{S})) \subset (\mathsf{PreSheaf}(X, \mathfrak{S})),$$

where "full" means that, for any two sheaves $\mathcal{F}$ and $\mathcal{G}$, every morphism $\mathcal{F} \longrightarrow \mathcal{G}$ in $(\mathsf{PreSheaf}(X, \mathfrak{S}))$ is also a morphism in $(\mathsf{Sheaf}(X, \mathfrak{S}))$. When the value category $\mathfrak{S}$ is clear from the context, we simply write

$$(\mathsf{PreSheaf}(X)) \qquad \text{and} \qquad (\mathsf{Sheaf}(X)).$$

Remark II.3.53. The categories of presheaves and sheaves are often denoted by $(\mathsf{PSh}(X))$ and $(\mathsf{Sh}(X))$, respectively. We avoid these abbreviations in this note because the spaces under consideration will frequently arise in complex analysis (for example, one may keep in mind a domain $X \subset \mathbb{C}$, although the relevant complex-analytic spaces will be introduced in the next chapter), and the sets of subharmonic functions and of their several-complex-variables generalization, plurisubharmonic functions, on such spaces are commonly denoted by $\mathrm{Sh}(X)$ and $\mathrm{Psh}(X)$, respectively. The notation adopted here is intended to avoid possible confusion with these sets of functions. □

The assignments $U \mapsto \mathcal{C}_T(U)$ and $U \mapsto \mathcal{C}^\infty_M(U)$ considered above define sheaves with values in (Ring). By forgetting multiplication, they may also be regarded as sheaves with values in (Vect). More generally, functions satisfying a local condition and admitting unique gluing frequently form a sheaf.

Example II.3.54. Let X be a topological space and let A be a nonzero abelian group. Consider the presheaf $\mathcal{F}$ defined by

$$\mathcal{F}(U) := \begin{cases} A & \text{if } U \neq \emptyset, \\ 0 & \text{if } U = \emptyset, \end{cases}$$

with the identity morphism as every restriction morphism between nonempty open subsets and the unique morphism to 0 as the restriction to the empty set. This is a well-defined presheaf, and it may at first appear to describe the "constant A-valued functions." Suppose, however, that X has two disjoint nonempty open subsets U and V. Then

$$\mathcal{F}(U \cup V) \cong A,$$

whereas the sheaf condition would require independent choices on U and V, corresponding to

$$\mathcal{F}(U) \oplus \mathcal{F}(V) \cong A \oplus A.$$

Indeed, if $a \in A \setminus \{0\}$, the section 0 on U and the section a on V agree on the empty intersection, but they cannot be obtained by restricting a single element of $\mathcal{F}(U \cup V)$. Thus

globally constant A-valued functions do not form a sheaf in general. Instead one considers the sheaf

$$\underline{A}_X(U) := \{f\colon U \to A \mid f \text{ is locally constant}\}.$$

This sheaf is called the *sheaf of A-valued locally constant functions* and will be denoted by $\underline{A}_X$ in this note. □

Let $\mathcal{F}$ be a presheaf on X. For an open subset $U \subset X$, an element of $\mathcal{F}(U)$ is called a *section of $\mathcal{F}$ over U*. We use the notation

$$\Gamma(U, \mathcal{F}) := \mathcal{F}(U).$$

In particular, $\Gamma(X, \mathcal{F})$ is the set, group, or module of *global sections*.

Let $K \subset X$ be an arbitrary subset. Using the directed set of open neighborhoods of K introduced in Example II.3.20, one may consider

$$\varinjlim_{U \in \mathcal{N}_X(K)} \Gamma(U, \mathcal{F}).$$

An element of this inductive limit is represented by a section defined on some open neighborhood of K, with two such sections identified when they agree on a possibly smaller neighborhood of K. The notation $\Gamma(K, \mathcal{F})$ is sometimes used for this object, but it may also be used for sections of the restricted sheaf on the subspace K. To avoid this ambiguity, we shall, whenever possible, display the inductive limit explicitly.

For a point $x \in X$, the inductive limit

$$\mathcal{F}_x := \varinjlim_{U \in \mathcal{N}_X(\{x\})} \mathcal{F}(U)$$

is called the *stalk of $\mathcal{F}$ at x*. Its elements are called *germs of sections at x*. Thus a germ is represented by a section defined on some neighborhood of x, and only its behavior on sufficiently small neighborhoods of x is retained.

Example II.3.55. Let $\Omega \subset \mathbb{C}$ be a domain containing the origin, and denote the standard coordinate on $\mathbb{C}$ by z. Denote by $\mathcal{O}_\Omega$ the sheaf of holomorphic functions on Ω. Then, by Mini-Exercise I.9.2, its stalk at the origin is isomorphic to the ring $\mathbb{C}\{z\}$ of convergent power series in z:

$$\mathcal{O}_{\Omega,0} \cong \mathbb{C}\{z\}.$$

□

There is a canonical method for turning a presheaf into a sheaf without changing its stalks.

Proposition II.3.56 (Sheafification). *For the value category $\mathfrak{S}$ as above, there is a functor*

$$a\colon (\mathsf{PreSheaf}(X, \mathfrak{S})) \xrightarrow{\text{functor}} (\mathsf{Sheaf}(X, \mathfrak{S})),$$

called the sheafification functor. *If*

$$\iota\colon (\mathsf{Sheaf}(X,\mathfrak{S})) \xrightarrow{\text{functor}} (\mathsf{PreSheaf}(X,\mathfrak{S}))$$

denotes the natural full inclusion, then $a \dashv \iota$. *Moreover, the canonical morphism*

$$\mathcal{F} \longrightarrow \iota(a(\mathcal{F}))$$

induces an isomorphism

$$\mathcal{F}_x \xrightarrow{\sim} (a(\mathcal{F}))_x$$

for every $x \in X$.

Proof. Refer to [I, Chapter II §2] and [St, Tag 007X and Tag 0088]. Here we describe only the idea of the construction.

As the discussion in Example II.3.54 suggests, sections of a sheaf must be determined by local conditions. Thus, informally, the sheafification is obtained by collecting all objects which locally look like sections of the original presheaf. For presheaves with algebraic structure, the corresponding operations are defined germwise (For example, if $\mathcal{F}$ is the presheaf of A-valued constant functions considered in Example II.3.54, then a section constructed in this way is precisely an A-valued locally constant function. Indeed, the sheafification of this presheaf is the sheaf $\underline{A}_X$ of A-valued locally constant functions). □

Remark II.3.57. There is another classical way to define/describe a sheaf $\mathcal{F}$ on a topological space X. In this approach, one forms a topological space whose fiber over each point $x \in X$ is the stalk $\mathcal{F}_x$, and regards local sections of the sheaf as continuous local sections of the resulting projection onto X. The topology on the total space is chosen so that this projection is a local homeomorphism. This sheaf-space description is equivalent to the description by sections over open subsets and restriction morphisms used in these notes. □

Sheaves of modules and sheaf cohomology

Let $\mathcal{R}$ be a sheaf of rings on a topological space X. A *sheaf of* $\mathcal{R}$*-modules* is a sheaf $\mathcal{F}$ of abelian groups together with an $\mathcal{R}(U)$-module structure on $\mathcal{F}(U)$ for every open subset $U \subset X$, such that all restriction morphisms are compatible with the module structures. The category of sheaves of $\mathcal{R}$-modules is denoted by

$$(\mathsf{Mod}(\mathcal{R})).$$

Taking $\mathcal{R} = \underline{R}_X$, the sheaf of R-valued locally constant functions for a ring R, recovers the category of sheaves of R-modules on X. More precisely, there is a natural identification

$$(\mathsf{Mod}(\underline{R}_X)) = (\mathsf{Sheaf}(X, (\mathsf{Mod}(R)))).$$

In particular, sheaves of abelian groups correspond to the case $R = \mathbb{Z}$:

$$(\mathsf{Mod}(\underline{\mathbb{Z}}_X)) = (\mathsf{Sheaf}(X, (\mathsf{Mod}(\mathbb{Z})))) = (\mathsf{Sheaf}(X, (\mathsf{Ab}))).$$

For later use, we also define the *support* of a sheaf of $\mathcal{R}$-modules $\mathcal{F}$ on X by $\operatorname{Supp}\mathcal{F} := \{x \in X \mid \mathcal{F}_x \neq 0\}$.

The following fact is the basic reason why ordinary homological algebra applies so effectively to sheaves of modules.

PROPOSITION II.3.58. *Let X be a topological space and let $\mathcal{R}$ be a sheaf of rings on X. Then $(\mathsf{Mod}(\mathcal{R}))$ is an abelian category and has enough injective objects. It also has all small limits and colimits. Kernels, images, cokernels, and exactness may be checked on stalks.*

PROOF. Refer to [I, Chapter II Theorems 2.5 and 3.1] and [St, Tag 01AG, Tag 01AH, and Tag 01DI]. □

More explicitly, for a topological space X and a sheaf $\mathcal{R}$ of rings on X, if

$$\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$$

is a morphism of sheaves of $\mathcal{R}$-modules, then the presheaf

$$U \longmapsto \operatorname{Ker}\big(\mathcal{F}(U) \to \mathcal{G}(U)\big)$$

is already a sheaf and defines $\operatorname{Ker}\varphi$. The sectionwise image and cokernel presheaves need not be sheaves; their sheafifications define $\operatorname{Image}\varphi$ and $\operatorname{Coker}\varphi$. Nevertheless,

$$(\operatorname{Ker}\varphi)_x \cong \operatorname{Ker}(\varphi_x), \qquad (\operatorname{Image}\varphi)_x \cong \operatorname{Image}(\varphi_x), \qquad (\operatorname{Coker}\varphi)_x \cong \operatorname{Coker}(\varphi_x).$$

A *subsheaf* $\mathcal{G}$ of an $\mathcal{R}$-module $\mathcal{F}$ is a sheaf such that

$$\mathcal{G}(U) \subset \mathcal{F}(U)$$

is an $\mathcal{R}(U)$-submodule for every open subset $U \subset X$, and the restriction morphisms of $\mathcal{G}$ are induced by those of $\mathcal{F}$. Subsheaves of $\mathcal{F}$ correspond to subobjects of $\mathcal{F}$ in $(\mathsf{Mod}(\mathcal{R}))$.

REMARK II.3.59. Strictly speaking, a subobject of $\mathcal{F}$ in a category is an equivalence class of monomorphisms into $\mathcal{F}$, rather than a literally specified inclusion. In $(\mathsf{Mod}(\mathcal{R}))$, every monomorphism $\mathcal{G} \to \mathcal{F}$ identifies $\mathcal{G}$ with its image subsheaf of $\mathcal{F}$. Thus, in the concrete situations considered in this note, it is convenient and harmless to speak of subsheaves using literal inclusions $\mathcal{G} \subset \mathcal{F}$. □

Let $\mathcal{G} \subset \mathcal{F}$ be a subsheaf. The presheaf

$$U \longmapsto \mathcal{F}(U)/\mathcal{G}(U)$$

need not be a sheaf. Its sheafification is called the *quotient sheaf* and is denoted by

$$\mathcal{F}/\mathcal{G}.$$

Thus there is a canonical morphism

$$\mathcal{F}(U)/\mathcal{G}(U) \longrightarrow \Gamma(U, \mathcal{F}/\mathcal{G}),$$

but this morphism need not be surjective. More concretely, every section

$$\sigma \in \Gamma(U, \mathcal{F}/\mathcal{G})$$

can be represented locally: there exist an open covering

$$U = \bigcup_{i \in I} U_i$$

and sections

$$s_i \in \mathcal{F}(U_i)$$

such that $\sigma|_{U_i}$ is represented by the class of s_i. On every overlap one has

$$s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \in \mathcal{G}(U_i \cap U_j).$$

Conversely, local sections satisfying this condition determine a unique section of $\mathcal{F}/\mathcal{G}$. Two such systems of local representatives determine the same section if, after passing to a common refinement, their differences are sections of $\mathcal{G}$. At the level of stalks one has a canonical isomorphism $(\mathcal{F}/\mathcal{G})_x \cong \mathcal{F}_x/\mathcal{G}_x$ (Similarly, if $\varphi\colon \mathcal{F} \to \mathcal{H}$ is a morphism of $\mathcal{R}$-modules, then $\operatorname{Image}\varphi$ is the subsheaf of $\mathcal{H}$ whose sections are locally in the image of φ. In other words, $s \in \Gamma(U, \mathcal{H})$ belongs to $\Gamma(U, \operatorname{Image}\varphi)$ if and only if there exists an open covering

$$U = \bigcup_{i \in I} U_i$$

and sections

$$t_i \in \mathcal{F}(U_i)$$

such that

$$\varphi(t_i) = s|_{U_i}$$

for every $i \in I$). A sequence

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}$$

of sheaves of $\mathcal{R}$-modules is exact if and only if

$$\mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x$$

is exact for every $x \in X$.

Remark II.3.60. Let $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves of $\mathcal{R}$-modules. The preceding stalkwise criterion implies that φ is injective if and only if $\varphi_x\colon \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is injective for every $x \in X$, and that φ is surjective if and only if φ_x is surjective for every $x \in X$.

For injectivity, there is little difference between the stalkwise and sectionwise viewpoints. Indeed, φ is injective if and only if $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is injective for every open subset $U \subset X$, see [I, Chapter II Lemma 2.2 (ii)].

For surjectivity, however, one must be more careful. Stalkwise surjectivity means that every section of $\mathcal{G}$ admits liftings locally: for every open subset $U \subset X$, every $s \in \mathcal{G}(U)$, and every $x \in U$, there exist an open neighborhood $V \subset U$ of x and a section $t \in \mathcal{F}(V)$ such that

$$\varphi(t) = s|_V.$$

It does not imply that s admits a lifting over all of U. Equivalently, the sectionwise image presheaf

$$U \longmapsto \varphi\big(\mathcal{F}(U)\big)$$

may fail to be a sheaf, and its sheafification is the image sheaf Image φ. Thus a morphism of sheaves may be surjective even though $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is not surjective for some open subset U (See Exercise II.2 (ii)). This distinction should be kept in mind in the discussion below. □

For an open subset $U \subset X$, the section functor

$$\Gamma(U, -)\colon (\mathsf{Mod}(\mathcal{R})) \xrightarrow{\text{functor}} (\mathsf{Mod}(\mathcal{R}(U)))$$

is left exact; see [St, Tag 0716]. Indeed, if

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''$$

is exact, then

$$\mathcal{F}' \cong \mathrm{Ker}\big(\mathcal{F} \to \mathcal{F}''\big).$$

Since kernels of sheaves of modules are computed sectionwise, it follows that

$$0 \longrightarrow \Gamma(U, \mathcal{F}') \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}'')$$

is exact. In general, however, $\Gamma(U, -)$ is not right exact, even for sheaves of abelian groups; see Exercise II.2 (ii).

Since $(\mathsf{Mod}(\mathcal{R}))$ has enough injectives (see Proposition II.3.58), the construction of right derived functors described in § 3.1 applies to the left exact functor

$$\Gamma(X, -)\colon (\mathsf{Mod}(\mathcal{R})) \xrightarrow{\text{functor}} (\mathsf{Mod}(\mathcal{R}(X))).$$

The resulting modules are called the *sheaf cohomology modules*:

Definition II.3.61. Let $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules. For every non-negative integer q, the *q-th sheaf cohomology module of $\mathcal{F}$ on X* is defined by

$$H^q(X, \mathcal{F}) := R^q\Gamma(X, \mathcal{F}).$$

Equivalently, for an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots$$

of $\mathcal{F}$, $H^q(X, \mathcal{F}) := H^q\big(\Gamma(X, \mathcal{I}^\bullet)\big)$. □

By the general properties of right derived functors summarized in Proposition II.3.47, this definition is independent, up to a canonical isomorphism, of the chosen injective resolution. Part (i) of that proposition gives

$$H^0(X,\mathcal{F}) \cong \Gamma(X,\mathcal{F}).$$

Moreover, by part (iii), every short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

of sheaves of $\mathcal{R}$-modules induces a natural long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{G}) \longrightarrow H^0(X,\mathcal{H}) \\ &\longrightarrow H^1(X,\mathcal{F}) \longrightarrow H^1(X,\mathcal{G}) \longrightarrow H^1(X,\mathcal{H}) \\ &\longrightarrow H^2(X,\mathcal{F}) \longrightarrow H^2(X,\mathcal{G}) \longrightarrow H^2(X,\mathcal{H}) \longrightarrow \cdots . \end{aligned}$$

Although injective resolutions enter the definition of sheaf cohomology, they are usually not the most convenient resolutions for explicit calculations. In the present setting, a sheaf $\mathcal{C}$ of $\mathcal{R}$-modules is $\Gamma(X,-)$-acyclic precisely when $H^q(X,\mathcal{C}) = 0$ for every integer $q > 0$. By the acyclicity theorem (Proposition II.3.49), if

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \cdots$$

is an exact resolution by $\Gamma(X,-)$-acyclic sheaves, then there are canonical isomorphisms

$$H^q(X,\mathcal{F}) \cong H^q\big(\Gamma(X,\mathcal{C}^\bullet)\big)$$

for every integer $q \geq 0$. Thus one may replace an abstract injective resolution by a more explicit acyclic resolution adapted to the problem under consideration. Different acyclic resolutions may give different concrete interpretations of the same sheaf cohomology modules and of their cohomology classes. We record three important classes of acyclic sheaves.

Definition II.3.62. Let $\mathcal{F}$ be a sheaf of abelian groups or modules on X.
(i) The sheaf $\mathcal{F}$ is called *flabby* (or *flasque*) if, for every pair of open subsets $V \subset U$, the restriction morphism

$$\Gamma(U,\mathcal{F}) \longrightarrow \Gamma(V,\mathcal{F})$$

is surjective.
(ii) The sheaf $\mathcal{F}$ is called *soft* if, for every closed subset $K \subset X$, the natural morphism

$$\Gamma(X,\mathcal{F}) \longrightarrow \varinjlim_{U \in \mathcal{N}_X(K)} \Gamma(U,\mathcal{F})$$

is surjective.
(iii) Let $\mathcal{R}$ be a sheaf of rings on X and $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules. The sheaf $\mathcal{F}$ is called *fine* if, for every locally finite open covering $\{U_i\}_{i\in I}$ of X, there exists a family of $\mathcal{R}$-linear sheaf endomorphisms

$$h_i \colon \mathcal{F} \longrightarrow \mathcal{F} \qquad (i \in I)$$

such that

$$\operatorname{Supp} h_i := \overline{\{x \in X \mid (h_i)_x \neq 0\}} \subset U_i$$

for every $i \in I$, and

$$\sum_{i \in I} h_i = \mathrm{id}_{\mathcal{F}}$$

(Here the sum is well defined because the family of supports is locally finite). □

The following proposition summarizes the basic acyclicity properties of these classes of sheaves.

Proposition II.3.63. *Let X be a topological space.*
(i) Every flabby sheaf is $\Gamma(U, -)$-acyclic for every open subset $U \subset X$. In particular, every flabby sheaf is $\Gamma(X, -)$-acyclic.
If, moreover, X is paracompact and Hausdorff, then the following additional statements hold:
(ii) Every soft sheaf is $\Gamma(X, -)$-acyclic.
(iii) Every fine sheaf of modules is soft, and hence $\Gamma(X, -)$-acyclic.
(iv) Let $\mathcal{R}$ be a soft sheaf of rings on X. Then every sheaf of $\mathcal{R}$-modules is soft, and hence $\Gamma(X, -)$-acyclic.

Proof. For (i), see [I, Chapter II Theorem 3.5], [De4, Chapter IV Theorem 4.4], and [St, Tag 09SY]. The assertion in (ii) follows from [De4, Chapter IV Theorem 4.15]. We prove (iii). Let $\mathcal{F}$ be a fine sheaf of $\mathcal{R}$-modules, $K \subset X$ a closed subset, and let

$$\sigma \in \varinjlim_{U \in \mathcal{N}_X(K)} \Gamma(U, \mathcal{F}).$$

Choose an open neighborhood U of K and a section $s \in \Gamma(U, \mathcal{F})$ representing σ. By applying the fineness of $\mathcal{F}$ to the finite open covering $X = U \cup (X \setminus K)$, one has $\mathcal{R}$-linear sheaf endomorphisms $h_1, h_2 \colon \mathcal{F} \to \mathcal{F}$ such that

$$h_1 + h_2 = \mathrm{id}_{\mathcal{F}}, \qquad \operatorname{Supp} h_1 \subset U, \qquad \operatorname{Supp} h_2 \subset X \setminus K.$$

The section $h_1(s) \in \Gamma(U, \mathcal{F})$ has support contained in the closed subset $\operatorname{Supp} h_1 \subset U$, and therefore extends by zero to a global section $\widetilde{s} \in \Gamma(X, \mathcal{F})$. Since $\operatorname{Supp} h_2$ is disjoint from K, the endomorphism h_2 vanishes on some open neighborhood of K. Hence $h_1 = \mathrm{id}_{\mathcal{F}}$ on that neighborhood, and $\widetilde{s}$ represents σ. Thus $\mathcal{F}$ is soft. Its $\Gamma(X, -)$-acyclicity follows from (ii). Finally, (iv) follows from [De4, Chapter IV, Lemma 4.18], together with (ii). □

Example II.3.64. Let X be a paracompact Hausdorff topological space, and let $\mathcal{C}_X$ denote the sheaf of complex-valued continuous functions on X. Ordinary partitions of unity show that every sheaf of $\mathcal{C}_X$-modules is fine. Put $\mathcal{R} := \mathcal{C}_X$, and let $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules. Indeed, by Mini-Exercise I.3.14 (ii), for every locally finite open covering $X = \bigcup_{i \in I} U_i$, there exists a family of global sections

$$\varphi_i \in \Gamma(X, \mathcal{R}) \qquad (i \in I)$$

such that

$$\operatorname{Supp}\varphi_i \subset U_i$$

for every $i \in I$, and

$$\sum_{i\in I} \varphi_i = 1.$$

Then the multiplication by φ_i defines an $\mathcal{R}$-linear sheaf endomorphism

$$h_i \colon \mathcal{F} \longrightarrow \mathcal{F}, \qquad h_i(s) := \varphi_i s.$$

One has

$$\operatorname{Supp} h_i \subset \operatorname{Supp}\varphi_i \subset U_i$$

and

$$\sum_{i\in I} h_i = \mathrm{id}_{\mathcal{F}}.$$

Thus $\mathcal{F}$ is fine. In particular, $\mathcal{C}_X$ is fine, and hence every sheaf of $\mathcal{C}_X$-modules is $\Gamma(X,-)$-acyclic, by Proposition II.3.63 (*iii*) (*iv*).

Similarly, if M is a C^∞ manifold and $\mathcal{C}^\infty_M$ denotes the sheaf of complex-valued C^∞ functions on M, then C^∞ partitions of unity show that every sheaf of $\mathcal{C}^\infty_M$-modules is soft and $\Gamma(M,-)$-acyclic, by Proposition II.3.63 (*iii*) (*iv*). □

Example II.3.65. Let M be a C^∞ manifold, r a positive integer, and $E \to M$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle equipped with its flat connection $D\colon \mathcal{A}^q(M,E) \to \mathcal{A}^{q+1}(M,E)$. Let $\mathbb{C}_M(E)$ denote the sheaf of local D-flat sections of E:

$$\Gamma(U, \mathbb{C}_M(E)) = \big\{ s \in C^\infty(U, E|_U) \;\big|\; Ds = 0 \big\}$$

for every open subset $U \subset M$. For every integer $q \geq 0$, let $\mathcal{A}^q_M(E)$ denote the sheaf of E-valued C^∞ differential q-forms on M:

$$\Gamma(U, \mathcal{A}^q_M(E)) = \mathcal{A}^q(U, E|_U)$$

for every open subset $U \subset M$. The Poincaré lemma, applied in local flat trivializations of E, gives an exact sequence of sheaves

$$0 \longrightarrow \mathbb{C}_M(E) \longrightarrow \mathcal{A}^0_M(E) \xrightarrow{D} \mathcal{A}^1_M(E) \xrightarrow{D} \mathcal{A}^2_M(E) \xrightarrow{D} \cdots;$$

recall Remark I.8.45.

For the statements involving currents in this example, assume that M is orientable and fix an orientation. Similarly, let $\mathcal{K}^q_M(E)$ denote the sheaf of q-currents with values in E:

$$\Gamma(U, \mathcal{K}^q_M(E)) = \mathcal{K}^q(U, E|_U)$$

for every open subset $U \subset M$. The Poincaré lemma for currents, again applied in local flat trivializations, gives another exact sequence

$$0 \longrightarrow \mathbb{C}_M(E) \longrightarrow \mathcal{K}^0_M(E) \xrightarrow{D} \mathcal{K}^1_M(E) \xrightarrow{D} \mathcal{K}^2_M(E) \xrightarrow{D} \cdots;$$

see Lemma I.8.52.

Each $\mathcal{A}^q_M(E)$ and $\mathcal{K}^q_M(E)$ is naturally a sheaf of $\mathcal{C}^\infty_M$-modules, where $\mathcal{C}^\infty_M$ denotes the sheaf of complex-valued C^∞ functions on M (For currents, multiplication by a smooth function is defined by $(fT)(\eta) := T(f\eta)$). Therefore Example II.3.64 shows that all the sheaves $\mathcal{A}^q_M(E)$ and $\mathcal{K}^q_M(E)$ are fine. By Proposition II.3.63 (*iii*), they are soft and $\Gamma(M,-)$-acyclic. Applying the acyclicity theorem (Proposition II.3.49) to the two resolutions above gives canonical isomorphisms

$$H^q(M,\mathbb{C}_M(E)) \cong H^q\big(\Gamma(M,\mathcal{A}^\bullet_M(E)),D\big) = H^q_{\mathrm{DR}}(M,E)$$

and

$$H^q(M,\mathbb{C}_M(E)) \cong H^q\big(\Gamma(M,\mathcal{K}^\bullet_M(E)),D\big) = H^q\big(\mathcal{K}^\bullet(M,E),D\big).$$

Consequently, there are canonical isomorphisms

$$H^q_{\mathrm{DR}}(M,E) \cong H^q(M,\mathbb{C}_M(E)) \cong H^q\big(\mathcal{K}^\bullet(M,E),D\big). \tag{85}$$

In particular, this proves the isomorphism announced in (45).

When $E \to M$ is a trivial complex line bundle with a fixed global D-flat frame, this frame gives the identifications

$$\mathbb{C}_M(E) = \underline{\mathbb{C}}_M, \qquad D = d.$$

Combining the preceding isomorphisms with the singular–de Rham correspondence (Theorem I.8.54), we obtain natural isomorphisms

$$H^q(M,\mathbb{C}) \cong H^q_{\mathrm{DR}}(M) \cong H^q\big(\mathcal{K}^\bullet(M),d\big). \tag{86}$$

□

Remark II.3.66. In the preceding example, the ambient abelian category in which the resolutions are considered was left implicit.

The complexes

$$\mathcal{A}^0_M(E) \xrightarrow{D} \mathcal{A}^1_M(E) \xrightarrow{D} \mathcal{A}^2_M(E) \xrightarrow{D} \cdots$$

and

$$\mathcal{K}^0_M(E) \xrightarrow{D} \mathcal{K}^1_M(E) \xrightarrow{D} \mathcal{K}^2_M(E) \xrightarrow{D} \cdots$$

are not complexes in $(\mathsf{Mod}(\mathcal{C}^\infty_M))$, since the flat connection D is not $\mathcal{C}^\infty_M$-linear (Recall $D(f\alpha) = df \wedge \alpha + fD\alpha$ for a smooth function f and an E-valued differential form α).

The operator D is, however, $\mathbb{C}$-linear and, more generally, linear with respect to locally constant complex-valued functions. Hence both resolutions may naturally be regarded as resolutions in

$$(\mathsf{Mod}(\underline{\mathbb{C}}_M)) = (\mathsf{Sheaf}(M,(\mathsf{Mod}(\mathbb{C})))) = (\mathsf{Sheaf}(M,(\mathsf{Vect}))).$$

The sheaf $\mathbb{C}_M(E)$ of local D-flat sections is naturally a sheaf of $\underline{\mathbb{C}}_M$-modules. After forgetting the complex vector space structures, the same resolutions may also be regarded as resolutions in

$$(\mathsf{Mod}(\underline{\mathbb{Z}}_M)) = (\mathsf{Sheaf}(M,(\mathsf{Mod}(\mathbb{Z})))) = (\mathsf{Sheaf}(M,(\mathsf{Ab}))).$$

The $\mathcal{C}^\infty_M$-module structures on the individual sheaves are used to construct the sheaf endomorphisms arising from smooth partitions of unity, and hence to prove that the terms of the resolutions are fine. These same endomorphisms remain available after forgetting part of the scalar structure. Thus the terms remain $\Gamma(M, -)$-acyclic in either of the module categories above. □

We next record the sheaf-cohomological Mayer–Vietoris sequence. In what follows, for a topological space X, a sheaf $\mathcal{F}$ on X, and an open subset $U \subset X$, we simply write

$$H^q(U, \mathcal{F}) := H^q(U, \mathcal{F}|_U) \qquad (q \in \mathbb{Z}_{\geq 0}),$$

where $\mathcal{F}|_U$ denotes the restriction of $\mathcal{F}$ to U, endowed with the relative topology (Since U is open in X, the category $(\mathsf{Open}(U))$ is naturally identified with the full subcategory of $(\mathsf{Open}(X))$ consisting of the open subsets contained in U, and $\mathcal{F}|_U$ is obtained by restricting the functor $\mathcal{F}$ from $(\mathsf{Open}(X))^{\mathrm{op}}$ to $(\mathsf{Open}(U))^{\mathrm{op}}$). Explicitly, for every open subset $W \subset U$, one has $(\mathcal{F}|_U)(W) = \mathcal{F}(W)$, with the restriction morphisms inherited from $\mathcal{F}$.

Proposition II.3.67 (Mayer–Vietoris sequence). *Let X be a topological space, let $\mathcal{R}$ be a sheaf of rings on X, and let $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules. Suppose that $U, V \subset X$ are open subsets such that $X = U \cup V$. Then there exists a natural long exact sequence*

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \longrightarrow H^0(U \cap V, \mathcal{F}) \quad .$$

$$\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(U, \mathcal{F}) \oplus H^1(V, \mathcal{F}) \longrightarrow H^1(U \cap V, \mathcal{F})$$

$$\longrightarrow H^2(X, \mathcal{F}) \longrightarrow H^2(U, \mathcal{F}) \oplus H^2(V, \mathcal{F}) \longrightarrow H^2(U \cap V, \mathcal{F}) \longrightarrow \dots$$

Proof. For the existence and exactness of the sequence, see [I, Chapter II Theorem 5.10] and [De4, Chapter IV Theorem 3.11]. We only describe here the morphisms appearing in the sequence.

The morphism

$$H^q(X, \mathcal{F}) \longrightarrow H^q(U, \mathcal{F}) \oplus H^q(V, \mathcal{F})$$

is induced by the two restriction morphisms. The morphism

$$H^q(U, \mathcal{F}) \oplus H^q(V, \mathcal{F}) \longrightarrow H^q(U \cap V, \mathcal{F})$$

is the difference of the restriction morphisms:

$$(\xi, \eta) \longmapsto \xi|_{U\cap V} - \eta|_{U\cap V}.$$

The resulting long exact sequence is functorial in $\mathcal{F}$.

We also describe the connecting morphism in a situation which frequently occurs in concrete computations. Suppose that there exists a resolution

$$0 \to \mathcal{F} \to \mathcal{L}^0 \xrightarrow{d} \mathcal{L}^1 \xrightarrow{d} \mathcal{L}^2 \xrightarrow{d} \cdots$$

by fine sheaves and that the complexes of sections of this resolution over X, U, V, and $U \cap V$ compute the corresponding sheaf cohomology groups. In this case, by the fineness of $\mathcal{L}^q$, there exist sheaf endomorphisms

$$h_U, h_V \colon \mathcal{L}^q \longrightarrow \mathcal{L}^q$$

such that

$$h_U + h_V = \mathrm{id}_{\mathcal{L}^q}, \qquad \operatorname{Supp} h_U \subset U, \qquad \operatorname{Supp} h_V \subset V.$$

For an element

$$[c] \in H^q(U \cap V, \mathcal{F})$$

represented by a section $c \in \Gamma(U \cap V, \mathcal{L}^q)$ satisfying

$$dc = 0,$$

the section $h_V(c)$ on $U \cap V$ extends by zero to a section

$$c_U \in \Gamma(U, \mathcal{L}^q),$$

and the section $-h_U(c)$ extends by zero to a section

$$c_V \in \Gamma(V, \mathcal{L}^q).$$

On $U \cap V$, one has

$$c_U - c_V = h_V(c) + h_U(c) = \mathrm{id}_{\mathcal{L}^q}(c) = c.$$

Since $dc = 0$, the sections

$$dc_U \in \Gamma(U, \mathcal{L}^{q+1}) \qquad \text{and} \qquad dc_V \in \Gamma(V, \mathcal{L}^{q+1})$$

coincide on $U \cap V$. Hence they glue to a global section

$$a \in \Gamma(X, \mathcal{L}^{q+1})$$

characterized by $a|_U = dc_U$ and $a|_V = dc_V$. With the sign convention used above, the connecting morphism

$$\delta^q \colon H^q(U \cap V, \mathcal{F}) \longrightarrow H^{q+1}(X, \mathcal{F})$$

is given by $\delta^q([c]) = [a]$. □

In the situation of Example II.3.65, applying this proposition to the sheaf $\mathbb{C}_M(E)$ of local flat sections and using the resolution by E-valued differential forms recovers the Mayer–Vietoris sequence for E-valued de Rham cohomology constructed in Chapter I.

The Mayer–Vietoris sequence is one manifestation of the close relation between sheaf cohomology and open coverings. After discussing local cohomology and cohomology with compact support, we shall return to this relation and formulate it systematically in terms of Čech cohomology in the next subsection §3.3.

Local cohomology and cohomology with compact support

We next consider variants of sheaf cohomology in which the supports of sections are restricted.

Definition II.3.68. A *family of supports* on a topological space X is a non-empty collection $\mathcal{S}$ of closed subsets of X satisfying the following conditions:
(i) If $K, L \in \mathcal{S}$, then $K \cup L \in \mathcal{S}$.
(ii) If $K \in \mathcal{S}$ and $L \subset K$ is closed in X, then $L \in \mathcal{S}$. □

Let X be a topological space, $\mathcal{S}$ a family of supports on X, $\mathcal{R}$ a sheaf of rings on X, and $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules on X. Set

$$\Gamma_{\mathcal{S}}(X, \mathcal{F}) := \big\{ s \in \Gamma(X, \mathcal{F}) \;\big|\; \operatorname{Supp} s \in \mathcal{S} \big\},$$

where

$$\operatorname{Supp} s := \overline{\{x \in X \mid s_x \neq 0\}} \qquad (s \in \Gamma(X, \mathcal{F})).$$

Lemma II.3.69. *For X, $\mathcal{S}$, and $\mathcal{R}$ as above, the functor*

$$\Gamma_{\mathcal{S}}(X, -) \colon (\mathsf{Mod}(\mathcal{R})) \xrightarrow{\text{functor}} (\mathsf{Mod}(\mathcal{R}(X)))$$

is left exact.

Proof. Let

$$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

be an exact sequence of sheaves of $\mathcal{R}$-modules. Since the support of the image of a section is contained in the support of that section, the morphisms φ and ψ induce a sequence

$$0 \longrightarrow \Gamma_{\mathcal{S}}(X, \mathcal{F}) \xrightarrow{\varphi} \Gamma_{\mathcal{S}}(X, \mathcal{G}) \xrightarrow{\psi} \Gamma_{\mathcal{S}}(X, \mathcal{H}).$$

The first morphism is injective because φ is a monomorphism.

Let

$$s \in \Gamma_{\mathcal{S}}(X, \mathcal{G})$$

satisfy $\psi(s) = 0$. Since the ordinary section functor $\Gamma(X, -)$ is left exact, there exists a unique section

$$t \in \Gamma(X, \mathcal{F})$$

such that $\varphi(t) = s$. Since the morphism

$$\varphi_x \colon \mathcal{F}_x \longrightarrow \mathcal{G}_x$$

is injective for every $x \in X$, one has that $t_x = 0$ if and only if $s_x = 0$. Consequently, $\operatorname{Supp} t = \operatorname{Supp} s \in \mathcal{S}$, from which we have $t \in \Gamma_{\mathcal{S}}(X, \mathcal{F})$. □

Let X, $\mathcal{S}$, $\mathcal{R}$, and $\mathcal{F}$ be as above. The right derived functors

$$H^q_{\mathcal{S}}(X,\mathcal{F}) := R^q\Gamma_{\mathcal{S}}(X,\mathcal{F}) \qquad (q \in \mathbb{Z}_{\geq 0})$$

are called the *cohomology groups of $\mathcal{F}$ with supports in $\mathcal{S}$*.

Let $Z \subset X$ be a closed subset. The family

$$\mathcal{S}_Z := \big\{K \subset X \;\big|\; K \text{ is closed in } X \text{ and } K \subset Z\big\}$$

is a family of supports. We write

$$\Gamma_Z(X,\mathcal{F}) := \Gamma_{\mathcal{S}_Z}(X,\mathcal{F})$$

and

$$H^q_Z(X,\mathcal{F}) := H^q_{\mathcal{S}_Z}(X,\mathcal{F}).$$

These groups are called the *cohomology groups of $\mathcal{F}$ with support in Z*, or the *local cohomology groups with support in Z*.

If X is Hausdorff, let c denote the family of compact subsets of X. Thus

$$\Gamma_c(X,\mathcal{F}) = \big\{s \in \Gamma(X,\mathcal{F}) \;\big|\; \operatorname{Supp} s \text{ is compact}\big\},$$

and we set

$$H^q_c(X,\mathcal{F}) := R^q\Gamma_c(X,\mathcal{F}).$$

These groups are called the *cohomology groups of $\mathcal{F}$ with compact support*.

Proposition II.3.70. *Let $\mathcal{S}$ be a family of supports on X. Then the following hold.*
(i) Every flabby sheaf is $\Gamma_{\mathcal{S}}(X,-)$-acyclic.
(ii) Suppose that X is paracompact and Hausdorff and that, for every $K \in \mathcal{S}$, there exists $L \in \mathcal{S}$ such that $K \subset \operatorname{Int} L$. Then every soft sheaf is $\Gamma_{\mathcal{S}}(X,-)$-acyclic. In particular, under the same assumptions, every fine sheaf is $\Gamma_{\mathcal{S}}(X,-)$-acyclic.
(iii) Suppose that

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}^0 \longrightarrow \mathcal{L}^1 \longrightarrow \mathcal{L}^2 \longrightarrow \cdots$$

is a resolution in which every $\mathcal{L}^p$ is $\Gamma_{\mathcal{S}}(X,-)$-acyclic. Then there are canonical isomorphisms

$$H^q_{\mathcal{S}}(X,\mathcal{F}) \cong H^q\big(\Gamma_{\mathcal{S}}(X,\mathcal{L}^\bullet)\big)$$

for every integer $q \geq 0$.

Proof. The assertions in (i) and (ii) are proved in [De4, Chapter IV, § 7]. Under the assumptions in (ii), every fine sheaf is soft by Proposition II.3.63 (iii); hence the assertion concerning fine sheaves follows from the one concerning soft sheaves. The assertion in (iii) follows from the acyclicity theorem (Proposition II.3.49) applied to the left exact functor $\Gamma_{\mathcal{S}}(X,-)$; see also [De4, Chapter IV, (7.8)]. □

Note that, if X is locally compact, paracompact, and Hausdorff, then every compact subset of X is contained in the interior of another compact subset. Consequently, every soft sheaf is $\Gamma_c(X, -)$-acyclic in this case by Proposition II.3.70 (ii) (Also refer to [I, Chapter III Theorem 2.7]).

PROPOSITION II.3.71. *Let $Z \subset X$ be a closed subset, and set $U := X \setminus Z$. For every sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on X, there is a natural long exact sequence*

$$0 \longrightarrow H^0_Z(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(U, \mathcal{F}) \quad .$$

$$\longrightarrow H^1_Z(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(U, \mathcal{F})$$

$$\longrightarrow H^2_Z(X, \mathcal{F}) \longrightarrow H^2(X, \mathcal{F}) \longrightarrow H^2(U, \mathcal{F}) \longrightarrow \dots$$

PROOF. Choose an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots .$$

Since every injective sheaf is flabby (see [St, Tag 09SX]), the restriction morphism

$$\Gamma(X, \mathcal{I}^p) \longrightarrow \Gamma(U, \mathcal{I}^p)$$

is surjective for every integer $p \geq 0$. As its kernel is $\Gamma_Z(X, \mathcal{I}^p)$, one has a short exact sequence of cochain complexes

$$0 \longrightarrow \Gamma_Z(X, \mathcal{I}^\bullet) \longrightarrow \Gamma(X, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet) \longrightarrow 0.$$

The associated long exact cohomology sequence gives the assertion. See also [I, Chapter II Proposition 9.2] and [St, Tag 0A39]. □

The preceding sequence can be used to compare ordinary cohomology with cohomology with compact support.

PROPOSITION II.3.72. *Let X be a Hausdorff topological space, $\mathcal{R}$ a sheaf of rings on X, $\mathcal{F}$ a sheaf of $\mathcal{R}$-modules on X, and let*

$$\mathfrak{K}_X := \big\{ K \subset X \;\big|\; K \text{ is compact} \big\},$$

ordered by inclusion. Then there is a natural long exact sequence

$$0 \longrightarrow H^0_c(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow \varinjlim_{K \in \mathfrak{K}_X} H^0(X \setminus K, \mathcal{F}) \quad ,$$

$$\longrightarrow H^1_c(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \varinjlim_{K \in \mathfrak{K}_X} H^1(X \setminus K, \mathcal{F})$$

$$\longrightarrow H^2_c(X, \mathcal{F}) \longrightarrow H^2(X, \mathcal{F}) \longrightarrow \varinjlim_{K \in \mathfrak{K}_X} H^2(X \setminus K, \mathcal{F}) \longrightarrow \dots$$

where, for compact subsets $K \subset L$ of X, the transition morphism

$$H^q(X \setminus K, \mathcal{F}) \longrightarrow H^q(X \setminus L, \mathcal{F})$$

is induced by restriction.

Proof. Choose an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots .$$

Since X is Hausdorff, every $K \in \mathfrak{K}_X$ is closed. The proof of Proposition II.3.71 therefore gives a short exact sequence of cochain complexes

$$0 \longrightarrow \Gamma_K(X, \mathcal{I}^\bullet) \longrightarrow \Gamma(X, \mathcal{I}^\bullet) \longrightarrow \Gamma(X \setminus K, \mathcal{I}^\bullet) \longrightarrow 0$$

for every $K \in \mathfrak{K}_X$. These sequences are compatible with inclusions of compact subsets.

The set $\mathfrak{K}_X$ is directed, since the union of two compact subsets is compact. Taking the inductive limit over $K \in \mathfrak{K}_X$ and using the exactness of filtered inductive limits (Proposition II.3.43) gives a short exact sequence

$$0 \longrightarrow \varinjlim_{K \in \mathfrak{K}_X} \Gamma_K(X, \mathcal{I}^\bullet) \longrightarrow \Gamma(X, \mathcal{I}^\bullet) \longrightarrow \varinjlim_{K \in \mathfrak{K}_X} \Gamma(X \setminus K, \mathcal{I}^\bullet) \longrightarrow 0.$$

As

$$\varinjlim_{K \in \mathfrak{K}_X} \Gamma_K(X, \mathcal{I}^\bullet) = \Gamma_c(X, \mathcal{I}^\bullet),$$

it follows from Corollary II.3.44 that

$$\begin{aligned} H^q\left(\varinjlim_{K \in \mathfrak{K}_X} \Gamma(X \setminus K, \mathcal{I}^\bullet)\right) &\cong \varinjlim_{K \in \mathfrak{K}_X} H^q\big(\Gamma(X \setminus K, \mathcal{I}^\bullet)\big) \\ &\cong \varinjlim_{K \in \mathfrak{K}_X} H^q(X \setminus K, \mathcal{F}). \end{aligned}$$

The cohomology of $\Gamma_c(X, \mathcal{I}^\bullet)$ is $H^q_c(X, \mathcal{F})$. The long exact cohomology sequence associated with the preceding short exact sequence of complexes now gives the assertion. □

Example II.3.73. Let M be a C^∞ manifold and $E \to M$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle equipped with the flat connection D. For the comparisons with currents, assume that M is orientable and fix an orientation. As shown in Example II.3.65, the sheaves $\mathcal{A}^q_M(E)$ and $\mathcal{K}^q_M(E)$ are fine and hence soft. Since M is locally compact, paracompact, and Hausdorff, these sheaves are also $\Gamma_c(M, -)$-acyclic by Proposition II.3.70 (*ii*). Applying compactly supported sections to the resolutions by differential forms and currents therefore gives canonical isomorphisms

$$\begin{aligned} H^q_c(M, \mathbb{C}_M(E)) &\cong H^q\big(\Gamma_c(M, \mathcal{A}^\bullet_M(E)), D\big) \\ &= H^q\big(\mathcal{A}^\bullet_c(M, E), D\big) \\ &= H^q_{\mathrm{DR},c}(M, E) \end{aligned}$$

and

$$H_c^q(M, \mathbb{C}_M(E)) \cong H^q\big(\Gamma_c(M, \mathcal{K}_M^\bullet(E)), D\big) = H^q\big(\mathcal{K}_c^\bullet(M, E), D\big).$$

Consequently, there are canonical isomorphisms

$$H_{\mathrm{DR},c}^q(M, E) \cong H_c^q(M, \mathbb{C}_M(E)) \cong H^q\big(\mathcal{K}_c^\bullet(M, E), D\big).$$

In particular, when $E \to M$ is a trivial complex line bundle with a fixed global D-flat frame, one obtains

$$H_{\mathrm{DR},c}^q(M) \cong H_c^q(M, \underline{\mathbb{C}}_M) \cong H^q\big(\mathcal{K}_c^\bullet(M), d\big).$$

□

In what follows, for a topological space X, a sheaf $\mathcal{F}$ on X, and an open subset $U \subset X$, we simply write

$$H_c^q(U, \mathcal{F}) := H_c^q(U, \mathcal{F}|_U) \qquad (q \in \mathbb{Z}_{\geq 0}).$$

Proposition II.3.74 (Mayer–Vietoris sequence with compact supports). *Let X be a locally compact Hausdorff space, let $\mathcal{R}$ be a sheaf of rings on X, and let $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules. Suppose that $U, V \subset X$ are open subsets such that $X = U \cup V$. Then there exists a natural long exact sequence*

$$0 \longrightarrow H_c^0(U \cap V, \mathcal{F}) \longrightarrow H_c^0(U, \mathcal{F}) \oplus H_c^0(V, \mathcal{F}) \longrightarrow H_c^0(X, \mathcal{F}) \qquad .$$

$$\longrightarrow H_c^1(U \cap V, \mathcal{F}) \longrightarrow H_c^1(U, \mathcal{F}) \oplus H_c^1(V, \mathcal{F}) \longrightarrow H_c^1(X, \mathcal{F})$$

$$\longrightarrow H_c^2(U \cap V, \mathcal{F}) \longrightarrow H_c^2(U, \mathcal{F}) \oplus H_c^2(V, \mathcal{F}) \longrightarrow H_c^2(X, \mathcal{F}) \longrightarrow \dots$$

Proof. For the existence and exactness of the sequence, see [I, Chapter III §7]. We only describe here the morphisms appearing in the sequence.

First, we introduce a notation for extension by zero. Let Y be a Hausdorff space, let $W \subset Y$ be an open subset, and let $\mathcal{G}$ be a sheaf on Y. If $s \in \Gamma_c(W, \mathcal{G})$, then its support is a compact subset of W and hence a closed subset of Y. Consequently, s extends by zero to a section on Y. We denote this extension by

$$\widetilde{s} \in \Gamma_c(Y, \mathcal{G}).$$

The ambient space to which the section is extended will always be clear from the context.

With this notation, the morphism

$$H_c^q(U \cap V, \mathcal{F}) \longrightarrow H_c^q(U, \mathcal{F}) \oplus H_c^q(V, \mathcal{F})$$

is induced by

$$s \longmapsto (\widetilde{s}, \widetilde{s}),$$

where the two components are the extensions by zero to U and V, respectively. The morphism

$$H_c^q(U,\mathcal{F}) \oplus H_c^q(V,\mathcal{F}) \longrightarrow H_c^q(X,\mathcal{F})$$

is induced by

$$(\xi,\eta) \longmapsto \widetilde{\xi} - \widetilde{\eta}.$$

We finally describe the connecting morphism in a situation that frequently occurs in concrete computations. Suppose, in addition, that X is paracompact and that there exists a resolution

$$0 \to \mathcal{F} \to \mathcal{L}^0 \xrightarrow{d} \mathcal{L}^1 \xrightarrow{d} \mathcal{L}^2 \xrightarrow{d} \cdots$$

by fine sheaves. For an open subset $W \subset X$, let $\mathcal{S}_W$ be the family of compact subsets of X contained in W. Extension by zero identifies

$$\Gamma_{\mathcal{S}_W}(X,\mathcal{G}) \cong \Gamma_c(W,\mathcal{G}|_W)$$

for every sheaf $\mathcal{G}$ of $\mathcal{R}$-modules on X. Applying this identification to a flabby resolution and its restriction to W gives

$$H^q_{\mathcal{S}_W}(X,\mathcal{F}) \cong H_c^q(W,\mathcal{F}|_W) \qquad (q \geq 0).$$

By local compactness, every $K \in \mathcal{S}_W$ is contained in the interior of another member of $\mathcal{S}_W$. Proposition II.3.70 (ii) and (iii), applied on X to $\mathcal{S}_W$, therefore show that the complex of compactly supported sections of the given fine resolution over W computes $H_c^q(W,\mathcal{F}|_W)$. Taking $W = X, U, V, U \cap V$ gives the four complexes needed below. Choose sheaf endomorphisms

$$h_U, h_V \colon \mathcal{L}^q \longrightarrow \mathcal{L}^q$$

subordinate to the covering $X = U \cup V$, so that

$$h_U + h_V = \mathrm{id}_{\mathcal{L}^q}, \qquad \operatorname{Supp} h_U \subset U, \qquad \operatorname{Supp} h_V \subset V.$$

Let $[c] \in H_c^q(X,\mathcal{F})$ be represented by a compactly supported section

$$c \in \Gamma_c(X,\mathcal{L}^q)$$

such that $dc = 0$. Set

$$c_U := h_U(c)|_U \in \Gamma_c(U,\mathcal{L}^q)$$

and

$$c_V := -h_V(c)|_V \in \Gamma_c(V,\mathcal{L}^q).$$

Then

$$\widetilde{c_U} - \widetilde{c_V} = h_U(c) + h_V(c) = \mathrm{id}_{\mathcal{L}^q}(c) = c.$$

Since $dc = 0$, one has

$$\widetilde{dc_U} - \widetilde{dc_V} = d\big(\widetilde{c_U} - \widetilde{c_V}\big) = dc = 0.$$

It follows that

$$dc_U|_{U\cap V} = dc_V|_{U\cap V}.$$

Let

$$a := dc_U|_{U\cap V} = dc_V|_{U\cap V}.$$

Then

$$a \in \Gamma_c(U \cap V, \mathcal{L}^{q+1}), \qquad da = 0.$$

With the sign convention used above, the connecting morphism

$$\delta_c^q \colon H_c^q(X, \mathcal{F}) \longrightarrow H_c^{q+1}(U \cap V, \mathcal{F})$$

is given by $\delta_c^q([c]) = [a]$. □

Let M be a C^∞ manifold, let $E \to M$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle, and let $M = U \cup V$ be an open covering. Under the canonical identifications

$$H_c^q(W, \mathbb{C}_M(E)) \cong H_{\mathrm{DR},c}^q(W, E) \qquad \big(W = M,\ U,\ V,\ U \cap V\big)$$

from Example II.3.73, the compact-support Mayer–Vietoris sequence of Proposition II.3.74 is identified with the Mayer–Vietoris sequence for compactly supported de Rham cohomology constructed in Chapter I.

3.3 Čech cohomology and applications

In Chapter I, we encountered several constructions involving local data on an open covering and their compatibility on pairwise and higher intersections. Examples include the comparison between singular and simplicial cohomology in Proposition I.7.24 and the proof of the singular–de Rham correspondence in Theorem I.8.54. The purpose of this subsection is to introduce the terminology and formalism of *Čech cohomology*, which provide a systematic framework for such constructions, and to explain its relation with sheaf cohomology.

Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X, and let

$$\mathcal{U} = \{U_j\}_{j\in J}$$

be an open covering of X. For an ordered tuple

$$(j_0, j_1, \dots, j_q) \in J^{q+1},$$

($q \in \mathbb{Z}_{\geq 0}$), we write

$$U_{j_0 j_1 \cdots j_q} := U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_q}.$$

Notice that, although

$$U_{jk} = U_{kj}$$

as open subsets of X, the ordered indices (j, k) and (k, j) are distinguished when labeling the components of a Čech cochain introduced below.

For every integer $q \geq 0$, the group of *Čech q-cochains of $\mathcal{F}$ with respect to $\mathcal{U}$* is

$$\check{C}^q(\mathcal{U}, \mathcal{F}) := \prod_{(j_0, j_1, \ldots, j_q) \in J^{q+1}} \mathcal{F}(U_{j_0 j_1 \cdots j_q}).$$

Thus an element of $\check{C}^q(\mathcal{U}, \mathcal{F})$ (a *Čech q-cochain of $\mathcal{F}$ with respect to $\mathcal{U}$*) may be written as

$$c = \left\{ (U_{j_0 j_1 \cdots j_q}, c_{j_0 j_1 \cdots j_q}) \right\}_{(j_0, j_1, \ldots, j_q) \in J^{q+1}},$$

where $c_{j_0 j_1 \cdots j_q} \in \mathcal{F}(U_{j_0 j_1 \cdots j_q})$.

Define the *Čech coboundary morphism*

$$\check{\delta}^q \colon \check{C}^q(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^{q+1}(\mathcal{U}, \mathcal{F})$$

by

$$\check{\delta}^q(c) := \left\{ \left(U_{j_0 j_1 \cdots j_{q+1}}, \sum_{\nu=0}^{q+1} (-1)^\nu c_{j_0 \cdots \widehat{j_\nu} \cdots j_{q+1}} \big|_{U_{j_0 j_1 \cdots j_{q+1}}} \right) \right\}_{(j_0, j_1, \ldots, j_{q+1}) \in J^{q+2}},$$

where the symbol $\widehat{j_\nu}$ means that the index j_ν is omitted (Thus $c_{j_0 \cdots \widehat{j_\nu} \cdots j_{q+1}} = c_{j_0 \cdots j_{\nu-1} j_{\nu+1} \cdots j_{q+1}}$, for example).

In what follows, for an element $\gamma \in \check{C}^{q+1}(\mathcal{U}, \mathcal{F})$, we denote its component over $U_{j_0 j_1 \cdots j_{q+1}}$ by

$$\gamma_{j_0 j_1 \cdots j_{q+1}} \qquad \text{or} \qquad (\gamma)_{j_0 j_1 \cdots j_{q+1}}.$$

With this notation, the preceding definition may be written more briefly as

$$(\check{\delta}^q c)_{j_0 j_1 \cdots j_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^\nu c_{j_0 \cdots \widehat{j_\nu} \cdots j_{q+1}} \big|_{U_{j_0 j_1 \cdots j_{q+1}}}.$$

Then one has

$$\check{\delta}^{q+1} \circ \check{\delta}^q = 0,$$

since, for every $c \in \check{C}^q(\mathcal{U}, \mathcal{F})$, one has

$$\left(\check{\delta}^{q+1}(\check{\delta}^q(c)) \right)_{j_0 j_1 \cdots j_{q+2}} = \sum_{0 \leq \mu < \nu \leq q+2} \left((-1)^{\mu+\nu-1} + (-1)^{\mu+\nu} \right) c_{j_0 \cdots \widehat{j_\mu} \cdots \widehat{j_\nu} \cdots j_{q+2}} \big|_{U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_{q+2}}} = 0.$$

Therefore, the groups $\check{C}^q(\mathcal{U}, \mathcal{F})$ and the morphisms $\check{\delta}^q$ form a cochain complex

$$0 \longrightarrow \check{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\check{\delta}^0} \check{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\check{\delta}^1} \check{C}^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\check{\delta}^2} \cdots.$$

We set

$$\check{Z}^q(\mathcal{U}, \mathcal{F}) := \operatorname{Ker} \check{\delta}^q, \qquad \check{B}^q(\mathcal{U}, \mathcal{F}) := \begin{cases} 0 & \text{if } q = 0 \\ \operatorname{Image} \check{\delta}^{q-1} & \text{otherwise} \end{cases}.$$

An element of $\check{Z}^q(\mathcal{U},\mathcal{F})$ is called a *Čech q-cocycle of $\mathcal{F}$ with respect to $\mathcal{U}$*, while an element of $\check{B}^q(\mathcal{U},\mathcal{F})$ is called a *Čech q-coboundary of $\mathcal{F}$ with respect to $\mathcal{U}$*. The quotient group

$$\check{H}^q(\mathcal{U},\mathcal{F}) := \frac{\check{Z}^q(\mathcal{U},\mathcal{F})}{\check{B}^q(\mathcal{U},\mathcal{F})}$$

is called the *q-th Čech cohomology group of $\mathcal{F}$ with respect to $\mathcal{U}$*.

Example II.3.75. In low degrees, the Čech coboundary morphisms are given by

$$(\check{\delta}^0 f)_{jk} = f_k|_{U_j\cap U_k} - f_j|_{U_j\cap U_k}$$

and

$$(\check{\delta}^1 g)_{jk\ell} = g_{k\ell}|_{U_j\cap U_k\cap U_\ell} - g_{j\ell}|_{U_j\cap U_k\cap U_\ell} + g_{jk}|_{U_j\cap U_k\cap U_\ell}.$$

For the ordered two-set covering (U,V), $\check{\delta}^0$ is second restriction minus first restriction. The difference map in the Mayer–Vietoris sequence (Proposition II.3.67) has the opposite order. Changing that difference map by -1 also changes the connecting morphisms by -1; it does not change the cohomology groups or the exactness of the sequence. Consequently, if

$$g = \{(U_{jk}, g_{jk})\} \in \check{Z}^1(\mathcal{U},\mathcal{F}),$$

then

$$g_{jj} = 0, \qquad g_{kj} = -g_{jk},$$

and

$$g_{jk} + g_{k\ell} + g_{\ell j} = 0$$

on $U_j \cap U_k \cap U_\ell$.

The sheaf condition gives a canonical isomorphism

$$\check{H}^0(\mathcal{U},\mathcal{F}) \cong \Gamma(X,\mathcal{F}).$$

Indeed, a Čech 0-cocycle is precisely a family $\{(U_j, f_j)\}$ such that $f_j|_{U_j\cap U_k} = f_k|_{U_j\cap U_k}$ for all $j,k \in J$. □

Definition II.3.76. Let $\mathcal{U} = \{U_j\}_{j\in J}$ and $\mathcal{V} = \{V_a\}_{a\in A}$ be open coverings of a topological space X. We say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if, for every $a \in A$, there exists $j \in J$ such that $V_a \subset U_j$. A map $\rho\colon A \to J$ such that $V_a \subset U_{\rho(a)}$ for every $a \in A$ is called a *refinement map*. □

Let $\mathcal{U} = \{U_j\}_{j\in J}$ and $\mathcal{V} = \{V_a\}_{a\in A}$ be open coverings of X such that $\mathcal{V}$ is a refinement of $\mathcal{U}$. A refinement map $\rho\colon A \to J$ induces a morphism of cochain complexes

$$\rho^*\colon \check{C}^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow \check{C}^\bullet(\mathcal{V},\mathcal{F})$$

defined by

$$(\rho^* c)_{a_0 a_1\cdots a_q} := c_{\rho(a_0)\rho(a_1)\cdots\rho(a_q)}\big|_{V_{a_0}\cap V_{a_1}\cap\cdots\cap V_{a_q}}.$$

It is known that different choices of the refinement map induce chain-homotopic morphisms and hence induce the same morphism

$$\check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^q(\mathcal{V}, \mathcal{F}),$$

see [St, Tag 09UY].

Let

$$(\mathsf{OpCov}(X))$$

denote the preorder category whose objects are open coverings of X and in which there is a unique morphism

$$\mathcal{U} \longrightarrow \mathcal{V}$$

if and only if $\mathcal{V}$ is a refinement of $\mathcal{U}$.

For every integer $q \geq 0$, the refinement morphisms constructed above define a functor

$$\check{H}^q(-, \mathcal{F})\colon (\mathsf{OpCov}(X)) \xrightarrow{\text{functor}} (\mathsf{Ab}), \qquad \mathcal{U} \longmapsto \check{H}^q(\mathcal{U}, \mathcal{F}),$$

since the morphism associated with a refinement is independent of the choice of a refinement map, and the morphisms induced by successive refinements are compatible with composition.

Note that the category $(\mathsf{OpCov}(X))$ is filtered. Indeed, any two open coverings $\mathcal{U}$ and $\mathcal{V}$ admit the common refinement $\big\{U \cap V \;\big|\; U \in \mathcal{U},\ V \in \mathcal{V},\ U \cap V \neq \emptyset\big\}$. Since $(\mathsf{OpCov}(X))$ is a preorder category, the remaining condition in the definition of a filtered category is automatic.

Definition II.3.77. Let X be a topological space and $\mathcal{F}$ be a sheaf of abelian groups on X. The *q-th Čech cohomology group of X with values in $\mathcal{F}$* is the inductive limit

$$\check{H}^q(X, \mathcal{F}) := \varinjlim_{\mathcal{U} \in \mathrm{ob}((\mathsf{OpCov}(X)))} \check{H}^q(\mathcal{U}, \mathcal{F}).$$

□

Whereas the definition of Čech cohomology group of X above involves all open coverings of X, for a suitably acyclic individual covering, it is not necessary to pass to the inductive limit. We first introduce the relevant condition.

Definition II.3.78. Let X be a topological space and $\mathcal{F}$ be a sheaf of abelian groups on X. An open covering $\{U_j\}_{j \in J}$ of X is called a *Leray covering for $\mathcal{F}$* if

$$H^r\big(U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_q}, \mathcal{F}\big) = 0$$

for every integer $r > 0$, every integer $q \geq 0$, and every ordered tuple $(j_0, j_1, \ldots, j_q) \in J^{q+1}$ for which the intersection is nonempty. □

THEOREM II.3.79. *Let X be a topological space and $\mathcal{F}$ be a sheaf of abelian groups on X.*
(i) For every open covering $\mathcal{U}$ of X and every integer $q \geq 0$, there is a canonical comparison morphism

$$\kappa^q_{\mathcal{U}} \colon \check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F}).$$

These morphisms are compatible with refinements. More precisely, if $\mathcal{V}$ is a refinement of $\mathcal{U}$ and

$$\rho^* \colon \check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^q(\mathcal{V}, \mathcal{F})$$

is the morphism induced by a refinement map ρ, then

$$\kappa^q_{\mathcal{V}} \circ \rho^* = \kappa^q_{\mathcal{U}}.$$

Consequently, the morphisms $\kappa^q_{\mathcal{U}}$ induce a canonical morphism

$$\kappa^q_X \colon \check{H}^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F}).$$

(ii) If $\mathcal{U}$ is a Leray covering for $\mathcal{F}$, then

$$\kappa^q_{\mathcal{U}} \colon \check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F})$$

is an isomorphism for every integer $q \geq 0$.
(iii) If X is paracompact and Hausdorff, then

$$\kappa^q_X \colon \check{H}^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F})$$

is an isomorphism for every integer $q \geq 0$.

PROOF. For the construction of the comparison morphisms and their compatibility with refinements, see [De4, Chapter IV, § 5.B, especially (5.13)–(5.15)] and [St, Tag 01EO]. For assertion *(ii)* (*Leray's theorem*) and assertion *(iii)* (the comparison theorem on paracompact spaces), see [De4, Chapter IV, Theorems 5.17 and 5.21]. Here we explain the double-complex argument underlying assertions *(i)* and *(ii)*. For assertion *(i)*, we use the horizontal part of the argument in the proof of Theorem A.1.1; under the Leray condition, both directions of that theorem apply.

Choose an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots$$

in $(\mathsf{Sheaf}(X, (\mathsf{Ab})))$. Let $\mathcal{U} = \{U_j\}_{j \in J}$ be an open covering of X, and set

$$C^{p,r}_{\mathcal{U}} := \check{C}^p(\mathcal{U}, \mathcal{I}^r) \qquad (p, r \in \mathbb{Z}_{\geq 0}).$$

The horizontal morphisms are the Čech coboundary morphisms, and the vertical morphisms are induced by the differential of the injective resolution. These two unsigned differentials commute. Writing d for the resolution differential, we use the total differential $\check{\delta} + (-1)^p d$ on $C^{p,r}_{\mathcal{U}}$, as in Appendix 1.1. Thus the comparison is induced by the two edge inclusions, with no additional degree-dependent sign.

The relevant augmented diagram has the following form:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma(X,\mathcal{I}^1) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{I}^1) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{I}^1) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{I}^1) & \longrightarrow \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma(X,\mathcal{I}^0) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{I}^0) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{I}^0) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{I}^0) & \longrightarrow \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma(X,\mathcal{F}) & \longrightarrow & \check{C}^0(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{C}^1(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{C}^2(\mathcal{U},\mathcal{F}) & \longrightarrow \cdots .
\end{array}$$

The bottom and left edges have familiar interpretations: For every integer $p \geq 0$, one has

$$\check{C}^p(\mathcal{U},\mathcal{F}) \cong \operatorname{Ker}\left(C_{\mathcal{U}}^{p,0} \longrightarrow C_{\mathcal{U}}^{p,1}\right),$$

while the sheaf condition gives, for every integer $r \geq 0$,

$$\Gamma(X,\mathcal{I}^r) \cong \operatorname{Ker}\left(C_{\mathcal{U}}^{0,r} \longrightarrow C_{\mathcal{U}}^{1,r}\right).$$

Thus, in the notation of Appendix A, the bottom edge $E^\bullet$ is naturally identified with

$$\check{C}^\bullet(\mathcal{U},\mathcal{F}),$$

and the left edge $F^\bullet$ is naturally identified with

$$\Gamma(X,\mathcal{I}^\bullet).$$

The inclusions of the two edge complexes into the total complex $\operatorname{Tot}(C_{\mathcal{U}}^{\bullet,\bullet})$ (see Appendix A for the definition) give morphisms of cochain complexes

$$\alpha_{\mathcal{U}} \colon \check{C}^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow \operatorname{Tot}\left(C_{\mathcal{U}}^{\bullet,\bullet}\right)$$

and

$$\beta_{\mathcal{U}} \colon \Gamma(X,\mathcal{I}^\bullet) \longrightarrow \operatorname{Tot}\left(C_{\mathcal{U}}^{\bullet,\bullet}\right).$$

For every integer $r \geq 0$, it is known that the augmented horizontal complex

$$0 \longrightarrow \Gamma(X,\mathcal{I}^r) \longrightarrow \check{C}^0(\mathcal{U},\mathcal{I}^r) \longrightarrow \check{C}^1(\mathcal{U},\mathcal{I}^r) \longrightarrow \check{C}^2(\mathcal{U},\mathcal{I}^r) \longrightarrow \cdots$$

is exact (see [St, Tag 01EP]). Therefore, the horizontal argument in the proof of Theorem A.1.1 shows that

$$H^q(\beta_{\mathcal{U}}) \colon H^q\left(\Gamma(X,\mathcal{I}^\bullet)\right) \longrightarrow H^q\left(\operatorname{Tot}\left(C_{\mathcal{U}}^{\bullet,\bullet}\right)\right)$$

is an isomorphism for every integer $q \geq 1$. The same assertion in degree zero follows directly from the sheaf condition.

Since $H^q\left(\Gamma(X,\mathcal{I}^\bullet)\right) = H^q(X,\mathcal{F})$, we define

$$\kappa_{\mathcal{U}}^q := H^q(\beta_{\mathcal{U}})^{-1} \circ H^q(\alpha_{\mathcal{U}}) \colon \check{H}^q(\mathcal{U},\mathcal{F}) \longrightarrow H^q(X,\mathcal{F}).$$

This construction also describes the comparison morphism concretely. Let

$$c \in \check{Z}^q(\mathcal{U}, \mathcal{F})$$

be a Čech cocycle. Through the inclusion $\mathcal{F} \longrightarrow \mathcal{I}^0$, we may regard c as an element of

$$C_{\mathcal{U}}^{q,0} \subset \mathrm{Tot}^q\big(C_{\mathcal{U}}^{\bullet,\bullet}\big).$$

It is a cocycle in the total complex. Its total cohomology class is $H^q(\alpha_{\mathcal{U}})([c])$. The class $\kappa_{\mathcal{U}}^q([c])$ is the unique class in

$$H^q\big(\Gamma(X, \mathcal{I}^\bullet)\big) = H^q(X, \mathcal{F})$$

whose image under $H^q(\beta_{\mathcal{U}})$ is this total cohomology class.

Equivalently, the horizontal exactness above permits the zig-zag procedure used in the proof of Theorem A.1.1: the total cocycle represented initially in the bottom row can be replaced, modulo a total coboundary, by a cocycle supported in the leftmost column. The resulting cocycle in

$$\Gamma(X, \mathcal{I}^q)$$

represents $\kappa_{\mathcal{U}}^q([c])$. In degree zero, this is simply the global section obtained by gluing the compatible local sections of a Čech 0-cocycle.

The standard comparison theorem for injective resolutions shows that the resulting morphism is independent of the chosen injective resolution and is natural in $\mathcal{F}$. Let

$$\mathcal{V} = \{V_a\}_{a \in A}$$

be a refinement of $\mathcal{U}$, and let $\rho\colon A \to J$ be a refinement map. The refinement morphisms give a morphism of the corresponding augmented double complexes

$$C_{\mathcal{U}}^{\bullet,\bullet} \longrightarrow C_{\mathcal{V}}^{\bullet,\bullet}.$$

This morphism commutes with $\alpha_{\mathcal{U}}$, $\alpha_{\mathcal{V}}$, $\beta_{\mathcal{U}}$, and $\beta_{\mathcal{V}}$. It follows that

$$\kappa_{\mathcal{V}}^q \circ \rho^* = \kappa_{\mathcal{U}}^q.$$

Thus the morphisms $\kappa_{\mathcal{U}}^q$ are compatible with refinements, and the universal property of the inductive limit gives a canonical morphism

$$\kappa_X^q\colon \check{H}^q(X, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F}).$$

This proves (i).

Suppose now that $\mathcal{U}$ is a Leray covering for $\mathcal{F}$. For every integer $p \geq 0$, consider the augmented vertical complex

$$0 \longrightarrow \check{C}^p(\mathcal{U}, \mathcal{F}) \longrightarrow C_{\mathcal{U}}^{p,0} \longrightarrow C_{\mathcal{U}}^{p,1} \longrightarrow C_{\mathcal{U}}^{p,2} \longrightarrow \cdots.$$

For every ordered tuple $(j_0, j_1, \ldots, j_p) \in J^{p+1}$, the corresponding component of this complex is

$$0 \longrightarrow \mathcal{F}\big(U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_p}\big) \longrightarrow \mathcal{I}^0\big(U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_p}\big) \longrightarrow \mathcal{I}^1\big(U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_p}\big) \longrightarrow \cdots .$$

Its cohomology in every positive degree r is

$$H^r\big(U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_p}, \mathcal{F}\big),$$

which vanishes by the Leray condition whenever the intersection is nonempty. If the intersection is empty, all the terms in the corresponding component are zero. Taking the product over all ordered tuples therefore shows that the augmented vertical complex is exact.

Thus, under the Leray condition, the double complex has the required exactness both horizontally and vertically. For every integer $q \geq 1$, Theorem A.1.1 therefore shows that both

$$H^q(\alpha_{\mathcal{U}})\colon \check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q\left(\mathrm{Tot}\big(C_{\mathcal{U}}^{\bullet,\bullet}\big)\right)$$

and

$$H^q(\beta_{\mathcal{U}})\colon H^q(X, \mathcal{F}) \longrightarrow H^q\left(\mathrm{Tot}\big(C_{\mathcal{U}}^{\bullet,\bullet}\big)\right)$$

are isomorphisms. Consequently,

$$\kappa_{\mathcal{U}}^q = H^q(\beta_{\mathcal{U}})^{-1} \circ H^q(\alpha_{\mathcal{U}})$$

is an isomorphism. For $q = 0$, the assertion follows directly from the canonical identifications

$$\check{H}^0(\mathcal{U}, \mathcal{F}) \cong \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F}).$$

This proves (ii). □

The preceding theorem is one reason that the refinement procedure is essential: a fixed open covering need not compute sheaf cohomology, whereas the inductive limit over all refinements does so on a paracompact Hausdorff space. This is the systematic version of the refinement principle discussed in Chapter I; compare Remark I.7.10.

Remark II.3.80. Let

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$$

be a short exact sequence in $(\mathsf{Sheaf}(X, (\mathsf{Ab})))$. Take an open covering $\mathcal{U} = \{U_j\}_{j \in J}$ which is a Leray covering for $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$. Then the corresponding Čech complexes compute the sheaf cohomology groups in the long exact cohomology sequence.

We first describe the connecting morphism $H^0(X, \mathcal{H}) \longrightarrow H^1(X, \mathcal{F})$. Let

$$h \in H^0(X, \mathcal{H}) = \Gamma(X, \mathcal{H}),$$

and write $h_j := h|_{U_j}$. Since $H^1(U_j, \mathcal{F}) = 0$, the morphism

$$\mathcal{G}(U_j) \longrightarrow \mathcal{H}(U_j)$$

is surjective. Choose $g_j \in \mathcal{G}(U_j)$ such that

$$\beta(g_j) = h_j.$$

On $U_j \cap U_k$, one has

$$\beta(g_k - g_j) = h_k - h_j = 0.$$

Therefore there exists a unique

$$f_{jk} \in \mathcal{F}(U_j \cap U_k)$$

such that

$$\alpha(f_{jk}) = g_k - g_j.$$

The family $\{(U_{jk}, f_{jk})\}$ is a Čech 1-cocycle, and the connecting morphism is given by

$$h \longmapsto \big[\{(U_{jk}, f_{jk})\}\big].$$

Similarly, let

$$\{(U_{jk}, h_{jk})\} \in \check{Z}^1(\mathcal{U}, \mathcal{H})$$

represent a class in $H^1(X, \mathcal{H})$. Choose $g_{jk} \in \mathcal{G}(U_j \cap U_k)$ such that

$$\beta(g_{jk}) = h_{jk}.$$

On $U_j \cap U_k \cap U_\ell$, one has

$$\beta\big(g_{k\ell} - g_{j\ell} + g_{jk}\big) = h_{k\ell} - h_{j\ell} + h_{jk} = 0.$$

Hence there exists a unique

$$f_{jk\ell} \in \mathcal{F}(U_j \cap U_k \cap U_\ell)$$

such that

$$\alpha(f_{jk\ell}) = g_{k\ell} - g_{j\ell} + g_{jk}.$$

The family $\{(U_{jk\ell}, f_{jk\ell})\}$ is a Čech 2-cocycle, and the connecting morphism

$$H^1(X, \mathcal{H}) \longrightarrow H^2(X, \mathcal{F})$$

is given by

$$\big[\{(U_{jk}, h_{jk})\}\big] \longmapsto \big[\{(U_{jk\ell}, f_{jk\ell})\}\big].$$

One can easily see that changing the chosen lifts changes the resulting cocycles by Čech coboundaries. Thus the cohomology classes above are independent of all auxiliary choices. □

We next compare the preceding constructions with the cohomology theories considered in Chapter I.

Proposition II.3.81. *Let X be a triangulable Hausdorff topological space of the type considered in Proposition I.7.24, and let G be an abelian group. Let $\underline{G}_X$ denote the sheaf of G-valued locally constant functions on X. Then there are natural isomorphisms*

$$\check{H}^q(X, \underline{G}_X) \cong H^q(X, \underline{G}_X) \cong H^q(X, G)$$

for every integer $q \geq 0$.

Moreover, for the open covering $\{U_v\}_{v \in X^0}$ constructed in Lemma I.7.20, one has

$$\check{H}^q\big(\{U_v\}_{v \in X^0}, \underline{G}_X\big) \cong H^q(X, G).$$

Under the natural identification of locally constant functions on the nonempty intersections with elements of G, this is precisely the isomorphism constructed in Proposition I.7.24.

Proof. The comparison between Čech and sheaf cohomology follows from Theorem II.3.79. The comparison with singular cohomology is the standard sheaf-theoretic form of the Čech–singular correspondence. For the specific covering used here, it is the correspondence proved in Proposition I.7.24. See also [De4, Chapter IV, §§ 5 and 6]. □

Example II.3.82. Let M be a C^∞ manifold. For the comparison with currents, assume that M is orientable and fix an orientation. Combining Theorem I.8.54, Example II.3.65, and Proposition II.3.81, one obtains natural isomorphisms

$$H^q_{\mathrm{DR}}(M) \cong H^q(M, \mathbb{C}) \cong \check{H}^q(M, \underline{\mathbb{C}}_M) \cong H^q(M, \underline{\mathbb{C}}_M) \cong H^q\big(\mathcal{K}^\bullet(M), d\big).$$

This also supplies the refinement-based interpretation anticipated in Remark I.8.55. □

Example II.3.83. Let M be a C^∞ manifold and $E \to M$ be a $\mathrm{GL}_r(\mathbb{C})$-flat vector bundle ($r \in \mathbb{Z}_{>0}$). For the comparison with currents, assume that M is orientable and fix an orientation. Let $\mathbb{C}_M(E)$ denote the sheaf of local D-flat sections of E, as in Example II.3.65. Then

$$\check{H}^q(M, \mathbb{C}_M(E)) \cong H^q(M, \mathbb{C}_M(E)) \cong H^q_{\mathrm{DR}}(M, E) \cong H^q\big(\mathcal{K}^\bullet(M, E), D\big).$$

The groups on the left are often described as cohomology groups with local coefficients determined by the flat bundle E. □

Example II.3.84. Let M be a C^∞ manifold and $\mathcal{U} = \{U_j\}_{j \in J}$ be an open covering such that every nonempty finite intersection $U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_q}$ is contractible; recall Lemma I.7.20. Let $\omega \in \Gamma(M, \mathcal{A}^1_M)$ be a closed 1-form. By Corollary I.8.43, one can take $f_j \in C^\infty(U_j)$ such that

$$\omega|_{U_j} = df_j.$$

Set

$$c_{jk} := f_j - f_k$$

on $U_j \cap U_k$. Then c_{jk} is locally constant and

$$c_{k\ell} - c_{j\ell} + c_{jk} = 0$$

on $U_j \cap U_k \cap U_\ell$. Thus $\{(U_{jk}, c_{jk})\}$ represents a class in $\check{H}^1\big(\mathcal{U}, \underline{\mathbb{C}}_M\big)$.

Conversely, let

$$\{(U_{jk}, c_{jk})\} \in \check{Z}^1\big(\mathcal{U}, \underline{\mathbb{C}}_M\big).$$

Choose a smooth partition of unity $\{(U_\lambda, \rho_\lambda)\}_{\lambda \in J}$ subordinate to $\mathcal{U}$, and define on U_j

$$f_j := \sum_{\lambda \in J} \rho_\lambda c_{j\lambda}.$$

Each term is extended by zero from $U_\lambda \cap U_j$ to U_j, and the sum is locally finite. The cocycle condition gives

$$f_j - f_k = c_{jk}$$

on $U_j \cap U_k$. Consequently, $df_j = df_k$ on $U_j \cap U_k$, and the forms df_j glue to a global closed 1-form $\omega \in \Gamma(M, \mathcal{A}^1_M)$.

These constructions realize, at the level of representatives, the isomorphism $H^1_{\mathrm{DR}}(M) \cong \check{H}^1\big(\mathcal{U}, \underline{\mathbb{C}}_M\big)$.

The sign here is fixed by the preceding total-complex convention: for $f = (f_j)$ one has $\check{\delta} f = -c$ and $df = \omega$, so $\Delta f = \omega - c$. In particular, the period of the form corresponding to c is its evaluation on the oriented singular cycle of Chapter I, §7.5. This is the degree-one case of Remark A.1.5. □

Remark II.3.85 (G-flat bundles and first Čech cohomology). Let G be an abelian group. A G-flat bundle is described by locally constant transition functions

$$g_{jk} \in \underline{G}_X(U_j \cap U_k)$$

satisfying $g_{jk} g_{k\ell} = g_{j\ell}$ on $U_j \cap U_k \cap U_\ell$ (Since G is considered here as a structure group, we use multiplicative notation for its group operation, even though G is abelian. Thus sums and additive inverses in the preceding Čech notation are written here as products and multiplicative inverses). A change of flat trivializations replaces them by

$$g'_{jk} = \psi_j g_{jk} \psi_k^{-1}$$

for locally constant functions

$$\psi_j \in \underline{G}_X(U_j).$$

Since G is abelian, this is exactly the equivalence relation given by Čech coboundaries. After passing to refinements, the classes of G-flat bundles are therefore described by $\check{H}^1(X, \underline{G}_X)$.

If X is a non-empty triangulable Hausdorff topological space, then Proposition II.3.81 gives a natural isomorphism

$$\check{H}^1(X, \underline{G}_X) \cong H^1(X, G),$$

which gives the classification of Lemma I.7.36 and Remark I.7.32. The two descriptions use inverse characters: transport from a chart j to a chart k multiplies fiber coordinates by g_{kj}, whereas an oriented edge from j to k evaluates the cocycle at g_{jk}. Thus the singular class

of (g_{jk}) corresponds to the inverse of the endpoint-transport monodromy used in Chapter I. This distinction does not change the classification, but fixes its sign in additive notation.

For a non-abelian group G, the same transition functions define a non-abelian first Čech cohomology set, but not in general an abelian cohomology group. We shall not develop that theory here. □

We now apply the preceding discussion to ordinary complex line bundles.

EXAMPLE II.3.86 (The exponential sequence and the first Chern class). Let X be a paracompact Hausdorff space. Let $\mathcal{C}_X$ denote the sheaf of complex-valued continuous functions on X, and let $\mathcal{C}_X^*$ denote the sheaf of nowhere-vanishing complex-valued continuous functions. The sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow \mathcal{C}_X \xrightarrow{\exp(2\pi\sqrt{-1}\,\bullet)} \mathcal{C}_X^* \longrightarrow 1$$

is exact in $(\mathsf{Sheaf}(X, (\mathsf{Ab})))$ (Recall that the surjectivity of the last morphism is a stalkwise statement: every nowhere-vanishing continuous function admits a continuous logarithm in a neighborhood of each point). The sheaf $\mathcal{C}_X$ is fine and hence $\Gamma(X, -)$-acyclic (recall Example II.3.64). Therefore

$$H^q(X, \mathcal{C}_X) = 0 \qquad (q > 0),$$

and the long exact cohomology sequence gives an isomorphism

$$\delta_{\exp}\colon H^1(X, \mathcal{C}_X^*) \xrightarrow{\sim} H^2(X, \underline{\mathbb{Z}}_X).$$

A complex line bundle $L \to X$ is described by transition functions

$$g_{jk} \in \mathcal{C}_X^*(U_j \cap U_k)$$

satisfying $g_{jk}g_{k\ell} = g_{j\ell}$ on $U_j \cap U_k \cap U_\ell$. Our convention is $e_k = g_{jk}e_j$ for local frames; equivalently, $s = s_j e_j = s_k e_k$ gives $s_j = g_{jk}s_k$. It is the fiber-coordinate convention of Chapter I, §7.6. Changing local trivializations changes $\{(U_{jk}, g_{jk})\}$ by a Čech coboundary. Consequently, tensor product induces a natural group isomorphism between the group of topological isomorphism classes of complex line bundles and

$$H^1(X, \mathcal{C}_X^*)$$

(Compare Lemma I.7.35).

With the sign convention used in this note, the *first Chern class* of L is defined by

$$c_1(L) := \delta_{\exp}([L]) \in H^2(X, \underline{\mathbb{Z}}_X).$$

Thus complex line bundles over X, modulo $\cong_{\text{top}}$, are classified by $H^2(X, \underline{\mathbb{Z}}_X)$.

This class can be described explicitly. After refining the trivializing covering so that logarithms exist on its pairwise intersections, choose continuous functions

$$f_{jk} \in \mathcal{C}_X(U_j \cap U_k)$$

such that

$$g_{jk} = \exp(2\pi\sqrt{-1}\, f_{jk}).$$

Then

$$n_{jk\ell} := f_{k\ell} - f_{j\ell} + f_{jk}$$

is a $\mathbb{Z}$-valued locally constant function on $U_j \cap U_k \cap U_\ell$. The family

$$\{(U_{jk\ell}, n_{jk\ell})\}$$

is a Čech 2-cocycle representing $\delta_{\exp}([L])$. Consequently,

$$c_1(L) = \big[\{(U_{jk\ell}, n_{jk\ell})\}\big].$$

If X is triangulable, or more generally lies in a class of spaces for which the singular–sheaf comparison holds, then

$$H^2(X, \underline{\mathbb{Z}}_X) \cong H^2(X, \mathbb{Z}).$$

Thus the topological classification announced in Remark I.7.37 is recovered.

We keep the exponential map and the alternating coboundary fixed as written above; an inverse transition cocycle describes the dual bundle and gives $c_1(L^*) = -c_1(L)$. With the complex orientation, the first Chern class of the dual of the tautological line bundle pairs to 1 with the fundamental class of $\mathbb{P}^1$. See Remark III.2.25 for a verification using Chern curvature. □

Example II.3.87. Let M be a C^∞ manifold. Denote by $\mathcal{C}^\infty_M$ the sheaf of complex-valued C^∞ functions, and by $(\mathcal{C}^\infty_M)^*$ the sheaf of nowhere-vanishing complex-valued C^∞ functions. The sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_M \longrightarrow \mathcal{C}^\infty_M \xrightarrow{\exp(2\pi\sqrt{-1}\,\bullet)} (\mathcal{C}^\infty_M)^* \longrightarrow 1$$

is exact. As the sheaf $\mathcal{C}^\infty_M$ is fine, the connecting morphism gives an isomorphism

$$H^1\big(M, (\mathcal{C}^\infty_M)^*\big) \xrightarrow{\sim} H^2(M, \underline{\mathbb{Z}}_M)$$

(See the arguments in Example II.3.86). The inclusion $(\mathcal{C}^\infty_M)^* \to \mathcal{C}^*_M$ fits into a commutative diagram of exponential sequences and hence induces a commutative diagram

$$\begin{array}{ccc} H^1\big(M, (\mathcal{C}^\infty_M)^*\big) & \xrightarrow{\sim} & H^2(M, \underline{\mathbb{Z}}_M) \\ \downarrow & & \| \\ H^1(M, \mathcal{C}^*_M) & \xrightarrow{\sim} & H^2(M, \underline{\mathbb{Z}}_M). \end{array}$$

Therefore the left vertical morphism is an isomorphism.

The upper-left group classifies complex C^∞ line bundles up to C^∞ isomorphism, while the lower-left group classifies complex topological line bundles up to $\cong_{\text{top}}$. The vertical morphism is induced by forgetting the C^∞ structure. Consequently:

- every topological complex line bundle on M admits a C^∞ structure;
- two C^∞ complex line bundles are topologically isomorphic if and only if they are C^∞-isomorphic;
- the first Chern class defined using the smooth exponential sequence agrees with the topological first Chern class.

□

Mini-Exercise II.3.88. Let X be a connected non-empty triangulable Hausdorff topological space. Assume that $H_q(X)$ is finitely generated for any $q \in \mathbb{Z}_{\geq 0}$, and that $H_1(X)$ is torsion-free.
(i) Show that the canonical morphism $H^2(X, \underline{\mathbb{Z}}_X) \to H^2(X, \underline{\mathbb{C}}_X)$ induced by the inclusion $\underline{\mathbb{Z}}_X \to \underline{\mathbb{C}}_X$ is injective.
(ii) Consider the exponential sequence of locally constant sheaves

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow \underline{\mathbb{C}}_X \xrightarrow{\exp(2\pi\sqrt{-1}\bullet)} \underline{\mathbb{C}^*}_X \longrightarrow 1$$

and the continuous exponential sequence in Example II.3.86. By comparing the induced long exact cohomology sequences, show the following: Every $\mathbb{C}^*$-flat line bundle over X is topologically trivial.

3.4 Ringed and locally ringed spaces

Most of the main geometric objects considered in Chapter I have underlying topological spaces. For example, a C^∞ manifold is, in particular, a topological space (What distinguishes it from an arbitrary topological space is the additional smooth structure). This additional structure can be described by specifying, for every open subset U, which functions on U are regarded as smooth. These functions form the sheaf $\mathcal{C}^\infty_M$ on a C^∞ manifold M. Similarly, many of the geometric structures considered in this note can be encoded by equipping an underlying topological space with a distinguished sheaf of rings of functions. This leads to the language of ringed spaces.

Definition II.3.89. A *ringed space* is a pair

$$(X, \mathcal{R}_X)$$

consisting of a topological space X and a sheaf of commutative rings $\mathcal{R}_X$ on X. The sheaf $\mathcal{R}_X$ is called the *structure sheaf* of the ringed space. A ringed space $(X, \mathcal{R}_X)$ is called a *locally ringed space* if the stalk $\mathcal{R}_{X,x}$ is a local ring for any $x \in X$. □

Remark II.3.90. The structure sheaf of a ringed space is commonly denoted by $\mathcal{O}_X$. In this note, however, we use the notation $\mathcal{R}_X$ for a general structure sheaf and reserve $\mathcal{O}_X$ for the sheaf of holomorphic functions on a complex manifold or a complex analytic space. □

The following are basic examples.

- If T is a topological space, then
$$(T, \mathcal{C}_T)$$
is a locally ringed space.
- If M is a C^∞ manifold, then
$$(M, \mathcal{C}^\infty_M)$$
is a locally ringed space.
- If $\Omega \subset \mathbb{C}$ is a domain and $\mathcal{O}_\Omega$ denotes the sheaf of holomorphic functions, then
$$(\Omega, \mathcal{O}_\Omega)$$
is a locally ringed space.

For each of the examples above, the unique maximal ideal of the stalk of the structure sheaf at a point x consists of the germs of functions vanishing at x. Indeed, a germ of a continuous, smooth, or holomorphic function at x is invertible if and only if its value at x is nonzero.

Remark II.3.91. Most of the sheaf-theoretic constructions in this subsection require only the structure of a ringed space. Nevertheless, we shall already use the local-ring condition in an elementary but important way to relate the structure sheaf to the points of the underlying space. Indeed, the unique maximal ideal $\mathfrak{m}_{X,x} \subset \mathcal{R}_{X,x}$ distinguishes the germs that are regarded as vanishing at x, and the residue field $\kappa(x) := \mathcal{R}_{X,x}/\mathfrak{m}_{X,x}$ records the values of germs modulo such vanishing (see the discussion below for the details).

Apart from these pointwise features, we shall not yet use the deeper geometric consequences of the local-ring condition; they will appear later in these notes. □

Before defining morphisms of locally ringed spaces, we recall one piece of terminology. Let A and B be local rings with unique maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$, respectively. A ring homomorphism $\varphi\colon A \to B$ is called a *local homomorphism* if $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$ (Equivalently, $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. Indeed, the inclusion $\mathfrak{m}_A \subset \varphi^{-1}(\mathfrak{m}_B)$ is precisely the condition $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$. Conversely, if $a \notin \mathfrak{m}_A$, then a is invertible in A, and hence $\varphi(a)$ is invertible in B; therefore $\varphi(a) \notin \mathfrak{m}_B$).

Definition II.3.92. A *morphism of ringed spaces*
$$(f, f^\sharp)\colon (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$$
consists of a continuous map
$$f\colon X \longrightarrow Y$$
and a morphism of sheaves of rings
$$f^\sharp\colon \mathcal{R}_Y \longrightarrow f_*\mathcal{R}_X,$$

where the *direct image sheaf* $f_*\mathcal{F}$ on Y for a sheaf $\mathcal{F}$ on X is defined by

$$(f_*\mathcal{F})(V) := \mathcal{F}\big(f^{-1}(V)\big)$$

for every open subset $V \subset Y$ (The restriction morphisms of $f_*\mathcal{F}$ are induced by those of $\mathcal{F}$. Note that it follows immediately from the sheaf condition for $\mathcal{F}$ that $f_*\mathcal{F}$ is a sheaf).

A *morphism of locally ringed spaces* is a morphism of ringed spaces for which the induced homomorphism $f_x^\sharp \colon \mathcal{R}_{Y,f(x)} \longrightarrow \mathcal{R}_{X,x}$ is a local homomorphism for every $x \in X$. □

Remark II.3.93. In this section, where we develop the general theory of ringed and locally ringed spaces, we shall usually write a morphism explicitly as $(f, f^\sharp) \colon (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$. In applications, however, it is customary to abbreviate this notation and write simply

$$f \colon (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y).$$

We shall follow this convention throughout this note whenever no confusion is likely to arise.

Under this abbreviated notation, the symbol f denotes the entire morphism of ringed spaces and is also used, by abuse of notation, for its underlying continuous map

$$f \colon X \longrightarrow Y.$$

The corresponding morphism of structure sheaves is still denoted by

$$f^\sharp \colon \mathcal{R}_Y \longrightarrow f_*\mathcal{R}_X.$$

We also use the same symbol $f^\sharp$ for the adjoint homomorphism

$$f^\sharp \colon f^{-1}\mathcal{R}_Y \longrightarrow \mathcal{R}_X,$$

when there is no danger of confusion.

The abbreviated notation will be used when the intended morphism of structure sheaves is clear from the context or is canonically induced by the geometric map under consideration. It should be kept in mind that, for general ringed spaces, the underlying continuous map alone does not necessarily determine the morphism of structure sheaves. □

The morphisms defined above form a category, which we denote by

$$(\mathsf{LocRingSp}).$$

See [St, Tag 01HA] for the details of this category.

We shall now single out the class of locally ringed spaces over $\mathbb{C}$ that will be used in this note. By a sheaf of $\mathbb{C}$-algebras on a topological space X, we mean a sheaf of rings $\mathcal{R}_X$ together with a morphism

$$\underline{\mathbb{C}}_X \longrightarrow \mathcal{R}_X$$

of sheaves of rings, where $\underline{\mathbb{C}}_X$ denotes the sheaf of $\mathbb{C}$-valued locally constant functions on X. Equivalently, every ring $\mathcal{R}_X(U)$ is endowed with a $\mathbb{C}$-algebra structure, compatibly with the restriction homomorphisms.

We use the notation

$$(\mathsf{LocRingSp}_{\mathbb{C}})$$

for the following class of objects and morphisms. Its objects are locally ringed spaces $(X, \mathcal{R}_X)$ such that $\mathcal{R}_X$ is a sheaf of $\mathbb{C}$-algebras and, for every $x \in X$, the canonical homomorphism

$$\mathbb{C} \longrightarrow \kappa(x) := \mathcal{R}_{X,x}/\mathfrak{m}_{X,x}$$

is an isomorphism, where $\mathfrak{m}_{X,x}$ denotes the unique maximal ideal of the local ring $\mathcal{R}_{X,x}$ ($\kappa(x)$ is called the *residue field* at x in general). The morphisms are the morphisms of locally ringed spaces

$$(f, f^\sharp)\colon (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$$

for which

$$f^\sharp\colon \mathcal{R}_Y \longrightarrow f_*\mathcal{R}_X$$

is a morphism of sheaves of $\mathbb{C}$-algebras.

Mini-Exercise II.3.94. Show that the objects and morphisms described above form a category: i.e., $(\mathsf{LocRingSp}_{\mathbb{C}})$ is well-defined as a category.

Remark II.3.95. Let $(X, \mathcal{R}_X)$ be an object of $(\mathsf{LocRingSp}_{\mathbb{C}})$. If $U \subset X$ is open, $s \in \mathcal{R}_X(U)$, and $x \in U$, we define the *value of s at x* by

$$s(x) := (\text{the image of } s_x \text{ under } \mathcal{R}_{X,x} \longrightarrow \kappa(x) \cong \mathbb{C}).$$

Thus

$$\mathfrak{m}_{X,x} = \{\, s_x \in \mathcal{R}_{X,x} \mid s(x) = 0 \,\}.$$

This definition makes sense even when $\mathcal{R}_X$ is not literally given as a sheaf of functions on X.

Moreover, every morphism

$$(f, f^\sharp)\colon (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$$

in $(\mathsf{LocRingSp}_{\mathbb{C}})$ preserves values in the following sense: for every open subset $V \subset Y$, every $s \in \mathcal{R}_Y(V)$, and every $x \in f^{-1}(V)$, one has

$$\big(f_V^\sharp(s)\big)(x) = s(f(x)).$$

Indeed, the local homomorphism

$$f_x^\sharp\colon \mathcal{R}_{Y,f(x)} \longrightarrow \mathcal{R}_{X,x}$$

induces a homomorphism of residue fields

$$\kappa(f(x)) \longrightarrow \kappa(x).$$

Under the canonical identifications of both residue fields with $\mathbb{C}$, this is a homomorphism of $\mathbb{C}$-algebras $\mathbb{C} \to \mathbb{C}$ and hence is the identity homomorphism. □

Remark II.3.96. The one-point space

$$\big(\{*\}, \underline{\mathbb{C}}_{\{*\}}\big)$$

is a terminal object of $(\mathsf{LocRingSp}_{\mathbb{C}})$. Indeed, for every object $(X, \mathcal{R}_X)$, there is a unique continuous map

$$f\colon X \longrightarrow \{*\}.$$

The $\mathbb{C}$-algebra structure of $\mathcal{R}_X$ gives the unique morphism of sheaves of $\mathbb{C}$-algebras

$$f^\sharp\colon \underline{\mathbb{C}}_{\{*\}} \longrightarrow f_*\mathcal{R}_X.$$

For every $x \in X$, the induced homomorphism

$$\mathbb{C} \longrightarrow \mathcal{R}_{X,x}$$

is local, since its composite with $\mathcal{R}_{X,x} \to \kappa(x)$ is an isomorphism. Thus $(f, f^\sharp)$ is the unique morphism to the displayed one-point space. □

Example II.3.97. Let M be a C^∞ manifold, and let $\mathcal{C}^\infty_M$ denote the sheaf of complex-valued C^∞ functions on M. For every $x \in M$, evaluation at x gives a surjective homomorphism of $\mathbb{C}$-algebras

$$\mathcal{C}^\infty_{M,x} \longrightarrow \mathbb{C}, \qquad h_x \longmapsto h(x).$$

Its kernel is

$$\mathfrak{m}_{M,x} := \{\, h_x \in \mathcal{C}^\infty_{M,x} \mid h(x) = 0 \,\}.$$

If $h(x) \neq 0$, then h is nowhere zero on some neighborhood of x, and $1/h$ is C^∞ on that neighborhood. Consequently, every germ outside $\mathfrak{m}_{M,x}$ is invertible, so $\mathcal{C}^\infty_{M,x}$ is a local ring with maximal ideal $\mathfrak{m}_{M,x}$ and residue field $\mathbb{C}$. Thus $(M, \mathcal{C}^\infty_M)$ is an object of $(\mathsf{LocRingSp}_{\mathbb{C}})$.

Let M and N be C^∞ manifolds, and let

$$f\colon M \longrightarrow N$$

be a C^∞ map. For every open subset $V \subset N$, define

$$f^\sharp_V\colon \mathcal{C}^\infty_N(V) \longrightarrow \mathcal{C}^\infty_M(f^{-1}(V))$$

by

$$f^\sharp_V(h) := h \circ f \qquad \big(h \in \mathcal{C}^\infty_N(V)\big).$$

These homomorphisms are compatible with restrictions and define a morphism of sheaves of $\mathbb{C}$-algebras

$$f^\sharp\colon \mathcal{C}^\infty_N \longrightarrow f_*\mathcal{C}^\infty_M.$$

For every $x \in M$, the induced homomorphism on stalks maps $\mathfrak{m}_{N,f(x)}$ into $\mathfrak{m}_{M,x}$, since

$$h(f(x)) = 0 \quad \Longrightarrow \quad (h \circ f)(x) = 0 \qquad \big(h \in \mathcal{C}^\infty_N(V)\big).$$

It is therefore a local homomorphism, and hence $(f, f^\sharp)\colon (M, \mathcal{C}^\infty_M) \longrightarrow (N, \mathcal{C}^\infty_N)$ is a morphism in $(\mathsf{LocRingSp}_{\mathbb{C}})$.

Conversely, suppose that

$$(f, f^\sharp)\colon (M, \mathcal{C}^\infty_M) \longrightarrow (N, \mathcal{C}^\infty_N)$$

is a morphism in $(\mathsf{LocRingSp}_{\mathbb{C}})$. By Remark II.3.95, for every open subset $V \subset N$, every $h \in \mathcal{C}^\infty_N(V)$, and every $x \in f^{-1}(V)$, one has $\big(f^\sharp_V(h)\big)(x) = h(f(x))$, from which it follows that

$$f^\sharp_V(h) = h \circ f.$$

Fix $x \in M$, and choose local coordinates $(y^1, y^2, \dots, y^n)$ on an open neighborhood $V \subset N$ of $f(x)$, where the real-valued coordinate functions are regarded as complex-valued functions. Since f is continuous, $f^{-1}(V)$ is an open neighborhood of x, and

$$y^a \circ f = f^\sharp_V(y^a) \in \mathcal{C}^\infty_M(f^{-1}(V))$$

for every $a = 1, 2, \dots, n$. Thus all coordinate components of f are C^∞ near x. Since x was arbitrary, f is a C^∞ map.

We have therefore obtained a bijection

$$\{\, f\colon M \to N \mid f \text{ is a } C^\infty \text{ map} \,\} \longrightarrow \mathrm{Hom}_{(\mathsf{LocRingSp}_{\mathbb{C}})}\big((M, \mathcal{C}^\infty_M), (N, \mathcal{C}^\infty_N)\big),$$

which sends f to $(f, f^\sharp)$. In particular, $(M, \mathcal{C}^\infty_M) \cong (N, \mathcal{C}^\infty_N)$ in $(\mathsf{LocRingSp}_{\mathbb{C}})$ if and only if M and N are C^∞-diffeomorphic. Thus the locally ringed space $(M, \mathcal{C}^\infty_M)$ remembers not only the topology of M, but also its entire C^∞ structure.

The preceding discussion also gives a characterization of C^∞ manifolds in terms of locally ringed spaces. Namely, for $n \in \mathbb{Z}_{\geq 0}$, an n-dimensional C^∞ manifold may equivalently be described as an object $(X, \mathcal{R}_X)$ of $(\mathsf{LocRingSp}_{\mathbb{C}})$ such that X is Hausdorff and second countable and, for every $x \in X$, there exist an open neighborhood $U \subset X$ of x, an open subset $V \subset \mathbb{R}^n$, and an isomorphism

$$(U, \mathcal{R}_X|_U) \cong (V, \mathcal{C}^\infty_V)$$

in $(\mathsf{LocRingSp}_{\mathbb{C}})$. Indeed, a local coordinate chart of a C^∞ manifold gives such an isomorphism. Conversely, the underlying homeomorphisms of these local isomorphisms give coordinate charts on X. On the intersection of two such charts, the corresponding transition map induces an isomorphism between the associated locally ringed spaces. By the result proved above, this transition map is a C^∞-diffeomorphism. The resulting charts therefore define a C^∞ structure on X, and the local isomorphisms identify $\mathcal{R}_X$ with the sheaf of complex-valued C^∞ functions for this structure. See also [St, Tag 01HA]. □

For a ringed space $(X, \mathcal{R}_X)$, a sheaf of $\mathcal{R}_X$-modules will usually be called simply an $\mathcal{R}_X$-module. By Proposition II.3.58, the category

$$(\mathsf{Mod}(\mathcal{R}_X))$$

is abelian, has enough injective objects, and admits all small limits and colimits.

Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{R}_X$-modules. Their tensor product

$$\mathcal{F} \otimes_{\mathcal{R}_X} \mathcal{G}$$

is the sheafification of the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{R}_X(U)} \mathcal{G}(U).$$

For every $x \in X$, there is a natural isomorphism

$$\big(\mathcal{F} \otimes_{\mathcal{R}_X} \mathcal{G}\big)_x \cong \mathcal{F}_x \otimes_{\mathcal{R}_{X,x}} \mathcal{G}_x.$$

The *sheaf Hom*

$$\mathcal{H}om_{\mathcal{R}_X}\big(\mathcal{F}, \mathcal{G}\big)$$

is defined by

$$\Gamma\big(U, \mathcal{H}om_{\mathcal{R}_X}(\mathcal{F}, \mathcal{G})\big) := \mathrm{Hom}_{\mathcal{R}_X|_U}\big(\mathcal{F}|_U, \mathcal{G}|_U\big)$$

for every open subset $U \subset X$. This assignment is already a sheaf. We also write

$$\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{R}_X}\big(\mathcal{F}, \mathcal{R}_X\big).$$

See [St, Tag 01CA and Tag 01CB].

In the preceding example, the structure morphism was written in the form

$$f^\sharp \colon \mathcal{R}_Y \longrightarrow f_*\mathcal{R}_X.$$

To describe pullbacks of module sheaves, it is useful to express the same data as a morphism between sheaves on X. For this purpose, we here introduce the *inverse image* of a sheaf. Let

$$f \colon X \longrightarrow Y$$

be a continuous map and $\mathcal{G}$ a sheaf on Y. For every open subset $U \subset X$, set

$$(f^{-1}_{\mathrm{pre}}\mathcal{G})(U) := \varinjlim_{V \in \mathcal{N}_Y(f(U))} \mathcal{G}(V),$$

where $\mathcal{N}_Y(f(U))$ is the directed set of open neighborhoods of $f(U)$ introduced in Example II.3.20, and the transition morphisms are induced by restrictions. The *inverse image sheaf* $f^{-1}\mathcal{G}$ of $\mathcal{G}$ is, by definition, the sheafification of $f^{-1}_{\mathrm{pre}}\mathcal{G}$. The same construction applies to sheaves of groups, rings, modules, and the other algebraic structures considered above, with the inductive limits and sheafification taken in the corresponding category. For every $x \in X$, there is a natural isomorphism

$$(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}.$$

The inverse image functor

$$f^{-1} \colon (\mathsf{Sheaf}(Y, \mathfrak{S})) \xrightarrow{\text{functor}} (\mathsf{Sheaf}(X, \mathfrak{S}))$$

is left adjoint to the direct image functor

$$f_*\colon (\mathsf{Sheaf}(X,\mathfrak{S})) \xrightarrow{\text{functor}} (\mathsf{Sheaf}(Y,\mathfrak{S})),$$

where $\mathfrak{S}$ is one of the categories of algebraic structures considered here, such as (Ab), (Ring), $(\mathsf{Mod}(R))$ for a ring R, or (Vect); that is,

$$f^{-1} \dashv f_*.$$

See [St, Tag 008P].

Applying this adjunction to sheaves of rings, the morphism

$$f^\sharp\colon \mathcal{R}_Y \longrightarrow f_*\mathcal{R}_X$$

associated with a morphism of ringed spaces is equivalent to a morphism of sheaves of rings

$$f^{-1}\mathcal{R}_Y \longrightarrow \mathcal{R}_X.$$

We shall denote the adjoint morphism also by $f^\sharp$ (On the stalk at $x \in X$, it also induces the homomorphism

$$f_x^\sharp\colon \mathcal{R}_{Y,f(x)} \longrightarrow \mathcal{R}_{X,x}$$

appearing in Definition II.3.92).

We can now define the *direct image* and *pullback* of module sheaves. Let

$$(f, f^\sharp)\colon (X,\mathcal{R}_X) \longrightarrow (Y,\mathcal{R}_Y)$$

be a morphism of ringed spaces. If $\mathcal{F}$ is an $\mathcal{R}_X$-module, then the direct image sheaf $f_*\mathcal{F}$ is naturally an $\mathcal{R}_Y$-module through

$$f^\sharp\colon \mathcal{R}_Y \longrightarrow f_*\mathcal{R}_X.$$

Thus one obtains a functor

$$f_*\colon (\mathsf{Mod}(\mathcal{R}_X)) \xrightarrow{\text{functor}} (\mathsf{Mod}(\mathcal{R}_Y)).$$

If $\mathcal{G}$ is an $\mathcal{R}_Y$-module, then $f^{-1}\mathcal{G}$ is naturally an $f^{-1}\mathcal{R}_Y$-module. The adjoint structure morphism

$$f^\sharp\colon f^{-1}\mathcal{R}_Y \longrightarrow \mathcal{R}_X$$

makes $\mathcal{R}_X$ an $f^{-1}\mathcal{R}_Y$-algebra, but $f^{-1}\mathcal{G}$ is not in general an $\mathcal{R}_X$-module. We therefore define the *pullback* of $\mathcal{G}$ by

$$f^*\mathcal{G} := \mathcal{R}_X \otimes_{f^{-1}\mathcal{R}_Y} f^{-1}\mathcal{G}.$$

This defines a functor

$$f^*\colon (\mathsf{Mod}(\mathcal{R}_Y)) \xrightarrow{\text{functor}} (\mathsf{Mod}(\mathcal{R}_X)).$$

Example II.3.98. Let $\Omega \subset \mathbb{C}$ be a domain containing the origin, and denote the standard coordinate on $\mathbb{C}$ by z. Here we denote by $\mathcal{O}_\Omega$ the sheaf of holomorphic functions on Ω. Consider the objects $(X, \mathcal{R}_X) := \big(\{*\}, \underline{\mathbb{C}}_{\{*\}}\big)$ and $(Y, \mathcal{R}_Y) := (\Omega, \mathcal{O}_\Omega)$ of $(\mathsf{LocRingSp}_{\mathbb{C}})$. Let

$$f\colon X \longrightarrow Y$$

be the continuous map defined by

$$f(*) = 0.$$

Then, by definition, one has

$$(f_*\underline{\mathbb{C}}_{\{*\}})(U) \cong \begin{cases} \mathbb{C} & \text{if } 0 \in U \\ 0 & \text{otherwise} \end{cases}$$

for each open subset $U \subset \Omega$ (the *skyscraper sheaf* supported at 0). Letting the structure morphism be the one induced by evaluation at the origin, we obtain the morphism $(f, f^\sharp)\colon \big(\{*\}, \underline{\mathbb{C}}_{\{*\}}\big) \to (\Omega, \mathcal{O}_\Omega)$ of ringed spaces:

$$f^\sharp(U)\colon \mathcal{O}_\Omega(U) \ni h \mapsto \begin{cases} h(0) & \text{if } 0 \in U \\ 0 & \text{otherwise} \end{cases} \in (f_*\underline{\mathbb{C}}_{\{*\}})(U) \qquad (U \subset \Omega : \text{ open}).$$

Since a sheaf on the one-point space $\{*\}$ is determined by its sections over $\{*\}$, one has

$$\Gamma\big(\{*\}, f^{-1}\mathcal{O}_\Omega\big) \cong \mathcal{O}_{\Omega,0} \cong \mathbb{C}\{z\}$$

(Recall Example II.3.55). Thus

$$f^{-1}\mathcal{O}_\Omega \cong \underline{\mathbb{C}\{z\}}_{\{*\}}.$$

On the other hand, the morphism

$$f^{-1}\mathcal{O}_\Omega \longrightarrow \underline{\mathbb{C}}_{\{*\}}$$

is induced by the local homomorphism

$$\mathbb{C}\{z\} \longrightarrow \mathbb{C}, \qquad h \longmapsto h(0).$$

Consequently,

$$f^*\mathcal{O}_\Omega = \underline{\mathbb{C}}_{\{*\}} \otimes_{f^{-1}\mathcal{O}_\Omega} f^{-1}\mathcal{O}_\Omega \cong \underline{\mathbb{C}}_{\{*\}}$$

(Here we note that $\mathbb{C}$ is regarded as a $\mathbb{C}\{z\}$-module through the evaluation homomorphism $\mathbb{C}\{z\} \ni h \mapsto h(0) \in \mathbb{C}$, and the last isomorphism is induced by the map $\mathbb{C} \otimes_{\mathbb{C}\{z\}} \mathbb{C}\{z\} \to \mathbb{C}$ defined by $c \otimes h \mapsto c\,h(0)$). In particular,

$$f^{-1}\mathcal{O}_\Omega \ncong f^*\mathcal{O}_\Omega;$$

the former retains the germs of holomorphic functions at the origin, whereas the latter remembers only their values at the origin.

Therefore, mainly for the inclusion/embedding of a closed subset, as in the present example, one must be careful with notation such as $\mathcal{O}_\Omega|_{\{0\}}$. Unlike the case of an open embedding, the inverse image sheaf and the pullback module sheaf need not coincide. In a sheaf-theoretic context, the notation above may denote the inverse image sheaf $f^{-1}\mathcal{O}_\Omega$, whereas in the context of module sheaves on ringed spaces, it may denote the pullback $f^*\mathcal{O}_\Omega$. Whenever this distinction matters, we shall avoid such notation and write f^{-1} or f^* explicitly. □

The basic exactness properties of the functors f^{-1} and f^* are summarized as follows.

Proposition II.3.99. *Let*

$$(f, f^\sharp)\colon (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$$

be a morphism of ringed spaces. Then the following hold.
(*i*) *The inverse image functor*

$$f^{-1}\colon (\mathsf{Sheaf}(Y, (\mathsf{Ab}))) \xrightarrow{\text{functor}} (\mathsf{Sheaf}(X, (\mathsf{Ab})))$$

defined in the same manner as above is exact.
(*ii*) *There is a natural adjunction*

$$f^* \dashv f_*.$$

More precisely, for every $\mathcal{R}_Y$-module $\mathcal{G}$ and every $\mathcal{R}_X$-module $\mathcal{F}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{R}_X}\big(f^*\mathcal{G}, \mathcal{F}\big) \cong \mathrm{Hom}_{\mathcal{R}_Y}\big(\mathcal{G}, f_*\mathcal{F}\big).$$

(*iii*) *The functor f_* is left exact, while f^* is right exact.*

Proof. The assertion in (*i*) follows from the natural isomorphisms

$$(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)},$$

since exactness of sequences of sheaves of abelian groups can be checked on stalks.

For (*ii*), regard $\mathcal{F}$ as an $f^{-1}\mathcal{R}_Y$-module through $f^\sharp$. The adjunction $f^{-1} \dashv f_*$ and the tensor–Hom adjunction give natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{R}_X}\big(f^*\mathcal{G}, \mathcal{F}\big) &= \mathrm{Hom}_{\mathcal{R}_X}\big(\mathcal{R}_X \otimes_{f^{-1}\mathcal{R}_Y} f^{-1}\mathcal{G}, \mathcal{F}\big)\\ &\cong \mathrm{Hom}_{f^{-1}\mathcal{R}_Y}\big(f^{-1}\mathcal{G}, \mathcal{F}\big)\\ &\cong \mathrm{Hom}_{\mathcal{R}_Y}\big(\mathcal{G}, f_*\mathcal{F}\big).\end{aligned}$$

Finally (*iii*) follows from (*ii*), since a right adjoint between abelian categories is left exact and a left adjoint is right exact (Proposition II.3.40). See also [St, Tag 0094 and Tag 01AJ]. □

Since $(\mathsf{Mod}(\mathcal{R}_X))$ has enough injective objects (Proposition II.3.58) and the direct image functor is left exact (Proposition II.3.99 (*iii*)), its right derived functors are defined.

Definition II.3.100. Let

$$(f, f^\sharp)\colon (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$$

be a morphism of ringed spaces. For every integer $q \geq 0$, the *q-th higher direct image* is the right derived functor

$$R^q f_*\colon (\mathsf{Mod}(\mathcal{R}_X)) \xrightarrow{\text{functor}} (\mathsf{Mod}(\mathcal{R}_Y)).$$

Thus, if

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots$$

is an injective resolution, then

$$R^q f_* \mathcal{F} := H^q\big(f_* \mathcal{I}^\bullet\big).$$

In particular, there is a natural isomorphism $R^0 f_* \cong f_*$. □

For every $\mathcal{R}_X$-module $\mathcal{F}$, the sheaf $R^q f_* \mathcal{F}$ is naturally isomorphic to the sheaf associated with the presheaf

$$V \longmapsto H^q\big(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}\big)$$

on Y. Thus the higher direct images record the cohomology of the inverse images $f^{-1}(V)$ as the open subset $V \subset Y$ varies. See [St, Tag 01E4].

In particular, let $(X, \mathcal{R}_X)$ be an object of $(\mathsf{LocRingSp}_{\mathbb{C}})$ and let

$$a_X\colon (X, \mathcal{R}_X) \longrightarrow \big(\{*\}, \underline{\mathbb{C}}_{\{*\}}\big)$$

be the unique morphism to the terminal object. After identifying sheaves of $\mathbb{C}$-modules on $\{*\}$ with complex vector spaces, one has

$$R^q (a_X)_* \mathcal{F} \cong H^q(X, \mathcal{F}).$$

Thus ordinary sheaf cohomology can be regarded as a special case of a higher direct image.

The following low-degree exact sequence will be useful later.

Proposition II.3.101 (Leray five-term exact sequence). *Let $(f, f^\sharp)\colon (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a morphism in $(\mathsf{LocRingSp}_{\mathbb{C}})$, and let $\mathcal{F}$ be an $\mathcal{R}_X$-module. Then there is a natural exact sequence*

$$0 \longrightarrow H^1(Y, f_*\mathcal{F}) \xrightarrow{\iota^1} H^1(X, \mathcal{F}) \xrightarrow{\epsilon} H^0(Y, R^1 f_* \mathcal{F}) \xrightarrow{\tau} H^2(Y, f_*\mathcal{F}) \xrightarrow{\iota^2} H^2(X, \mathcal{F}).$$

Proof. Take an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \mathcal{I}^2 \xrightarrow{d^2} \cdots$$

in $(\mathsf{Mod}(\mathcal{R}_X))$, and set

$$\mathcal{C}^\bullet := f_* \mathcal{I}^\bullet.$$

We use the same notation d^q for the differentials of $\mathcal{C}^\bullet$.

The proof below uses only the following three properties of this resolution:

(P1) Each $\mathcal{I}^q$ is $\Gamma(X,-)$-acyclic; that is,

$$H^s(X,\mathcal{I}^q)=0 \qquad (s\geq 1).$$

(P2) Each $\mathcal{I}^q$ is f_*-acyclic; that is,

$$R^s f_*\mathcal{I}^q=0 \qquad (s\geq 1).$$

(P3) Each $\mathcal{C}^q=f_*\mathcal{I}^q$ is $\Gamma(Y,-)$-acyclic; that is,

$$H^s(Y,\mathcal{C}^q)=0 \qquad (s\geq 1).$$

Properties (P1) and (P2) follow from Proposition II.3.47 (ii), applied to the left exact functors $\Gamma(X,-)$ and f_*, respectively. To verify (P3), recall that every injective $\mathcal{R}_X$-module is flabby; see [St, Tag 09SX]. Moreover, the direct image of a flabby sheaf is again flabby. Indeed, for open subsets $V'\subset V\subset Y$, the restriction morphism $(f_*\mathcal{I}^q)(V)=\mathcal{I}^q\left(f^{-1}(V)\right)\longrightarrow \mathcal{I}^q\left(f^{-1}(V')\right)=(f_*\mathcal{I}^q)(V')$ is surjective. Thus $\mathcal{C}^q=f_*\mathcal{I}^q$ is flabby and hence $\Gamma(Y,-)$-acyclic by Proposition II.3.63 (i).

For every integer $q\geq 0$, set

$$\mathcal{Z}^q:=\operatorname{Ker}\left(d^q\colon \mathcal{C}^q\longrightarrow \mathcal{C}^{q+1}\right), \qquad \mathcal{B}^q:=\begin{cases}0 & \text{if } q=0,\\ \operatorname{Image}\left(d^{q-1}\colon \mathcal{C}^{q-1}\longrightarrow \mathcal{C}^q\right) & \text{if } q\geq 1,\end{cases}$$

and

$$\mathcal{H}^q:=\mathcal{Z}^q/\mathcal{B}^q.$$

By property (P2) and the acyclicity theorem (Proposition II.3.49),

$$\mathcal{H}^q\cong R^q f_*\mathcal{F}.$$

In particular, $\mathcal{H}^0\cong f_*\mathcal{F}$. Moreover,

$$\Gamma\left(Y,\mathcal{C}^\bullet\right)=\Gamma\left(Y,f_*\mathcal{I}^\bullet\right)=\Gamma\left(X,\mathcal{I}^\bullet\right)$$

by definition of the functor f_*. Therefore, by property (P1) and the acyclicity theorem (Proposition II.3.49), one has

$$H^q\left(\Gamma\left(Y,\mathcal{C}^\bullet\right)\right)\cong H^q(X,\mathcal{F}) \qquad (q\geq 0).$$

We use the following three short exact sequences:

$$0\longrightarrow \mathcal{H}^0\longrightarrow \mathcal{C}^0\longrightarrow \mathcal{B}^1\longrightarrow 0,$$

$$0\longrightarrow \mathcal{B}^1\longrightarrow \mathcal{Z}^1\longrightarrow \mathcal{H}^1\longrightarrow 0,$$

and

$$0\longrightarrow \mathcal{Z}^1\longrightarrow \mathcal{C}^1\longrightarrow \mathcal{B}^2\longrightarrow 0$$

(Here the last morphism is the corestriction of $\mathcal{C}^1 \to \mathcal{C}^2$ to its image $\mathcal{B}^2 = \text{Image}(d^1)$. Thus its kernel is precisely $\mathcal{Z}^1$).

Definition of ι^1 and its injectivity. By Property (P3), the long exact cohomology sequence associated with the first short exact sequence gives a natural isomorphism

$$H^1\left(Y, \mathcal{H}^0\right) \cong \frac{\Gamma(Y, \mathcal{B}^1)}{\text{Image}\left(\Gamma(Y, \mathcal{C}^0) \longrightarrow \Gamma(Y, \mathcal{B}^1)\right)}.$$

The inclusion $\mathcal{B}^1 \to \mathcal{Z}^1$ therefore induces a natural injective morphism

$$\iota^1 \colon H^1\left(Y, \mathcal{H}^0\right) \longrightarrow \frac{\Gamma(Y, \mathcal{Z}^1)}{\text{Image}\left(\Gamma(Y, \mathcal{C}^0) \longrightarrow \Gamma(Y, \mathcal{Z}^1)\right)}.$$

The group on the right is $H^1\left(\Gamma\left(Y, \mathcal{C}^\bullet\right)\right)$. Under the identifications $\mathcal{H}^0 \cong f_*\mathcal{F}$ and $H^1\left(\Gamma\left(Y, \mathcal{C}^\bullet\right)\right) \cong H^1(X, \mathcal{F})$, this defines the morphism

$$\iota^1 \colon H^1(Y, f_*\mathcal{F}) \longrightarrow H^1(X, \mathcal{F}),$$

which is injective by construction.

Definition of ϵ and the exactness at $H^1(X, \mathcal{F})$. The quotient morphism $\mathcal{Z}^1 \to \mathcal{H}^1$ induces a natural morphism

$$\epsilon \colon H^1\left(\Gamma\left(Y, \mathcal{C}^\bullet\right)\right) \longrightarrow \Gamma(Y, \mathcal{H}^1).$$

More explicitly, if $a \in \Gamma(Y, \mathcal{Z}^1)$ represents a class in $H^1(\Gamma(Y, \mathcal{C}^\bullet))$, then $\epsilon([a])$ is the image of a in $\Gamma(Y, \mathcal{H}^1)$. Under the identification $\mathcal{H}^1 \cong R^1 f_*\mathcal{F}$, this gives

$$\epsilon \colon H^1(X, \mathcal{F}) \longrightarrow H^0(Y, R^1 f_*\mathcal{F}).$$

The exactness at $H^1(X, \mathcal{F})$ is clear, since

$$\operatorname{Ker} \epsilon = \frac{\Gamma(Y, \mathcal{B}^1)}{\text{Image}\left(\Gamma(Y, \mathcal{C}^0) \longrightarrow \Gamma(Y, \mathcal{B}^1)\right)} = \text{Image}\, \iota^1.$$

Definition of τ and the exactness at $H^0(Y, R^1 f_*\mathcal{F})$. The second short exact sequence gives a connecting morphism

$$\partial \colon \Gamma(Y, \mathcal{H}^1) \longrightarrow H^1(Y, \mathcal{B}^1).$$

Since $\mathcal{C}^0$ is acyclic, the first short exact sequence also gives a natural isomorphism

$$H^1(Y, \mathcal{B}^1) \cong H^2(Y, \mathcal{H}^0).$$

We define

$$\tau \colon \Gamma(Y, \mathcal{H}^1) \longrightarrow H^2(Y, \mathcal{H}^0)$$

to be the composite of ∂ with this isomorphism. Using $\mathcal{H}^0 \cong f_*\mathcal{F}$ and $\mathcal{H}^1 \cong R^1 f_*\mathcal{F}$, this becomes

$$\tau \colon H^0(Y, R^1 f_*\mathcal{F}) \longrightarrow H^2(Y, f_*\mathcal{F}).$$

By the long exact sequence associated with the second short exact sequence, one has

$$\operatorname{Ker}\partial = \operatorname{Image}\left(\Gamma(Y,\mathcal{Z}^1) \longrightarrow \Gamma(Y,\mathcal{H}^1)\right),$$

which is precisely the image of ϵ. Since the morphism $H^1(Y,\mathcal{B}^1) \to H^2(Y,\mathcal{H}^0)$ used in the definition of τ is an isomorphism, it follows that

$$\operatorname{Ker}\tau = \operatorname{Image}\epsilon.$$

Definition of ι^2 and the exactness at $H^2(Y, f_*\mathcal{F})$. The first short exact sequence gives a natural isomorphism

$$H^2(Y,\mathcal{H}^0) \cong H^1(Y,\mathcal{B}^1).$$

The inclusion $\mathcal{B}^1 \to \mathcal{Z}^1$ induces

$$H^1(Y,\mathcal{B}^1) \longrightarrow H^1(Y,\mathcal{Z}^1).$$

Since $\mathcal{C}^1$ is acyclic, the third short exact sequence gives a natural isomorphism

$$H^1(Y,\mathcal{Z}^1) \cong \frac{\Gamma(Y,\mathcal{B}^2)}{\operatorname{Image}\left(\Gamma(Y,\mathcal{C}^1) \longrightarrow \Gamma(Y,\mathcal{B}^2)\right)}.$$

The inclusion $\mathcal{B}^2 \to \mathcal{Z}^2$ induces an injective morphism

$$\frac{\Gamma(Y,\mathcal{B}^2)}{\operatorname{Image}\left(\Gamma(Y,\mathcal{C}^1) \longrightarrow \Gamma(Y,\mathcal{B}^2)\right)} \longrightarrow \frac{\Gamma(Y,\mathcal{Z}^2)}{\operatorname{Image}\left(\Gamma(Y,\mathcal{C}^1) \longrightarrow \Gamma(Y,\mathcal{Z}^2)\right)}.$$

The group on the right is $H^2\left(\Gamma\left(Y,\mathcal{C}^\bullet\right)\right)$. The resulting composite is the morphism

$$\iota^2 \colon H^2(Y,\mathcal{H}^0) \longrightarrow H^2\left(\Gamma\left(Y,\mathcal{C}^\bullet\right)\right),$$

which is identified with

$$\iota^2 \colon H^2(Y, f_*\mathcal{F}) \longrightarrow H^2(X,\mathcal{F}).$$

The long exact sequence associated with the second short exact sequence gives

$$\operatorname{Image}\partial = \operatorname{Ker}\left(H^1(Y,\mathcal{B}^1) \longrightarrow H^1(Y,\mathcal{Z}^1)\right).$$

The natural morphism

$$H^1(Y,\mathcal{Z}^1) \longrightarrow H^2\left(\Gamma\left(Y,\mathcal{C}^\bullet\right)\right)$$

constructed above is injective. Therefore

$$\operatorname{Ker}\iota^2 = \operatorname{Image}\tau,$$

which proves the exactness of the sequence.

All the constructions are natural in $\mathcal{F}$. The comparison theorem for injective resolutions shows that they are independent of the chosen injective resolution. Note that the exact sequence obtained above is also the low-degree part of the *Leray spectral sequence*; See [St, Tag 01F2]. □

Remark II.3.102. The proof of Proposition II.3.101 does not use the injectivity of the resolution directly. It uses only properties (P1)–(P3) listed in the proof. Consequently, the same proof applies if the injective resolution is replaced by a resolution

$$0 \to \mathcal{F} \to \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \xrightarrow{d^1} \mathcal{A}^2 \xrightarrow{d^2} \cdots$$

such that

$$H^s(X, \mathcal{A}^q) = 0, \qquad R^s f_* \mathcal{A}^q = 0, \qquad H^s(Y, f_* \mathcal{A}^q) = 0$$

for every $q \geq 0$ and every $s \geq 1$.

We record one useful situation in which these conditions are satisfied. Suppose that f is a closed map, that Y is paracompact and Hausdorff, and that, for every open subset $V \subset Y$, the space $f^{-1}(V)$ is paracompact and Hausdorff and $\mathcal{A}^q|_{f^{-1}(V)}$ is fine for every $q \geq 0$. Then the acyclicity of fine sheaves gives

$$H^s\left(f^{-1}(V), \mathcal{A}^q\right) = 0 \qquad (s \geq 1).$$

Taking $V = Y$ gives property (P1), and the vanishing for all open subsets $V \subset Y$ gives property (P2).

Moreover, the direct image of a fine sheaf by a closed map is fine. Indeed, let $\{V_\lambda\}_{\lambda \in \Lambda}$ be a locally finite open covering of Y. Then $\{f^{-1}(V_\lambda)\}_{\lambda \in \Lambda}$ is a locally finite open covering of X. Choose endomorphisms $\theta_\lambda \colon \mathcal{A}^q \to \mathcal{A}^q$ subordinate to this covering. Their direct images

$$f_*\theta_\lambda \colon f_*\mathcal{A}^q \longrightarrow f_*\mathcal{A}^q$$

sum to the identity. Furthermore,

$$\operatorname{Supp}(f_*\theta_\lambda) \subset f\left(\operatorname{Supp}(\theta_\lambda)\right) \subset V_\lambda,$$

where the first set on the right is closed because f is a closed map. Thus $f_*\mathcal{A}^q$ is fine. Since Y is paracompact and Hausdorff, it is $\Gamma(Y, -)$-acyclic by Proposition II.3.63 (*iii*). This proves property (P3).

Hence, in this situation, the morphisms in the Leray five-term exact sequence may be constructed and computed using the resolution $\mathcal{A}^\bullet$ itself. □

In Chapter I, we encountered vector bundles over topological spaces and smooth vector bundles over C^∞ manifolds. Their sheaf-theoretic counterpart on a ringed space is the notion of a *locally free module*. We conclude this subsection by introducing locally free modules and the related notion of *flatness*.

Definition II.3.103. Let $(X, \mathcal{R}_X)$ be a ringed space. An $\mathcal{R}_X$-module $\mathcal{E}$ is called *locally free* if every point $x \in X$ has an open neighborhood U such that

$$\mathcal{E}|_U \cong \bigoplus_{a \in A} \mathcal{R}_X|_U$$

as $\mathcal{R}_X|_U$-modules for some index set A. It is called *finite locally free*, or *locally free of finite rank*, if the index set A may be chosen finite around every point. It is called *locally free of rank* r if one may take

$$A = \{1, 2, \ldots, r\}$$

around every point ($r \in \mathbb{Z}_{\geq 0}$). A locally free $\mathcal{R}_X$-module of rank one is also called an *invertible $\mathcal{R}_X$-module*. □

For a locally ringed space with nonzero stalks, the local rank of a finite locally free module $\mathcal{E}$ is $r(x) := \dim_{\kappa(x)}(\mathcal{E}_x \otimes_{\mathcal{R}_{X,x}} \kappa(x))$. On a trivializing neighborhood with r basis sections this dimension is r at every point, so the local rank is locally constant. Thus finite rank here is a local condition: on a disconnected space it need not have one constant value, or even a uniform bound. The phrase *of rank* r fixes that value on all of X.

The preceding definition is the sheaf-theoretic analogue of the local triviality of a vector bundle. Indeed, let $\mathcal{E}$ be a locally free $\mathcal{R}_X$-module of rank r. After choosing an open covering $\mathcal{U} = \{U_j\}_{j\in J}$ of X and isomorphisms

$$\tau_j \colon \mathcal{E}|_{U_j} \xrightarrow{\sim} \big(\mathcal{R}_X|_{U_j}\big)^{\oplus r},$$

the changes of trivialization

$$g_{jk} := \tau_j \circ \tau_k^{-1}$$

on $U_j \cap U_k$ are represented by matrices

$$g_{jk} \in \mathrm{GL}_r\big(\mathcal{R}_X(U_j \cap U_k)\big)$$

satisfying

$$g_{jk} g_{k\ell} = g_{j\ell}$$

on $U_j \cap U_k \cap U_\ell$. Conversely, such transition data glue the trivial modules

$$\big(\mathcal{R}_X|_{U_j}\big)^{\oplus r}$$

to a locally free $\mathcal{R}_X$-module of rank r. Changing the local trivializations replaces the transition matrices by

$$g'_{jk} = \psi_j g_{jk} \psi_k^{-1}$$

for suitable

$$\psi_j \in \mathrm{GL}_r\big(\mathcal{R}_X(U_j)\big).$$

Thus locally free modules of rank r are described by $\mathrm{GL}_r(\mathcal{R}_X)$-valued Čech-type 1-cocycle data, in the same manner as vector bundles are described by transition functions in Lemma I.7.35. For $r > 1$, the groups GL_r are generally non-abelian, so these data do not form an ordinary Čech cohomology group of the kind introduced above.

When $r = 1$, the transition functions take values in the sheaf $\mathcal{R}_X^*$ of units of $\mathcal{R}_X$. Since $\mathcal{R}_X^*$ is abelian, the preceding description becomes the usual multiplicative Čech 1-cocycle description, and isomorphism classes of invertible $\mathcal{R}_X$-modules are identified with $\check{H}^1\big(X, \mathcal{R}_X^*\big)$ (Compare Example II.3.86).

In the geometric examples considered in this note, this correspondence has its familiar vector-bundle interpretation. If T is a topological space and $E \to T$ is a continuous complex vector bundle of rank r, then the sheaf $\mathcal{C}_T(E)$ of continuous sections of E is a locally free $\mathcal{C}_T$-module of rank r. Conversely, every locally free $\mathcal{C}_T$-module of rank r arises in this way from a continuous complex vector bundle. Morphisms of such locally free modules correspond to continuous, fiber-preserving maps between the corresponding vector bundles which are complex-linear on every fiber, without any assumption that the fiber rank is locally constant. Thus they generalize the vector-bundle morphisms of Chapter I, §7.7 by dropping the rank condition. Similarly, if M is a C^∞ manifold, the sheaf $\mathcal{C}^\infty_M(E)$ of C^∞ sections of a complex C^∞ vector bundle $E \to M$ is a locally free $\mathcal{C}^\infty_M$-module of finite rank, and every finite locally free $\mathcal{C}^\infty_M$-module of constant rank arises from a complex C^∞ vector bundle of that rank. If the local rank varies, use this correspondence on each open and closed constant-rank subset. The same interpretation of morphisms applies, with continuous maps replaced by C^∞ maps and with no locally constant-rank condition. In particular, invertible modules in these two settings correspond to complex line bundles. For examples and further discussion of the distinction between these classes of maps in the holomorphic setting, see Remark III.2.13.

Motivated by the duality of vector bundles, if $\mathcal{F}$ is a finite locally free $\mathcal{R}_X$-module, we also write

$$\mathcal{F}^* := \mathcal{F}^\vee = \mathcal{H}om_{\mathcal{R}_X}\big(\mathcal{F}, \mathcal{R}_X\big)$$

and call it the *dual* of $\mathcal{F}$. It is again a finite locally free $\mathcal{R}_X$-module of the same rank.

In particular, if $\mathcal{F}$ is invertible, then so is $\mathcal{F}^*$, and the natural evaluation morphism

$$\mathcal{F} \otimes_{\mathcal{R}_X} \mathcal{F}^* \longrightarrow \mathcal{R}_X$$

is an isomorphism. The set of isomorphism classes of invertible $\mathcal{R}_X$-modules is denoted by

$$\mathrm{Pic}(X, \mathcal{R}_X)$$

and is called the *Picard group* of the ringed space $(X, \mathcal{R}_X)$. Its group operation is induced by the tensor product:

$$[\mathcal{F}] + [\mathcal{G}] := \big[\mathcal{F} \otimes_{\mathcal{R}_X} \mathcal{G}\big].$$

The identity element is $[\mathcal{R}_X]$, and the inverse of $[\mathcal{F}]$ is $[\mathcal{F}^*]$. For this reason, the dual of an invertible module is also written

$$\mathcal{F}^{-1} := \mathcal{F}^*.$$

When the structure sheaf is clear from the context, we shall also write

$$\mathrm{Pic}(X) := \mathrm{Pic}(X, \mathcal{R}_X).$$

Under the correspondence above, the dual of the sheaf of sections of a vector bundle E is naturally identified with the sheaf of sections of the dual vector bundle E^*.

Remark II.3.104. Let $\mathcal{R}_X^*$ denote the sheaf of units of $\mathcal{R}_X$. Since $\mathrm{GL}_1(\mathcal{R}_X) = \mathcal{R}_X^*$, from the argument above, one has a canonical group isomorphism

$$\mathrm{Pic}(X, \mathcal{R}_X) \cong \check{H}^1(X, \mathcal{R}_X^*)$$

(See also [St, Tag 09NT and Tag 09V1]).

Here the symbol $*$ in $\mathcal{R}_X^*$ denotes the sheaf of units, whereas the same symbol in $\mathcal{F}^*$ denotes the dual module. This customary overlap of notation should cause no confusion, since the dual of $\mathcal{R}_X$ as an $\mathcal{R}_X$-module is canonically isomorphic to $\mathcal{R}_X$ itself.

In the geometric situations described above, one obtains

$$\mathrm{Pic}(T, \mathcal{C}_T) \cong \check{H}^1(T, \mathcal{C}_T^*)$$

for a topological space T, and

$$\mathrm{Pic}(M, \mathcal{C}_M^\infty) \cong \check{H}^1(M, (\mathcal{C}_M^\infty)^*)$$

for a C^∞ manifold M. By the correspondences described above, these groups classify continuous complex line bundles on T and complex C^∞ line bundles on M, respectively. For this reason, particularly in geometric and algebraic-geometric literature, an invertible module is often simply called a *line bundle*.

In complex analytic terminology, such an $\mathcal{O}_X^*$-valued Čech 1-cocycle is traditionally called *multiplicative Cousin data*. A closely related description in algebraic geometry is given by *Cartier divisors*. We shall return to these notions in a later chapter. □

For any $\mathcal{R}_X$-modules $\mathcal{F}$ and $\mathcal{E}$ and every $x \in X$, there is a natural isomorphism

$$(\mathcal{F} \otimes_{\mathcal{R}_X} \mathcal{E})_x \cong \mathcal{F}_x \otimes_{\mathcal{R}_{X,x}} \mathcal{E}_x.$$

Indeed, the tensor product sheaf is the sheafification of the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{R}_X(U)} \mathcal{E}(U),$$

and sheafification does not change stalks. Hence the left-hand side is naturally isomorphic to

$$\varinjlim_{U \in \mathcal{N}_X(\{x\})} \left(\mathcal{F}(U) \otimes_{\mathcal{R}_X(U)} \mathcal{E}(U)\right).$$

The homomorphisms

$$\mathcal{F}(U) \otimes_{\mathcal{R}_X(U)} \mathcal{E}(U) \longrightarrow \mathcal{F}_x \otimes_{\mathcal{R}_{X,x}} \mathcal{E}_x, \qquad s \otimes t \longmapsto s_x \otimes t_x,$$

are compatible with restrictions and therefore induce a natural homomorphism

$$(\mathcal{F} \otimes_{\mathcal{R}_X} \mathcal{E})_x \longrightarrow \mathcal{F}_x \otimes_{\mathcal{R}_{X,x}} \mathcal{E}_x.$$

This homomorphism is an isomorphism: every finite sum of pure tensors on the right-hand side can be represented on one common neighborhood of x, and every finite tensor relation holding between germs already holds after restriction to a sufficiently small common neighborhood of x (See also [St, Tag 01CB]).

If $\mathcal{E}$ is locally free, then $\mathcal{E}_x$ is a free and hence flat $\mathcal{R}_{X,x}$-module for every $x \in X$. Since exactness of sequences of $\mathcal{R}_X$-modules can be checked on stalks and tensor products commute with taking stalks, the functor

$$- \otimes_{\mathcal{R}_X} \mathcal{E} \colon (\mathsf{Mod}(\mathcal{R}_X)) \xrightarrow{\text{functor}} (\mathsf{Mod}(\mathcal{R}_X))$$

is exact.

This argument motivates the following definition.

Definition II.3.105. Let $(X, \mathcal{R}_X)$ be a ringed space. An $\mathcal{R}_X$-module $\mathcal{E}$ is called *flat* if the functor

$$- \otimes_{\mathcal{R}_X} \mathcal{E} \colon (\mathsf{Mod}(\mathcal{R}_X)) \xrightarrow{\text{functor}} (\mathsf{Mod}(\mathcal{R}_X))$$

is exact. □

By the argument above, every locally free $\mathcal{R}_X$-module is flat.

Remark II.3.106. The word "flat" in Definition II.3.105 refers to an algebraic exactness property of tensor products. It should not be confused with the notion of a G-flat vector bundle considered, for example, in Remark I.7.32. □

The flatness condition also gives a useful criterion for the exactness of pullback.

Proposition II.3.107. *Let*

$$(f, f^\sharp) \colon (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$$

be a morphism of ringed spaces. Regard $\mathcal{R}_X$ as an $f^{-1}\mathcal{R}_Y$-module through $f^\sharp$. If $\mathcal{R}_X$ is flat as an $f^{-1}\mathcal{R}_Y$-module, then

$$f^* \colon (\mathsf{Mod}(\mathcal{R}_Y)) \xrightarrow{\text{functor}} (\mathsf{Mod}(\mathcal{R}_X))$$

is exact.

Proof. The functor f^{-1} is exact by Proposition II.3.99 (i). The assertion therefore follows from $f^*\mathcal{G} = \mathcal{R}_X \otimes_{f^{-1}\mathcal{R}_Y} f^{-1}\mathcal{G}$ and the flatness of $\mathcal{R}_X$ as an $f^{-1}\mathcal{R}_Y$-module. □

We next give an application of the preceding homological formalism. Lemma I.8.5 gave a direct proof of the splitting of short exact sequences of C^∞ real vector bundles by choosing a metric and taking an orthogonal complement. The same argument, using a Hermitian metric, applies to complex vector bundles. The following gives an alternative sheaf-cohomological proof in the complex continuous and C^∞ categories.

Proposition II.3.108. *The following hold.*
(i) *Let T be a second-countable paracompact Hausdorff space, $E \to T$ a (continuous) vector bundle, and let $E' \subset E$ be a vector subbundle. Denote by $E'' := E/E'$ the quotient bundle and by $q \colon E \to E''$ the quotient morphism. Then the canonical short exact sequence*

$$0 \longrightarrow E' \longrightarrow E \xrightarrow{q} E'' \longrightarrow 0$$

splits continuously. Equivalently, there exists a vector bundle morphism $s \colon E'' \to E$ such that $q \circ s = \mathrm{id}_{E''}$, and consequently $E = E' \oplus s(E'')$.
(ii) *Let M be a C^∞ manifold, $E \to M$ a C^∞ vector bundle, and let $E' \subset E$ be a C^∞ vector subbundle. Denote by $E'' := E/E'$ the quotient bundle and by $q \colon E \to E''$ the quotient morphism. Then the canonical short exact sequence*

$$0 \longrightarrow E' \longrightarrow E \xrightarrow{q} E'' \longrightarrow 0$$

splits in the C^∞ category. Equivalently, there exists a C^∞ vector bundle morphism $s \colon E'' \to E$ such that $q \circ s = \mathrm{id}_{E''}$, and consequently $E = E' \oplus s(E'')$.

PROOF. We prove the two assertions simultaneously. In case (i), set

$$X := T, \qquad \mathcal{S}_X := \mathcal{C}_X,$$

and let $\mathcal{S}_X(F)$ denote the sheaf of continuous sections of a complex vector bundle $F \to X$. In case (ii), set

$$X := M, \qquad \mathcal{S}_X := \mathcal{C}^\infty_M,$$

and let $\mathcal{S}_X(F)$ denote the sheaf of C^∞ sections of a C^∞ vector bundle $F \to X$.

Set

$$\mathcal{E}' := \mathcal{S}_X(E'), \qquad \mathcal{E} := \mathcal{S}_X(E), \qquad \mathcal{E}'' := \mathcal{S}_X(E'').$$

The canonical sequence of section sheaves

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \xrightarrow{q} \mathcal{E}'' \longrightarrow 0$$

is exact. Indeed, this may be checked on stalks, where it follows from the local triviality of the vector bundles and the fact that $E'' = E/E'$.

Since $\mathcal{E}''$ is finite locally free, there is a natural isomorphism of functors

$$\mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', -\big) \cong (\mathcal{E}'')^* \otimes_{\mathcal{S}_X} -.$$

The module $(\mathcal{E}'')^*$ is locally free and hence flat. Therefore the functor

$$\mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', -\big) \colon (\mathsf{Mod}(\mathcal{S}_X)) \xrightarrow{\text{functor}} (\mathsf{Mod}(\mathcal{S}_X))$$

is exact.

Applying this functor to the preceding short exact sequence gives another short exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', \mathcal{E}'\big) \longrightarrow \mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', \mathcal{E}\big) \longrightarrow \mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', \mathcal{E}''\big) \longrightarrow 0,$$

where the last morphism is induced by postcomposition with $q \colon \mathcal{E} \to \mathcal{E}''$. The associated long exact cohomology sequence contains

$$\Gamma\left(X, \mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', \mathcal{E}\big)\right) \longrightarrow \Gamma\left(X, \mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', \mathcal{E}''\big)\right) \longrightarrow H^1\left(X, \mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', \mathcal{E}'\big)\right).$$

Moreover, one has a natural isomorphism

$$\mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', \mathcal{E}'\big) \cong \mathcal{S}_X\big((E'')^* \otimes E'\big).$$

This is a sheaf of $\mathcal{S}_X$-modules and is therefore fine and $\Gamma(X, -)$-acyclic by Example II.3.64 and Proposition II.3.63. Consequently, $H^1\left(X, \mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', \mathcal{E}'\big)\right) = 0$, from which it follows that the identity section

$$\mathrm{id}_{\mathcal{E}''} \in \Gamma\left(X, \mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', \mathcal{E}''\big)\right)$$

lifts to a section

$$s \in \Gamma\left(X, \mathcal{H}om_{\mathcal{S}_X}\big(\mathcal{E}'', \mathcal{E}\big)\right),$$

which proves both assertions. □

REMARK II.3.109 (For readers familiar with schemes). The main examples of ringed spaces in this note come from complex analysis rather than from algebraic geometry. Although *schemes* and the spaces considered here are both described by means of ringed spaces, the resulting ringed spaces are generally different.

For example, let $\Omega \subset \mathbb{C}$ be a domain and consider the ringed space $(\Omega, \mathcal{O}_\Omega)$. This ringed space is *not* a scheme. Indeed, every point of a scheme has an affine open neighborhood, and the underlying topological space of an affine scheme is compact. Consequently, every point of a scheme has a compact open neighborhood. On the other hand, Ω has no nonempty compact open subset. To see this, let $V \subset \Omega$ be compact and open. Since Ω is Hausdorff, the compact subset V is closed in Ω. Hence V is both open and closed in Ω. Since Ω is connected, it follows that $V = \Omega$ whenever V is nonempty. However, Ω itself is not compact (Indeed, if it were compact, then it would be closed in $\mathbb{C}$, because $\mathbb{C}$ is Hausdorff. Since Ω is also nonempty and open in the connected space $\mathbb{C}$, this would imply $\Omega = \mathbb{C}$, which is again not compact). Thus no point of Ω has a compact open neighborhood, and $(\Omega, \mathcal{O}_\Omega)$ cannot be a scheme.

Another point to keep in mind is that, in the analytic setting of these notes, rings of sections are not assumed to be Noetherian. For example, Exercise I.7 shows that $\mathcal{O}_{\mathbb{C}}(U)$ is not Noetherian when $U \subset \mathbb{C}$ is a simply connected domain.

Note that these differences do not affect the general sheaf-theoretic constructions discussed in this subsection. □

3.5 Cohomology over shrinking neighborhoods of a compact subset

The inverse image construction has an important cohomological application. Let X be a topological space, let $Z \subset X$ be a compact subset, and let

$$j_Z \colon Z \longrightarrow X$$

denote the inclusion. For every sheaf $\mathcal{F}$ of abelian groups on X, every open neighborhood V of Z in X, and every integer $q \geq 0$, restriction induces a canonical morphism

$$\mathrm{res}^q_{V,Z} \colon H^q(V, \mathcal{F}) \longrightarrow H^q\left(Z, j_Z^{-1}\mathcal{F}\right).$$

These morphisms are compatible with restrictions to smaller neighborhoods. Hence they induce a canonical morphism

$$\Theta^q_{Z,\mathcal{F}} \colon \varinjlim_{V \in \mathcal{N}_X(Z)} H^q(V, \mathcal{F}) \longrightarrow H^q\left(Z, j_Z^{-1}\mathcal{F}\right),$$

where $\mathcal{N}_X(Z)$ is ordered by reverse inclusion as in Example II.3.20.

REMARK II.3.110. In this note, it is important that we consider the inverse image sheaf $j_Z^{-1}\mathcal{F}$, rather than a pullback sheaf denoted by $j_Z^*\mathcal{F}$ (Note that, at this stage, X and Z are merely topological spaces and $j_Z \colon Z \to X$ is merely a continuous map. Thus the inverse

image sheaf $j_Z^{-1}\mathcal{F}$ is well-defined, whereas the notation $j_Z^*\mathcal{F}$ does not yet have a meaning: the latter requires compatible ringed space structures and a sheaf of modules. In later complex-analytic applications, X and Z will often carry natural structures as objects of $(\mathsf{LocRingSp}_{\mathbb{C}})$, and the inclusion j_Z will be a morphism of locally ringed spaces). Indeed, in many examples treated in this note, the pull-back j_Z^* of the structure sheaf of X is isomorphic to the structure sheaf of Z, whereas the inverse image j_Z^{-1} of the structure sheaf of X retains germs of ambient holomorphic functions along Z, and it is this latter type of information that will be important in the neighborhood theory developed later in this note. □

We introduce a concrete condition under which the canonical morphism above can be studied directly by means of finite Čech complexes.

Definition II.3.111. Let X be a topological space, let $Z \subset X$, and let $\mathcal{F}$ be a sheaf of abelian groups on X. We say that $\mathcal{F}$ satisfies the *cofinal finite Leray refinement condition along* Z if the following holds.

For every $V \in \mathcal{N}_X(Z)$ and every finite open covering $\mathcal{B} = \{B_\ell\}_{\ell\in L}$ of V, there exist an open neighborhood $N \in \mathcal{N}_X(Z)$ with $N \subset V$, a finite open covering $\mathcal{U} = \{U_j\}_{j\in J}$ of N, and a map $\rho\colon J \to L$ such that

$$U_j \subset B_{\rho(j)} \cap N$$

for every $j \in J$, and such that $\mathcal{U}$ is a Leray covering for $\mathcal{F}|_N$. □

The main theorem explained in this subsection is the following.

Theorem II.3.112. *Let X be a topological space satisfying the separation axiom T_3, $Z \subset X$ be a compact subset, and let*

$$j_Z\colon Z \longrightarrow X$$

be the inclusion. Let $\mathcal{F}$ be a sheaf of abelian groups on X satisfying the cofinal finite Leray refinement condition along Z. Then, for every integer $q \geq 0$, the canonical morphism

$$\Theta^q_{Z,\mathcal{F}}\colon \varinjlim_{V\in\mathcal{N}_X(Z)} H^q(V,\mathcal{F}) \longrightarrow H^q\left(Z, j_Z^{-1}\mathcal{F}\right)$$

is an isomorphism.

Remark II.3.113. The conclusion of Theorem II.3.112 is known under weaker assumptions and without the finite Leray refinement condition. More precisely, it suffices that Z be compact and that any two distinct points of Z admit disjoint open neighborhoods in X. See [St, Tag 09V3].

We retain the stronger hypotheses above because they give a direct proof using the finite Čech formalism developed in this chapter and arise naturally in the complex-analytic applications considered later.

The separation axiom T_3 is also satisfied in the principal topological situations considered in this note. For example, a CW complex is paracompact and Hausdorff by the facts recalled in § 7.3; hence it satisfies T_4, and therefore T_3, by Lemma I.3.12. □

In what follows, for a finite open covering $\mathcal{U} = \{U_j\}_{j\in J}$ of an open neighborhood of Z, we write

$$\mathcal{U}|_Z := \{U_j \cap Z\}_{j\in J},$$

where empty members may be omitted.

We shall also use the naturality of the comparison morphisms with respect to restriction to Z. More precisely, if $\mathcal{U} = \{U_j\}_{j\in J}$ is an open covering of an open neighborhood V of Z, then the diagram

$$\begin{array}{ccc} \check{H}^q(\mathcal{U}, \mathcal{F}) & \xrightarrow{\kappa^q_{\mathcal{U}}} & H^q(V, \mathcal{F}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{res}^q_{V,Z}} \\ \check{H}^q\left(\mathcal{U}|_Z, j_Z^{-1}\mathcal{F}\right) & \xrightarrow{\kappa^q_{\mathcal{U}|_Z}} & H^q\left(Z, j_Z^{-1}\mathcal{F}\right) \end{array}$$

is commutative. Here the horizontal morphisms are those introduced in Theorem II.3.79. The commutativity follows from the functorial construction of the comparison morphisms described in the proof of that theorem.

The proof of Theorem II.3.112 uses two elementary principles. First, finitely many open sets can be shrunk simultaneously so that finitely many prescribed conditions on their intersections hold (Lemma II.3.114). Second, a finite amount of Čech data with values in $j_Z^{-1}\mathcal{F}$ can be represented, after refinement, by sections of $\mathcal{F}$ on an open neighborhood of Z (Lemma II.3.115).

We record these principles before proving the theorem.

Lemma II.3.114. *Let X be a topological space satisfying the separation axiom T_3, let $Z \subset X$ be compact, and let $\{U_j\}_{j\in J}$ be a finite family of open subsets of X such that*

$$Z \subset \bigcup_{j\in J} U_j.$$

Fix an integer $p \geq 0$. For every tuple $(j_0, j_1, \ldots, j_p) \in J^{p+1}$, suppose that an open subset

$$W_{j_0 j_1 \ldots j_p} \subset U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_p}$$

is given and satisfies

$$W_{j_0 j_1 \ldots j_p} \cap Z = U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_p} \cap Z.$$

Then there exist open subsets $U'_j \subset U_j$ for each $j \in J$ such that

$$Z \subset \bigcup_{j\in J} U'_j, \qquad \overline{U'_j} \subset U_j \quad (j \in J),$$

and

$$U'_{j_0} \cap U'_{j_1} \cap \cdots \cap U'_{j_p} \subset W_{j_0 j_1 \ldots j_p}$$

for every ordered tuple $(j_0, j_1, \ldots, j_p) \in J^{p+1}$.

PROOF. The case $p = 0$ is established in Mini-Exercise I.3.10 (ii) in Chapter I. Parts (iii) and (iv) of the same Mini-Exercise prove the case $p = 1$ by starting from the case $p = 0$ and successively removing the closed subsets on which one of the prescribed intersection conditions fails. The removal procedure used there is not specific to intersections of two open subsets: For any fixed integer $p \geq 1$, the same procedure may be applied successively to the finitely many tuples in J^{p+1}. It therefore proves the assertion for arbitrary p without any essentially new argument.

For completeness, although the argument is essentially a repetition of Mini-Exercise I.3.10 (iii)–(iv), we here explain explicitly how the same procedure works in the case $p = 2$.

Applying the case $p = 0$ to the open subsets

$$W_{jjj} \subset U_j \qquad (j \in J),$$

we obtain open subsets $V_j \subset W_{jjj}$ such that $Z \subset \bigcup_{j\in J} V_j$ and $\overline{V_j} \subset U_j$ for each $j \in J$.

Consider a triple $(j, k, \ell) \in J^3$ for which

$$V_j \cap V_k \cap V_\ell \not\subset W_{jk\ell}.$$

Set

$$T_{jk\ell} := \overline{V_j \cap V_k \cap V_\ell \setminus W_{jk\ell}}.$$

Since $W_{jk\ell}$ is open, one has

$$T_{jk\ell} \subset \overline{V_j} \cap \overline{V_k} \cap \overline{V_\ell} \setminus W_{jk\ell}.$$

We claim that $T_{jk\ell} \cap Z = \emptyset$. Indeed, if $z \in T_{jk\ell} \cap Z$, then

$$z \in \overline{V_j} \cap \overline{V_k} \cap \overline{V_\ell} \cap Z \subset U_j \cap U_k \cap U_\ell \cap Z.$$

By the hypothesis on $W_{jk\ell}$, this implies $z \in W_{jk\ell} \cap Z$, contrary to $T_{jk\ell} \subset X \setminus W_{jk\ell}$.

We may therefore replace V_j with $V_j \setminus T_{jk\ell}$. This is an open subset of V_j, removes no point of Z, and hence does not destroy the covering property. Moreover,

$$\overline{V_j \setminus T_{jk\ell}} \subset \overline{V_j} \subset U_j,$$

and

$$(V_j \setminus T_{jk\ell}) \cap V_k \cap V_\ell \subset W_{jk\ell}.$$

Since all subsequent modifications consist only of shrinking the open subsets, none of the inclusions already obtained is destroyed.

As the set J^3 is finite, repeating this operation for every ordered triple proves the assertion for $p = 2$.

For an arbitrary integer $p \geq 0$, one proceeds in exactly the same manner, by considering

$$T_{\mathbf{j}} := \overline{V_{j_0} \cap V_{j_1} \cap \cdots \cap V_{j_p} \setminus W_{j_0 j_1 \ldots j_p}}$$

for every tuple $\mathbf{j} = (j_0, j_1, \ldots, j_p) \in J^{p+1}$: As the same argument gives $T_{\mathbf{j}} \cap Z = \emptyset$, removing $T_{\mathbf{j}}$ from one of the open subsets occurring in the intersection enforces the required inclusion

for $\mathbf{j}$ without destroying the covering property or any condition already obtained. Since J^{p+1} is finite, the procedure terminates. See also [St, Tag 0CY5], where a version under weaker separation assumptions, with the weaker closure condition $\overline{U'_j} \cap Z \subset U_j$, is proved in the same manner. □

LEMMA II.3.115. *Let X be a topological space satisfying the separation axiom T_3, let $Z \subset X$ be compact, let*

$$j_Z \colon Z \longrightarrow X$$

be the inclusion, and let $\mathcal{F}$ be a sheaf of abelian groups on X.
(i) Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a finite open covering of Z, let $p \geq 0$, and let

$$c \in \check{C}^p\left(\mathcal{W}, j_Z^{-1}\mathcal{F}\right).$$

Then there exist an open neighborhood $N \in \mathcal{N}_X(Z)$, a finite open covering $\mathcal{V} = \{V_a\}_{a \in A}$ of N, a refinement map $\rho \colon A \to I$ from $\mathcal{V}|_Z$ to $\mathcal{W}$, and a cochain

$$\widetilde{c} \in \check{C}^p(\mathcal{V}, \mathcal{F})$$

such that

$$\widetilde{c}|_Z = \rho^* c.$$

If $\mathcal{W}$ is a refinement of $\mathcal{B}|_Z$ for some finite open covering $\mathcal{B} = \{B_\ell\}_{\ell \in L}$ of an open neighborhood of Z, then $\mathcal{V}$ may be chosen to refine $\mathcal{B}$.
(ii) Let $\mathcal{V} = \{V_a\}_{a \in A}$ be a finite open covering of an open neighborhood N of Z, and let

$$\gamma \in \check{C}^p(\mathcal{V}, \mathcal{F}).$$

Suppose that $\gamma|_Z = 0$ in $\check{C}^p\left(\mathcal{V}|_Z, j_Z^{-1}\mathcal{F}\right)$. Then there exist open subsets $V'_a \subset V_a$ for each $a \in A$ such that

$$Z \subset N' := \bigcup_{a \in A} V'_a \qquad \text{and} \qquad \gamma|_{\mathcal{V}'} = 0 \in \check{C}^p(\mathcal{V}', \mathcal{F}),$$

where $\mathcal{V}' := \{V'_a\}_{a \in A}$.
(iii) In (i), if $c \in \check{Z}^p\left(\mathcal{W}, j_Z^{-1}\mathcal{F}\right)$, then, after replacing N and $\mathcal{V}$ by smaller ones if necessary, the cochain $\widetilde{c}$ may be chosen so that $\widetilde{c} \in \check{Z}^p(\mathcal{V}, \mathcal{F})$.

PROOF. We first prove (i). For every tuple $\mathbf{i} = (i_0, i_1, \ldots, i_p) \in I^{p+1}$, set

$$W_{\mathbf{i}} := W_{i_0} \cap W_{i_1} \cap \cdots \cap W_{i_p},$$

and denote by

$$c_{\mathbf{i}} \in (j_Z^{-1}\mathcal{F})(W_{\mathbf{i}})$$

the corresponding component of c.

In the argument below, a further refinement of $\mathcal{W}$ is needed, since $j_Z^{-1}\mathcal{F}$ is obtained by sheafifying the inverse image presheaf. Consequently, the section $c_{\mathbf{i}}$ need not be represented

by a single section of $\mathcal{F}$ on an open neighborhood in X of the whole set $W_{\mathbf{i}}$. It is only locally represented in this way after $W_{\mathbf{i}}$ has been covered by smaller open subsets.

We shall therefore first choose, on a suitable open covering of each $W_{\mathbf{i}}$, sections of $\mathcal{F}$ representing the local restrictions of $c_{\mathbf{i}}$. We then apply Lemma II.3.116 below to obtain a finite refinement of $\mathcal{W}$ such that each of its $(p+1)$-fold intersections is contained in one of the open subsets on which a chosen local representative is available. Finally, Lemma II.3.114 allows us to extend this covering to a covering of an open neighborhood of Z in such a way that the corresponding intersections in X are contained in the domains of those local representatives. Their restrictions then form the required cochain $\widetilde{c}$.

For every $i \in I$, choose an open subset

$$\widetilde{W}_i \subset X$$

such that

$$W_i = \widetilde{W}_i \cap Z.$$

When $\mathcal{W}$ refines $\mathcal{B}|_Z$ for some $\mathcal{B} = \{B_\ell\}_{\ell \in L}$, fix a refinement map $\tau\colon I \to L$ and choose $\widetilde{W}_i$ so that, in addition, $\widetilde{W}_i \subset B_{\tau(i)}$. By the local description of the sheafification of the inverse image presheaf, for every tuple $\mathbf{i} \in I^{p+1}$, we may choose a nonempty index set $\Lambda_{\mathbf{i}}$ and an open covering

$$W_{\mathbf{i}} = \bigcup_{\lambda \in \Lambda_{\mathbf{i}}} W_{\mathbf{i},\lambda}$$

with the following property: For every $\lambda \in \Lambda_{\mathbf{i}}$, there exist an open subset

$$\widetilde{W}_{\mathbf{i},\lambda} \subset \widetilde{W}_{i_0} \cap \widetilde{W}_{i_1} \cap \cdots \cap \widetilde{W}_{i_p}$$

and a section

$$\widetilde{c}_{\mathbf{i},\lambda} \in \mathcal{F}\left(\widetilde{W}_{\mathbf{i},\lambda}\right)$$

such that

$$\widetilde{W}_{\mathbf{i},\lambda} \cap Z = W_{\mathbf{i},\lambda} \qquad \text{and} \qquad c_{\mathbf{i}}|_{W_{\mathbf{i},\lambda}} = \widetilde{c}_{\mathbf{i},\lambda}|_{W_{\mathbf{i},\lambda}}$$

(If $W_{\mathbf{i}} = \emptyset$, there is no nontrivial local condition to impose on this component. For uniformity of notation, set

$$\Lambda_{\mathbf{i}} := \{0\}, \qquad W_{\mathbf{i},0} := \emptyset, \qquad \widetilde{W}_{\mathbf{i},0} := \emptyset,$$

and let $\widetilde{c}_{\mathbf{i},0} \in \mathcal{F}(\emptyset)$ be the unique section (recall Remark II.3.51) in what follows).

By Lemma II.3.116 below, one has a finite open covering $\mathcal{W}' = \{W'_a\}_{a \in A}$ of Z and a refinement map $\rho\colon A \to I$ such that $\overline{W'_a} \subset W_{\rho(a)}$ for each $a \in A$ and that, for every ordered tuple $\mathbf{a} = (a_0, a_1, \ldots, a_p) \in A^{p+1}$, the intersection

$$W'_{a_0} \cap W'_{a_1} \cap \cdots \cap W'_{a_p}$$

is contained in one of the open subsets $W_{\rho(a_0)\rho(a_1)\ldots\rho(a_p),\lambda}$.

For every $a \in A$, choose an open subset $P_a \subset X$ such that $P_a \cap Z = W'_a$ and $P_a \subset \widetilde{W}_{\rho(a)}$. For every tuple $\mathbf{a} = (a_0, a_1, \ldots, a_p)$, choose an index

$$\lambda(\mathbf{a}) \in \Lambda_{\rho(a_0)\rho(a_1)\ldots\rho(a_p)}$$

such that

$$W'_{a_0} \cap W'_{a_1} \cap \cdots \cap W'_{a_p} \subset W_{\rho(a_0)\rho(a_1)\ldots\rho(a_p),\lambda(\mathbf{a})}.$$

Set

$$Q_{\mathbf{a}} := P_{a_0} \cap P_{a_1} \cap \cdots \cap P_{a_p} \cap \widetilde{W}_{\rho(a_0)\rho(a_1)\ldots\rho(a_p),\lambda(\mathbf{a})}.$$

Then

$$Q_{\mathbf{a}} \cap Z = P_{a_0} \cap P_{a_1} \cap \cdots \cap P_{a_p} \cap Z.$$

Applying Lemma II.3.114 to the finite family $\{P_a\}_{a\in A}$, we obtain open subsets

$$V_a \subset P_a \qquad (a \in A)$$

which still cover Z and satisfy

$$V_{a_0} \cap V_{a_1} \cap \cdots \cap V_{a_p} \subset Q_{\mathbf{a}}$$

for every ordered tuple $\mathbf{a} \in A^{p+1}$.

Set

$$N := \bigcup_{a\in A} V_a, \qquad \mathcal{V} := \{V_a\}_{a\in A}.$$

For every tuple $\mathbf{a} = (a_0, a_1, \ldots, a_p)$, the section

$$\widetilde{c}_{\rho(a_0)\rho(a_1)\ldots\rho(a_p),\lambda(\mathbf{a})}$$

may be restricted to $V_{a_0} \cap V_{a_1} \cap \cdots \cap V_{a_p}$. These restrictions define a cochain

$$\widetilde{c} \in \check{C}^p(\mathcal{V}, \mathcal{F}).$$

By construction, $\widetilde{c}|_Z = \rho^* c$, which proves the first assertion of (i). If $\mathcal{W}$ refines $\mathcal{B}|_Z$, then

$$V_a \subset P_a \subset \widetilde{W}_{\rho(a)} \subset B_{\tau(\rho(a))}.$$

Thus $\mathcal{V}$ refines $\mathcal{B}$, which proves the final assertion of (i).

We next prove (ii). For every tuple $\mathbf{a} = (a_0, a_1, \ldots, a_p) \in A^{p+1}$, the component

$$\gamma_{\mathbf{a}} \in \mathcal{F}\left(V_{a_0} \cap V_{a_1} \cap \cdots \cap V_{a_p}\right)$$

has zero restriction to

$$V_{a_0} \cap V_{a_1} \cap \cdots \cap V_{a_p} \cap Z.$$

By the definition of the inverse image sheaf, there exists an open subset

$$Q_{\mathbf{a}} \subset V_{a_0} \cap V_{a_1} \cap \cdots \cap V_{a_p}$$

such that

$$Q_{\mathbf{a}} \cap Z = V_{a_0} \cap V_{a_1} \cap \cdots \cap V_{a_p} \cap Z \qquad \text{and} \qquad \gamma_{\mathbf{a}}|_{Q_{\mathbf{a}}} = 0.$$

Applying Lemma II.3.114 to the finite family $\{V_a\}_{a\in A}$, we obtain open subsets $V'_a \subset V_a$ which still cover Z and satisfy

$$V'_{a_0} \cap V'_{a_1} \cap \cdots \cap V'_{a_p} \subset Q_{\mathbf{a}}.$$

Hence $\gamma|_{\mathcal{V}'} = 0$, which proves (ii).

Finally, suppose that $c \in \check{C}^p\left(\mathcal{W}, j_Z^{-1}\mathcal{F}\right)$ is a cocycle. After applying (i), the cochain

$$\check{\delta}(\widetilde{c}) \in \check{C}^{p+1}(\mathcal{V}, \mathcal{F})$$

restricts to $\rho^*(\check{\delta}(c)) = 0$ on Z. Applying (ii) to $\check{\delta}(\widetilde{c})$ and shrinking $\mathcal{V}$ once more around Z, we obtain $\check{\delta}(\widetilde{c}) = 0$, which proves (iii). □

Lemma II.3.116. *Let Z be a compact Hausdorff topological space and $\mathcal{U} = \{U_i\}_{i\in I}$ be a finite open covering of Z. Fix an integer $p \geq 0$. For every tuple $(i_0, i_1, \ldots, i_p) \in I^{p+1}$, let $\left\{W_{i_0 i_1 \ldots i_p, \lambda}\right\}_{\lambda \in \Lambda_{i_0 i_1 \ldots i_p}}$ be an open covering of $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}$, where each index set $\Lambda_{i_0 i_1 \ldots i_p}$ is assumed to be nonempty. Then there exist a finite open covering $\mathcal{V} = \{V_a\}_{a\in A}$ of Z and a map $\rho\colon A \to I$ such that*

$$\overline{V_a} \subset U_{\rho(a)} \qquad (a \in A),$$

where the closure is taken in Z, and such that, for every tuple $(a_0, a_1, \ldots, a_p) \in A^{p+1}$, there exists an index $\lambda \in \Lambda_{\rho(a_0)\rho(a_1)\ldots\rho(a_p)}$ for which

$$V_{a_0} \cap V_{a_1} \cap \cdots \cap V_{a_p} \subset W_{\rho(a_0)\rho(a_1)\ldots\rho(a_p), \lambda}.$$

Proof. This is the finite-covering case of [St, Tag 09UW]. The proof proceeds by induction on p. For $p = 0$, one takes a finite shrinking refinement subordinate to the prescribed open coverings. In the induction step, one first passes to suitable common refinements in order to treat tuples for which some indices coincide. For the remaining tuples, whose relevant indices are pairwise distinct, one subdivides one member of the covering into finitely many open subsets so that the offending intersection becomes subordinate to the prescribed covering. Since only finitely many tuples occur, this process terminates. We refer to the cited lemma for the complete argument. □

Proof of Theorem II.3.112. We first consider the case $q = 0$. Let $s \in \Gamma\left(Z, j_Z^{-1}\mathcal{F}\right)$. Regard s as a Čech 0-cocycle for the one-member covering $\{Z\}$. By Lemma II.3.115 (i) and (iii), there exist an open neighborhood N of Z, a finite open covering $\mathcal{V}$ of N, and a Čech 0-cocycle $\widetilde{s} \in \check{Z}^0(\mathcal{V}, \mathcal{F})$ whose restriction to Z is s. The sheaf condition glues the components of $\widetilde{s}$ to a section in $\Gamma(N, \mathcal{F})$ which restricts to s. Thus $\Theta^0_{Z,\mathcal{F}}$ is surjective.

On the other hand, let $t \in \Gamma(N, \mathcal{F})$ for some $N \in \mathcal{N}_X(Z)$, and suppose that $t|_Z = 0$. Apply Lemma II.3.115 (ii) to the one-member covering $\{N\}$ and the 0-cochain t. Then t vanishes on some smaller open neighborhood of Z. Hence t represents zero in $\varinjlim_{N\in\mathcal{N}_X(Z)} \Gamma(N, \mathcal{F})$, and $\Theta^0_{Z,\mathcal{F}}$ is injective.

We now fix an integer $q > 0$. To prove surjectivity, take $\eta \in H^q\left(Z, j_Z^{-1}\mathcal{F}\right)$. Since X satisfies T_3, it is Hausdorff. Hence Z is compact Hausdorff and therefore paracompact. By Theorem II.3.79 (iii), the canonical morphism

$$\kappa^q_Z\colon \check{H}^q\left(Z, j_Z^{-1}\mathcal{F}\right) \xrightarrow{\sim} H^q\left(Z, j_Z^{-1}\mathcal{F}\right)$$

is an isomorphism.

Choose a finite open covering $\mathcal{W} = \{W_i\}_{i\in I}$ of Z and a cocycle $c \in \check{Z}^q\left(\mathcal{W}, j_Z^{-1}\mathcal{F}\right)$ whose class represents $(\kappa_Z^q)^{-1}(\eta)$. Such a finite covering may be chosen because Z is compact. By Lemma II.3.115 (i) and (iii), there exist an open neighborhood N of Z, a finite open covering $\mathcal{V} = \{V_a\}_{a\in A}$ of N, and a cocycle $\widetilde{c} \in \check{Z}^q(\mathcal{V}, \mathcal{F})$ whose restriction to Z represents the same class as c.

Apply the cofinal finite Leray refinement condition along Z to N and $\mathcal{V}$. Then we obtain an open neighborhood $N' \subset N$ of Z and a finite Leray covering $\mathcal{U} = \{U_j\}_{j\in J}$ of N' which refines $\mathcal{V}|_{N'}$. Let $\widetilde{c}_{\mathcal{U}} \in \check{Z}^q(\mathcal{U}, \mathcal{F})$ be the cocycle induced by this refinement. By Theorem II.3.79 (ii), the class

$$[\widetilde{c}_{\mathcal{U}}] \in \check{H}^q(\mathcal{U}, \mathcal{F})$$

determines a class

$$\xi_{N'} := \kappa_{\mathcal{U}}^q\left([\widetilde{c}_{\mathcal{U}}]\right) \in H^q(N', \mathcal{F}).$$

By the naturality diagram above, the restriction of $\xi_{N'}$ to Z is η. Thus $\Theta_{Z,\mathcal{F}}^q$ is surjective.

We next prove injectivity. Let

$$\xi \in \varinjlim_{N\in\mathcal{N}_X(Z)} H^q(N, \mathcal{F})$$

and suppose that $\Theta_{Z,\mathcal{F}}^q(\xi) = 0$. Choose $N \in \mathcal{N}_X(Z)$ and a representative $\xi_N \in H^q(N, \mathcal{F})$ of ξ. Apply the cofinal finite Leray refinement condition along Z to N and the one-member covering $\{N\}$. After replacing N with a smaller neighborhood of Z, we may assume that N has a finite Leray covering $\mathcal{U} = \{U_j\}_{j\in J}$. By Theorem II.3.79 (ii), there exists a cocycle $c \in \check{Z}^q(\mathcal{U}, \mathcal{F})$ such that $\kappa_{\mathcal{U}}^q([c]) = \xi_N$.

Let

$$c|_Z \in \check{Z}^q\left(\mathcal{U}|_Z, j_Z^{-1}\mathcal{F}\right)$$

be the restricted cocycle. Its image in $H^q\left(Z, j_Z^{-1}\mathcal{F}\right)$ is zero. Since κ_Z^q is an isomorphism, the image of $[c|_Z]$ in $\check{H}^q\left(Z, j_Z^{-1}\mathcal{F}\right)$ is also zero. By the definition of Čech cohomology as an inductive limit over open coverings, there exist a finite open covering $\mathcal{W} = \{W_a\}_{a\in A}$ of Z, a refinement map $\rho\colon A \to J$ from $\mathcal{W}$ to $\mathcal{U}|_Z$, and a cochain $b \in \check{C}^{q-1}\left(\mathcal{W}, j_Z^{-1}\mathcal{F}\right)$ such that

$$\rho^*(c|_Z) = \check{\delta}(b).$$

Again, the covering $\mathcal{W}$ may be chosen finite because Z is compact. By Lemma II.3.115 (i), applied relative to the ambient covering $\mathcal{U}$, there exist an open neighborhood $N' \subset N$ of Z, a finite open covering $\mathcal{V} = \{V_\alpha\}_{\alpha\in B}$ of N' refining $\mathcal{U}|_{N'}$, and a cochain

$$\widetilde{b} \in \check{C}^{q-1}(\mathcal{V}, \mathcal{F})$$

whose restriction to Z is the pullback of b.

Let $c_{\mathcal{V}} \in \check{Z}^q(\mathcal{V}, \mathcal{F})$ be the refinement of c. Then

$$\left(c_{\mathcal{V}} - \check{\delta}(\widetilde{b})\right)\Big|_Z = 0.$$

Applying Lemma II.3.115 (*ii*) to the cochain $c_{\mathcal{V}} - \check{\delta}(\widetilde{b})$, and shrinking $\mathcal{V}$ once more around Z, we obtain

$$c_{\mathcal{V}} = \check{\delta}(\widetilde{b}).$$

Consequently, the restriction of ξ_N to a sufficiently small open neighborhood of Z is zero. Therefore ξ is zero in $\varinjlim_{N \in \mathcal{N}_X(Z)} H^q(N, \mathcal{F})$, which proves injectivity and completes the proof. □

Example II.3.117 (G-flat extensions to neighborhoods). Let X be a triangulable CW complex and $Z \subset X$ be a compact subcomplex with respect to a fixed triangulation. Let $G \subset \mathbb{C}^*$ be a subgroup; the principal examples to keep in mind are

$$G = \mathrm{U}(1) \qquad \text{and} \qquad G = \mathbb{C}^*.$$

Denote by $\underline{G}_X$ and $\underline{G}_Z$ the sheaves of locally constant G-valued functions on X and Z, respectively. For the inclusion $j_Z \colon Z \to X$, one has

$$j_Z^{-1} \underline{G}_X \cong \underline{G}_Z,$$

since, for every $z \in Z$, the stalk of $j_Z^{-1}\underline{G}_X$ at z is canonically identified with the stalk of $\underline{G}_X$ at z, and both are naturally isomorphic to G.

Recall from Remark I.7.32 that G-flat line bundles are classified, up to G-flat isomorphism, by the first cohomology group with coefficients in G. In the present setting, Proposition II.3.81 gives natural isomorphisms $H^1(V, G) \cong H^1(V, \underline{G}_V)$ for the spaces V considered below, and similarly for Z.

Let L be a G-flat line bundle on Z. If V is an open neighborhood of Z in X, a G-flat line bundle $\widetilde{L}$ on V such that

$$\widetilde{L}|_Z \cong_{G-\mathrm{flat}} L$$

will be called a *G-flat extension* of L to V. When $G = \mathrm{U}(1)$, we also call it a *unitary flat extension*. The conclusion of Theorem II.3.112, together with Remark II.3.113, applied to $\underline{G}_X$, gives

$$\varinjlim_{V \in \mathcal{N}_X(Z)} H^1(V, \underline{G}_V) \cong H^1(Z, \underline{G}_Z).$$

Consequently, every G-flat line bundle on Z admits a G-flat extension to some open neighborhood of Z in X. Moreover, if two G-flat extensions are given on possibly different open neighborhoods of Z, then their restrictions to some sufficiently small common open neighborhood of Z are G-flat isomorphic.

In the present CW-complex setting, this conclusion can also be seen more directly from the facts recalled in § 7.3. Indeed, Z admits a fundamental system of open neighborhoods V such that Z is a deformation retract of V. For every such V, the inclusion $Z \to V$ is a homotopy equivalence, and hence induces an isomorphism $H^1(V, G) \cong H^1(Z, G)$. Equivalently, $H^1(V, \underline{G}_V) \cong H^1(Z, \underline{G}_Z)$. It follows that, for every V in this fundamental system, L admits a G-flat extension to V, and its G-flat isomorphism class is uniquely determined by L (More concretely, if $r \colon V \to Z$ is a deformation retraction, then $r^*L \to V$ is the G-flat extension of L). □

Remark II.3.118. Suppose, for example, that X is a locally finite CW complex and that Z is a compact subcomplex. More generally, the following discussion applies whenever X is locally compact and metrizable on an open neighborhood of Z. Assume that $\mathcal{F}$ is a sheaf of abelian groups on X satisfying the cofinal finite Leray refinement condition along Z.

In this situation, the neighborhoods and the finite Leray coverings appearing in the proof of Theorem II.3.112 may be chosen recursively as follows: There exist a decreasing cofinal sequence

$$V_0 \supset V_1 \supset V_2 \supset \cdots$$

in $\mathcal{N}_X(Z)$, finite Leray coverings

$$\mathcal{V}_\nu = \{V_{\nu,j}\}_{j \in J_\nu}$$

of V_ν, and refinement maps

$$\rho_\nu \colon J_{\nu+1} \longrightarrow J_\nu$$

such that

$$\overline{V_{\nu+1}} \text{ is compact and contained in } V_\nu$$

and

$$\overline{V_{\nu+1,a}} \text{ is compact and contained in } V_{\nu,\rho_\nu(a)} \qquad (a \in J_{\nu+1})$$

for every $\nu \in \mathbb{Z}_{\geq 0}$.

Indeed, after choosing a decreasing countable fundamental system of relatively compact open neighborhoods of Z, one proceeds recursively. At each stage, choose a finite open covering of a smaller neighborhood of Z whose members have compact closures contained in members of the preceding Leray covering, and then apply the cofinal finite Leray refinement condition.

Suppose moreover that $\mathcal{F}$ is a sheaf of vector spaces and that, for every ν, every integer $r \geq 0$, and every tuple $(j_0, j_1, \ldots, j_r) \in J_\nu^{r+1}$, the space

$$\mathcal{F}\left(V_{\nu,j_0} \cap V_{\nu,j_1} \cap \cdots \cap V_{\nu,j_r}\right)$$

carries a natural Fréchet space structure, and that all the restriction morphisms under consideration are continuous linear maps. Then each $\check{C}^r\left(\mathcal{V}_\nu, \mathcal{F}\right)$ is a Fréchet space, since it is a finite product of Fréchet spaces, and the Čech differential is continuous.

Endow

$$\check{H}^q\left(\mathcal{V}_\nu, \mathcal{F}\right) = \check{Z}^q\left(\mathcal{V}_\nu, \mathcal{F}\right) / \check{B}^q\left(\mathcal{V}_\nu, \mathcal{F}\right)$$

with the quotient locally convex topology. This is an LCTVS, but it need not be Hausdorff. If it is Hausdorff, then $\check{B}^q\left(\mathcal{V}_\nu, \mathcal{F}\right)$ is closed in $\check{Z}^q\left(\mathcal{V}_\nu, \mathcal{F}\right)$, and hence $\check{H}^q\left(\mathcal{V}_\nu, \mathcal{F}\right)$ is a Fréchet space by Mini-Exercise I.6.42 (*iii*). Via the Leray isomorphism, the same topology may be transported to $H^q(V_\nu, \mathcal{F})$.

The refinement maps induce a countable inductive sequence of LCTVSs, and Theorem II.3.112 gives an isomorphism of vector spaces

$$H^q\left(Z, j_Z^{-1}\mathcal{F}\right) \cong \varinjlim_\nu H^q(V_\nu, \mathcal{F}).$$

Thus the chosen sequence and coverings realize this cohomology group as a countable inductive limit of LCTVSs. Endowing the inductive limit with the locally convex final topology gives an LCTVS structure, which may a priori depend on the choices made above.

If, in addition, all the spaces $H^q(V_\nu, \mathcal{F})$ are Hausdorff and the induced restriction morphisms

$$H^q(V_\nu, \mathcal{F}) \longrightarrow H^q(V_{\nu+1}, \mathcal{F})$$

form a strict inductive sequence of Fréchet spaces in the sense of Chapter I, then the resulting LCTVS is a strict countable LF-space. □

Remark II.3.119. Theorem II.3.112 concerns an inductive limit over shrinking open neighborhoods. We also record a different comparison involving projective systems. Unlike the preceding theorem, it is not a statement valid for arbitrary projective systems of sheaves. Under suitable exactness conditions, however, the comparison follows formally from the homological algebra of projective limits.

For simplicity, consider a projective system of sheaves of abelian groups on a topological space Z,

$$\cdots \longrightarrow \mathcal{F}_{n+1} \longrightarrow \mathcal{F}_n \longrightarrow \cdots \longrightarrow \mathcal{F}_0.$$

The category of sheaves of abelian groups on Z admits projective limits: In particular, the projective limit

$$\varprojlim_n \mathcal{F}_n$$

exists as a sheaf of abelian groups on Z. Moreover, projective limits of sheaves are computed sectionwise: for every open subset $U \subset Z$, there is a canonical isomorphism

$$\left(\varprojlim_n \mathcal{F}_n\right)(U) \cong \varprojlim_n \mathcal{F}_n(U). \tag{87}$$

Indeed, the sectionwise projective limit of the underlying presheaves is again a sheaf: the sheaf condition is expressed by products and equalizers, and projective limits commute with these limits. For details, see [St, Tag 009E and Tag 03CO].

For every integer $n \geq 0$, the canonical projection

$$\varprojlim_m \mathcal{F}_m \longrightarrow \mathcal{F}_n$$

induces, by the functoriality of sheaf cohomology, a morphism

$$H^q\left(Z, \varprojlim_m \mathcal{F}_m\right) \longrightarrow H^q(Z, \mathcal{F}_n) \qquad (q \in \mathbb{Z}_{\geq 0}).$$

These morphisms are compatible with the transition morphisms of the projective system $\{H^q(Z, \mathcal{F}_n)\}_{n\geq 0}$. The universal property of the projective limit therefore gives a canonical comparison morphism

$$H^q\left(Z, \varprojlim_n \mathcal{F}_n\right) \longrightarrow \varprojlim_n H^q(Z, \mathcal{F}_n).$$

For $q = 0$, this comparison morphism is always an isomorphism. Indeed, it follows from (87) that

$$H^0\left(Z,\ \varprojlim_n \mathcal{F}_n\right) = \Gamma\left(Z,\ \varprojlim_n \mathcal{F}_n\right) \cong \varprojlim_n \Gamma(Z, \mathcal{F}_n) = \varprojlim_n H^0(Z, \mathcal{F}_n).$$

For higher cohomology groups, however, the sectionwise description alone does not determine whether the comparison morphism is an isomorphism.

By Corollary II.3.42, applied with $R = \mathbb{Z}$, the projective-limit functor

$$\varprojlim_n \colon \mathrm{Fun}\big((\mathbb{Z}_{\geq 0})^{\mathrm{op}}, (\mathsf{Ab})\big) \xrightarrow{\text{functor}} (\mathsf{Ab})$$

is left exact. Its right derived functors are denoted by

$$R^p \varprojlim_n \qquad (p \geq 0).$$

Recall that a projective system of abelian groups $\{A_n\}_{n\geq 0}$ satisfies the *Mittag–Leffler condition* if, for every integer $n \geq 0$, the images $\mathrm{Image}\big(A_m \to A_n\big)$ are independent of m for all sufficiently large $m \geq n$. For example, this condition holds if all the transition morphisms $A_{n+1} \longrightarrow A_n$ are surjective. For a projective system indexed by $\mathbb{Z}_{\geq 0}$, the Mittag–Leffler condition implies

$$R^1 \varprojlim_n A_n = 0$$

(Refer to [St, Tag 02MY and Tag 07KV]).

More generally, let

$$\{C_n^\bullet\}_{n\geq 0}$$

be a projective system of bounded-below cochain complexes of abelian groups. Suppose that, for every integer p, the projective system $\{C_n^p\}_{n\geq 0}$ satisfies the Mittag–Leffler condition. Then the termwise projective-limit complex $\varprojlim_n C_n^\bullet$ computes the derived projective limit, and there is a canonical short exact sequence

$$0 \longrightarrow R^1 \varprojlim_n H^{q-1}(C_n^\bullet) \longrightarrow H^q\left(\varprojlim_n C_n^\bullet\right) \longrightarrow \varprojlim_n H^q(C_n^\bullet) \longrightarrow 0. \tag{88}$$

See [St, Tag 015E, Tag 07KV, and Tag 07KY]. Consequently, if the projective system $\{H^{q-1}(C_n^\bullet)\}_{n\geq 0}$ also satisfies the Mittag–Leffler condition, then the canonical comparison morphism is an isomorphism:

$$H^q\left(\varprojlim_n C_n^\bullet\right) \cong \varprojlim_n H^q(C_n^\bullet).$$

For a fixed finite open covering $\mathcal{U}$ of Z, projective limits commute with the products occurring in each degree of the Čech complex (as both projective limits and products are limits, recall Proposition II.3.34). Hence there are canonical isomorphisms

$$\check{C}^p\left(\mathcal{U},\ \varprojlim_n \mathcal{F}_n\right) \cong \varprojlim_n \check{C}^p\left(\mathcal{U}, \mathcal{F}_n\right).$$

To state explicitly a criterion which will be used later, fix an integer $q \geq 1$ and set

$$C_n^\bullet := \check{C}^\bullet\left(\mathcal{U}, \mathcal{F}_n\right) \qquad (n \geq 0).$$

The preceding degreewise isomorphisms are compatible with the Čech coboundary morphisms, and therefore give a canonical isomorphism of cochain complexes

$$\check{C}^\bullet\left(\mathcal{U},\ \varprojlim_n \mathcal{F}_n\right) \cong \varprojlim_n C_n^\bullet. \tag{89}$$

Assume first that, for every integer $p \geq 0$, the following holds:

(Assumption A): $\left\{\check{C}^p\left(\mathcal{U}, \mathcal{F}_n\right)\right\}_{n\geq 0}$ satisfies the Mittag–Leffler condition.

Applying the preceding short exact sequence to $C_n^\bullet = \check{C}^\bullet\left(\mathcal{U}, \mathcal{F}_n\right)$ and using (88) and (89), we obtain a canonical short exact sequence

$$0 \longrightarrow R^1 \varprojlim_n \check{H}^{q-1}\left(\mathcal{U}, \mathcal{F}_n\right) \longrightarrow \check{H}^q\left(\mathcal{U},\ \varprojlim_n \mathcal{F}_n\right) \longrightarrow \varprojlim_n \check{H}^q\left(\mathcal{U}, \mathcal{F}_n\right) \longrightarrow 0. \tag{90}$$

Assume, in addition, the following:

(Assumption B): $\left\{\check{H}^{q-1}\left(\mathcal{U}, \mathcal{F}_n\right)\right\}_{n\geq 0}$ satisfies the Mittag–Leffler condition.

Then

$$R^1 \varprojlim_n \check{H}^{q-1}\left(\mathcal{U}, \mathcal{F}_n\right) = 0,$$

and hence (90) gives a canonical isomorphism

$$\check{H}^q\left(\mathcal{U},\ \varprojlim_n \mathcal{F}_n\right) \cong \varprojlim_n \check{H}^q\left(\mathcal{U}, \mathcal{F}_n\right).$$

Thus, under **(Assumption A)** and **(Assumption B)**, if the covering $\mathcal{U}$ is chosen so that the canonical comparison morphisms from Čech cohomology to sheaf cohomology are isomorphisms for $\varprojlim_n \mathcal{F}_n$ and for every $\mathcal{F}_n$, then the canonical comparison morphism constructed above is an isomorphism:

$$H^q\left(Z, \varprojlim_n \mathcal{F}_n\right) \cong \varprojlim_n H^q\left(Z, \mathcal{F}_n\right).$$

□

Exercises for Chapter II

Exercise II.1. Fix the standard complex coordinate z on $\mathbb{C}$. For a domain $\Omega \subset \mathbb{C}$, let $\mathcal{O}_\Omega$ denote the sheaf of holomorphic functions on Ω, $\mathcal{A}^0_\Omega$ the sheaf of complex-valued C^∞ functions, and $\mathcal{A}^{0,1}_\Omega$ the sheaf defined by

$$\mathcal{A}^{0,1}_\Omega(U) := \left\{ g\, d\overline{z} \;\middle|\; g \in \mathcal{A}^0_\Omega(U) \right\}$$

for every open subset $U \subset \Omega$, with the usual restriction morphisms.
(i) Let $\varphi \in C^\infty_c(\mathbb{C})$. By applying the Lelong–Poincaré formula (Mini-Exercise I.9.5), show that the function

$$(P\varphi)(z) := -\frac{1}{\pi} \int_{\mathbb{C}} \varphi(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta} \right) d\lambda(\zeta)$$

is of class C^∞ on $\mathbb{C}$ and satisfies

$$(P\varphi)(0) = 0$$

and

$$\frac{\partial}{\partial \overline{z}} (P\varphi)(z) = \varphi(z),$$

where $d\lambda$ denotes the Lebesgue measure on $\mathbb{C}$.
(ii) Show that, for every domain $\Omega \subset \mathbb{C}$, the sequence

$$0 \longrightarrow \mathcal{O}_\Omega \longrightarrow \mathcal{A}^0_\Omega \xrightarrow{\overline{\partial}} \mathcal{A}^{0,1}_\Omega \longrightarrow 0$$

is exact in $(\mathsf{Sheaf}(\Omega, (\mathsf{Ab})))$.
(iii) Show that there is a canonical isomorphism

$$H^1(\Omega, \mathcal{O}_\Omega) \cong \frac{\mathcal{A}^{0,1}_\Omega(\Omega)}{\mathrm{Image}\big(\overline{\partial} \colon \mathcal{A}^0_\Omega(\Omega) \longrightarrow \mathcal{A}^{0,1}_\Omega(\Omega)\big)}.$$

Exercise II.2. Let

$$\Omega := \left\{ z \in \mathbb{C} \;\middle|\; \frac{1}{2} < |z| < 2 \right\}.$$

Let $\mathcal{O}_\Omega$ denote the sheaf of holomorphic functions on Ω and $\mathcal{O}^*_\Omega$ the sheaf of nowhere vanishing holomorphic functions. Consider the exponential exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_\Omega \longrightarrow \mathcal{O}_\Omega \xrightarrow{\exp(2\pi\sqrt{-1}\,\bullet)} \mathcal{O}^*_\Omega \longrightarrow 1$$

of sheaves of abelian groups, and let

$$\delta_{\exp} \colon H^0(\Omega, \mathcal{O}^*_\Omega) \longrightarrow H^1(\Omega, \underline{\mathbb{Z}}_\Omega)$$

be the induced connecting homomorphism. We identify

$$H^1(\Omega, \underline{\mathbb{Z}}_\Omega) \cong \mathbb{Z}$$

by using the radial deformation retraction

$$\Omega \longrightarrow \mathrm{U}(1), \qquad z \longmapsto \frac{z}{|z|},$$

and the positively oriented generator of $\pi_1(\mathrm{U}(1), 1)$ determined by the winding-number isomorphism introduced in Chapter I.
(*i*) Let $f \in H^0(\Omega, \mathcal{O}_\Omega^*)$. Show that, under the above identification, the integer corresponding to $\delta_{\exp}(f)$ is the negative of the winding number of the map

$$\mathrm{U}(1) \longrightarrow \mathrm{U}(1), \qquad z \longmapsto \frac{f(z)}{|f(z)|}.$$

(*ii*) Show that there does not exist a holomorphic function $F \in \mathcal{O}_\Omega(\Omega)$ such that $e^{F(z)} = z$ for every $z \in \Omega$.

Exercise II.3. Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X, and let $\mathcal{U} = \{U_j\}_{j\in J}$ and $\mathcal{V} = \{V_a\}_{a\in A}$ be open coverings of X. Assume that $\mathcal{V}$ is a refinement of $\mathcal{U}$, and let $\rho\colon A \to J$ be a refinement map. In what follows, we use the notation in Theorem II.3.79.
(*i*) Show that the induced morphism

$$\rho^*\colon \check{H}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^1(\mathcal{V}, \mathcal{F})$$

is injective.
(*ii*) Show that, for every open covering $\mathcal{U}$ of X, the canonical morphism

$$\check{H}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^1(X, \mathcal{F})$$

is injective.
(*iii*) Give an example showing that the morphism ρ^* in (*i*) need not be surjective.
(*iv*) Let $q \geq 0$. Suppose that the comparison morphism

$$\kappa^q_{\mathcal{U}}\colon \check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F})$$

is injective. Show that

$$\rho^*\colon \check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^q(\mathcal{V}, \mathcal{F})$$

is injective.
(*v*) Let $q \geq 0$. Suppose that $\mathcal{U}$ is a Leray covering for $\mathcal{F}$. Show that

$$\rho^*\colon \check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^q(\mathcal{V}, \mathcal{F})$$

is injective.

Chapter III

An introduction to complex analytic geometry and complex dynamics

In this chapter, we collect known results in complex analytic geometry and complex dynamics that will be used in the later chapters, drawing on Chapters I and II as needed.

Section III.1 develops the local theory on domains in complex Euclidean spaces, from differential forms and holomorphic functions to local analytic algebra and coherence. We also examine how these constructions behave under holomorphic changes of coordinates.

Section III.2 uses this local theory to study complex manifolds, holomorphic vector bundles, and complex spaces. With these preparations, we survey the complex analytic geometry of domains and complex spaces under various pseudoconvexity conditions.

Section III.3 turns to local dynamics near a fixed point in one complex variable. We focus on the iteration of holomorphic germs, especially linearization, small cycles, and invariant compact sets, together with the arithmetic conditions that arise in these questions.

1 Complex analysis on domains in complex Euclidean spaces

Let n be a positive integer. As in the case of one complex variable, a subset $\Omega \subset \mathbb{C}^n$ is called a *domain* if it is open and connected. Throughout this section, let $\Omega \subset \mathbb{C}^n$ be an open subset, and $z = (z^1, z^2, \dots, z^n)$ be the standard complex coordinates of $\mathbb{C}^n$. We let $x^j := \operatorname{Re} z^j$ and $y^j := \operatorname{Im} z^j$ for each $j \in \{1, 2, \dots, n\}$ in this section.

1.1 Forms on complex domains

Complexified tangent bundles on complex domains

Since $\mathbb{C}^n$ is a real Euclidean space of real dimension $2n$, the open subset Ω has a natural structure of a C^∞ manifold. In this note, the tensor product of the tangent bundle $T\Omega$ in this sense and the trivial complex line bundle on Ω is denoted by $T^{\mathbb{C}}\Omega$. This is a trivial complex

vector bundle of rank $2n$ on Ω, and its trivialization is given by the frame

$$\frac{\partial}{\partial z^1},\ \frac{\partial}{\partial z^2},\ \dots,\ \frac{\partial}{\partial z^n},\ \frac{\partial}{\partial \overline{z^1}},\ \frac{\partial}{\partial \overline{z^2}},\ \dots,\ \frac{\partial}{\partial \overline{z^n}}.$$

Here these vector fields are written in the same sense as in (35), that is, in the sense of the Wirtinger derivatives. More precisely, for each $j \in \{1, 2, \dots, n\}$, we set

$$\frac{\partial}{\partial z^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \sqrt{-1}\,\frac{\partial}{\partial y^j}\right), \qquad \frac{\partial}{\partial \overline{z^j}} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} + \sqrt{-1}\,\frac{\partial}{\partial y^j}\right).$$

We define complex vector subbundles $T^{1,0}\Omega$ and $T^{0,1}\Omega$ of $T^{\mathbb{C}}\Omega$ of rank n by requiring them to be spanned by

$$\frac{\partial}{\partial z^1},\ \frac{\partial}{\partial z^2},\ \dots,\ \frac{\partial}{\partial z^n},$$

and

$$\frac{\partial}{\partial \overline{z^1}},\ \frac{\partial}{\partial \overline{z^2}},\ \dots,\ \frac{\partial}{\partial \overline{z^n}},$$

respectively. In other words, their total spaces are

$$T^{1,0}\Omega := \Omega \times \bigoplus_{j=1}^{n} \mathbb{C}\,\frac{\partial}{\partial z^j}, \quad T^{0,1}\Omega := \Omega \times \bigoplus_{j=1}^{n} \mathbb{C}\,\frac{\partial}{\partial \overline{z^j}},$$

and these are regarded as complex vector bundles over Ω by the first projection. Then we have the direct sum decomposition

$$T^{\mathbb{C}}\Omega = T^{1,0}\Omega \oplus T^{0,1}\Omega. \tag{91}$$

Note that the complexified tangent bundle $T^{\mathbb{C}}\Omega = T\Omega \otimes_{\mathbb{R}} \mathbb{C}$ carries the natural complex conjugation defined by

$$\overline{v \otimes a} := v \otimes \overline{a} \qquad (v \in T\Omega,\ a \in \mathbb{C}).$$

After identifying $T\Omega$ with its natural real subbundle of $T^{\mathbb{C}}\Omega$, the fixed-point subbundle of this conjugation is precisely $T\Omega$. Moreover,

$$\overline{\left(\frac{\partial}{\partial z^j}\right)} = \frac{\partial}{\partial \overline{z^j}}, \qquad \overline{T^{1,0}\Omega} = T^{0,1}\Omega.$$

Note also that the decomposition (91) can be described by using the bundle isomorphism $J\colon T\Omega \to T\Omega$ defined by

$$J\left(\frac{\partial}{\partial x^j}\right) := \frac{\partial}{\partial y^j}, \quad J\left(\frac{\partial}{\partial y^j}\right) := -\frac{\partial}{\partial x^j} \tag{92}$$

for each $j \in \{1, 2, \dots, n\}$. This J naturally induces a bundle isomorphism $J\colon T^{\mathbb{C}}\Omega \to T^{\mathbb{C}}\Omega$. Then $T^{1,0}\Omega$ is the vector subbundle consisting of the $\sqrt{-1}$-eigenvectors of this induced bundle isomorphism, while $T^{0,1}\Omega$ is the vector subbundle consisting of its $-\sqrt{-1}$-eigenvectors.

Remark III.1.1. In some texts or contexts, the role played by the complex vector bundle $T^{1,0}\Omega \to \Omega$ in the arguments below is instead played by the real vector bundle $T\Omega \to \Omega$ equipped with the complex structure induced by J. More precisely, one regards the real vector bundle $T\Omega \to \Omega$ as a complex vector bundle by defining, for each point $p \in \Omega$ and each tangent vector $v \in T_p\Omega$, $\sqrt{-1} \cdot v := J(v)$. In this note, we denote this complex vector bundle by $H\Omega$:

$$H\Omega := (T\Omega, J).$$

There is a natural isomorphism of complex vector bundles on Ω

$$H\Omega \cong T^{1,0}\Omega.$$

Indeed, for each point $p \in \Omega$, the maps

$$H_p\Omega \ni v \mapsto v - \sqrt{-1}\, J(v) \in T^{1,0}_p\Omega, \qquad T^{1,0}_p\Omega \ni w \mapsto \frac{w + \overline{w}}{2} \in H_p\Omega$$

are mutually inverse complex-linear isomorphisms. On the other hand, this isomorphism should not be confused with an equality inside $T^{\mathbb{C}}\Omega$: After identifying $T\Omega$ with its natural image as a real subbundle of $T^{\mathbb{C}}\Omega$, the real subbundle $T\Omega$ and the complex subbundle $T^{1,0}\Omega$ are different subbundles of $T^{\mathbb{C}}\Omega$. □

(p, q)-forms and (p, q)-currents

For the vector bundles $T^{1,0}\Omega$ and $T^{0,1}\Omega$ of rank n, we use the following notation for the frames of their dual bundles. For the dual bundle $(T^{1,0}\Omega)^*$ of $T^{1,0}\Omega$, we denote by

$$dz^1,\ dz^2,\ \ldots,\ dz^n$$

the dual frame of

$$\frac{\partial}{\partial z^1},\ \frac{\partial}{\partial z^2},\ \cdots,\ \frac{\partial}{\partial z^n}.$$

For the dual bundle $(T^{0,1}\Omega)^*$ of $T^{0,1}\Omega$, we denote by

$$d\overline{z^1},\ d\overline{z^2},\ \ldots,\ d\overline{z^n}$$

the dual frame of

$$\frac{\partial}{\partial \overline{z^1}},\ \frac{\partial}{\partial \overline{z^2}},\ \cdots,\ \frac{\partial}{\partial \overline{z^n}}.$$

For non-negative integers p and q, we define a C^∞ complex vector bundle $\Lambda^{p,q}\Omega \to \Omega$ by

$$\Lambda^{p,q}\Omega := \left(\bigwedge^p (T^{1,0}\Omega)^*\right) \wedge \left(\bigwedge^q (T^{0,1}\Omega)^*\right).$$

If p or q is greater than n, then this is clearly a vector bundle of rank 0. In this sense, we mainly consider the case where $0 \leq p, q \leq n$. We set

$$\mathcal{A}^{p,q}(\Omega) := C^\infty(\Omega, \Lambda^{p,q}\Omega) = \mathcal{A}^0(\Omega, \Lambda^{p,q}\Omega).$$

An element of $\mathcal{A}^{p,q}(\Omega)$ is called a (p,q)-*form* on Ω. On the other hand, we set

$$\mathcal{K}^{p,q}(\Omega) := \mathcal{K}^0(\Omega, \Lambda^{p,q}\Omega).$$

An element of $\mathcal{K}^{p,q}(\Omega)$ is called a (p,q)-*current* on Ω. For a (p,q)-current $T \in \mathcal{K}^{p,q}(\Omega)$, its complex conjugate $\overline{T}$ is the (q,p)-current defined by

$$\overline{T}(\eta) := \overline{T(\overline{\eta})}$$

for every compactly supported smooth test form η of bidegree $(n-q, n-p)$ (See Mini-Exercise III.1.2 below for the compatibility of this definition with the complex conjugation of smooth forms).

Mini-Exercise III.1.2. Let $\alpha \in \mathcal{A}^{p,q}(\Omega)$. Regarding α as a (p,q)-current, still denoted by α, by

$$\alpha(\eta) := \int_\Omega \alpha \wedge \eta$$

for every compactly supported smooth form η of bidegree $(n-p, n-q)$, show that the conjugate current $\overline{\alpha}$ defined above coincides with the (q,p)-current induced by the smooth form $\overline{\alpha}$.

For every non-negative integer r, the decomposition (91) induces the direct sum decomposition

$$\bigwedge^r (T^{\mathbb{C}}\Omega)^* = \bigoplus_{\substack{p+q=r \\ 0\le p,q\le n}} \Lambda^{p,q}\Omega.$$

Consequently, we have

$$\mathcal{A}^r(\Omega) = \bigoplus_{\substack{p+q=r \\ 0\le p,q\le n}} \mathcal{A}^{p,q}(\Omega), \qquad \mathcal{K}^r(\Omega) = \bigoplus_{\substack{p+q=r \\ 0\le p,q\le n}} \mathcal{K}^{p,q}(\Omega).$$

For $j_1, j_2, \ldots, j_p \in \{1, 2, \ldots, n\}$ and $k_1, k_2, \ldots, k_q \in \{1, 2, \ldots, n\}$,

$$dz^{j_1} \wedge dz^{j_2} \wedge \cdots \wedge dz^{j_p} \wedge d\overline{z^{k_1}} \wedge d\overline{z^{k_2}} \wedge \cdots \wedge d\overline{z^{k_q}}$$

is a section of $\Lambda^{p,q}\Omega \to \Omega$, which is often abbreviated by using *multi-indices* $J = (j_1, j_2, \ldots, j_p) \in \{1, 2, \ldots, n\}^p$ and $K = (k_1, k_2, \ldots, k_q) \in \{1, 2, \ldots, n\}^q$ as

$$dz^J \wedge d\overline{z^K} := dz^{j_1} \wedge dz^{j_2} \wedge \cdots \wedge dz^{j_p} \wedge d\overline{z^{k_1}} \wedge d\overline{z^{k_2}} \wedge \cdots \wedge d\overline{z^{k_q}}.$$

Using this notation, a (p,q)-form α on Ω can be written as

$$\alpha = \sum_{\#J=p, \#K=q} f_{J,\overline{K}}(z^1, z^2, \ldots, z^n)\, dz^J \wedge d\overline{z^K}, \tag{93}$$

where $f_{J,\overline{K}} \colon \Omega \to \mathbb{C}$ is a C^∞ function (By a slight extension of the notation fixed in Notations and Conventions, when J is a finite ordered tuple, here $\#J$ denotes the number of its entries,

counted with repetitions. Thus $\#J = p$ and $\#K = q$ mean that $J \in \{1, 2, \dots, n\}^p$ and $K \in \{1, 2, \dots, n\}^q$, respectively). For $\sigma \in S_p$ and $J = (j_1, j_2, \dots, j_p) \in \{1, 2, \dots, n\}^p$, we set

$$\sigma(J) := (j_{\sigma(1)},\ j_{\sigma(2)},\ \dots,\ j_{\sigma(p)}).$$

Similarly, for $\tau \in S_q$ and $K = (k_1, k_2, \dots, k_q) \in \{1, 2, \dots, n\}^q$, we define $\tau(K)$. Then

$$dz^{\sigma(J)} \wedge d\overline{z^{\tau(K)}} = \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)\, dz^J \wedge d\overline{z^K}.$$

Therefore, when we write an expression such as (93), we can avoid redundant terms, for example, by one of the following conventions:

- we restrict the range of the summation by requiring $J = (j_1, j_2, \dots, j_p) \in \{1, 2, \dots, n\}^p$ and $K = (k_1, k_2, \dots, k_q) \in \{1, 2, \dots, n\}^q$ to satisfy $j_1 < j_2 < \cdots < j_p$ and $k_1 < k_2 < \cdots < k_q$; or
- we impose the condition $f_{\sigma(J),\overline{\tau(K)}} = \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)\, f_{J,\overline{K}}$ for any $\sigma \in S_p$ and $\tau \in S_q$ on the coefficient functions.

Whenever we restrict the range of multi-indices in this way, we explicitly say so. If $\#J = 1$, then $J = (j)$ for some $j \in \{1, 2, \dots, n\}$, and we sometimes omit the parentheses and simply write $J = j$, provided that no confusion can occur. For example, an expression of a $(p, 1)$-form based on (93) is written as

$$\alpha = \sum_{\#J=p} \sum_{k=1}^{n} f_{J,\overline{k}}(z)\, dz^J \wedge d\overline{z^k}.$$

(p, p)-forms, (p, p)-currents, and their semi-positivity

For $p \in \{1, 2, \dots, n\}$, a (p, p)-form ω on an open subset $\Omega \subset \mathbb{C}^n$ is said to be *real* if $\overline{\omega} = \omega$. The same terminology is used for (p, p)-currents. Let us write a (p, p)-form ω as in (93):

$$\omega = \sum_{\#J=\#K=p} f_{J,\overline{K}}(z)\, dz^J \wedge d\overline{z^K}.$$

Put $h_{J,\overline{K}} := (-\sqrt{-1})^{p^2} f_{J,\overline{K}}$. Then

$$\omega = (\sqrt{-1})^{p^2} \sum_{\#J=\#K=p} h_{J,\overline{K}}(z)\, dz^J \wedge d\overline{z^K}. \tag{94}$$

In this notation, we have

$$\overline{\omega} = (-\sqrt{-1})^{p^2} \sum_{\#J=\#K=p} \overline{h_{J,\overline{K}}(z)}\, d\overline{z^J} \wedge dz^K = (\sqrt{-1})^{p^2} \sum_{\#J=\#K=p} \overline{h_{J,\overline{K}}(z)}\, dz^K \wedge d\overline{z^J}.$$

Thus, if the range of J and K is chosen without redundancy, for example by using strictly increasing multi-indices, then ω is real if and only if $h_{K,\overline{J}}(z) = \overline{h_{J,\overline{K}}(z)}$ for every point $z \in \Omega$. In particular, when $p = 1$, for an expression of the form

$$\omega = \sqrt{-1} \sum_{j=1}^{n} \sum_{k=1}^{n} h_{j\overline{k}}(z) dz^j \wedge d\overline{z^k},$$

the correspondence

$$\omega \mapsto (h_{j\overline{k}})_{j,k=1}^{n} \tag{95}$$

gives a one-to-one correspondence between real $(1,1)$-forms on Ω and Hermitian matrix-valued C^∞ functions on Ω.

Remark III.1.3. Let us consider a real $(1,1)$-form on an open subset $\Omega \subset \mathbb{C}^n$ of the form

$$\omega = \sqrt{-1} \sum_{j=1}^{n} \sum_{k=1}^{n} h_{j\overline{k}}(z)\, dz^j \wedge d\overline{z^k},$$

and assume that the Hermitian matrix $(h_{j\overline{k}}(z))_{j,k=1}^{n}$ is positive definite at every point of Ω. Concerning the correspondence (95) and using the normalization

$$dz^j \wedge d\overline{z^k} = dz^j \otimes d\overline{z^k} - d\overline{z^k} \otimes dz^j$$

for the wedge product of 1-forms, for each $j, k \in \{1, 2, \ldots, n\}$, we have

$$\omega\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right) = 0, \quad \omega\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z^k}}\right) = \sqrt{-1}\, h_{j\overline{k}}, \quad \omega\left(\frac{\partial}{\partial \overline{z^j}}, \frac{\partial}{\partial \overline{z^k}}\right) = 0.$$

Thus the matrix $(h_{j\overline{k}})$ defines a C^∞ Hermitian metric h on $T^{1,0}\Omega$ by

$$h\left(\sum_{j=1}^{n} a_j \frac{\partial}{\partial z^j}, \sum_{k=1}^{n} b_k \frac{\partial}{\partial z^k}\right) = (a_1, a_2, \ldots, a_n) \begin{pmatrix} h_{1\overline{1}} & h_{1\overline{2}} & \ldots & h_{1\overline{n}} \\ h_{2\overline{1}} & h_{2\overline{2}} & \ldots & h_{2\overline{n}} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n\overline{1}} & h_{n\overline{2}} & \ldots & h_{n\overline{n}} \end{pmatrix} \begin{pmatrix} \overline{b_1} \\ \overline{b_2} \\ \vdots \\ \overline{b_n} \end{pmatrix}.$$

For $v, w \in T^{1,0}\Omega$, this relation can be written as

$$h(v, w) = -\sqrt{-1}\, \omega(v, \overline{w}).$$

The Hermitian metric h also induces a Hermitian metric $h^{\mathbb{C}}$ on $T^{\mathbb{C}}\Omega = T^{1,0}\Omega \oplus T^{0,1}\Omega$ by

$$h^{\mathbb{C}} := h \oplus \overline{h}.$$

We define a Riemannian metric g on $T\Omega$ by restricting this Hermitian metric to the natural real subbundle $T\Omega \subset T^{\mathbb{C}}\Omega$:

$$g := h^{\mathbb{C}}|_{T\Omega}$$

(Note that, for real tangent vectors $u, v \in T\Omega$, the value $h^{\mathbb{C}}(u, v)$ is real, and hence this restriction is a real symmetric bilinear form).

In this manner, a real $(1,1)$-form whose corresponding Hermitian matrices are positive definite corresponds to a C^∞ Hermitian metric on $T^{1,0}\Omega$, and it induces a Riemannian structure on Ω. □

Mini-Exercise III.1.4. Use the notation in Remark III.1.3. Show that, for each $j, k \in \{1, 2, \ldots, n\}$,

$$g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = 2\,\mathrm{Re}\, h_{j\overline{k}}, \quad g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\right) = 2\,\mathrm{Im}\, h_{j\overline{k}}, \quad g\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = 2\,\mathrm{Re}\, h_{j\overline{k}}.$$

Also show that

$$g(u, v) = \omega(u, Jv) \qquad (u, v \in T\Omega),$$

holds (J is as in (92)).

Definition III.1.5. Let n be a positive integer and let $p \in \{1, 2, \ldots, n\}$. Let $\Omega \subset \mathbb{C}^n$ be an open subset. Let ω be a real (p,p)-form on Ω and let $x \in \Omega$. We say that ω is *semi-positive* at x if, for any $(1,0)$-forms $a_1, a_2, \ldots, a_{n-p}$ defined on a neighborhood of x, there exists $c \in \mathbb{R}_{\geq 0}$ such that

$$\omega \wedge \bigwedge_{j=1}^{n-p} (\sqrt{-1} a_j \wedge \overline{a_j}) = c \bigwedge_{j=1}^{n} (\sqrt{-1} dz^j \wedge d\overline{z^j}) \quad \text{at } x.$$

If ω is semi-positive at every point $x \in \Omega$, then we say that ω is semi-positive on Ω and write $\omega \geq 0$. Similarly, a real (p,p)-current T on Ω is said to be a *semi-positive* (p,p)-current on Ω, and we write $T \geq 0$, if, for any smooth $(1,0)$-forms $a_1, a_2, \ldots, a_{n-p}$ on Ω,

$$T \wedge \bigwedge_{j=1}^{n-p} (\sqrt{-1} a_j \wedge \overline{a_j})$$

is a semi-positive measure on Ω. □

Remark III.1.6. When $1 < p < n$, several distinct positivity notions for real (p,p)-forms and (p,p)-currents are used in the literature, and some care is required with their terminology. The notion called semi-positive above is often called *weakly positive*, for example. In this note, however, we shall mainly apply this terminology when $p = 1$ or $p = n$. In these cases, the standard positivity notions in question coincide, and hence no ambiguity will arise. □

As follows from Lemma III.1.7 below, under the correspondence (95), real semi-positive $(1,1)$-forms on Ω correspond to positive semi-definite Hermitian matrix-valued C^∞ functions on Ω. Based on this observation, we also use the following terminology in this note: for a real $(1,1)$-form ω on Ω and a point $x \in \Omega$, we say that ω is *positive* at x, or write $\omega > 0$ at x, if the matrix $(h_{j\overline{k}}(x))_{j,k=1}^n$ in the notation of (95) is positive definite. If ω is positive at every point $x \in \Omega$, then we say that ω is positive on Ω and write $\omega > 0$.

LEMMA III.1.7. *Let n be a positive integer and let $p \in \{1, 2, \dots, n\}$. Let $\Omega \subset \mathbb{C}^n$ be an open subset. Let ω be a real (p,p)-form on Ω, and let $h_{J,\overline{K}}$ be the coefficient functions in the expression (94). For each point $x = (z_0^1, z_0^2, \dots, z_0^n) \in \Omega$, the following conditions are equivalent:*
(i) ω is semi-positive at x.
(ii) For any $n \times p$ matrix $M = (m_b^a)_{a,b}$, define the map $f_M \colon \mathbb{C}^p \to \mathbb{C}^n$ by

$$f_M(w^1, w^2, \dots, w^p) := \left(z_0^1 + \sum_{b=1}^{p} m_b^1 w^b,\ z_0^2 + \sum_{b=1}^{p} m_b^2 w^b,\ \dots,\ z_0^n + \sum_{b=1}^{p} m_b^n w^b \right).$$

Then there exists $c \in \mathbb{R}_{\geq 0}$ such that

$$f_M^* \omega = c \bigwedge_{j=1}^{p} (\sqrt{-1} dw^j \wedge d\overline{w^j}) \quad \text{at } 0.$$

Furthermore, when $p = 1$, these conditions are also equivalent to the following condition:
(iii) The matrix $(h_{j\overline{k}}(x))_{j,k=1}^n$ is positive semi-definite.

PROOF. We first prove $(i) \implies (ii)$. Assume that ω is semi-positive at x. Let $M = (m_b^a)_{a,b}$ be an $n \times p$ matrix. For $J = (j_1, j_2, \dots, j_p)$, we have

$$f_M^*(dz^{j_1} \wedge dz^{j_2} \wedge \cdots \wedge dz^{j_p}) = \bigwedge_{\nu=1}^{p} \left(\sum_{b=1}^{p} m_b^{j_\nu}\, dw^b \right) = \det M^J\, dw^1 \wedge dw^2 \wedge \cdots \wedge dw^p,$$

where $M^J := (m_b^{j_\nu})_{\nu,b=1}^p$. If $\operatorname{rank} M < p$, then $\det M^J = 0$ for every $J \in \{1, 2, \dots, n\}^p$, and hence $f_M^* \omega = 0$. Thus the assertion holds with $c = 0$.

Assume that $\operatorname{rank} M = p$. The columns of M span a p-dimensional complex vector subspace of $\mathbb{C}^n$. By a unitary change of the standard coordinates of $\mathbb{C}^n$, and by a change of basis of $\mathbb{C}^p$, it is enough to consider the case where f_M is the standard affine inclusion

$$(w^1, \dots, w^p) \mapsto (z_0^1 + w^1, \dots, z_0^p + w^p, z_0^{p+1}, \dots, z_0^n).$$

Indeed, these changes only multiply the relevant volume forms by positive real constants. In this case, the coefficient of $\bigwedge_{j=1}^p (\sqrt{-1} dw^j \wedge d\overline{w^j})$ in $f_M^* \omega$ is, up to a positive real constant, the coefficient of $\bigwedge_{j=1}^n (\sqrt{-1} dz^j \wedge d\overline{z^j})$ in

$$\omega \wedge \bigwedge_{j=p+1}^{n} (\sqrt{-1} dz^j \wedge d\overline{z^j})$$

at x. By the semi-positivity of ω at x, this coefficient is non-negative. This proves (ii).

Next we prove $(ii) \implies (i)$. Let $a_1, a_2, \dots, a_{n-p}$ be $(1,0)$-forms defined on a neighborhood of x. It is enough to look at their values at x. If $a_1, \dots, a_{n-p}$ are linearly dependent at x, then $\bigwedge_{j=1}^{n-p} (\sqrt{-1} a_j \wedge \overline{a_j})$ vanishes at x, and there is nothing to prove. Assume that they are linearly

independent at x. The common kernel of $a_1, \ldots, a_{n-p}$ in $T^{1,0}_x\Omega$ is a p-dimensional complex vector subspace. After a linear change of the standard coordinates, and after multiplying the form $\bigwedge_{j=1}^{n-p}(\sqrt{-1}a_j \wedge \overline{a_j})$ by a positive real factor, we may assume that this common kernel is spanned by $\partial/\partial z^1, \ldots, \partial/\partial z^p$ at x and that the forms $a_1, \ldots, a_{n-p}$ are $dz^{p+1}, \ldots, dz^n$ at x. Then the required non-negativity is exactly the non-negativity obtained from (ii) for the standard affine inclusion of this p-plane. Thus (i) follows.

Finally, assume $p = 1$. In this case

$$\omega = \sqrt{-1}\sum_{j,k=1}^{n} h_{j\bar{k}}(z)\, dz^j \wedge d\overline{z^k}.$$

For each $M = (m^1, m^2, \ldots, m^n) \in \mathbb{C}^n$, define $f_M \colon \mathbb{C} \to \mathbb{C}^n$ by

$$f_M(w) := (z_0^1 + m^1 w,\ z_0^2 + m^2 w,\ \ldots,\ z_0^n + m^n w).$$

Then

$$f_M^*\omega = \left(\sum_{j,k=1}^{n} m^j h_{j\bar{k}}(x)\overline{m^k}\right)\sqrt{-1}\, dw \wedge d\overline{w} \quad \text{at } 0.$$

Therefore (ii) is equivalent to the positive semi-definiteness of the Hermitian matrix $(h_{j\bar{k}}(x))_{j,k=1}^n$. □

Normal forms of real and positive $(1,1)$-forms at a point

We first prove the following:

Lemma III.1.8. *Let n be a positive integer and let $\Omega \subset \mathbb{C}^n$ be an open subset. Let ω be a real $(1,1)$-form on Ω. For each point $x = (z_0^1, z_0^2, \ldots, z_0^n) \in \Omega$, the following holds: there exists a unitary matrix $U = (u_\mu^\nu)_{\nu,\mu=1}^n \in \mathrm{U}(n)$ such that, for the map $f \colon \mathbb{C}^n \to \mathbb{C}^n$ defined by*

$$f(w^1, w^2, \ldots, w^n) := \left(z_0^1 + \sum_{\mu=1}^{n} u_\mu^1 w^\mu,\ z_0^2 + \sum_{\mu=1}^{n} u_\mu^2 w^\mu,\ \ldots,\ z_0^n + \sum_{\mu=1}^{n} u_\mu^n w^\mu\right),$$

there exist $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{R}$ such that

$$f^*\omega = \sqrt{-1}\sum_{j=1}^{n} \gamma_j\, dw^j \wedge d\overline{w^j} \quad \text{at } 0. \tag{96}$$

Furthermore, if ω is positive at x, then all $\gamma_1, \gamma_2, \ldots, \gamma_n$ are positive, and if ω is semi-positive at x, then all $\gamma_1, \gamma_2, \ldots, \gamma_n$ are non-negative.

Proof. Consider the expression

$$\omega = \sqrt{-1}\sum_{j,k=1}^{n} h_{j\bar{k}}(z)\, dz^j \wedge d\overline{z^k}.$$

Since ω is real, the matrix $(h_{j\overline{k}}(x))_{j,k=1}^n$ is Hermitian. For a unitary matrix $U = (u_\mu^j)_{j,\mu=1}^n$, define $f(w) = x + Uw$, that is,

$$f(w^1, w^2, \dots, w^n) := \left(z_0^1 + \sum_{\mu=1}^n u_\mu^1 w^\mu,\ z_0^2 + \sum_{\mu=1}^n u_\mu^2 w^\mu,\ \dots,\ z_0^n + \sum_{\mu=1}^n u_\mu^n w^\mu \right).$$

Then $f^* dz^j = \sum_{\mu=1}^n u_\mu^j \, dw^\mu$, and hence, at 0,

$$f^*\omega = \sqrt{-1} \sum_{\mu,\nu=1}^n \left(\sum_{j,k=1}^n h_{j\overline{k}}(x) u_\mu^j \overline{u_\nu^k} \right) dw^\mu \wedge d\overline{w^\nu}.$$

Choose $U \in \mathrm{U}(n)$ so that the Hermitian matrix $(h_{j\overline{k}}(x))_{j,k=1}^n$ is diagonalized in these coordinates. Then the desired expression follows. The assertions concerning positivity and semi-positivity follow from Lemma III.1.7. □

Let n be a positive integer, let $\Omega \subset \mathbb{C}^n$ be an open subset, and let ω be a real $(1,1)$-form on Ω. We now consider a normal form of ω at a point $x \in \Omega$ under suitable changes of coordinates, assuming that ω is positive at x. By translation, we may assume for simplicity that $x = 0$.

By Lemma III.1.8, after a unitary change of coordinates, we may assume that, for some $\gamma_1, \gamma_2, \dots, \gamma_n \in \mathbb{R}_{>0}$,

$$\omega = \sqrt{-1} \sum_{j=1}^n \gamma_j \, dz^j \wedge d\overline{z^j} \quad \text{at } 0.$$

By further replacing each z^j by $z^j/\sqrt{\gamma_j}$, we obtain the normal form at the level of the 0-jet:

$$\omega = \sqrt{-1} \sum_{j=1}^n dz^j \wedge d\overline{z^j} \quad \text{at } 0. \tag{97}$$

So far, we have only used affine changes of coordinates. If we also allow coordinate changes involving quadratic terms, then we obtain the following normal form up to the first order.

Lemma III.1.9. *Let ω be a real $(1,1)$-form defined in a neighborhood of a point $x = (z_0^1, z_0^2, \dots, z_0^n)$ of $\mathbb{C}^n$. Assume that $\omega > 0$ at x. Then the following are equivalent.*
(i) There exist an invertible matrix $(a_\nu^\mu)_{\nu,\mu=1}^n$ of order n and $b_{pq}^\ell \in \mathbb{C}$ for any $\ell, p, q \in \{1, 2, \dots, n\}$ such that the map $f\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$f(w^1, w^2, \dots, w^n) =$$
$$\left(z_0^1 + \sum_{\nu=1}^n a_\nu^1 w^\nu + \sum_{p,q=1}^n b_{pq}^1 w^p w^q,\ z_0^2 + \sum_{\nu=1}^n a_\nu^2 w^\nu + \sum_{p,q=1}^n b_{pq}^2 w^p w^q,\ \dots,\ z_0^n + \sum_{\nu=1}^n a_\nu^n w^\nu + \sum_{p,q=1}^n b_{pq}^n w^p w^q \right)$$

satisfies the following: for C^∞ functions $g_{j\overline{k}}$ defined on a neighborhood of the origin of $\mathbb{C}^n$ with

$$f^*\omega = \sqrt{-1} \sum_{j=1}^n \sum_{k=1}^n g_{j\overline{k}}(w) \, dw^j \wedge d\overline{w^k},$$

$$g_{j\bar{k}}(w) = \begin{cases} 1 + O(|w|^2) & \text{if } j = k, \\ O(|w|^2) & \text{otherwise} \end{cases} \tag{98}$$

holds as $w = (w^1, w^2, \dots, w^n) \to 0$, *where* $|w|^2 := |w^1|^2 + |w^2|^2 + \cdots + |w^n|^2$.
(*ii*) $d\omega$ *vanishes at* x.

Proof. By the preceding discussion, after a complex affine change of coordinates, we may assume that $x = 0$ and

$$\omega = \sqrt{-1} \sum_{j,k=1}^{n} h_{j\bar{k}}(z)\, dz^j \wedge d\overline{z^k}, \qquad h_{j\bar{k}}(0) = \delta_{jk} := \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases}.$$

We first describe the condition $d\omega = 0$ at 0 in terms of the first derivatives of $h_{j\bar{k}}$. A direct computation gives

$$d\omega = \sqrt{-1} \sum_{j,k,\ell=1}^{n} \frac{\partial h_{j\bar{k}}}{\partial z^\ell}\, dz^\ell \wedge dz^j \wedge d\overline{z^k} + \sqrt{-1} \sum_{j,k,\ell=1}^{n} \frac{\partial h_{j\bar{k}}}{\partial \overline{z^\ell}}\, d\overline{z^\ell} \wedge dz^j \wedge d\overline{z^k}.$$

Thus $d\omega = 0$ at 0 is equivalent to

$$\frac{\partial h_{j\bar{k}}}{\partial z^\ell}(0) = \frac{\partial h_{\ell\bar{k}}}{\partial z^j}(0) \qquad (1 \le j, k, \ell \le n),$$

together with the complex conjugate conditions. Since ω is real, the latter are the complex conjugates of the former.

We first prove (*i*) $\implies$ (*ii*). If (*i*) holds, then in the coordinate w we have $g_{j\bar{k}}(w) = \delta_{jk} + O(|w|^2)$. Therefore $d(f^*\omega)$ vanishes at 0. Since the Jacobian of f is invertible at 0, this implies that $d\omega$ vanishes at x.

We prove the converse. Assume that $d\omega = 0$ at 0. Write the Taylor expansion of $h_{j\bar{k}}$ as

$$h_{j\bar{k}}(z) = \delta_{jk} + \sum_{\ell=1}^{n} \left(a_{j\bar{k}\ell} z^\ell + \overline{a_{k\bar{j}\ell}}\, \overline{z^\ell} \right) + O(|z|^2),$$

where $a_{j\bar{k}\ell} := \frac{\partial h_{j\bar{k}}}{\partial z^\ell}(0)$, and the reality condition $h_{j\bar{k}} = \overline{h_{k\bar{j}}}$ has been used. By the condition $d\omega = 0$ at 0, we have

$$a_{j\bar{k}\ell} = a_{\ell\bar{k}j} \qquad (1 \le j, k, \ell \le n).$$

We consider a change of coordinates of the form

$$z^r = w^r + \sum_{p,q=1}^{n} b^r_{pq} w^p w^q \qquad (r = 1, 2, \dots, n),$$

where we may and shall assume $b^r_{pq} = b^r_{qp}$. The Jacobian matrix at the origin is the identity, so the normalization $h_{j\bar{k}}(0) = \delta_{jk}$ is preserved. Let $g_{j\bar{k}}$ be defined by

$$f^*\omega = \sqrt{-1} \sum_{j,k=1}^{n} g_{j\bar{k}}(w)\, dw^j \wedge d\overline{w^k}.$$

Since

$$\frac{\partial z^r}{\partial w^j} = \delta_{rj} + 2\sum_{q=1}^{n} b^r_{jq} w^q,$$

the linear part of $g_{j\overline{k}}$ is

$$\sum_{\ell=1}^{n} \left(a_{j\overline{k}\ell} + 2b^k_{j\ell}\right) w^\ell + \sum_{\ell=1}^{n} \overline{\left(a_{k\overline{j}\ell} + 2b^j_{k\ell}\right)}\,\overline{w^\ell}.$$

Therefore, if we choose

$$b^k_{j\ell} := -\frac{1}{2} a_{j\overline{k}\ell},$$

then all the linear terms vanish. This choice is compatible with the required symmetry $b^k_{j\ell} = b^k_{\ell j}$ because $a_{j\overline{k}\ell} = a_{\ell\overline{k}j}$. Hence, in the new coordinates, we obtain

$$g_{j\overline{k}}(w) = \delta_{jk} + O(|w|^2).$$

This is precisely the assertion in (i). □

The complex exterior derivatives of type $(1,0)$ **and** $(0,1)$

Let α be a (p,q)-form on an open subset $\Omega \subset \mathbb{C}^n$, and write

$$\alpha = \sum_{\#J=p,\ \#K=q} f_{J,\overline{K}}\, dz^J \wedge d\overline{z^K}.$$

We define

$$\partial\alpha := \sum_{\#J=p,\ \#K=q} \left(\sum_{\nu=1}^{n} \frac{\partial f_{J,\overline{K}}}{\partial z^\nu}\, dz^\nu\right) \wedge dz^J \wedge d\overline{z^K}$$

and

$$\overline{\partial}\alpha := \sum_{\#J=p,\ \#K=q} \left(\sum_{\nu=1}^{n} \frac{\partial f_{J,\overline{K}}}{\partial \overline{z^\nu}}\, d\overline{z^\nu}\right) \wedge dz^J \wedge d\overline{z^K}.$$

Thus we obtain maps

$$\partial\colon \mathcal{A}^{p,q}(\Omega) \to \mathcal{A}^{p+1,q}(\Omega), \qquad \overline{\partial}\colon \mathcal{A}^{p,q}(\Omega) \to \mathcal{A}^{p,q+1}(\Omega).$$

By Proposition I.6.49, these operators naturally extend to currents, giving maps

$$\partial\colon \mathcal{K}^{p,q}(\Omega) \to \mathcal{K}^{p+1,q}(\Omega), \qquad \overline{\partial}\colon \mathcal{K}^{p,q}(\Omega) \to \mathcal{K}^{p,q+1}(\Omega).$$

Mini-Exercise III.1.10. Show that

$$\partial\partial \equiv 0, \qquad \overline{\partial\partial} \equiv 0 \tag{99}$$

on both (p,q)-forms and (p,q)-currents on Ω.

The complex Hessian, Levi form, and the operator d^c

Let $\Omega \subset \mathbb{C}^n$ be an open subset and $f\colon \Omega \to \mathbb{C}$ be a C^2 function. Then

$$\omega := \sqrt{-1}\partial\overline{\partial} f$$

is a continuous $(1,1)$-form on Ω (i.e., it is a continuous section of $\Lambda^{1,1}\Omega \to \Omega$). Moreover, since

$$\overline{\omega} = -\sqrt{-1}\,\overline{\partial}\partial\,\overline{f} = \sqrt{-1}\partial\overline{\partial}\,\overline{f},$$

the form ω is real when f is real-valued (Here we have used the identity $\partial\overline{\partial} = -\overline{\partial}\partial$, which follows from $d = \partial + \overline{\partial}$, $d^2 = 0$, and (99)). In this case, the matrix-valued function corresponding to ω in the sense of (95) is

$$\Omega \ni p \mapsto H(f)(p) := \left(\frac{\partial^2 f}{\partial z^j \partial \overline{z^k}}(p)\right)_{1\leq j,k\leq n}.$$

This matrix $H(f)(p)$ is called the *complex Hessian matrix* of f at p. When f is real-valued, the corresponding Hermitian form on $T^{1,0}\Omega$ is called the *Levi form* of f.

We define the operator d^c by

$$d^c := \frac{1}{4\pi\sqrt{-1}}(\partial - \overline{\partial}).$$

If ξ is a real form on Ω, then $\overline{\partial\xi} = \overline{\partial}\xi$ and $\overline{\overline{\partial}\xi} = \partial\xi$, and hence

$$\overline{d^c\xi} = \frac{-1}{4\pi\sqrt{-1}}(\overline{\partial}\xi - \partial\xi) = d^c\xi.$$

Thus d^c is a real operator in this sense. Furthermore, using these identities, we obtain

$$dd^c = \frac{1}{4\pi\sqrt{-1}}(\partial + \overline{\partial})(\partial - \overline{\partial}) = \frac{\sqrt{-1}}{2\pi}\,\partial\overline{\partial}.$$

Remark III.1.11. Let f be a C^2 complex-valued function on Ω. We write the complex Hessian matrix of f at a point of Ω as

$$H(f) := \left(\frac{\partial^2 f}{\partial z^j \partial \overline{z^k}}\right)_{1\leq j,k\leq n}.$$

Then, by (36), its trace is given by

$$\operatorname{tr} H(f) = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial z^j \partial \overline{z^j}} = -\frac{1}{4}\Delta f.$$

Therefore, if $\partial\overline{\partial} f \equiv 0$, or equivalently $dd^c f \equiv 0$, then $\Delta f \equiv 0$. In particular, if $\overline{\partial} f \equiv 0$ or $\partial f \equiv 0$, then $\Delta f \equiv 0$. □

Real hypersurfaces and their Levi forms

Let n be a positive integer, $\Omega \subset \mathbb{C}^n$ an open subset, and $M \subset \Omega$ be a C^2 real hypersurface, that is, a C^2 submanifold of real codimension one, and let $p \in M$. By the definition of a submanifold, after shrinking a neighborhood $U \subset \Omega$ of p, one can choose C^2 real local coordinates

$$\xi = (\xi^1, \xi^2, \ldots, \xi^{2n})$$

on U such that

$$M \cap U = \{q \in U \mid \xi^1(q) = 0\}.$$

Then the function

$$\rho := \xi^1$$

is a C^2 real-valued function on U satisfying

$$M \cap U = \{\rho = 0\}, \qquad d\rho \neq 0 \quad \text{on } M \cap U.$$

Such a function ρ is called a *local defining function* of M near p.

In this note, we often treat the case where M is locally the boundary of a domain. More precisely, suppose that $D \subset \mathbb{C}^n$ is a domain such that, after shrinking U, $M \cap U = \partial D \cap U$. We also assume that, for U sufficiently small, $D \cap U$ is one of the two connected components of $U \setminus M$ (the one-sided condition). Then, after replacing ρ by $-\rho$ if necessary and after shrinking U again, one may assume that

$$D \cap U = \{q \in U \mid \rho(q) < 0\}.$$

In this case, ρ is also called a *local defining function of the domain* D near p. In this note, defining functions for domains will always be chosen with this sign convention.

Let ρ be a local defining function of M near p. We define

$$T^{1,0}_p M := \left\{ w \in T^{1,0}_p \Omega \;\middle|\; \langle (\partial\rho)_p, w \rangle = 0 \right\}, \tag{100}$$

where $\langle -, - \rangle$ denotes the natural pairing between covectors and tangent vectors. Since $(\partial\rho)_p \neq 0$, this is a complex vector subspace of $T^{1,0}_p\Omega$ of complex dimension $n-1$. As p varies in M, these subspaces form a complex vector bundle $T^{1,0}M \to M$ of class C^1.

The definition (100) also admits the following intrinsic description. For $w \in T^{1,0}_p\Omega$, we have $\langle (d\rho)_p, w \rangle = \langle (\partial\rho)_p, w \rangle$, since $(\overline{\partial}\rho)_p$ vanishes on $T^{1,0}_p\Omega$. Therefore,

$$T^{1,0}_p M = (T_p M \otimes_{\mathbb{R}} \mathbb{C}) \cap T^{1,0}_p \Omega,$$

where both spaces on the right-hand side are regarded as complex vector subspaces of $T^{\mathbb{C}}_p\Omega$.

Let

$$w = \sum_{j=1}^{n} a^j \frac{\partial}{\partial z^j}, \qquad w' = \sum_{k=1}^{n} b^k \frac{\partial}{\partial z^k}$$

be elements of $T^{1,0}_p M$. We define a Hermitian form $H_{M,p}(\rho)$ on $T^{1,0}_p M$ by

$$H_{M,p}(\rho)(w, w') := \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z^j \partial \overline{z^k}}(p) a^j \overline{b^k}. \tag{101}$$

In other words, $H_{M,p}(\rho)$ is the restriction to $T^{1,0}_p M$ of the Levi form of the function ρ at p. It is called the *Levi form of M at p with respect to ρ.*

We next give a real description of the complex tangent space. Regarding $T_p M$ as a real vector subspace of $T_p\Omega$, set

$$\begin{aligned} H_p M &:= T_p M \cap J(T_p M) \\ &= \{v \in T_p\Omega \mid v,\ J(v) \in T_p M\} \\ &= \{v \in T_p\Omega \mid \langle (d\rho)_p, v\rangle = \langle (d\rho)_p, J(v)\rangle = 0\}\,. \end{aligned} \tag{102}$$

The space $H_p M$ is invariant under J. Thus it is naturally regarded as a complex vector space by defining

$$\sqrt{-1} \cdot v := J(v) \qquad (v \in H_p M).$$

As p varies in M, these spaces form a real vector bundle $HM \to M$ of class C^1, equipped with a complex structure induced by J.

The isomorphism in Remark III.1.1 restricts to an isomorphism

$$H_p M \cong T^{1,0}_p M.$$

Indeed, for $v \in H_p\Omega$, the vector $v - \sqrt{-1}\, J(v)$ belongs to $T^{1,0}_p M$ if and only if $\left\langle (\partial\rho)_p,\ v - \sqrt{-1}\, J(v)\right\rangle = 0$. Since $(\overline{\partial}\rho)_p$ vanishes on $T^{1,0}_p\Omega$, this condition is equivalent to $\left\langle (d\rho)_p,\ v - \sqrt{-1}\, J(v)\right\rangle = 0$, or equivalently,

$$\langle (d\rho)_p, v\rangle - \sqrt{-1}\, \langle (d\rho)_p, J(v)\rangle = 0.$$

Since the two pairings in the last expression are real, this is equivalent to

$$\langle (d\rho)_p, v\rangle = \langle (d\rho)_p, J(v)\rangle = 0.$$

Thus the maps

$$H_p M \ni v \longmapsto v - \sqrt{-1}\, J(v) \in T^{1,0}_p M, \qquad T^{1,0}_p M \ni w \longmapsto \frac{w + \overline{w}}{2} \in H_p M$$

are mutually inverse complex-linear isomorphisms.

Mini-Exercise III.1.12. Let n be an integer with $n \geq 2$, $\Omega \subset \mathbb{C}^n$ an open subset, and $M \subset \Omega$ be a C^2 real hypersurface, and let ρ and $\widetilde{\rho}$ be two C^2 local defining functions of M on a neighborhood U of a point $p \in M$.
(i) Show that, after shrinking U if necessary, there exists a nowhere-vanishing real-valued C^1 function a on U such that $\widetilde{\rho} = a\rho$. Also show that a can be chosen to be of class C^∞ if M,

ρ, and $\widetilde{\rho}$ are of class C^∞.
(*ii*) Show that, for every $q \in M \cap U$,

$$(d\widetilde{\rho})_q = a(q)(d\rho)_q, \qquad (\partial\widetilde{\rho})_q = a(q)(\partial\rho)_q.$$

Conclude that the space $T^{1,0}_q M$ in (100) is independent of the choice of a local defining function.
(*iii*) Show that

$$H_{M,q}(\widetilde{\rho}) = a(q) H_{M,q}(\rho) \qquad (q \in M \cap U).$$

(*iv*) Suppose moreover that $M \cap U = \partial D \cap U$ for a domain $D \subset \mathbb{C}^n$, and that $D \cap U = \{\rho < 0\} = \{\widetilde{\rho} < 0\}$. Show that a is positive after shrinking U if necessary. Conclude that the numbers of positive, negative, and zero eigenvalues of $H_{M,q}(\rho)$ are independent of the choice of a local defining function satisfying this sign convention.

Remark III.1.13. For a real hypersurface without a specified choice of side, its Levi form is determined only up to multiplication by a nowhere-zero real-valued function (refer to Mini-Exercise III.1.12). Consequently, its rank and the property of vanishing identically are intrinsic, whereas its positive and negative indices may be interchanged by replacing a local defining function ρ by $-\rho$.

Whenever one side of a real hypersurface has been specified, we use only local defining functions which are negative on that side. In particular, when M is locally the boundary of a domain D, we use the sign convention $D = \{\rho < 0\}$. With this convention, the inertia of the Levi form of a real hypersurface is well-defined. □

In Lemmata III.1.8 and III.1.9, we considered how far the local expression of a real $(1,1)$-form can be simplified at a point by affine and quadratic polynomial changes of coordinates. We now record a similar second-order normal form for a real hypersurface.

Lemma III.1.14. *Let n be an integer with $n \geq 2$, $\Omega \subset \mathbb{C}^n$ an open subset, and $M \subset \Omega$ be a C^2 real hypersurface, $p = (z_0^1, z_0^2, \ldots, z_0^n) \in M$ a point, and ρ be a C^2 local defining function of M near p. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ the eigenvalues of the Hermitian form $H_{M,p}(\rho)$, where $T^{1,0}_p M$ is equipped with the Hermitian inner product induced by the standard coordinates of $\mathbb{C}^n$. Then there exist a unitary matrix $U = (u^\nu_\mu)^n_{\nu,\mu=1} \in \mathrm{U}(n)$, a positive real number τ, and complex numbers $b^\nu_{\mu\kappa}$ for each $\nu, \mu, \kappa \in \{1, 2, \ldots, n\}$ such that, for the polynomial map $f\colon \mathbb{C}^n \to \mathbb{C}^n$ from the complex Euclidean space with complex coordinates $w = (w^1, w^2, \ldots, w^n)$ defined by*

$$\begin{aligned} f(w^1, w^2, \ldots, w^n) := \Bigg(& z_0^1 + \sum_{\mu=1}^{n-1} u^1_\mu w^\mu + \tau u^1_n w^n + \sum_{\mu,\kappa=1}^{n} b^1_{\mu\kappa} w^\mu w^\kappa, \\ & z_0^2 + \sum_{\mu=1}^{n-1} u^2_\mu w^\mu + \tau u^2_n w^n + \sum_{\mu,\kappa=1}^{n} b^2_{\mu\kappa} w^\mu w^\kappa, \\ & \ldots, \\ & z_0^n + \sum_{\mu=1}^{n-1} u^n_\mu w^\mu + \tau u^n_n w^n + \sum_{\mu,\kappa=1}^{n} b^n_{\mu\kappa} w^\mu w^\kappa \Bigg), \end{aligned}$$

the following holds: After restricting f to a sufficiently small neighborhood of the origin, it is a C^∞ diffeomorphism onto a neighborhood of p, and the inverse image of M is expressed near the origin as the set of points satisfying an equation of the form

$$\operatorname{Im} w^n = \sum_{j=1}^{n-1} \lambda_j |w^j|^2 + R(w^1, \ldots, w^{n-1}, \operatorname{Re} w^n), \tag{103}$$

where R is a real-valued C^2 function defined near the origin such that

$$R(w^1, \ldots, w^{n-1}, \operatorname{Re} w^n) = o\left(\sum_{j=1}^{n-1} |w^j|^2 + |\operatorname{Re} w^n|^2\right)$$

as $(w^1, \ldots, w^{n-1}, \operatorname{Re} w^n) \to (0, 0, \ldots, 0)$. If M and ρ are of class C^3, then

$$R(w^1, \ldots, w^{n-1}, \operatorname{Re} w^n) = O\left(\left(\sum_{j=1}^{n-1} |w^j|^2 + |\operatorname{Re} w^n|^2\right)^{3/2}\right)$$

as $(w^1, \ldots, w^{n-1}, \operatorname{Re} w^n) \to (0, 0, \ldots, 0)$.

Proof. Choose an orthonormal basis $e_1, e_2, \ldots, e_{n-1}$ of $T^{1,0}_p M$ consisting of eigenvectors of $H_{M,p}(\rho)$. Thus

$$H_{M,p}(\rho)(e_j, e_k) = \lambda_j \delta_{jk} \qquad (j, k \in \{1, 2, \ldots, n-1\}).$$

Choose a unit vector

$$e_n \in T^{1,0}_p \Omega$$

orthogonal to $T^{1,0}_p M$. Since $(\partial\rho)_p$ vanishes on $T^{1,0}_p M$ and is non-zero, we have

$$\langle (\partial\rho)_p, e_n \rangle \neq 0.$$

After multiplying e_n by a complex number of absolute value one, we may assume that

$$\langle (\partial\rho)_p, e_n \rangle = \sqrt{-1}\, c$$

for some positive real number c. Set

$$\tau := \frac{1}{2c}.$$

Write

$$e_\mu = \sum_{\nu=1}^{n} u^\nu_\mu \frac{\partial}{\partial z^\nu} \qquad (\mu \in \{1, 2, \ldots, n\}).$$

Since $e_1, e_2, \ldots, e_n$ form an orthonormal basis, the matrix $U = (u^\nu_\mu)^n_{\nu,\mu=1}$ is unitary.

Consider first the complex affine map $f_0\colon \mathbb{C}^n \to \mathbb{C}^n$ from the complex Euclidean space with complex coordinates $\zeta = (\zeta^1, \zeta^2, \ldots, \zeta^n)$ defined by

$$f_0(\zeta) := p + \sum_{\mu=1}^{n-1} e_\mu \zeta^\mu + \tau e_n \zeta^n.$$

In terms of the standard coordinates, this means

$$f_0(\zeta^1, \zeta^2, \ldots, \zeta^n) = \left(z_0^1 + \sum_{\mu=1}^{n-1} u_\mu^1 \zeta^\mu + \tau u_n^1 \zeta^n, z_0^2 + \sum_{\mu=1}^{n-1} u_\mu^2 \zeta^\mu + \tau u_n^2 \zeta^n, \ldots, z_0^n + \sum_{\mu=1}^{n-1} u_\mu^n \zeta^\mu + \tau u_n^n \zeta^n \right).$$

Set

$$\widetilde{\rho} := \rho \circ f_0.$$

Then

$$T_0^{1,0}(f_0^{-1}(M)) = \bigoplus_{j=1}^{n-1} \mathbb{C} \frac{\partial}{\partial \zeta^j}.$$

Moreover,

$$\frac{\partial \widetilde{\rho}}{\partial \zeta^n}(0) = \tau \langle (\partial \rho)_p, e_n \rangle = \frac{\sqrt{-1}}{2}.$$

Since $\widetilde{\rho}$ is real-valued, it follows that

$$(d\widetilde{\rho})_0 = -d(\operatorname{Im} \zeta^n)_0.$$

Write

$$\zeta^n = x^n + \sqrt{-1}\, y^n$$

and

$$\zeta' := (\zeta^1, \zeta^2, \ldots, \zeta^{n-1}).$$

By the implicit function theorem, there exists a real-valued C^2 function $\varphi(\zeta', x^n)$ defined near the origin such that

$$f_0^{-1}(M) = \{(\zeta', \zeta^n) \mid y^n = \varphi(\zeta', x^n)\}.$$

Moreover, by construction, one has

$$\varphi(0) = 0, \qquad d\varphi(0) = 0. \tag{104}$$

The function

$$\rho_0(\zeta) := \varphi(\zeta', x^n) - y^n$$

is a local defining function of $f_0^{-1}(M)$ near the origin. Since $(d\rho_0)_0 = -dy^n = (d\widetilde{\rho})_0$, Mini-Exercise III.1.12 gives $H_{f_0^{-1}(M),0}(\rho_0) = H_{f_0^{-1}(M),0}(\widetilde{\rho})$. By the choice of the vectors $e_1, e_2, \ldots, e_{n-1}$, we therefore have

$$\frac{\partial^2 \varphi}{\partial \zeta^j \partial \overline{\zeta^k}}(0) = \lambda_j \delta_{jk} \qquad (j, k \in \{1, 2, \ldots, n-1\}).$$

From (104), it follows that the Taylor expansion of φ at the origin can be written as

$$\varphi(\zeta', x^n) = P_2(\zeta', \overline{\zeta'}, x^n) + o\left(|\zeta'|^2 + |x^n|^2\right), \tag{105}$$

where $P_2(\zeta', \overline{\zeta'}, x^n)$ is a real-valued homogeneous polynomial of degree two in the real variables

$$\operatorname{Re}\zeta^1,\ \operatorname{Im}\zeta^1,\ \ldots,\ \operatorname{Re}\zeta^{n-1},\ \operatorname{Im}\zeta^{n-1},\ x^n.$$

Using the complex variables, we can write P_2 in the form

$$\begin{aligned} P_2(\zeta', \overline{\zeta'}, x^n) = &\sum_{j,k=1}^{n-1} h_{j\overline{k}}\,\zeta^j\overline{\zeta^k} \\ &+ 2\operatorname{Re}\left(\sum_{j,k=1}^{n-1} q_{jk}\zeta^j\zeta^k + \sum_{j=1}^{n-1} q_{jn}\zeta^j x^n + q_{nn}(x^n)^2\right), \end{aligned}$$

where $(h_{j\overline{k}})_{j,k=1}^{n-1}$ is a Hermitian matrix and the q_{jk}, q_{jn}, and q_{nn} are complex numbers (Indeed, the terms containing both ζ^j and $\overline{\zeta^k}$ form a Hermitian quadratic form, while the remaining terms occur together with their complex conjugates because P_2 is real-valued).

Define a homogeneous polynomial of degree two in the complex variables $\zeta^1, \zeta^2, \ldots, \zeta^n$ by

$$Q(\zeta^1, \zeta^2, \ldots, \zeta^n) := \sum_{j,k=1}^{n-1} q_{jk}\zeta^j\zeta^k + \sum_{j=1}^{n-1} q_{jn}\zeta^j\zeta^n + q_{nn}(\zeta^n)^2.$$

Then

$$P_2(\zeta', \overline{\zeta'}, x^n) = \sum_{j,k=1}^{n-1} h_{j\overline{k}}\,\zeta^j\overline{\zeta^k} + 2\operatorname{Re} Q(\zeta', x^n),$$

where, in the last expression, Q is evaluated at $\zeta^n = x^n$. Now it is clear that

$$h_{j\overline{k}} = \frac{\partial^2\varphi}{\partial\zeta^j\partial\overline{\zeta^k}}(0) = \lambda_j\delta_{jk} \qquad (j, k \in \{1, 2, \ldots, n-1\}).$$

Combining this equality with (105), we obtain

$$\varphi(\zeta', x^n) = \sum_{j=1}^{n-1} \lambda_j|\zeta^j|^2 + 2\operatorname{Re} Q(\zeta', x^n) + o\left(|\zeta'|^2 + |x^n|^2\right). \tag{106}$$

Define a polynomial map $g\colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$g(w^1, w^2, \ldots, w^n) := \left(w^1, \ldots, w^{n-1}, w^n + 2\sqrt{-1}\,Q(w^1, w^2, \ldots, w^n)\right).$$

Its differential at the origin is the identity. Hence it is a C^∞ diffeomorphism between suitable neighborhoods of the origin.

Set

$$f := f_0 \circ g.$$

Since f_0 is affine and g is a polynomial map of degree two, f has the form stated in the lemma for suitable complex numbers $b^{\nu}_{\mu\kappa}$.

We now determine the expression of $f^{-1}(M)$. Since the tangent hyperplane of $g^{-1}(f_0^{-1}(M)) = f^{-1}(M)$ at the origin is $\{\operatorname{Im} w^n = 0\}$, we have

$$\operatorname{Im} w^n = O(|w|^2) \quad \text{on } f^{-1}(M).$$

Since Q is homogeneous of degree two, it follows that, on $f^{-1}(M)$,

$$Q(w^1, w^2, \ldots, w^n) = Q(w^1, \ldots, w^{n-1}, \operatorname{Re} w^n) + o(|w|^2).$$

Put $\zeta = g(w)$. Then

$$\zeta^j = w^j \qquad (j \in \{1, 2, \ldots, n-1\}) \qquad \text{and} \qquad \zeta^n = w^n + 2\sqrt{-1}\, Q(w).$$

Consequently,

$$\operatorname{Im} \zeta^n = \operatorname{Im} w^n + 2 \operatorname{Re} Q(w) \qquad \text{and} \qquad \operatorname{Re} \zeta^n = \operatorname{Re} w^n - 2 \operatorname{Im} Q(w).$$

Since $Q(w) = O(|w|^2)$, we have

$$\operatorname{Re} \zeta^n = \operatorname{Re} w^n + O(|w|^2).$$

On the other hand, since $\zeta = g(w) \in f_0^{-1}(M)$ for $w \in f^{-1}(M)$, we have $\operatorname{Im} \zeta^n = \varphi(\zeta', \operatorname{Re} \zeta^n)$. Using (106) and the fact that $g(w) = w + O(|w|^2)$, we obtain

$$\begin{aligned} \operatorname{Im} w^n + 2 \operatorname{Re} Q(w) &= \varphi\left(w', \operatorname{Re} w^n - 2 \operatorname{Im} Q(w)\right) \\ &= \sum_{j=1}^{n-1} \lambda_j |w^j|^2 + 2 \operatorname{Re} Q(w', \operatorname{Re} w^n) + o(|w|^2), \end{aligned}$$

where $w' := (w^1, w^2, \ldots, w^{n-1})$. Moreover,

$$2 \operatorname{Re} Q(w) = 2 \operatorname{Re} Q(w', \operatorname{Re} w^n) + o(|w|^2)$$

on $f^{-1}(M)$. Cancelling these terms, we obtain

$$\operatorname{Im} w^n = \sum_{j=1}^{n-1} \lambda_j |w^j|^2 + o(|w|^2),$$

which proves (103).

If M and ρ are of class C^3, Taylor's theorem gives an $O(|\zeta|^3)$ remainder in (106). Since $\operatorname{Im} w^n = O(|w|^2)$ on $f^{-1}(M)$, the same calculation gives an $O(|w|^3)$ remainder in (103). $\square$

Remark III.1.15. The assumption $n \geq 2$ in Lemma III.1.14 is imposed only to avoid notation involving an empty list of eigenvalues. When $n = 1$, the same argument shows that there exist a complex number u with $|u| = 1$, a positive real number $\tau > 0$, and a complex number b such that, for the polynomial map

$$f\colon \mathbb{C} \to \mathbb{C}, \qquad f(w) := z_0 + \tau u w + b w^2,$$

the restriction of f to a sufficiently small neighborhood of the origin is a C^∞ diffeomorphism onto a neighborhood of p, and

$$f^{-1}(M) = \{w \mid \operatorname{Im} w = R(\operatorname{Re} w)\}$$

near the origin, where R is a real-valued C^2 function satisfying $R(t) = o(|t|^2)$ as $t \to 0$. If M and ρ are of class C^3, then $R(t) = O(|t|^3)$ as $t \to 0$.

The restriction to $n \geq 2$ in Mini-Exercise III.1.12 was made for a similar reason: Indeed, if $n = 1$, then $T^{1,0}_p M = \{0\}$, and hence the Levi form is a Hermitian form on the zero-dimensional complex vector space. Thus there are no eigenvalues whose signs could carry any information.

These observations are also related to the role of the *pseudoconvexity*, which will be discussed later in this chapter. For domains in one complex variable, pseudoconvexity does not impose a genuine restriction, whereas in complex dimension at least two it becomes a fundamental and non-trivial condition. □

Let n be a positive integer and $\Omega \subset \mathbb{C}^n$ be an open subset. A C^2 real hypersurface $M \subset \Omega$ is said to be *Levi-flat at* $p \in M$ if

$$H_{M,p}(\rho) \equiv 0$$

for one, and hence for every, local defining function ρ of M near p (refer to Mini-Exercise III.1.12). It is said to be *Levi-flat* if it is Levi-flat at every point.

Mini-Exercise III.1.16. Let $f\colon \Omega \to \mathbb{C}$ be a C^∞ function and $M \subset \Omega$ be a C^∞ real hypersurface.
(*i*) Show that, for every real tangent vector $v \in T\Omega$,

$$\langle d^c f, v\rangle = -\frac{1}{4\pi}\langle df, J(v)\rangle.$$

(*ii*) Let ρ be a real-valued C^∞ function on Ω, and let u and v be real C^∞ vector fields on Ω. Show that

$$4\pi(dd^c\rho)(u,v) = -u\big(\langle d\rho, J(v)\rangle\big) + v\big(\langle d\rho, J(u)\rangle\big) + \langle d\rho, J([u,v])\rangle. \tag{107}$$

(*iii*) Let $p \in M$, and let ρ be a C^∞ local defining function of M near p. For $u \in H_pM$, set

$$w := u - \sqrt{-1}\, J(u) \in T^{1,0}_p M.$$

Show that

$$H_{M,p}(\rho)(w,w) = 4\pi(dd^c\rho)_p(u, J(u)).$$

Conclude that $H_{M,p}(\rho) \equiv 0$ if and only if $(dd^c\rho)_p|_{H_pM\times H_pM} \equiv 0$.
(iv) Let u and v be C^∞ sections of HM over an open subset V of M. Show that

$$4\pi(dd^c\rho)(u,v) = \langle d\rho, J([u,v])\rangle \quad \text{on } V,$$

and that

$$\langle d\rho, [u,v]\rangle = 0 \quad \text{on } V.$$

Conclude that M is Levi-flat if and only if the real distribution $HM \subset TM$ is involutive.

Assume now that M is a C^∞ Levi-flat real hypersurface. By Mini-Exercise III.1.16 (iv), the C^∞ real distribution $HM \subset TM$ is involutive. Since HM has constant real rank $2n-2$, the Frobenius theorem recalled in Chapter I §8.10 yields a C^∞ foliation $\mathcal{F}$ on M such that

$$T\mathcal{F} = HM.$$

This foliation has real codimension one in M. Since M itself has real codimension one in Ω, every leaf of $\mathcal{F}$ has real codimension two in Ω (as an immersed submanifold). The foliation $\mathcal{F}$ is called the *Levi foliation* of M.

If $\mathcal{L}$ is a leaf of $\mathcal{F}$, then

$$T_p\mathcal{L} = H_pM \qquad (p \in \mathcal{L}).$$

In particular, the tangent space $T_p\mathcal{L}$ is invariant under the complex structure J of Ω at every point $p \in \mathcal{L}$. This suggests that the leaves of the Levi foliation are compatible with the complex structure of the ambient space in a stronger sense (see Mini-Exercise III.2.10 for details).

Remark III.1.17. In this note, the definition of Levi-flatness given above also includes the case $n = 1$. Indeed, if $\Omega \subset \mathbb{C}$ is an open subset and $M \subset \Omega$ is a C^2 real hypersurface, then $T^{1,0}_pM = \{0\}$ for any $p \in M$. Consequently, for every local defining function ρ of M near p, the Hermitian form $H_{M,p}(\rho)$ on $T^{1,0}_pM$ vanishes identically. Thus every C^2 real hypersurface in an open subset of $\mathbb{C}$ is Levi-flat. If M is of class C^∞, then the Levi foliation of M is the zero-dimensional foliation whose leaves are the singletons. □

1.2 Holomorphic functions of several variables

Definition and characterizations of holomorphic functions

Let n be a positive integer and let $\Omega \subset \mathbb{C}^n$ be an open subset. Let $f\colon \Omega \to \mathbb{C}$ be a function. We say that f is *holomorphic* on Ω if, for every $a = (a^1, a^2, \ldots, a^n) \in \Omega$ and every $j \in \{1, 2, \ldots, n\}$, the function

$$w \longmapsto f(a^1, \ldots, a^{j-1}, w, a^{j+1}, \ldots, a^n)$$

is holomorphic on a neighborhood of $w = a^j \in \mathbb{C}$ as a function of one complex variable.

In this definition, we require only holomorphicity with respect to each variable separately. In particular, no continuity or other joint regularity assumption on f is imposed a priori.

In the setting of real functions and real partial derivatives, separate regularity alone gives no reason to expect good regularity with respect to all the variables together (Recall Mini-Exercise I.4.1). For functions of complex variables, however, the situation is completely different. In fact, the following theorem holds.

THEOREM III.1.18. *Let $f\colon \Omega \to \mathbb{C}$ be a function. Then the following conditions are equivalent:*
(i) The function f is holomorphic on Ω.
(ii) For every point $a = (a^1, a^2, \ldots, a^n) \in \Omega$, there exist an open neighborhood $V \subset \Omega$ of a and complex numbers $c_{k_1,k_2,\ldots,k_n}$ for each $k_1, k_2, \ldots, k_n \in \mathbb{Z}_{\geq 0}$ such that

$$f(z) = \lim_{N\to\infty} \sum_{\substack{k_1,k_2,\ldots,k_n \in \mathbb{Z}_{\geq 0} \\ k_1+k_2+\cdots+k_n \leq N}} c_{k_1,k_2,\ldots,k_n} (z^1 - a^1)^{k_1} (z^2 - a^2)^{k_2} \cdots (z^n - a^n)^{k_n}$$

uniformly for $z \in V$.
(iii) The function f is of class C^∞ on Ω and satisfies $\overline{\partial} f \equiv 0$.
(iv) The function f is an element of $L^2_{\mathrm{loc}}(\Omega)$ and satisfies $\overline{\partial} f \equiv 0$ in the sense of currents (See also Remark III.1.19 below for this condition).

A function satisfying condition *(ii)* of Theorem III.1.18 is said to be *(complex) analytic*. Thus Theorem III.1.18 shows that a function on an open subset of a complex Euclidean space is holomorphic if and only if it is complex analytic.

For functions of complex variables, we usually use the term "holomorphic", following the terminology in Notation and Conventions.

REMARK III.1.19. Condition *(iv)* in Theorem III.1.18 is understood in accordance with Remark I.6.28. Thus an element of $L^2_{\mathrm{loc}}(\Omega)$ is an equivalence class of measurable functions modulo equality almost everywhere. Strictly speaking, the implication from *(iv)* to the other conditions means that an L^2_{loc} equivalence class satisfying $\overline{\partial} f \equiv 0$ in the sense of currents admits a unique holomorphic representative. As usual, we use the same symbol f for the equivalence class and for this representative.

Since $L^2_{\mathrm{loc}}(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$, such an element defines a 0-current (recall Example I.8.19), and the equation $\overline{\partial} f \equiv 0$ in condition *(iv)* is understood in the sense of currents. □

Before proving the theorem, we fix some notation. For every positive number ℓ, set

$$\Delta_\ell := \{z \in \mathbb{C} \mid |z| < \ell\}.$$

To simplify the power series notation in condition *(ii)* and in the proof below, for

$$k = (k_1, k_2, \ldots, k_n) \in (\mathbb{Z}_{\geq 0})^n,$$

we set

$$|k| := k_1 + k_2 + \cdots + k_n$$

and

$$(z - a)^k := (z^1 - a^1)^{k_1} (z^2 - a^2)^{k_2} \cdots (z^n - a^n)^{k_n}.$$

We also write

$$c_k := c_{k_1,k_2,\dots,k_n}.$$

With this notation, the power series representation in condition (ii) is written as

$$f(z) = \lim_{N\to\infty} \sum_{\substack{k\in(\mathbb{Z}_{\geq 0})^n \\ |k|\leq N}} c_k (z-a)^k.$$

When condition (ii) holds, we also write

$$f(z) = \sum_{k\in(\mathbb{Z}_{\geq 0})^n} c_k \, (z-a)^k.$$

Remark III.1.20. As emphasized above, no continuity or other joint regularity assumption is imposed a priori on a holomorphic function. This point should be kept in mind in the proof below: joint continuity cannot be used before it has been established. □

Proof of Theorem III.1.18. For Hartogs' separate analyticity theorem in arbitrary dimension, see [No2, Theorem 4.2.9]. The implication $(iii) \implies (iv)$ is immediate. Conversely, if (iv) holds, Theorem I.8.37, applied with

$$\frac{\partial f}{\partial \overline{z^j}} = 0 \qquad (j \in \{1, 2, \dots, n\}),$$

shows that f has a C^∞ representative. Thus $(iv) \implies (iii)$ holds in the sense explained in Remark III.1.19.

The implication $(ii) \implies (iii)$ follows from the standard properties of convergent power series. Indeed, after shrinking the neighborhood under consideration, the series and the series obtained by differentiating it term by term converge uniformly. Consequently, f is of class C^∞, and, since no monomial in the series contains any $\overline{z^j}$, we have $\overline{\partial} f \equiv 0$.

The implication $(iii) \implies (i)$ follows by restricting f to each complex coordinate line and applying the one-variable Cauchy–Riemann criterion.

It remains to prove *Hartogs' separate analyticity theorem* (Theorem III.1.18, $(i) \implies (ii)$). For the convenience of the reader, we give a detailed proof in the case $n = 2$. This proof also exhibits the main analytic mechanism of the general case; see Remark III.1.22 below.

Write the standard coordinates of $\mathbb{C}^2$ as (z, w). It is enough to prove the assertion near an arbitrary point of Ω. After a translation, we may assume that this point is the origin.

We first consider the case where there exist positive numbers r and ℓ and a positive constant M such that

$$\Delta_r \times \Delta_\ell \subset \Omega \qquad \text{and} \qquad \sup_{\Delta_r\times\Delta_\ell} |f| \leq M. \tag{108}$$

As the function $w \mapsto f(z, w)$ is holomorphic on Δ_ℓ for every fixed $z \in \Delta_r$, one has the Taylor expansion

$$f(z, w) = \sum_{k=0}^{\infty} a_k(z) \, w^k,$$

where, at this stage, the regularity of the functions $a_k\colon \Delta_r \to \mathbb{C}$ has not yet been established. By the (one-variable) Cauchy estimate in the w-variable, we have

$$\sup_{\Delta_r} |a_k| \leq \frac{M}{\ell^k} \qquad (k \in \mathbb{Z}_{\geq 0}). \tag{109}$$

From this, one can show that each a_k is holomorphic (see Mini-Exercise III.1.21 below), and hence admits the Taylor expansion

$$a_k(z) = \sum_{j=0}^{\infty} c_{jk} z^j.$$

By the Cauchy estimate and (109), we obtain

$$|c_{jk}| \leq \frac{M}{r^j \ell^k} \qquad (j, k \in \mathbb{Z}_{\geq 0})$$

(More precisely, this estimate is first obtained with radii strictly smaller than r and ℓ, and the displayed estimate follows by letting those radii increase to r and ℓ). Therefore,

$$f(z, w) = \sum_{j,k=0}^{\infty} c_{jk} z^j w^k,$$

and this series converges absolutely and uniformly on $\Delta_\rho \times \Delta_\sigma$ for every pair of positive numbers ρ and σ satisfying $\rho < r$ and $\sigma < \ell$, which proves condition (ii) under the additional boundedness assumption (108).

We now remove this boundedness assumption. Choose positive numbers R and L such that

$$\overline{\Delta_R \times \Delta_L} \subset \Omega,$$

and take positive numbers r and ℓ satisfying

$$r < R, \qquad \ell < L.$$

For every positive integer ν, set

$$E_\nu := \left\{ w \in \overline{\Delta_\ell} \;\middle|\; |f(z, w)| \leq \nu \text{ for every } z \in \overline{\Delta_r} \right\}.$$

As

$$E_\nu = \bigcap_{z \in \overline{\Delta_r}} \left\{ w \in \overline{\Delta_\ell} \;\middle|\; |f(z, w)| \leq \nu \right\}$$

and the function $w \mapsto f(z, w)$ is continuous for each fixed $z \in \overline{\Delta_r}$ by assumption, we have that E_ν is closed in $\overline{\Delta_\ell}$ for each $\nu \in \mathbb{Z}_{>0}$. Moreover, the sequence $\{E_\nu\}_{\nu \geq 1}$ is increasing and

$$\bigcup_{\nu=1}^{\infty} E_\nu = \overline{\Delta_\ell}$$

(Indeed, for each fixed $w \in \overline{\Delta_\ell}$, the function $z \mapsto f(z,w)$ is holomorphic and hence continuous on a neighborhood of $\overline{\Delta_r}$, and is therefore bounded on $\overline{\Delta_r}$). As the closed disc $\overline{\Delta_\ell}$ is a complete metric space, it follows from the Baire category theorem that there exists a positive integer N such that E_N has non-empty interior relative to $\overline{\Delta_\ell}$. We may therefore choose a point $a \in \Delta_\ell$ and a positive number $\delta > 0$ such that

$$a + \Delta_\delta \subset E_N.$$

Choose an automorphism $\psi \in \mathrm{Aut}(\Delta_L)$ such that $\psi(0) = a$, and define

$$g(z,w) := f(z, \psi(w)) \qquad ((z,w) \in \Delta_R \times \Delta_L).$$

After choosing a sufficiently small positive number

$$\ell_* < L,$$

we have

$$\psi(\Delta_{\ell_*}) \subset a + \Delta_\delta.$$

Consequently,

$$\sup_{\Delta_r \times \Delta_{\ell_*}} |g| \le N.$$

For each fixed $z \in \Delta_r$, expand g in the w-variable:

$$g(z,w) = \sum_{k=0}^{\infty} b_k(z)\, w^k.$$

Applying the bounded case proved above, with f, ℓ, and M replaced by g, ℓ_*, and N, respectively, we obtain the following facts:

- each function $b_k \colon \Delta_r \to \mathbb{C}$ is holomorphic;
- the inequality
$$\sup_{\Delta_r} |b_k| \le \frac{N}{\ell_*^k} \qquad (k \in \mathbb{Z}_{\ge 0}) \tag{110}$$
holds;
- the function g is represented by a convergent power series on $\Delta_r \times \Delta_{\ell_*}$.

Note that, however, the origin for the original function f corresponds, under the change of variables by ψ, to the point $(0, \psi^{-1}(0))$ for the function g. Thus the power-series representation only on $\Delta_r \times \Delta_{\ell_*}$ is not yet sufficient. For this reason, in what follows we shall show that the same series represents g locally uniformly not only on $\Delta_r \times \Delta_{\ell_*}$ but also on $\Delta_r \times \Delta_L$.

For every positive integer k, set

$$u_k := \frac{1}{k} \log |b_k|.$$

Since b_k is holomorphic, u_k is a subharmonic function on Δ_r (Recall Mini-Exercise I.9.5). By (110),

$$\sup_{\Delta_r} u_k \leq -\log \ell_* + \frac{\log N}{k}.$$

In particular, these suprema are bounded above independently of k.

On the other hand, by assumption, the function $w \mapsto g(z,w)$ is holomorphic on Δ_L for every fixed $z \in \Delta_r$. The Cauchy–Hadamard formula therefore gives

$$\limsup_{k\to\infty} |b_k(z)|^{1/k} \leq \frac{1}{L},$$

and hence

$$\limsup_{k\to\infty} u_k(z) \leq -\log L \qquad (z \in \Delta_r).$$

Fix positive numbers ρ_0 and σ_0 such that

$$\rho_0 < r, \qquad \sigma_0 < L.$$

Choose positive numbers ρ, λ, and σ such that

$$\rho_0 < \rho < \lambda < r, \qquad \sigma_0 < \sigma < L.$$

Apply Mini-Exercise I.9.7 (ii) to the sequence $\{u_k\}$ with

$$A := -\log L, \qquad \alpha := -\log \sigma.$$

Since $\sigma < L$, we have $\alpha > A$. It follows that there exists a positive integer N' such that

$$\sup_{\Delta_\rho} u_k \leq -\log \sigma$$

for every $k > N'$. Equivalently,

$$\sup_{\Delta_\rho} |b_k| \leq \frac{1}{\sigma^k} \qquad (k > N'). \tag{111}$$

For every $k \in \mathbb{Z}_{\geq 0}$, write the Taylor expansion

$$b_k(z) = \sum_{j=0}^{\infty} e_{jk} z^j.$$

By the Cauchy estimate and (111), we have

$$|e_{jk}| \leq \frac{1}{\rho^j \sigma^k} \qquad (j \in \mathbb{Z}_{\geq 0},\ k > N').$$

Therefore,

$$g(z,w) = \sum_{k=0}^{N'} b_k(z) w^k + \sum_{\substack{j\geq 0 \\ k>N'}} e_{jk} z^j w^k.$$

The second series converges absolutely and uniformly on $\Delta_{\rho_0} \times \Delta_{\sigma_0}$, because

$$\sum_{\substack{j\geq 0 \\ k>N'}} \frac{\rho_0^j \sigma_0^k}{\rho^j \sigma^k} < \infty.$$

The first sum contains only finitely many holomorphic functions b_k, each of which has a Taylor expansion converging locally uniformly on Δ_r. It follows that g is represented by a convergent power series on $\Delta_{\rho_0} \times \Delta_{\sigma_0}$.

Since $\rho_0 < r$ and $\sigma_0 < L$ were arbitrary, the series represents g locally uniformly throughout $\Delta_r \times \Delta_L$. In particular, it represents g on a neighborhood of $(0, \psi^{-1}(0))$. Since ψ^{-1} is a holomorphic function of one variable and hence is locally represented by a convergent power series, the identity $f(z,w) = g(z, \psi^{-1}(w))$ shows that f is represented by a convergent power series on a neighborhood of $(0,0)$, which proves $(i) \implies (ii)$ when $n = 2$. See also Remark III.1.22 below. □

Mini-Exercise III.1.21. Under the condition (i) of Theorem III.1.18 and the assumption (108), show by induction on k that each function $a_k \colon \Delta_r \to \mathbb{C}$ in the proof above is holomorphic.

Remark III.1.22. The proof of Hartogs' separate analyticity theorem (Theorem III.1.18) in arbitrary dimension follows the same general strategy. In the general argument, the one-variable subharmonic functions used in the proof above are replaced by their several-variable counterparts, called *plurisubharmonic functions*. We shall introduce these functions and study their basic properties later in this chapter. □

Remark III.1.23. The proof also gives the following more precise statement. Suppose that $\Delta_{r_1} \times \Delta_{r_2} \times \cdots \times \Delta_{r_n} \subset \Omega$ for $r_1, r_2, \ldots, r_n \in \mathbb{R}_{>0}$. Then a holomorphic function on Ω admits, on this polydisc, a power-series representation

$$f(z) = \sum_{k \in (\mathbb{Z}_{\geq 0})^n} c_k z^k$$

which converges locally uniformly on $\Delta_{r_1} \times \Delta_{r_2} \times \cdots \times \Delta_{r_n}$. Indeed, Hartogs' separate analyticity theorem (Theorem III.1.18) first shows that f is continuous. On every smaller polydisc whose closure is contained in the displayed polydisc, the function f is therefore bounded. The bounded argument used in the proof above, applied successively to the variables, gives the asserted expansion and its locally uniform convergence. □

Expansion with respect to selected variables

We shall frequently use power-series expansions with respect to selected variables only. For this purpose, let d and m be positive integers with $d+m=n$, and set

$$w^1 := z^{d+1},\ w^2 := z^{d+2},\ \ldots,\ w^m := z^n,$$

and denote by

$$z' = (z^1, z^2, \ldots, z^d) \in \mathbb{C}^d, \qquad w = (w^1, w^2, \ldots, w^m) \in \mathbb{C}^m.$$

Thus we simply distinguish the last m variables among the standard coordinates of $\mathbb{C}^n$ and write a point of $\mathbb{C}^n$ as $z = (z', w)$.

For $r > 0$, we write

$$\mathbb{B}^m_r := \{w \in \mathbb{C}^m \mid |w| < r\}.$$

Let $U \subset \mathbb{C}^d$ be a bounded domain and set

$$V_r := U \times \mathbb{B}^m_r.$$

Suppose that f is holomorphic on a neighborhood of

$$\overline{V_r} = \overline{U} \times \overline{\mathbb{B}^m_r}.$$

For each fixed z', the function $w \mapsto f(z', w)$ admits a Taylor expansion at $w = 0$, by virtue of Hartogs' separate analyticity theorem (Theorem III.1.18). Hence we may write

$$f(z', w) = \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^m} a_\alpha(z')\, w^\alpha, \tag{112}$$

where $w^\alpha := (w^1)^{\alpha_1} \cdot (w^2)^{\alpha_2} \cdots (w^m)^{\alpha_m}$ $(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m))$. Note that the coefficients a_α are holomorphic functions of z'. We shall refer to (112) simply as the *Taylor expansion of f with respect to the w-variables.*

It is often convenient to group together the terms of the same degree. Set

$$f_\nu(z', w) := \sum_{|\alpha| = \nu} a_\alpha(z')\, w^\alpha \qquad (\nu \in \mathbb{Z}_{\geq 0}).$$

Then

$$f(z', w) = \sum_{\nu=0}^{\infty} f_\nu(z', w),$$

where $f_\nu(z', \bullet)$ is a homogeneous polynomial of degree ν in w.

We next introduce a norm which is particularly convenient for such expansions. Let $d\sigma_r$ denote the volume form of the sphere $\partial\mathbb{B}^m_r$ induced by the standard Euclidean metric on $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Thus

$$\int_{\partial \mathbb{B}^m_r} d\sigma_r = \frac{2\pi^m}{(m-1)!} r^{2m-1}.$$

For a holomorphic function f on a neighborhood of $\overline{V_r}$, set

$$\|f\|_{U,r} := \sup_{z' \in U} \left(\int_{\partial \mathbb{B}_r^m} |f(z', w)|^2 \, d\sigma_r(w) \right)^{1/2}. \tag{113}$$

We also use the usual supremum norm

$$\|f\|_{L^\infty(V_r)} := \sup_{V_r} |f|.$$

The quantity $\|\bullet\|_{U,r}$ is a norm on the vector space of holomorphic functions on a neighborhood of $\overline{V_r}$. Indeed, homogeneity and the triangle inequality follow from the corresponding properties of the L^2-norm on $\partial \mathbb{B}_r^m$. If $\|f\|_{U,r} = 0$, then, for every fixed $z' \in U$, the continuity of $f(z', \bullet)$ gives

$$f(z', w) = 0 \qquad (w \in \partial \mathbb{B}_r^m).$$

For every $\xi \in \partial \mathbb{B}_1^m$, the one-variable holomorphic function $\lambda \mapsto f(z', \lambda \xi)$ therefore vanishes for $|\lambda| = r$. By the (one-variable) maximum modulus principle, it vanishes for $|\lambda| < r$. Since every point of $\mathbb{B}_r^m$ lies on such a complex line through the origin, it follows that $f(z', \bullet) \equiv 0$ on $\mathbb{B}_r^m$. Thus $f \equiv 0$ on V_r.

Since both $\partial \mathbb{B}_r^m$ and $d\sigma_r$ are invariant under unitary transformations of $\mathbb{C}^m$, the norm $\| \bullet \|_{U,r}$ is unchanged under a unitary change of the w-coordinates. More precisely, for every $A \in U(m)$,

$$\|f(z', A\bullet)\|_{U,r} = \|f\|_{U,r}.$$

Recall the decomposition

$$f(z', w) = \sum_{\nu=0}^{\infty} f_\nu(z', w),$$

where each $f_\nu(z', \bullet)$ is homogeneous of degree ν in w. The homogeneous terms of different degrees are mutually orthogonal on $\partial \mathbb{B}_r^m$. Indeed, for $\theta \in \mathbb{R}$, the transformation

$$w \longmapsto e^{\sqrt{-1}\theta} w$$

preserves both $\partial \mathbb{B}_r^m$ and $d\sigma_r$, whereas

$$f_\nu(z', e^{\sqrt{-1}\theta} w) = e^{\sqrt{-1}\nu\theta} f_\nu(z', w).$$

It follows that

$$\int_{\partial \mathbb{B}_r^m} f_\nu(z', w) \overline{f_\mu(z', w)} \, d\sigma_r(w) = 0 \qquad (\nu \neq \mu).$$

Consequently,

$$\int_{\partial \mathbb{B}_r^m} |f(z', w)|^2 \, d\sigma_r(w) = \sum_{\nu=0}^{\infty} \int_{\partial \mathbb{B}_r^m} |f_\nu(z', w)|^2 \, d\sigma_r(w). \tag{114}$$

More explicitly, the monomials w^α are mutually orthogonal on $\partial\mathbb{B}_r^m$, and one has

$$\int_{\partial\mathbb{B}_r^m} |w^\alpha|^2 \, d\sigma_r(w) = \frac{2\pi^m \alpha!}{(|\alpha| + m - 1)!} r^{2|\alpha|+2m-1}, \tag{115}$$

where

$$\alpha! := \alpha_1! \cdot \alpha_2! \cdots \alpha_m!.$$

Hence, for every $\nu \in \mathbb{Z}_{\geq 0}$,

$$\int_{\partial\mathbb{B}_r^m} |f_\nu(z', w)|^2 \, d\sigma_r(w) = \frac{2\pi^m r^{2\nu+2m-1}}{(\nu + m - 1)!} \sum_{|\alpha|=\nu} \alpha! \, |a_\alpha(z')|^2. \tag{116}$$

For a positive integer N, set

$$P_{<N} f := \sum_{\nu=0}^{N-1} f_\nu \qquad \text{and} \qquad P_{\geq N} f := \sum_{\nu=N}^{\infty} f_\nu.$$

By the orthogonality above, these truncation operators satisfy

$$\|P_{<N} f\|_{U,r} \leq \|f\|_{U,r}, \qquad \|P_{\geq N} f\|_{U,r} \leq \|f\|_{U,r}. \tag{117}$$

Another elementary estimate which will be useful later is

$$\|fg\|_{U,r} \leq \|f\|_{L^\infty(V_r)} \|g\|_{U,r}, \tag{118}$$

whenever f and g are holomorphic on a neighborhood of $\overline{V_r}$. Indeed, for every fixed $z' \in U$,

$$|f(z', w) g(z', w)| \leq \|f\|_{L^\infty(V_r)} |g(z', w)| \qquad (w \in \partial\mathbb{B}_r^m),$$

and (118) follows immediately by squaring, integrating, and taking the supremum over $z' \in U$.

Remark III.1.24. When $m = 1$, the preceding formulas take a particularly simple form. In this case

$$\mathbb{B}_r^1 = \Delta_r, \qquad d\sigma_r = r \, d\theta \quad \text{on } w = re^{\sqrt{-1}\theta}.$$

Writing

$$f(z', w) = \sum_{\nu=0}^{\infty} a_\nu(z') \, w^\nu,$$

we obtain

$$\|f\|_{U,r}^2 = 2\pi r \sup_{z' \in U} \sum_{\nu=0}^{\infty} |a_\nu(z')|^2 r^{2\nu}.$$

In particular, if

$$P_{<N} f = \sum_{\nu=0}^{N-1} a_\nu(z') w^\nu,$$

then the remaining part can be written uniquely as

$$P_{\geq N} f = w^N Q_N f,$$

where

$$Q_N f = \sum_{\nu=N}^{\infty} a_\nu(z') w^{\nu-N},$$

and

$$\|Q_N f\|_{U,r} \leq r^{-N} \|f\|_{U,r}. \tag{119}$$

This observation will be useful below in the proof of the Weierstraß division theorem (Theorem III.1.28). □

Some fundamental properties of holomorphic functions

Having established Theorem III.1.18, we briefly record several fundamental properties of holomorphic functions on complex Euclidean spaces. Most of them follow from the corresponding results in one variable by applying them successively to the coordinate variables, or directly from the analyticity obtained by Hartogs' separate analyticity theorem (Theorem III.1.18). For standard versions of the statements recalled below, see [No1, §§1.2.1–1.2.2]; Liouville's theorem is also listed in [No1, Chapter 1, Exercise 6].

Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets, where $n, m \in \mathbb{Z}_{\geq 0}$. A map

$$F = (F^1, F^2, \ldots, F^m) \colon U \longrightarrow V$$

is said to be *holomorphic* if each component $F^\mu \colon U \to \mathbb{C}$ is holomorphic. Here, as usual, we regard $\mathbb{C}^0$ as the one-point space. Thus, when $n = 0$ or $m = 0$, every map as above is constant (whenever the source is non-empty), and we regard such a map as holomorphic. With this terminology, the following familiar properties hold.

For $a = (a^1, a^2, \ldots, a^n) \in \mathbb{C}^n$ and positive numbers $r_1, r_2, \ldots, r_n$, write

$$P(a; r_1, r_2, \ldots, r_n) := \left\{ z \in \mathbb{C}^n \;\middle|\; |z^j - a^j| < r_j \text{ for every } j \right\}.$$

For a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$, set

$$\alpha! := \alpha_1! \alpha_2! \cdots \alpha_n!, \qquad \partial_z^\alpha := \left(\frac{\partial}{\partial z^1}\right)^{\alpha_1} \left(\frac{\partial}{\partial z^2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial z^n}\right)^{\alpha_n}.$$

- The composition of holomorphic maps is holomorphic. More precisely, if

  $$F \colon U \longrightarrow V \qquad \text{and} \qquad G \colon V \longrightarrow W$$

  are holomorphic maps between open subsets of complex Euclidean spaces, then

  $$G \circ F \colon U \longrightarrow W$$

  is holomorphic.

- Suppose that
$$\overline{P(a; r_1, r_2, \ldots, r_n)} \subset U$$
and that f is holomorphic on a neighborhood of this closed polydisc. Then, for every $z \in P(a; r_1, r_2, \ldots, r_n)$, one has the Cauchy integral representation
$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\{|\zeta^1 - a^1| = r_1\}} \cdots \int_{\{|\zeta^n - a^n| = r_n\}} \frac{f(\zeta^1, \zeta^2, \ldots, \zeta^n)}{(\zeta^1 - z^1)(\zeta^2 - z^2) \cdots (\zeta^n - z^n)} \, d\zeta^n \cdots d\zeta^1,$$
where all circles are positively oriented.
Moreover, for every $\alpha \in (\mathbb{Z}_{\geq 0})^n$,
$$(\partial_z^\alpha f)(a) = \frac{\alpha!}{(2\pi\sqrt{-1})^n} \int_{\{|\zeta^1 - a^1| = r_1\}} \cdots \int_{\{|\zeta^n - a^n| = r_n\}} \frac{f(\zeta^1, \zeta^2, \ldots, \zeta^n)}{(\zeta^1 - a^1)^{\alpha_1 + 1} \cdots (\zeta^n - a^n)^{\alpha_n + 1}} \, d\zeta^n \cdots d\zeta^1.$$
In particular, the Cauchy inequality
$$|(\partial_z^\alpha f)(a)| \leq \frac{\alpha!}{r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n}} \sup_{P(a; r_1, r_2, \ldots, r_n)} |f| \tag{120}$$
holds.

- A locally uniform limit of holomorphic functions is holomorphic (the *Weierstraß convergence theorem*). Moreover, if $f_\nu \to f$ locally uniformly on U, then, for every $\alpha \in (\mathbb{Z}_{\geq 0})^n$,
$$\partial_z^\alpha f_\nu \longrightarrow \partial_z^\alpha f$$
locally uniformly on U.

- The identity principle holds in the following form: If U is a domain and two holomorphic functions on U coincide on a non-empty open subset of U, then they coincide on all of U.

- The maximum modulus principle holds: If U is a domain, f is holomorphic on U, and $|f|$ has a local maximum at a point of U, then f is constant on U.

- The several-variable version of Liouville's theorem holds: Every bounded holomorphic function $f \colon \mathbb{C}^n \to \mathbb{C}$ is constant.

- Montel's theorem holds: Let $\mathcal{F}$ be a family of holomorphic functions on an open set $U \subset \mathbb{C}^n$. If $\mathcal{F}$ is locally uniformly bounded, in the sense that
$$\sup_{f \in \mathcal{F}} \sup_K |f| < +\infty$$
for every compact subset $K \subset U$, then every sequence in $\mathcal{F}$ admits a subsequence which converges locally uniformly on U to a holomorphic function on U.

Let us briefly indicate why these assertions follow from what has already been established. The first assertion follows, for example, by using the local power-series representations given by Hartogs' separate analyticity theorem (Theorem III.1.18): A composition of locally convergent power series is again locally represented by a convergent power series.

The Cauchy integral representation is obtained by applying the one-variable Cauchy integral formula successively to $z^1, z^2, \ldots, z^n$. Differentiating the resulting formula with respect to the variables z^j gives the Cauchy integral representation of partial derivatives, and estimating the integrals immediately gives (120). Thus these formulas are direct several-variable analogues of the corresponding formulas recalled in § I.9.

The assertion concerning locally uniform limits follows in the same way. Indeed, on every sufficiently small closed polydisc contained in U, one may apply the Cauchy integral representation to the functions f_ν. Local uniform convergence then permits passage to the limit under the integrals. In particular, the limit satisfies the same Cauchy integral formula and is holomorphic.

For the identity principle, let f be a holomorphic function on a domain U which vanishes on a non-empty open subset. Consider the set of points $p \in U$ for which f vanishes on a neighborhood of p. This set is open. If p_ν is a sequence of such points converging to $p \in U$, then all derivatives $\partial_z^\alpha f$ vanish at p_ν. Their continuity therefore gives

$$(\partial_z^\alpha f)(p) = 0 \qquad \text{for every } \alpha \in (\mathbb{Z}_{\geq 0})^n.$$

The Taylor expansion of f at p then shows that f vanishes on a neighborhood of p. Thus the above set is also closed, and the connectedness of U proves the assertion.

Note that there is an important difference from the familiar one-variable form of the identity theorem: In several variables, the existence of an accumulation point of the zero set does not imply that the function is identically zero. For example, the holomorphic function $(z^1, z^2) \mapsto z^1$ on $\mathbb{C}^2$ vanishes identically on the complex hyperplane $\{z^1 = 0\}$ without being the zero function.

The maximum modulus principle is also reduced immediately to the one-variable case. If $|f|$ has a local maximum at $a \in U$, restrict f to small complex lines through a. The one-variable maximum modulus principle shows that these restrictions are constant near a. Hence f is constant on a neighborhood of a, and the identity principle shows that it is constant on the whole domain U.

Liouville's theorem follows directly from the Cauchy estimate. Indeed, if $|f| \leq M$ on $\mathbb{C}^n$, then, for every $a \in \mathbb{C}^n$ and every positive number R, (120) applied to the polydisc $P(a; R, R, \ldots, R)$ gives

$$\left|\frac{\partial f}{\partial z^j}(a)\right| \leq \frac{M}{R} \qquad (j = 1, 2, \ldots, n).$$

Letting $R \to +\infty$, we obtain

$$\frac{\partial f}{\partial z^j} \equiv 0 \qquad (j = 1, 2, \ldots, n),$$

and therefore f is constant.

Finally, the Cauchy estimates give uniform bounds for the first derivatives on every relatively compact subset of U for a locally uniformly bounded family of holomorphic functions. Such a family is therefore locally equicontinuous. Montel's theorem follows from the Arzelà–Ascoli theorem, an exhaustion of U by compact subsets K_ν (i.e., $K_\nu \subset \operatorname{Int} K_{\nu+1}$ and $U = \bigcup_{\nu\geq 0} K_\nu$), and the usual diagonal argument, together with the assertion above concerning locally uniform limits of holomorphic functions.

We now introduce notation for the space of holomorphic functions. For an open set $U \subset \mathbb{C}^n$, let

$$\mathcal{O}_{\mathbb{C}^n}(U)$$

denote the $\mathbb{C}$-algebra of holomorphic functions on U. As in the one-variable case discussed in § I.9, we endow $\mathcal{O}_{\mathbb{C}^n}(U)$ with the topology of locally uniform convergence. For every non-empty compact subset $K \subset U$, consider the seminorm

$$\mathcal{O}_{\mathbb{C}^n}(U) \ni f \longmapsto \max_K |f|.$$

These seminorms define a locally convex TVS topology (For $U = \emptyset$, we use the unique topology on $\mathcal{O}_{\mathbb{C}^n}(U) = \{0\}$). Choosing an exhaustion of U by a sequence of compact subsets shows that this topology is defined by a countable family of seminorms.

The space $\mathcal{O}_{\mathbb{C}^n}(U)$ is complete with respect to this topology. Indeed, a Cauchy sequence converges uniformly on every compact subset, and the resulting locally uniform limit is holomorphic by the observation above. Consequently, $\mathcal{O}_{\mathbb{C}^n}(U)$ is a Fréchet space. Moreover, Montel's theorem, together with the characterization of TVS boundedness in Mini-Exercise I.6.36, shows that every closed and TVS bounded subset of $\mathcal{O}_{\mathbb{C}^n}(U)$ is compact. Thus $\mathcal{O}_{\mathbb{C}^n}(U)$ has the Montel property, just as in the one-variable case.

Theorem III.1.18 also identifies this Fréchet space algebraically with

$$\operatorname{Ker}\left(\overline{\partial}\colon C^\infty(U) \longrightarrow \mathcal{A}^{0,1}(U)\right).$$

This identification is also compatible with the natural topologies. Indeed, the topology induced from $C^\infty(U)$ is a priori stronger than the topology of locally uniform convergence, whereas the Cauchy estimates show that all derivatives on relatively compact subsets are controlled by supremum norms on slightly larger compact subsets. Thus the two topologies coincide on the space of holomorphic functions.

We next discuss the differential of a holomorphic map. Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets, where $n, m \in \mathbb{Z}_{\geq 0}$. Let

$$F = (F^1, F^2, \ldots, F^m)\colon U \longrightarrow V$$

be holomorphic, and write

$$z = (z^1, z^2, \ldots, z^n), \qquad w = (w^1, w^2, \ldots, w^m)$$

for the standard coordinates on the source and target, respectively. We define the *holomorphic Jacobian matrix* of F at z by

$$(D^{\mathrm{hol}}F)(z) := \left(\frac{\partial F^\mu}{\partial z^j}(z)\right)_{\substack{1\leq\mu\leq m\\ 1\leq j\leq n}}.$$

At each point $z \in U$, this matrix represents the complex-linear map $T^{1,0}_z U \longrightarrow T^{1,0}_{F(z)} V$ given by

$$\frac{\partial}{\partial z^j} \longmapsto \sum_{\mu=1}^{m} \frac{\partial F^\mu}{\partial z^j}(z) \frac{\partial}{\partial w^\mu}.$$

Thus $D^{\mathrm{hol}} F$ may equivalently be regarded as a fiberwise complex-linear C^∞ map

$$T^{1,0} U \longrightarrow F^* T^{1,0} V$$

over U.

For holomorphic maps

$$U \xrightarrow{F} V \xrightarrow{G} \mathbb{C}^\ell$$

($\ell \in \mathbb{Z}_{\geq 0}$), the ordinary chain rule gives

$$D^{\mathrm{hol}}(G \circ F)(z) = (D^{\mathrm{hol}} G)(F(z))\, (D^{\mathrm{hol}} F)(z).$$

Let us compare this notation with the real Jacobian matrix DF introduced in Chapter I. A C^1 map

$$F \colon U \longrightarrow V$$

may also be regarded as a C^1 map between open subsets of $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$. Its real differential

$$DF \colon TU \longrightarrow F^* TV$$

therefore has the complexification

$$D^{\mathbb{C}} F \colon T^{\mathbb{C}} U \longrightarrow F^* T^{\mathbb{C}} V.$$

In terms of the frames introduced above, one has

$$D^{\mathbb{C}} F \left(\frac{\partial}{\partial z^j} \right) = \sum_{\mu=1}^{m} \frac{\partial F^\mu}{\partial z^j} \frac{\partial}{\partial w^\mu} + \sum_{\mu=1}^{m} \frac{\partial \overline{F^\mu}}{\partial z^j} \frac{\partial}{\partial \overline{w^\mu}},$$

$$D^{\mathbb{C}} F \left(\frac{\partial}{\partial \overline{z^j}} \right) = \sum_{\mu=1}^{m} \frac{\partial F^\mu}{\partial \overline{z^j}} \frac{\partial}{\partial w^\mu} + \sum_{\mu=1}^{m} \frac{\partial \overline{F^\mu}}{\partial \overline{z^j}} \frac{\partial}{\partial \overline{w^\mu}}.$$

In particular, when F is holomorphic,

$$D^{\mathbb{C}} F(T^{1,0} U) \subset F^* T^{1,0} V, \qquad D^{\mathbb{C}} F(T^{0,1} U) \subset F^* T^{0,1} V,$$

and

$$D^{\mathrm{hol}} F = D^{\mathbb{C}} F|_{T^{1,0} U}.$$

It is useful here to spell out the chain rule in Wirtinger coordinates. Let

$$F = (F^1, F^2, \ldots, F^m) \colon U \longrightarrow V$$

and

$$G\colon V \longrightarrow \mathbb{C}$$

be C^1 maps. Then

$$\frac{\partial(G\circ F)}{\partial z^j} = \sum_{\mu=1}^{m}\left(\frac{\partial G}{\partial w^\mu}(F)\frac{\partial F^\mu}{\partial z^j} + \frac{\partial G}{\partial \overline{w^\mu}}(F)\frac{\partial \overline{F^\mu}}{\partial z^j}\right), \tag{121}$$

$$\frac{\partial(G\circ F)}{\partial \overline{z^j}} = \sum_{\mu=1}^{m}\left(\frac{\partial G}{\partial w^\mu}(F)\frac{\partial F^\mu}{\partial \overline{z^j}} + \frac{\partial G}{\partial \overline{w^\mu}}(F)\frac{\partial \overline{F^\mu}}{\partial \overline{z^j}}\right). \tag{122}$$

These formulas are simply the ordinary real chain rule rewritten in terms of the Wirtinger derivatives.

This also explains a frequently used notation which may look slightly confusing at first. For a general smooth function on an open subset of $\mathbb{C}^n$, one often writes

$$f = f(z,\overline{z}).$$

This notation does not mean that z and $\overline{z}$ are actually independent variables: of course they are related by complex conjugation. Rather, it emphasizes that, after rewriting a function of the real variables

$$x^1, y^1, \ldots, x^n, y^n$$

in terms of

$$z^1, \overline{z^1}, \ldots, z^n, \overline{z^n},$$

the Wirtinger derivatives behave formally as differentiations with respect to independent variables, as illustrated by (121) and (122). For example,

$$|z|^2 = z\overline{z}, \qquad \frac{\partial |z|^2}{\partial z} = \overline{z}, \qquad \frac{\partial |z|^2}{\partial \overline{z}} = z.$$

When f is holomorphic, on the other hand,

$$\frac{\partial f}{\partial \overline{z^j}} = 0 \qquad (j = 1, 2, \ldots, n),$$

so the usual notation $f = f(z)$ is entirely consistent with this point of view.

We also record a consequence for holomorphic vector fields. A *holomorphic vector field* on an open set $U \subset \mathbb{C}^n$ is a vector field of the form

$$v = \sum_{j=1}^{n} v^j(z)\frac{\partial}{\partial z^j}, \qquad v^j \in \mathcal{O}_{\mathbb{C}^n}(U).$$

Using the complex conjugation on $T^{\mathbb{C}}U$, consider the real vector field

$$v_{\mathbb{R}} := v + \overline{v}.$$

If $v^j = a^j + \sqrt{-1}\, b^j$ for real-valued functions a^j and b^j, then

$$v_{\mathbb{R}} = \sum_{j=1}^{n} \left(a^j \frac{\partial}{\partial x^j} + b^j \frac{\partial}{\partial y^j} \right).$$

This is a C^∞ real vector field on U. Hence, by Proposition I.8.2, for every point $p \in U$ there exist a positive number $\varepsilon > 0$, an open neighborhood $W \subset U$ of p, and a C^∞ local flow

$$\varphi \colon (-\varepsilon, \varepsilon) \times W \longrightarrow U$$

of $v_{\mathbb{R}}$.

Let us write

$$\varphi(t,z) = \big(\varphi^1(t,z), \ldots, \varphi^n(t,z)\big), \qquad \varphi^j(t,z) = x^j(t,z) + \sqrt{-1}\, y^j(t,z).$$

Since φ is the flow of $v_{\mathbb{R}}$, its defining differential equation is

$$\frac{dx^j}{dt}(t,z) = a^j(\varphi(t,z)), \qquad \frac{dy^j}{dt}(t,z) = b^j(\varphi(t,z)) \qquad (j = 1, 2, \ldots, n).$$

Therefore,

$$\frac{d\varphi^j}{dt}(t,z) = a^j(\varphi(t,z)) + \sqrt{-1}\, b^j(\varphi(t,z)) = v^j(\varphi(t,z)).$$

Thus, after identifying U with an open subset of $\mathbb{C}^n$, the real flow equation can equivalently be written as

$$\frac{d\varphi^j}{dt}(t,z) = v^j(\varphi(t,z)) \qquad (j = 1, 2, \ldots, n), \qquad \varphi(0,z) = z. \tag{123}$$

For every fixed real t, the map

$$\varphi_t := \varphi(t, \bullet)$$

is holomorphic wherever it is defined. Indeed, set

$$B(t,z) := \left(\frac{\partial \varphi^\mu}{\partial \overline{z^j}}(t,z) \right)_{\mu,j}.$$

Differentiating (123) with respect to $\overline{z^j}$ and using the Wirtinger chain rule gives

$$\frac{\partial B}{\partial t}(t,z) = (D^{\mathrm{hol}} v)(\varphi(t,z))\, B(t,z), \qquad B(0,z) = 0.$$

The uniqueness assertion of the Cauchy–Lipschitz theorem / Picard–Lindelöf theorem (Theorem I.4.12) therefore gives

$$B(t,z) \equiv 0.$$

Thus

$$\overline{\partial} \varphi_t \equiv 0,$$

and Theorem III.1.18 shows that φ_t is holomorphic. The flow property also gives

$$\varphi_{-t} \circ \varphi_t = \mathrm{id}$$

whenever both sides are defined.

We finally record the holomorphic versions of the inverse and implicit function theorems; see [No1, Theorems 1.2.41 and 1.2.43]. They follow readily from their real counterparts recalled in Chapter I and the characterization of holomorphicity in Theorem III.1.18.

- Let $U \subset \mathbb{C}^n$ be an open subset and

  $$F\colon U \longrightarrow \mathbb{C}^n$$

  be a holomorphic map. Let $a \in U$, and assume that

  $$\det(D^{\mathrm{hol}}F)(a) \neq 0.$$

  Then there exist open neighborhoods $U_0 \subset U$ of a and $V_0 \subset \mathbb{C}^n$ of $F(a)$ such that

  $$F|_{U_0}\colon U_0 \longrightarrow V_0$$

  is bijective and its inverse

  $$(F|_{U_0})^{-1}\colon V_0 \longrightarrow U_0$$

  is holomorphic. Moreover,

  $$D^{\mathrm{hol}}(F|_{U_0}^{-1})(F(z)) = \bigl(D^{\mathrm{hol}}F(z)\bigr)^{-1} \qquad (z \in U_0).$$

- Let $\Omega \subset \mathbb{C}^n \times \mathbb{C}^m$ be open, let

  $$F = (F^1, F^2, \ldots, F^m)\colon \Omega \longrightarrow \mathbb{C}^m$$

  be holomorphic, and suppose that $(a, b) \in \Omega$ ($a \in \mathbb{C}^n$, $b \in \mathbb{C}^m$) satisfies

  $$F(a, b) = 0$$

  and

  $$\det\left(\frac{\partial F^\mu}{\partial w^\nu}(a, b)\right)_{1 \leq \mu, \nu \leq m} \neq 0$$

  (Here we use $z = (z^1, z^2, \ldots, z^n)$ and $w = (w^1, w^2, \ldots, w^m)$ for the standard coordinates on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively). Then there exist open neighborhoods $U_0 \subset \mathbb{C}^n$ of a and $V_0 \subset \mathbb{C}^m$ of b, and a unique holomorphic map

  $$g\colon U_0 \longrightarrow V_0$$

  such that

  $$F(z, w) = 0 \qquad ((z, w) \in U_0 \times V_0)$$

  if and only if $w = g(z)$.

For completeness, let us indicate why no additional analytic difficulty occurs in these statements. If F is holomorphic, its real differential is complex-linear. In particular,

$$\det{}_{\mathbb{R}}(DF)(a) = \left|\det(D^{\mathrm{hol}}F)(a)\right|^2. \tag{124}$$

Thus the assumption in the first assertion implies that $DF(a)$ is invertible as a real linear map. The real inverse function theorem gives a C^∞ local inverse G. Since $DG = (DF)^{-1}$ is again complex-linear, one has $\overline{\partial}G \equiv 0$. Hence G is holomorphic by Theorem III.1.18.

For the implicit function theorem, consider the holomorphic map

$$\Phi(z, w) := (z, F(z, w)).$$

Its holomorphic Jacobian matrix at (a, b) has the block form

$$(D^{\mathrm{hol}}\Phi)(a,b) = \begin{pmatrix} I_n & 0 \\ \dfrac{\partial F}{\partial z}(a,b) & \dfrac{\partial F}{\partial w}(a,b) \end{pmatrix},$$

which is invertible by the assumption, where I_n denotes the identity matrix of order n, and

$$\frac{\partial F}{\partial z}(a,b) := \left(\frac{\partial F^\mu}{\partial z^j}(a,b)\right)_{\substack{1\le\mu\le m\\ 1\le j\le n}}, \qquad \frac{\partial F}{\partial w}(a,b) := \left(\frac{\partial F^\mu}{\partial w^\nu}(a,b)\right)_{1\le\mu,\nu\le m}.$$

Applying the holomorphic inverse function theorem to Φ, after shrinking the neighborhoods if necessary, we obtain a holomorphic local inverse Ψ of Φ near $(a, 0)$. Since the first component of $\Phi(z, w)$ is z, the inverse map is necessarily of the form

$$\Psi(z, \eta) = \big(z, G(z, \eta)\big)$$

for a holomorphic map G. Set

$$g(z) := G(z, 0).$$

Then

$$\Phi(z, g(z)) = (z, 0),$$

and hence

$$F(z, g(z)) = 0.$$

Conversely, if (z, w) is sufficiently close to (a, b) and $F(z, w) = 0$, then

$$\Phi(z, w) = (z, 0),$$

so the uniqueness of the local inverse gives

$$(z, w) = \Psi(z, 0) = (z, g(z)).$$

Thus $w = g(z)$ is precisely the unique local solution of $F(z, w) = 0$.

1.3 Biholomorphic maps and invariance under holomorphic changes of coordinates

We have so far introduced several notions on open subsets of complex Euclidean spaces by using their standard complex coordinates. We now discuss their behavior under holomorphic changes of coordinates.

Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open subsets. A holomorphic map

$$F\colon U \longrightarrow V$$

is said to be *biholomorphic* if it is bijective and its inverse

$$F^{-1}\colon V \longrightarrow U$$

is also holomorphic. If there exists a biholomorphic map between U and V, we say that U and V are *biholomorphic* to each other (or *biholomorphically equivalent*), and write

$$U \cong V.$$

Notice that biholomorphically equivalent domains necessarily have the same complex dimension. Indeed, suppose that $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are domains, where $n, m \geq 1$, and let

$$F\colon U \longrightarrow V$$

be biholomorphic. Since holomorphic maps are continuous, both F and F^{-1} are continuous, and hence F is a homeomorphism. Fix a point $p \in U$, and put $q := F(p)$. Regarding U and V as open subsets of $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$, respectively, choose a Euclidean ball $B_2 \subset \mathbb{R}^{2n}$ centered at p with radius $r_2 > 0$ whose closure is contained in U. Since $F(B_2)$ is an open neighborhood of q, there exists $s > 0$ such that the ball $B_{\mathbb{R}^{2m}}(q, s) \subset \mathbb{R}^{2m}$ centered at q with radius s satisfies $B_{\mathbb{R}^{2m}}(q, s) \subset F(B_2)$. Set

$$W := F^{-1}\big(B_{\mathbb{R}^{2m}}(q, s)\big).$$

Then W is an open neighborhood of p contained in B_2. Hence we may choose $r_1 \in (0, r_2)$ such that the ball $B_1 \subset \mathbb{R}^{2n}$ centered at p with radius r_1 satisfies $B_1 \subset W$.

Consider the inclusions

$$B_1 \setminus \{p\} \longrightarrow W \setminus \{p\} \longrightarrow B_2 \setminus \{p\}.$$

Their composition is the natural inclusion

$$B_1 \setminus \{p\} \longrightarrow B_2 \setminus \{p\},$$

which is a homotopy equivalence. Since $B_\nu \setminus \{p\}$ is homotopy equivalent to the sphere S^{2n-1} of dimension $2n-1$ for $\nu \in \{1, 2\}$, the induced maps on homology are isomorphisms. It follows that

$$H_{2n-1}(W \setminus \{p\}) \neq 0.$$

On the other hand, the restriction of F gives a homeomorphism

$$W \setminus \{p\} \cong B_{\mathbb{R}^{2m}}(q, s) \setminus \{q\},$$

and the latter space is homotopy equivalent to S^{2m-1}. Therefore, by the computation of the homology of spheres in Mini-Exercise I.7.17,

$$H_{2n-1}(S^{2m-1}) \neq 0.$$

Since $2n - 1 > 0$, this implies $2n - 1 = 2m - 1$, and hence $n = m$.

Let us now fix a biholomorphic map

$$F \colon U \longrightarrow V$$

between open subsets of $\mathbb{C}^n$. Write

$$z = (z^1, z^2, \ldots, z^n) \qquad \text{and} \qquad w = (w^1, w^2, \ldots, w^n)$$

for the standard coordinates on U and V, respectively. By the holomorphic inverse function theorem, or directly by applying the chain rule to $F^{-1} \circ F = \mathrm{id}_U$, the holomorphic Jacobian $(D^{\mathrm{hol}}F)(z)$ is invertible at every point $z \in U$, and

$$(D^{\mathrm{hol}}F^{-1})(F(z)) = \big((D^{\mathrm{hol}}F)(z)\big)^{-1}.$$

In particular, the differential of F induces complex-linear isomorphisms

$$T^{1,0}_z U \longrightarrow T^{1,0}_{F(z)} V$$

and

$$T^{0,1}_z U \longrightarrow T^{0,1}_{F(z)} V.$$

More explicitly,

$$F^*(dw^\mu) = \sum_{j=1}^{n} \frac{\partial F^\mu}{\partial z^j}\, dz^j,$$

and

$$F^*(d\overline{w^\mu}) = \sum_{j=1}^{n} \overline{\left(\frac{\partial F^\mu}{\partial z^j}\right)}\, d\overline{z^j}.$$

Thus a biholomorphic map preserves the decomposition into types. In particular,

$$F^* \colon \mathcal{A}^{p,q}(V) \longrightarrow \mathcal{A}^{p,q}(U)$$

for every $p, q \in \mathbb{Z}_{\geq 0}$.

Several notions introduced above are consequently invariant under biholomorphic changes of coordinates. We record the basic ones here.

- For a function $f\colon V \to \mathbb{C}$,

$$f \text{ is holomorphic} \quad \Longleftrightarrow \quad f \circ F \text{ is holomorphic.}$$

More generally, holomorphicity of maps between open subsets of complex Euclidean spaces is invariant under biholomorphic changes of coordinates on the source and target.

In particular, pullback gives an isomorphism of $\mathbb{C}$-algebras

$$F^*\colon \mathcal{O}_{\mathbb{C}^n}(V) \longrightarrow \mathcal{O}_{\mathbb{C}^n}(U), \qquad f \longmapsto f \circ F.$$

With respect to the Fréchet topologies of locally uniform convergence introduced above, this is also an isomorphism of Fréchet spaces.

- The bidegree of a smooth form is invariant under biholomorphic changes of coordinates. Moreover,

$$F^*(\overline{\alpha}) = \overline{F^*\alpha}$$

for every smooth complex-valued form α. Consequently, reality of (p,p)-forms is preserved by pullback.

The same assertions hold for currents. Indeed, if $T \in \mathcal{K}^{p,q}(V)$, we define its pullback by the biholomorphic map F by

$$(F^*T)(\eta) := T\big((F^{-1})^*\eta\big)$$

for every compactly supported smooth test form η of bidegree $(n-p, n-q)$ on U. Then

$$F^*T \in \mathcal{K}^{p,q}(U).$$

- The operators ∂ and $\overline{\partial}$ commute with biholomorphic pullback:

$$F^*(\partial\alpha) = \partial(F^*\alpha), \qquad F^*(\overline{\partial}\alpha) = \overline{\partial}(F^*\alpha). \tag{125}$$

The same formulas hold for currents.

Indeed, pullback commutes with the exterior derivative d, and $d = \partial + \overline{\partial}$. Since biholomorphic pullback preserves bidegree, comparison of the components of different bidegrees gives (125). Consequently,

$$F^*(d^c\alpha) = d^c(F^*\alpha), \qquad F^*(dd^c\alpha) = dd^c(F^*\alpha).$$

- Semi-positivity of real (p,p)-forms is invariant under biholomorphic changes of coordinates:

$$\omega \geq 0 \quad \Longleftrightarrow \quad F^*\omega \geq 0.$$

The same is true for semi-positive (p,p)-currents. In particular, when $p = 1$, positivity and semi-positivity of real $(1,1)$-forms are biholomorphically invariant.

Let us explain the last assertion more explicitly in the case of real $(1,1)$-forms. Write

$$\omega = \sqrt{-1}\sum_{\mu,\nu=1}^{n} h_{\mu\overline{\nu}}(w)\, dw^{\mu}\wedge d\overline{w^{\nu}}$$

on V, and set

$$A(z) := (D^{\mathrm{hol}}F)(z) = \left(\frac{\partial F^{\mu}}{\partial z^{j}}(z)\right)_{\mu,j=1}^{n}.$$

Then

$$F^{*}\omega = \sqrt{-1}\sum_{j,k=1}^{n} g_{j\overline{k}}(z)\, dz^{j}\wedge d\overline{z^{k}},$$

where

$$\big(g_{j\overline{k}}(z)\big)_{j,k=1}^{n} = {}^{t}A(z)\big(h_{\mu\overline{\nu}}(F(z))\big)_{\mu,\nu=1}^{n}\overline{A(z)}. \tag{126}$$

Since $A(z)$ is invertible, the two Hermitian forms have the same numbers of positive, negative, and zero directions. Thus their inertia is unchanged by a biholomorphic change of coordinates.

Notice that the numerical values of their eigenvalues need not be preserved by an arbitrary biholomorphic change of coordinates. What is invariant here is the Hermitian form itself, up to the natural complex-linear identification of tangent spaces, and hence its rank, positivity, semi-positivity, and inertia.

This observation also clarifies the normal forms considered earlier. The affine coordinate changes in Lemma III.1.8 are biholomorphic. Moreover, the polynomial maps used in Lemma III.1.9 are holomorphic and have invertible holomorphic Jacobian at the point under consideration. Hence, by the holomorphic inverse function theorem, after restricting to sufficiently small neighborhoods, they are biholomorphic changes of coordinates. Thus the normal forms obtained there are normal forms with respect to holomorphic changes of coordinates. Notice also that

$$d(F^{*}\omega) = F^{*}(d\omega),$$

so the condition that $d\omega$ vanishes at a given point, which appears in Lemma III.1.9, is itself invariant under such coordinate changes.

We next apply the same observation to complex Hessians and Levi forms. Let u be a real-valued C^{2} function on V. By (125),

$$\sqrt{-1}\partial\overline{\partial}(u\circ F) = F^{*}(\sqrt{-1}\partial\overline{\partial}u).$$

Equivalently, if

$$H(u) = \left(\frac{\partial^{2}u}{\partial w^{\mu}\partial\overline{w^{\nu}}}\right)_{\mu,\nu=1}^{n}$$

denotes the complex Hessian matrix of u, then

$$H(u\circ F)(z) = {}^{t}A(z)H(u)(F(z))\overline{A(z)}. \tag{127}$$

Consequently, the rank and the numbers of positive, negative, and zero eigenvalues of the Levi form of a real-valued function are independent of the choice of holomorphic coordinates. In particular, the conditions

$$\sqrt{-1}\partial\overline{\partial}u \geq 0, \qquad \sqrt{-1}\partial\overline{\partial}u > 0, \qquad \partial\overline{\partial}u = 0$$

are biholomorphically invariant.

There is a completely parallel statement for real hypersurfaces. Let $M \subset V$ be a C^2 real hypersurface and set

$$\widetilde{M} := F^{-1}(M) \subset U.$$

If ρ is a local defining function of M, then

$$\widetilde{\rho} := \rho \circ F$$

is a local defining function of $\widetilde{M}$. Moreover,

$$(D^{\mathrm{hol}}F)_z \left(T^{1,0}_z \widetilde{M}\right) = T^{1,0}_{F(z)} M$$

for every $z \in \widetilde{M}$. For $\xi, \eta \in T^{1,0}_z \widetilde{M}$, one has

$$H_{\widetilde{M},z}(\widetilde{\rho})(\xi,\eta) = H_{M,F(z)}(\rho)\left((D^{\mathrm{hol}}F)_z(\xi), (D^{\mathrm{hol}}F)_z(\eta)\right). \tag{128}$$

Hence the rank and inertia of the Levi form are unchanged under holomorphic changes of coordinates.

This coordinate invariance is independent of the separate issue of the choice of a defining function discussed in Mini-Exercise III.1.12. In particular, if a side of the hypersurface has been fixed and the sign convention for defining functions is chosen as above, the numbers of positive, negative, and zero eigenvalues of the Levi form are independent both of the defining function and of the holomorphic coordinates. The property of being Levi-flat is therefore also invariant under biholomorphic changes of coordinates.

The same point applies to the second-order normal form of real hypersurfaces in Lemma III.1.14. The polynomial map used there is holomorphic and has invertible holomorphic Jacobian at the origin. Thus, after shrinking the neighborhoods, it may be regarded as a biholomorphic change of coordinates by the holomorphic inverse function theorem.

Finally, holomorphic vector fields are also preserved by biholomorphic changes of coordinates. If

$$v = \sum_{j=1}^{n} v^j(z) \frac{\partial}{\partial z^j}$$

is a holomorphic vector field on U, then its push-forward by F, defined by

$$(F_* v)_{F(z)} := (D^{\mathrm{hol}}F)_z(v_z),$$

is a holomorphic vector field on V. Thus the notion of a holomorphic vector field, like the notions recorded above, does not depend on the choice of holomorphic coordinates.

REMARK III.1.25. Let ω be a positive real $(1,1)$-form on Ω. By Lemma III.1.9 and the holomorphic inverse function theorem, if $d\omega = 0$, then for every point $p \in \Omega$, there exist holomorphic local coordinates $w = (w^1, w^2, \dots, w^n)$ on a neighborhood of p such that $p = (0, 0, \dots, 0)$ and, in the sense of the correspondence (95), the positive definite Hermitian matrix corresponding to ω is of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} + O(|w|^2) \quad \text{as } w \to 0. \tag{129}$$

Conversely, if such holomorphic coordinates exist at p, then $d\omega$ vanishes at p.

The normal form (129) is useful because many computations reduce to Euclidean computations at the point p. For example, let g be the Riemannian metric induced by ω as in Remark III.1.3. Put $\xi^j := \operatorname{Re} w^j$ and $\eta^j := \operatorname{Im} w^j$ for each $j \in \{1, 2, \dots, n\}$. By Mini-Exercise III.1.4, we have

$$g = 2\sum_{j=1}^{n} d\xi^j \otimes d\xi^j + 2\sum_{j=1}^{n} d\eta^j \otimes d\eta^j \quad \text{at } p.$$

Hence the Riemannian volume form dV_g associated with g satisfies

$$dV_g = 2^n \bigwedge_{j=1}^{n} (d\xi^j \wedge d\eta^j) \quad \text{at } p.$$

On the other hand,

$$\omega^{\wedge n} = n! \bigwedge_{j=1}^{n} (\sqrt{-1}\, dw^j \wedge d\overline{w^j}) = n! \bigwedge_{j=1}^{n} (2\, d\xi^j \wedge d\eta^j) \quad \text{at } p.$$

Therefore

$$dV_g = \frac{\omega^{\wedge n}}{n!} \quad \text{at } p.$$

Since p was arbitrary, this identity holds on Ω. □

1.4 From stalkwise algebra to coherence

Several arguments in this subsection proceed by induction on the number of variables. It is therefore convenient to distinguish the last variable. Let n be a positive integer and write $z = (z', w)$, where $z' = (z^1, z^2, \dots, z^{n-1})$ and $w = z^n$. We use the stalk notation of Example II.3.55. By Hartogs' separate analyticity theorem (Theorem III.1.18), the ring $\mathcal{O}_{\mathbb{C}^n,0}$ is identified with $\mathbb{C}\{z', w\}$, the ring of convergent power series at the origin. When $n = 1$, there are no z'-variables, and expressions involving only z' are understood to be constants; in particular, $\mathbb{C}\{z'\} = \mathbb{C}$ in this case. A germ u is a unit if and only if $u(0) \neq 0$: in that case a representative of u has a holomorphic reciprocal near the origin, and the converse follows by evaluating an inverse at 0.

Weierstraß division and preparation

The elementary division of a power series by w^N in Remark III.1.24 has a useful extension: a holomorphic function which is sufficiently close to w^N can also serve as a divisor, with a remainder polynomial in w of degree less than N. This extension is formulated in the Weierstraß division theorem (Theorem III.1.28). We prove it by the Banach fixed point theorem (Theorem I.3.8), using the norm already introduced in (113). For the classical statements on suitable polydiscs and a different proof using Cauchy's integral formula, see [De4, Chapter II, § 2.A, Theorems (2.1) and (2.3)].

Definition III.1.26. Let N be a non-negative integer. A germ $f \in \mathbb{C}\{z', w\}$ is said to be *regular of order N in w at the origin* if

$$f(0, w) = cw^N + O(w^{N+1}) \qquad (c \in \mathbb{C}^*).$$

A *Weierstraß polynomial of degree N in w* is a polynomial

$$P(z', w) = w^N + \sum_{j=0}^{N-1} a_j(z')w^j, \qquad a_j \in \mathbb{C}\{z'\}, \quad a_j(0) = 0.$$

For $N = 0$, this means $P = 1$. □

Note that when $n = 1$, the only Weierstraß polynomial of degree N is w^N.

Remark III.1.27. In this remark, we return to the several-variable setting used in (113). Let d, m be positive integers, let $U \subset \mathbb{C}^d$ be a bounded domain, and let $r > 0$. Write $\mathbb{B}_r^m := \{w \in \mathbb{C}^m \mid |w| < r\}$ and $V_r := U \times \mathbb{B}_r^m$, and use the norm $\| \bullet \|_{U,r}$ defined there. For the later use of the Banach fixed point theorem (Theorem I.3.8), we briefly describe a natural complete space associated with this norm. This allows us to take a limit without requiring holomorphic extension across the boundary of V_r.

First, let $A(\mathbb{B}_r^m)$ denote the Hilbert space obtained by completing the holomorphic polynomials in w with respect to the norm

$$\|h\|_{A(\mathbb{B}_r^m)} := \left(\int_{\partial \mathbb{B}_r^m} |h(w)|^2 \, d\sigma_r(w) \right)^{1/2}.$$

Thus, for every function f holomorphic on a neighborhood of $\overline{V_r}$ and every fixed $z' \in U$, the function $f(z', \bullet)$ naturally defines an element of $A(\mathbb{B}_r^m)$.

Let $\mathcal{A}_{U,r}$ be the vector space of functions on V_r which extend holomorphically to a neighborhood of $\overline{V_r}$, equipped with the norm $\| \bullet \|_{U,r}$ defined in (113). We denote by $X_{U,r}$ the completion of this normed vector space. Thus $X_{U,r}$ is a Banach space by definition.

We note that elements of $X_{U,r}$ can naturally be represented by holomorphic functions on V_r. Indeed, for every $\rho \in (0, r)$, there exists a constant $C_{m,\rho,r} > 0$ such that

$$\sup_{(z',w) \in U \times \mathbb{B}_\rho^m} |f(z', w)| \leq C_{m,\rho,r} \, \|f\|_{U,r} \qquad (f \in \mathcal{A}_{U,r}). \tag{130}$$

For example, by (115) and the Cauchy–Schwarz inequality, one may take

$$C_{m,\rho,r} = \left(\frac{(m-1)!}{2\pi^m r^{2m-1}}\right)^{1/2} \left(1 - \frac{\rho^2}{r^2}\right)^{-m/2}.$$

Hence every Cauchy sequence in $\mathcal{A}_{U,r}$ with respect to $\| \bullet \|_{U,r}$ converges uniformly on $U \times \mathbb{B}^m_\rho$ for every $\rho < r$. Its limit is therefore holomorphic on V_r by the theorem on locally uniform limits of holomorphic functions. The limits obtained for different values of ρ clearly agree on their common domains, and hence determine a unique holomorphic function on V_r. This correspondence does not depend on the choice of the Cauchy sequence, again by (130). It is also injective, as follows. For a Cauchy sequence $\{f_k\}$ in $\mathcal{A}_{U,r}$, the elements $f_k(z', \bullet)$ converge in $A(\mathbb{B}^m_r)$ to an element $H_{z'}$, uniformly for $z' \in U$. The norm of the element of $X_{U,r}$ represented by $\{f_k\}$ is therefore

$$\lim_{k\to\infty} \|f_k\|_{U,r} = \sup_{z'\in U} \|H_{z'}\|_{A(\mathbb{B}^m_r)}.$$

By the orthogonality of the monomials, each monomial coefficient is a continuous linear function on $A(\mathbb{B}^m_r)$. Cauchy's formula and the convergence on smaller balls show that these coefficients of $H_{z'}$ are precisely the Taylor coefficients of the associated holomorphic function at $(z', 0)$ in the w-variables. If that function vanishes, all these coefficients are zero. Since the polynomials are dense in $A(\mathbb{B}^m_r)$, this gives $H_{z'} = 0$ for every $z' \in U$, and the displayed identity shows that the element of $X_{U,r}$ is zero. In this sense, we regard $X_{U,r}$ as a Banach space of holomorphic functions on V_r.

Finally, any map $T\colon \mathcal{A}_{U,r} \to \mathcal{A}_{U,r}$ satisfying an estimate of the form

$$\|T(f) - T(g)\|_{U,r} \le C\|f - g\|_{U,r}$$

extends uniquely to a map on $X_{U,r}$ with the same estimate. Indeed, the Lipschitz estimate defines the extension on limits of Cauchy sequences, and density gives uniqueness. In particular, if $C < 1$, the resulting map on $X_{U,r}$ is a contraction, so that the Banach fixed point theorem (Theorem I.3.8) applies. □

For the proof below, we use this completion only when $n \ge 2$ and $m = 1$, so that $V_r = U \times \Delta_r$ with $U \subset \mathbb{C}^{n-1}$. We keep the notation $\| \bullet \|_{U,r}$ for the norm on $X_{U,r}$. Fix a positive integer N. The operators $P_{<N}$ and Q_N of Remark III.1.24 map $\mathcal{A}_{U,r}$ to itself. Indeed, the Taylor coefficients extend holomorphically to a neighborhood of $\overline{U}$, and removing the terms of degree less than N makes division by w^N holomorphic also at $w = 0$. The estimates (117) and (119) extend these operators continuously to $X_{U,r}$, with

$$\|P_{<N}h\|_{U,r} \le \|h\|_{U,r}, \qquad \|Q_N h\|_{U,r} \le r^{-N}\|h\|_{U,r}.$$

Multiplication by any fixed $b \in \mathcal{A}_{U,r}$ likewise extends to $X_{U,r}$ by (118), and satisfies

$$\|bh\|_{U,r} \le \|b\|_{L^\infty(V_r)}\|h\|_{U,r}.$$

On the associated holomorphic functions, these extensions are still Taylor truncation, division of the Taylor tail by w^N, and multiplication by b, respectively. This follows by passing to locally uniform limits, using Cauchy's formula for the Taylor coefficients. In particular, the identities

$$h = P_{<N}h + w^N Q_N h, \qquad Q_N(w^N h) = h$$

hold for every $h \in X_{U,r}$, first on the dense subspace $\mathcal{A}_{U,r}$ and then by continuity. Thus $Q_N h = 0$ implies that h is a polynomial in w of degree less than N, whose coefficients are holomorphic on U.

THEOREM III.1.28 (Weierstraß division theorem). *Let $f \in \mathbb{C}\{z', w\}$ be regular of order N in w at the origin. For every $g \in \mathbb{C}\{z', w\}$, there exist unique germs $q \in \mathbb{C}\{z', w\}$ and $r_j \in \mathbb{C}\{z'\}$ $(0 \le j < N)$ such that*

$$g = fq + R, \qquad R(z', w) = \sum_{j=0}^{N-1} r_j(z') w^j. \tag{131}$$

When $N = 0$, the remainder is $R = 0$.

PROOF. If $N = 0$, then f is a unit, and the assertion holds with $q = g/f$ and $R = 0$. Assume henceforth that $N \ge 1$.

We first treat $n = 1$ directly, without using the completion above. Write $f(w) = w^N v(w)$, where $v(0) \ne 0$. The Taylor expansion of g gives $g = w^N t + R$, where t is holomorphic near 0 and R is the Taylor polynomial of g of degree less than N. Then $q = t/v$ gives (131). For uniqueness, the difference of two remainders would be both a polynomial of degree less than N and divisible by w^N as a holomorphic germ, so it must vanish; the corresponding quotients then agree as well.

Suppose now that $n \ge 2$. Choose representatives of f and g holomorphic near the origin, and let $c \ne 0$ be the coefficient of w^N in $f(0, w)$. Put

$$\widehat{f} := c^{-1} f, \qquad E(z', w) := w^N - \widehat{f}(z', w).$$

Since $E(0, w) = O(w^{N+1})$, we can choose $r > 0$ sufficiently small that the representatives are defined near $\{0\} \times \overline{\Delta_r}$ and

$$\sup_{|w| \le r} |E(0, w)| \le \frac{1}{4} r^N.$$

Keeping this r fixed, continuity on the compact disc allows us to choose a polydisc U centered at 0 in $\mathbb{C}^{n-1}$ such that f and g are holomorphic on a neighborhood of $\overline{V_r} = \overline{U} \times \overline{\Delta_r}$ and

$$\sup_{z' \in U,\, |w| \le r} |E(z', w) - E(0, w)| \le \frac{1}{4} r^N.$$

Consequently,

$$\kappa := r^{-N} \|E\|_{L^\infty(V_r)} \le \frac{1}{2} < 1.$$

Here the choice of V_r may depend on the representatives of both f and g; no common domain for all germs g is being asserted.

Consider the map

$$T\colon X_{U,r} \longrightarrow X_{U,r}, \qquad T(h) := Q_N g + Q_N(Eh).$$

Here Q_N is the continuous extension of the operator in Remark III.1.24, as described above. All the operations take place on the same Banach space. The preceding estimates give

$$\|T(h_1) - T(h_2)\|_{U,r} \le r^{-N}\|E\|_{L^\infty(V_r)}\|h_1 - h_2\|_{U,r} = \kappa\|h_1 - h_2\|_{U,r}.$$

By the Banach fixed point theorem (Theorem I.3.8), T has a unique fixed point $h \in X_{U,r}$. By Remark III.1.27, this is a holomorphic function on V_r. Set $q := c^{-1}h$ and $R := g - \widehat{f}h = g - fq$. The fixed point identity implies

$$Q_N R = Q_N g - Q_N(w^N h) + Q_N(Eh) = Q_N g - h + Q_N(Eh) = 0.$$

Therefore $R = P_{<N}R$, which has the required form, with holomorphic coefficients on U. Taking germs at the origin proves existence.

It remains to check uniqueness as germs, not merely uniqueness in the particular space $X_{U,r}$ just used. Suppose that $g = fq_1 + R_1 = fq_2 + R_2$ as germs, with both remainders of degree less than N in w. Repeat the choice of r and U inside a common neighborhood on which all these germs have holomorphic representatives, so that $\kappa < 1$ still holds and all representatives extend across $\overline{V_r}$. Put $h := c(q_1 - q_2)$ and $S := R_1 - R_2$. Then $(w^N - E)h + S = 0$ and $Q_N S = 0$, whence

$$h = Q_N(Eh), \qquad \|h\|_{U,r} \le \kappa\|h\|_{U,r}.$$

Thus $h = 0$ since $\kappa < 1$, and then $S = 0$. The Taylor coefficients of S vanish as well, which proves all the asserted uniqueness statements. □

We now apply division to $g = w^N$. This produces a distinguished polynomial factor without choosing or labeling the individual roots.

THEOREM III.1.29 (Weierstraß preparation theorem). *Let $f \in \mathbb{C}\{z', w\}$ be regular of order N in w at the origin. There exist a unique unit $u \in \mathbb{C}\{z', w\}$ and a unique Weierstraß polynomial P of degree N in w such that*

$$f = uP. \tag{132}$$

PROOF. For $N = 0$, take $P = 1$ and $u = f$. Assume that $N \ge 1$. By the Weierstraß division theorem (Theorem III.1.28), write

$$w^N = fq + R, \qquad R = \sum_{j=0}^{N-1} r_j(z')w^j.$$

Write $f(0,w) = w^N v(w)$ for a one-variable unit v, and put $c := v(0)$. Restriction to $z' = 0$ gives

$$R(0,w) = w^N\big(1 - v(w)q(0,w)\big).$$

The left-hand side is a polynomial of degree less than N, and hence is zero. It follows that $r_j(0) = 0$ for all j and that $v(w)q(0,w) = 1$. In particular, $q(0) = c^{-1} \neq 0$. Thus $u := q^{-1}$ is a unit and $P := w^N - R$ is a Weierstraß polynomial, with $f = uP$ and $u(0) = c$.

For uniqueness, suppose also that $f = \widetilde{u}\widetilde{P}$, where $\widetilde{u}$ is a unit and $\widetilde{P}$ is a Weierstraß polynomial of degree N. There are two divisions of P by $\widetilde{P}$:

$$P = \widetilde{P}\,(u^{-1}\widetilde{u}) + 0 = \widetilde{P}\cdot 1 + (P - \widetilde{P}).$$

The polynomial $P - \widetilde{P}$ has degree less than N because both polynomials are monic of degree N. The uniqueness in the Weierstraß division theorem (Theorem III.1.28) therefore gives $P = \widetilde{P}$ and $u^{-1}\widetilde{u} = 1$. □

Local analytic algebra

We now apply Weierstraß division and preparation to the rings of holomorphic germs. Write

$$R_n := \mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z^1, z^2, \ldots, z^n\}, \qquad R_0 := \mathbb{C}.$$

By the description of germs and units above, R_n is a local ring with maximal ideal

$$\mathfrak{m}_n := \{f \in R_n \mid f(0) = 0\}.$$

Evaluation at 0 identifies $R_n/\mathfrak{m}_n$ with $\mathbb{C}$. For $n \geq 1$, grouping the nonconstant monomials of a convergent Taylor series according to a variable which divides them gives $\mathfrak{m}_n = (z^1, z^2, \ldots, z^n)$; the grouped series still converge on a smaller polydisc. The identity theorem on a common connected neighborhood of the origin shows that $fg = 0$ implies $f = 0$ or $g = 0$ in R_n; hence R_n is an integral domain.

Lemma III.1.30. *Let $n \geq 1$ and $f \in R_n \setminus \{0\}$. There exists an invertible complex-linear map $L\colon \mathbb{C}^n \to \mathbb{C}^n$ such that the germ $f \circ L$ is regular in its last variable at the origin.*

Proof. Let H_d be the first nonzero homogeneous term in the Taylor expansion of f, where $d \geq 0$ is its total degree. Choose a nonzero vector $v \in \mathbb{C}^n$ with $H_d(v) \neq 0$, and extend v to a basis in which it is the last vector. Let L be the linear map having these basis vectors as its columns. Writing the variables in its domain as (z', w), we obtain

$$(f \circ L)(0,w) = f(wv) = H_d(v)w^d + O(w^{d+1}).$$

This is precisely regularity of order d in w. □

Composition with L is a ring automorphism of R_n, with inverse composition with L^{-1}. Thus such a coordinate change, chosen for the germ under consideration, preserves the assertions about ideals and factorization below.

The next consequence of division records the quotient ring and its finite free module structure. For the corresponding comparison of polynomial and analytic divisibility, see [No1, Lemma 2.2.15] and [De4, Chapter II, Lemma (2.8)]. We again use the notation $z = (z', w)$ fixed at the beginning of this subsection.

PROPOSITION III.1.31. *Let $n \geq 1$, put $A := \mathbb{C}\{z'\}$ and $B := \mathbb{C}\{z', w\}$, and let $P \in A[w]$ be a Weierstraß polynomial of degree $N \geq 1$ in w. For $n = 1$, the ring A means $\mathbb{C}$. The inclusion $A[w] \subset B$ induces an isomorphism of A-algebras*

$$A[w]/PA[w] \;\cong\; B/PB. \tag{133}$$

Moreover, the map

$$A^{\oplus N} \longrightarrow B/PB, \qquad (c_0, c_1, \ldots, c_{N-1}) \longmapsto \left[\sum_{j=0}^{N-1} c_j w^j\right] \tag{134}$$

is an isomorphism of A-modules. In particular, B/PB is a free A-module with basis given by the classes of $1, w, \ldots, w^{N-1}$.

PROOF. By the Weierstraß division theorem (Theorem III.1.28), every $g \in B$ has a unique expression $g = Pq + R$, where $q \in B$ and $R \in A[w]$ has degree less than N (with $R = 0$ allowed). The coefficients of this unique remainder give the inverse of the A-linear map (134). For $H \in A[w]$, ordinary polynomial division by the monic polynomial P gives the same remainder, by uniqueness of analytic division. Thus $H \in PB$ holds exactly when $H \in PA[w]$. This identifies the kernel of the surjective ring homomorphism $A[w] \to B/PB$ and proves (133). □

The map in (134) is an isomorphism of modules, not of rings with componentwise multiplication: products of remainders must be reduced modulo P. For $n = 2$, it replaces a germ in (z, w) modulo P by N germs in z alone. This reduction is the key to the next proof.

THEOREM III.1.32 (Noetherianity of the local rings of holomorphic germs). *For every $n \geq 0$ and every $a \in \mathbb{C}^n$, the local ring $\mathcal{O}_{\mathbb{C}^n,a}$ is Noetherian.*

PROOF. See [No1, Theorem 2.2.20] and [De4, Chapter II, Theorem (2.7)]. We use Demailly's argument with remainder coefficients, expressed through the preceding proposition. Translation of a to the origin identifies $\mathcal{O}_{\mathbb{C}^n,a}$ with R_n, so it suffices to prove the assertion for R_n. The case $n = 0$ is the field case. For $n = 1$, Mini-Exercise I.9.2 (ii) shows that R_1 is a PID, hence is Noetherian.

Suppose that $n \geq 2$ and R_{n-1} is Noetherian. It suffices to consider a nonzero proper ideal $I \subset R_n$. Choose $f \in I \setminus \{0\}$. Lemma III.1.30 and the Weierstraß preparation theorem (Theorem III.1.29) allow us, after a linear coordinate change, to write $f = uP$, where P is a Weierstraß polynomial of positive degree N. Put $A := \mathbb{C}\{z'\}$ and $B := \mathbb{C}\{z', w\}$. Since u is a unit, $P \in I$, and Proposition III.1.31 gives

$$I/PB \;\subset\; B/PB \;\cong\; A^{\oplus N} \qquad \text{as } A\text{-modules.}$$

The induction hypothesis and Mini-Exercise I.5.4 (i) show that I/PB is finitely generated over A. Lift a finite set of generators to $g_1, g_2, \dots, g_r \in I$. Subtracting an A-linear combination of these lifts from any element of I leaves a multiple of P, so

$$I = (P, g_1, g_2, \dots, g_r).$$

The coordinate change transports these generators back to the original ideal. □

For factoriality, we use Gauß' theorem: if A is a UFD, then $A[w]$ is a UFD (see [Lan2, Chapter IV, Theorem 2.3] and [No1, Theorem 2.2.1]). The analytic step is to compare factorizations of a Weierstraß polynomial in $A[w]$ with factorizations among holomorphic germs.

THEOREM III.1.33 (Factoriality of the local rings of holomorphic germs). *For every $n \geq 0$ and every $a \in \mathbb{C}^n$, the ring $\mathcal{O}_{\mathbb{C}^n,a}$ is a UFD. Thus each nonzero nonunit is a finite product of irreducible germs, and this factorization is unique up to order and multiplication of the factors by units.*

PROOF. We outline the argument of [No1, Theorem 2.2.12 and Lemma 2.2.14]; see also [De4, Chapter II, Lemma (2.9) and Theorem (2.10)]. After translation, argue by induction on n. The case $n = 0$ is the field case, and for $n = 1$, Mini-Exercise I.9.2 (ii) shows that R_1 is a PID, hence is a UFD. For $n \geq 2$, put $A := R_{n-1}$ and $B := \mathbb{C}\{z', w\}$, and assume that A is a UFD. By Lemma III.1.30 and the Weierstraß preparation theorem (Theorem III.1.29), a nonzero nonunit of R_n becomes, after a linear coordinate change,

$$f = uP,$$

where $u \in B$ is a unit and $P \in A[w]$ is a Weierstraß polynomial of positive degree N.

In a factorization of P into irreducibles in the UFD $A[w]$, the leading coefficients are units in A since P is monic; normalizing them gives

$$P = Q_1 Q_2 \cdots Q_s$$

with monic irreducible $Q_j \in A[w]$ of positive degrees d_j. Each $Q_j(0, w)$ is a monic divisor of $P(0, w) = w^N$, hence equals w^{d_j}. Thus each Q_j is a Weierstraß polynomial.

The key observation is that analytic factors give no additional factorizations. If a Weierstraß polynomial Q satisfies $Q = gh$ with nonunits $g, h \in B$, then $g(0, w)$ and $h(0, w)$ are nonzero, because their product is a power of w. Applying the Weierstraß preparation theorem (Theorem III.1.29) to both factors, write

$$g = u_1 P_1, \qquad h = u_2 P_2,$$

where $P_1, P_2 \in A[w]$ are Weierstraß polynomials of positive degree. The uniqueness in that theorem gives

$$Q = P_1 P_2, \qquad u_1 u_2 = 1.$$

In particular, each Q_j above remains irreducible in B. Conversely, a nontrivial polynomial factorization of a Weierstraß polynomial has nonunit analytic factors, by the normalization above. Preparing a factorization of P into irreducible germs therefore gives a factorization into irreducible polynomials; their product is P by uniqueness of preparation. Uniqueness in $A[w]$ now gives uniqueness in B, up to units and order. This proves the required factorization statement, and the linear coordinate change transfers it back to the original germ. □

REMARK III.1.34. Every quotient R_n/I is Noetherian, since ascending chains of ideals lift to R_n. Factoriality need not pass to a quotient: the nonzero class of w in $\mathbb{C}\{w\}/(w^2)$ has square zero, so this quotient is not even an integral domain. □

Finite systems of holomorphic relations

We next consider finite systems of functions on a common open set, rather than only their germs at one fixed point. We use the language of sheaves of modules from Chapter II, §3.2, and of ringed spaces from Definition II.3.89. Let $(X, \mathcal{R}_X)$ be a ringed space and let $\mathcal{F}$ be a sheaf of $\mathcal{R}_X$-modules. For an open subset $V \subset X$ and sections $s_1, s_2, \ldots, s_q \in \Gamma(V, \mathcal{F})$, the morphism

$$(\mathcal{R}_X|_V)^{\oplus q} \longrightarrow \mathcal{F}|_V, \qquad (g_1, g_2, \ldots, g_q) \longmapsto \sum_{j=1}^{q} g_j s_j,$$

has a kernel, called the *sheaf of relations* among $s_1, s_2, \ldots, s_q$ and denoted by $\mathcal{R}(s_1, s_2, \ldots, s_q)$. Thus its sections on an open set $W \subset V$ are the tuples satisfying $\sum_{j=1}^{q} g_j(s_j|_W) = 0$ on W. By Proposition II.3.58, its stalk at $x \in V$ is the module of relations among the germs $s_{1,x}, s_{2,x}, \ldots, s_{q,x}$.

REMARK III.1.35. Here the same letter $\mathcal{R}$ serves two purposes: "ring" and "relation" both begin with R. Thus $\mathcal{R}_X$ and $\mathcal{R}(s_1, s_2, \ldots, s_q)$ should not be confused. In most of the complex-analytic situations below, the structure sheaf will be written as $\mathcal{O}_X$, while $\mathcal{R}(s_1, s_2, \ldots, s_q)$ will still denote a sheaf of relations, so this overlap should cause little confusion. □

DEFINITION III.1.36. Let $(X, \mathcal{R}_X)$ be a ringed space and let $\mathcal{F}$ be a sheaf of $\mathcal{R}_X$-modules. We say that $\mathcal{F}$ is *locally finitely generated* if, for every $a \in X$, there exist an open neighborhood U of a and finitely many sections $s_1, s_2, \ldots, s_r \in \Gamma(U, \mathcal{F})$ such that

$$\mathcal{F}_x = \sum_{j=1}^{r} \mathcal{R}_{X,x} s_{j,x} \qquad \text{for every } x \in U.$$

The empty family is allowed when $\mathcal{F}|_U = 0$. We say that $\mathcal{F}$ is *coherent* if it is locally finitely generated and, for every open subset $V \subset X$ and every finite family $s_1, s_2, \ldots, s_q \in \Gamma(V, \mathcal{F})$, the sheaf $\mathcal{R}(s_1, s_2, \ldots, s_q)$ is locally finitely generated on V. A sheaf of rings is called *coherent* when it is coherent as a module over itself. See [De4, Chapter II, (3.8) and (3.11)]. □

REMARK III.1.37 (Viewing coherence through local and global questions). Here and in § 2.11, we try to explain some basic features of coherence and its role in complex analysis.

As Noguchi emphasizes, the implications of Oka's coherence theorems are "broad and deep", and their importance cannot be explained in a few lines [No2, §2.2.2, p. 39].

The modern theory rests on Oka's fundamental ideas and coherence theorems, with Cartan playing a major role in its sheaf-theoretic formulation and further development. The author learned the subject through its modern sheaf-theoretic presentation, rather than through a historical study of the original papers. For the historical development and the respective contributions of Oka and Cartan, we refer to Noguchi's accounts in [No2, §2.2.2] and [No1, Chapter 9, pp. 367–373].

To explain its meaning, we distinguish viewpoints according to the information retained. These are not mutually exclusive classes of properties or a hierarchy of strengths.

Semi-positivity of a given real $(1,1)$-form is *pointwise*: one tests its value on the tangent space at each point, as in Lemma III.1.7. This is separate from smoothness. Being C^∞ or holomorphic is *local*: it cannot be decided from a single function value, but can be checked on small coordinate balls. By *global* we mean on the whole specified space; a section on U is global when U itself is that space.

The ring $\mathcal{O}_{\Omega,x}$ retains holomorphic germs, not just values: representatives are identified after shrinking their neighborhoods, as in Chapter II, §3.2 and Example II.3.55. Its Noetherian and factorial properties in Theorems III.1.32 and III.1.33 are *stalkwise*, or informally *semi-pointwise*, properties. They are not assertions about the ring of functions on a fixed open set; compare Exercise I.7.

For $f_1, f_2, \ldots, f_s \in \mathcal{O}_\Omega(\Omega)$, put $\mathcal{R} := \mathcal{R}(f_1, f_2, \ldots, f_s)$, where $\mathcal{O}_\Omega := \mathcal{O}_{\mathbb{C}^n}|_\Omega$. Noetherianity and Mini-Exercise I.5.4 give finite generation of each $\mathcal{R}_a$. Its generators can be represented by relations near a. Coherence requires more: one open neighborhood U of a and sections $\sigma_1, \sigma_2, \ldots, \sigma_r \in \Gamma(U, \mathcal{R})$ with

$$\mathcal{R}_x = \sum_{j=1}^{r} \mathcal{O}_{\Omega,x}\, \sigma_{j,x} \qquad \text{for every } x \in U. \tag{135}$$

The neighborhood and sections are fixed before x varies, although the coefficients remain germs at x. We call this viewpoint *semi-stalkwise*: nearby stalks are considered together through common neighborhood data, not through independently chosen generators. For $\mathcal{O}_\Omega$, Oka's first coherence theorem (Theorem III.1.41) below will establish this assertion for every finite family of holomorphic functions.

By Proposition II.3.58, (135) is equivalent to surjectivity of the morphism $\mathcal{O}_U^{\oplus r} \to \mathcal{R}|_U$ defined by these sections. As explained there, this need not give surjectivity on sections over the same U: an individual section lifts after shrinking around a point, but lifting over the whole fixed U is a separate question. The point of the term *semi-stalkwise* is that one keeps a common neighborhood and the same relation sections while considering all nearby stalks. Demailly calls coherence semi-local in [De4, Chapter II, §3.2]; here we reserve that explanatory name for the following viewpoint.

Chapter II, §3.2 also considered germs of sections near an entire subset Y, not necessarily a point. Chapter IV onwards studies such germs of neighborhoods geometrically. We call this viewpoint *semi-local*: shrink the ambient neighborhood, but retain all of Y and information

lost by restriction to Y itself; compare Remark II.3.110. Which global problems can be understood from this information is a question to investigate, not an automatic consequence of the terminology.

Viewpoints used for explanation in this note. The descriptions are illustrative, not additional mathematical definitions. The viewpoints are neither mutually exclusive nor ordered by strength; their use depends on the aspect of an argument under discussion.

Viewpoint	**Information retained and an example**
pointwise	The value at a fixed point: for example, testing the semi-positivity of a given real $(1,1)$-form on the tangent space at that point.
semi-pointwise (= *stalkwise*)	Germs at a fixed point, with representatives identified after shrinking neighborhoods: for example, the Noetherian and factorial properties of $\mathcal{O}_{\Omega,x}$.
semi-stalkwise	Common neighborhood data while nearby stalks are considered together: for example, one finite family of relation sections whose germs generate the relations at every point of that neighborhood.
local	Objects or conditions on a neighborhood of a point: for example, being C^∞ or holomorphic can be checked on small coordinate balls, but not from a single function value.
semi-local	Germs of ambient neighborhoods of an entire subset Y, not necessarily a point. Shrinking is allowed, but information not captured by restriction to Y itself is retained.
global	The whole specified space: for example, the existence of one solution on Ω, rather than separate solutions near its points.

□

Example III.1.38 (Vanishing ideal sheaves which are not coherent). For a closed subset $S \subset \mathbb{C}$, set

$$\mathcal{I}_S(U) := \{f \in \mathcal{O}_{\mathbb{C}}(U) \mid f|_{S\cap U} = 0\} \qquad (U \subset \mathbb{C} \text{ open}).$$

Restriction and gluing are inherited from $\mathcal{O}_{\mathbb{C}}$, so this defines an ideal sheaf; compare [No1, §1.3.3(5)]. If $\mathcal{I}_S$ is locally finitely generated near a and $(\mathcal{I}_S)_a = 0$, its finitely many local generators have zero germs at a. They therefore vanish on a common smaller neighborhood, and so does the sheaf.

(*i*) Let $S_1 = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_{>0}\}$. The identity theorem in one variable (Chapter I, §I.9) gives

$$(\mathcal{I}_{S_1})_x = \begin{cases} 0, & x = 0, \\ (z-1/n)\mathcal{O}_{\mathbb{C},1/n}, & x = 1/n, \quad n \in \mathbb{Z}_{>0}, \\ \mathcal{O}_{\mathbb{C},x}, & x \notin S_1. \end{cases}$$

Indeed, a holomorphic function near 0 vanishing on S_1 has zeros accumulating at 0 and is zero near 0. At an isolated point $1/n$, vanishing is equivalent to divisibility by $z-1/n$; outside S_1

a small disk misses S_1. Every neighborhood of 0 contains points outside S_1, where the stalk is nonzero. Thus $\mathcal{I}_{S_1}$ is not locally finitely generated at 0, and hence is not coherent.

(*ii*) Let $S_2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the closed unit disk. Then

$$(\mathcal{I}_{S_2})_x = \begin{cases} 0, & |x| \leq 1, \\ \mathcal{O}_{\mathbb{C},x}, & |x| > 1. \end{cases}$$

At a boundary point, every sufficiently small disk meets the interior of S_2, so the identity theorem again forces any such representative to vanish. Every neighborhood of that point also meets the exterior, where the stalk is nonzero. Thus $\mathcal{I}_{S_2}$ is not locally finitely generated at any boundary point and is not coherent; compare [No1, Example 2.4.13].

All stalk ideals in these examples are principal, but the sheaves are not locally finitely generated. Moreover, $\mathcal{I}_{S_1}(\mathbb{C}) = \mathcal{I}_{S_2}(\mathbb{C}) = 0$: the ideal sheaf generated only by entire functions vanishing on S_1 or S_2 would be the zero sheaf, not the one constructed above. □

We now turn to Oka's first coherence theorem (Theorem III.1.41) for the sheaf of holomorphic functions and to some consequences for its ideal sheaves. Before entering the analytic argument, we record a few facts which hold on arbitrary ringed spaces. They explain how coherence behaves under elementary operations; establishing the coherence of the structure sheaf of holomorphic functions will be the separate analytic task.

Proposition III.1.39 (Three Lemma for coherent sheaves). *Let $(X, \mathcal{R}_X)$ be a ringed space, and let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of sheaves of $\mathcal{R}_X$-modules. If two of $\mathcal{F}'$, $\mathcal{F}$, and $\mathcal{F}''$ are coherent, then so is the third. No coherence assumption on $\mathcal{R}_X$ itself is required.

Proof. For a proof, see [GrR, Annex, §4.1, pp. 236–237]. □

Proposition III.1.39 immediately implies that finite direct sums of coherent modules are coherent, by repeated application to the split sequences

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow 0.$$

We also use the fact that a locally finitely generated subsheaf of a coherent module is coherent; see [GrR, Annex, §3.3]. Indeed, its finite families of sections have the same relations when regarded as sections of the ambient coherent module. The following lemma applies these facts to a finite system of equations. Unlike the Three Lemma for coherent sheaves (Proposition III.1.39), it assumes that the structure sheaf itself is coherent.

Lemma III.1.40. *Let $(X, \mathcal{R}_X)$ be a ringed space whose structure sheaf is coherent, let $U \subset X$ be open, and let A be a $p \times q$ matrix with entries in $\mathcal{R}_X(U)$, where $p, q \in \mathbb{Z}_{\geq 0}$. Then the kernel of the morphism*

$$A\colon (\mathcal{R}_X|_U)^{\oplus q} \longrightarrow (\mathcal{R}_X|_U)^{\oplus p}$$

is locally finitely generated.

PROOF. The source and target are coherent by the finite-direct-sum consequence of the Three Lemma for coherent sheaves (Proposition III.1.39), with the zero sheaf allowed. The image sheaf of A is generated by the images of the standard coordinate sections of the source. It is therefore a locally finitely generated subsheaf of the coherent target, hence is coherent by the subsheaf criterion following Proposition III.1.39. Apply the Three Lemma for coherent sheaves (Proposition III.1.39) to the short exact sequence

$$0 \longrightarrow \operatorname{Ker} A \longrightarrow (\mathcal{R}_X|_U)^{\oplus q} \longrightarrow \operatorname{Image} A \longrightarrow 0.$$

The kernel is coherent and, in particular, locally finitely generated. □

We return to the analytic setting. For an open subset $\Omega \subset \mathbb{C}^n$, write $\mathcal{O}_\Omega$ for its sheaf of holomorphic functions. The result for which we have been preparing is the following.

THEOREM III.1.41 (Oka's first coherence theorem). *For every open subset $\Omega \subset \mathbb{C}^n$, where $n \geq 0$, the sheaf $\mathcal{O}_\Omega$ is coherent.*

We will prove this theorem by induction on n. The next lemma provides the analytic reduction needed for that induction: it shows how Weierstraß division controls relations at every point of a product neighborhood, not only at its center. The reduction does not assume coherence. After proving the lemma, we will apply Lemma III.1.40 to a coefficient system in one fewer variable, where coherence is already known by the induction hypothesis.

LEMMA III.1.42. *Let $n, s, N \geq 1$ be integers. Let $V \subset \mathbb{C}^{n-1}$ be open, let $D \subset \mathbb{C}$ be an open disc with coordinate w, and put $U := V \times D$. Let $P_1, P_2, \ldots, P_s \in \mathcal{O}_V(V)[w]$ be monic polynomials of degrees at most N, with P_s of degree N in w. For $x = (a, b) \in U$, the module $\mathcal{R}(P_1, P_2, \ldots, P_s)_x$ is generated over $\mathcal{O}_{U,x}$ by polynomial relations whose components belong to $\mathcal{O}_{V,a}[w]$ and have degree at most N.*

Before proving the lemma, let us clarify the generation assertion. Every relation at x is to be written as a finite $\mathcal{O}_{U,x}$-linear combination of polynomial relations whose components belong to $\mathcal{O}_{V,a}[w]$ and have degree at most N in w. The coefficients of the linear combination are holomorphic germs and need not be polynomials.

The statement starts with $n = 1$. In the zero-dimensional base case of Oka's first coherence theorem, a nonempty open subset of $\mathbb{C}^0$ is a point and the relation module is a finite-dimensional complex vector space, so this reduction is unnecessary.

For a concrete example, take $n = 1$, $s = 2$, $N = 4$, and $b = 0$, with D an open disc containing 0. Put

$$P_1(w) = w^3 - w^2, \qquad P_2(w) = w^4 + w^3.$$

Factoring out w^2, we write

$$P_1 = w^2 A_1, \qquad P_2 = w^2 A_2, \qquad A_1(w) = w - 1, \qquad A_2(w) = w(w+1).$$

Since $A_1(0) = -1$, its germ at 0 is a unit. The polynomial vector

$$T(w) := (-A_2(w), A_1(w)) = (-w(w+1), w - 1)$$

is a relation, since $P_1(-A_2) + P_2 A_1 = 0$. Its components have degrees 2 and 1, both at most $N = 4$. Now let $g = (g_1, g_2) \in \mathcal{O}_{\mathbb{C},0}^{\oplus 2}$ satisfy $P_1 g_1 + P_2 g_2 = 0$. Since $\mathcal{O}_{\mathbb{C},0}$ is an integral domain, cancelling w^2 gives $A_1 g_1 + A_2 g_2 = 0$. Hence

$$g_1 = -\frac{A_2}{A_1} g_2, \qquad g = \frac{g_2}{A_1} T.$$

Thus

$$\mathcal{R}(P_1, P_2)_0 = \mathcal{O}_{\mathbb{C},0}\, T.$$

Here a single polynomial relation generates the whole relation module. In general, finite generation at a fixed stalk is already known from Noetherianity. The point of the lemma is the degree bound on the polynomial relations, not the choice of a common finite family on a neighborhood. The polynomial relations used may depend on the given polynomials and the fixed point $x = (a, b)$; for the given polynomials, the same bound N applies at every x. Expanding their components in powers of w will lead to the finite coefficient system (137) in one fewer variable. Coherence in that dimension will then provide one finite family valid throughout a smaller product neighborhood.

Proof of Lemma III.1.42. For the general argument, see [De4, Chapter II, Lemma (3.20) and its proof, pp. 89–90]; a related reduction with finer degree bounds is given in [No1, Lemma 2.5.8]. We explain the cases $n = 1$ and $n = 2$, keeping s and N arbitrary. Write $e_1, e_2, \dots, e_s$ for the standard basis vectors.

For $n = 1$, V is a point, so the P_j belong to $\mathbb{C}[w]$. Fix $b \in D$ and put

$$m := \min_{1 \le j \le s} \operatorname{ord}_b(P_j), \qquad A_j := \frac{P_j}{(w-b)^m} \in \mathbb{C}[w].$$

Choose k with $A_k(b) \neq 0$. For $j \neq k$, the vector $T_j := A_k e_j - A_j e_k$ is a polynomial relation among the P_j, of degree at most N. If $g = (g_1, g_2, \dots, g_s)$ is a holomorphic relation at b, cancelling $(w-b)^m$ gives $\sum_j A_j g_j = 0$. Since $A_k(b) \neq 0$, this gives $g_k = -\sum_{j \neq k} A_j g_j / A_k$. Comparing components, we obtain

$$g = \sum_{j \neq k} \frac{g_j}{A_k}\, T_j.$$

The coefficients are holomorphic at b because $A_k(b) \neq 0$. This proves the assertion for $n = 1$; for $s = 1$, the relation module is zero and the sum is empty.

For $n = 2$, the coefficients also depend on a base variable, and we use Weierstraß preparation and division for the reduction. Write the coordinates as (z, w), and fix an arbitrary point $x = (a, b) \in V \times D$. Put $A := \mathcal{O}_{V,a}$ and $B := \mathcal{O}_{U,x}$. Pullback under the projection $U \to V$ identifies $A[w]$ with a subring of B. If $P_s(x) \neq 0$, every relation is a B-linear combination of $P_s e_j - P_j e_s$ for $j < s$, since P_s is a unit in B. We may therefore assume that $P_s(x) = 0$.

Since $P_s(a, w)$ is monic, its zero at $w = b$ has a finite order d with $1 \le d \le N$. Apply the Weierstraß preparation theorem (Theorem III.1.29) after translating (a, b) to the origin. It gives $P_s = Qu$, where $Q \in A[w]$ is monic of degree d, $Q(a, w) = (w-b)^d$, and $u \in B$

is a unit. In fact, u belongs to $A[w]$. Indeed, polynomial division gives $P_s = QH + S$ with $H, S \in A[w]$ and $\deg_w S < d$, where $\deg_w$ denotes degree in the variable w; a zero remainder is also allowed. Comparison with $P_s = Qu + 0$ and uniqueness in the Weierstraß division theorem (Theorem III.1.28) give $S = 0$ and $H = u$. Thus

$$P_s = QH, \qquad H \in A[w], \qquad \deg_w H = N - d, \qquad H(x) \neq 0.$$

Take a relation $g = (g_1, g_2, \ldots, g_s) \in B^{\oplus s}$. For $j < s$, divide g_j by Q to obtain $g_j = Qv_j + r_j$, where $v_j \in B$ and $r_j \in A[w]$ has degree less than d. Set $t_j := v_j/H$ and $r_s := g_s + \sum_{j<s} P_j t_j$. For $r := (r_1, r_2, \ldots, r_s)$, one then has

$$g = \sum_{j<s} t_j(P_s e_j - P_j e_s) + r, \qquad \sum_{j<s} P_j r_j + QHr_s = 0. \tag{136}$$

The polynomial $T := \sum_{j<s} P_j r_j$ has degree less than $N + d$. Divide T by Q in $A[w]$. The second equality in (136) and uniqueness of analytic division show that the remainder is zero and that Hr_s is a polynomial of degree less than N. For $j < s$, Hr_j also has degree less than N. Thus Hr is a polynomial relation of the required degree, and $r = H^{-1}(Hr)$. The first equality in (136) now proves the assertion for $n = 2$.

For $n \geq 3$, replace the single base variable z by the tuple $z' = (z^1, z^2, \ldots, z^{n-1})$. The same preparation, division, and degree estimates apply at an arbitrary (a, b), as in the cited proof. None of these steps uses coherence of $\mathcal{O}_U$. □

Proof of Oka's first coherence theorem (Theorem III.1.41). We follow [De4, Chapter II, Theorem (3.19)], using Weierstraß division and induction on n. The constant section 1 generates $\mathcal{O}_\Omega$. Thus it remains to prove that the relations among any finite family of holomorphic functions are locally finitely generated. We write e_j for the standard coordinate sections of the finite free sheaf in question. We first treat $n = 0, 1, 2$ explicitly.

For $n = 0$, a nonempty open set in $\mathbb{C}^0$ is a point, and the relations form a finite-dimensional vector space over $\mathbb{C}$. For $n = 1$, fix a point, translate it to 0, and let $f_1, f_2, \ldots, f_s$ be holomorphic near 0. If all the germs are zero, the functions vanish on a common smaller disc, and their relations form the whole free sheaf. Otherwise, after reordering, let f_s have the smallest vanishing order d among the nonzero germs. On a sufficiently small disc D, one has $f_s = w^d u$ with u nowhere zero, and each f_j is divisible by w^d. Hence $h_j := f_j/f_s$ extends holomorphically to D for $j \in \{1, 2, \ldots, s-1\}$. For every $x \in D$, the germ of f_s is nonzero and $\mathcal{O}_{D,x}$ is an integral domain. It follows that every relation $(g_1, g_2, \ldots, g_s)$ among the germs $f_{1,x}, f_{2,x}, \ldots, f_{s,x}$ satisfies $g_s = -\sum_{j\in\{1,2,\ldots,s-1\}} h_{j,x} g_j$. Thus the same sections $e_j - h_j e_s$ for $j \in \{1, 2, \ldots, s-1\}$ generate the relations at every point of D. This proves the assertion for $n = 1$.

Now consider $n = 2$, with coordinates denoted by (z, w). Fix the origin and functions $f_1, f_2, \ldots, f_s$ on a neighborhood of it. A function whose germ is zero can be taken to be identically zero after shrinking; its coordinate vector is an additional free generator of the relation sheaf. We may discard these functions, treating the case in which all are zero as above. If some $f_j(0) \neq 0$, division by that function gives relations $e_k - (f_k/f_j)e_j$ on a smaller

neighborhood, so this case is also settled. We can therefore assume that all the germs are nonzero and vanish at 0.

Choose linear coordinates in which all these germs are regular in w simultaneously. This uses the argument of Lemma III.1.30: if F_j is the first nonzero homogeneous term of f_j, choose the last coordinate direction v with $\prod_j F_j(v) \neq 0$. Such a direction exists since this product is a nonzero polynomial. By the Weierstraß preparation theorem (Theorem III.1.29), on a common polydisc $V \times D$ centered at 0 one has $f_j = u_j P_j$, where each u_j is nowhere zero and each P_j is a Weierstraß polynomial in w with coefficients holomorphic on the z-disc V. The invertible diagonal map $(g_j)_j \mapsto (u_j g_j)_j$ identifies the relations among the f_j with those among the P_j. It suffices to study the latter.

Put $N := \max_j \deg_w P_j$, and reorder so that P_s has degree N. Write

$$P_j(z,w) = \sum_{m=0}^{N} p_{jm}(z) w^m, \qquad p_{jm} \in \mathcal{O}_V(V),$$

padding with zero coefficients where necessary. For $x = (a,b) \in V \times D$, a polynomial tuple of degree at most N has the form $G_j(w) = \sum_{k=0}^{N} h_{jk} w^k$, with $h_{jk} \in \mathcal{O}_{V,a}$. The equality $\sum_j P_j G_j = 0$ is equivalent to the finite system

$$\sum_{j=1}^{s} \sum_{k=0}^{N} p_{j,\ell-k}\, h_{jk} = 0 \qquad (0 \leq \ell \leq 2N), \tag{137}$$

where $p_{j,m} := 0$ for $m < 0$ or $m > N$ and the coefficients are read as germs at a. There are $2N+1$ equations and $s(N+1)$ unknowns. Their coefficients depend on z, not on w.

The one-variable result already proved says that $\mathcal{O}_V$ is coherent. Here the ringed-space lemma is applied on V, not on $V \times D$. Lemma III.1.40 therefore supplies an open neighborhood $V' \subset V$ of 0 and finitely many solutions $h^{(1)}, h^{(2)}, \ldots, h^{(t)}$ of this system on V' whose germs generate its kernel at every $a \in V'$. We may shrink V' to an open disc. Form the polynomial relations

$$\sigma_\nu(z,w) := \left(\sum_{k=0}^{N} h_{jk}^{(\nu)}(z) w^k \right)_{j=1}^{s} \qquad (1 \leq \nu \leq t).$$

These are sections of $\mathcal{R}(P_1, P_2, \ldots, P_s)$ on $V' \times D$. At any $x = (a,b) \in V' \times D$, every bounded-degree polynomial relation is a linear combination of their germs with coefficients in $\mathcal{O}_{V,a}$, by (137). By Lemma III.1.42, such polynomial relations generate all the analytic relations at x over $\mathcal{O}_{V' \times D, x}$. Consequently, the same finite family $\sigma_1, \sigma_2, \ldots, \sigma_t$ generates the relation sheaf throughout $V' \times D$. Multiplying its j-th components by u_j^{-1} gives generators for the relations among the original f_j. This proves the assertion for $n = 2$.

For $n \geq 3$, assume the assertion in dimension $n-1$. The preceding reduction works with $z' = (z^1, z^2, \ldots, z^{n-1})$ in place of z. The zero and unit cases, the simultaneous choice of coordinates, and the preparation of the finite family are unchanged. The coefficients in (137) are now holomorphic on a polydisc $V \subset \mathbb{C}^{n-1}$. By the induction hypothesis and

Lemma III.1.40, their solution sheaf has finitely many generators on one smaller neighborhood V' of the origin. Lemma III.1.42 then gives generators of the original relation sheaf at every point of $V' \times D$, exactly as in the two-variable argument. This completes the induction. □

Coherent ideals and the strong Noetherian property

An *ideal sheaf* in $\mathcal{O}_\Omega$ means a sheaf of $\mathcal{O}_\Omega$-submodules of $\mathcal{O}_\Omega$.

PROPOSITION III.1.43. *Let $\Omega \subset \mathbb{C}^n$ be open and let $\mathcal{F} \subset \mathcal{O}_\Omega^{\oplus p}$ be a locally finitely generated subsheaf, where $p \geq 1$. Then $\mathcal{F}$ is coherent. In particular, every locally finitely generated ideal sheaf in $\mathcal{O}_\Omega$ is coherent.*

PROOF. By Oka's first coherence theorem (Theorem III.1.41), $\mathcal{O}_\Omega$ is coherent. Its finite direct sums are coherent by the Three Lemma for coherent sheaves (Proposition III.1.39), applied to the split sequences above. The locally finitely generated subsheaf criterion following Proposition III.1.39 now applies to $\mathcal{F} \subset \mathcal{O}_\Omega^{\oplus p}$. □

The following result provides uniform control of an increasing sequence of coherent submodules near a compact set.

THEOREM III.1.44 (Strong Noetherian property). *Let $\Omega \subset \mathbb{C}^n$ be open, where $n \in \mathbb{Z}_{\geq 0}$, let $\mathcal{F}$ be a coherent $\mathcal{O}_\Omega$-module, and let*

$$\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{F}$$

be an increasing sequence of coherent $\mathcal{O}_\Omega$-submodules. For every compact subset $K \subset \Omega$, there exist an open neighborhood $U \subset \Omega$ of K and an integer $\nu_0 \geq 1$ such that

$$\mathcal{G}_\nu|_U = \mathcal{G}_{\nu_0}|_U \qquad \text{for every } \nu \geq \nu_0.$$

In particular, the same conclusion holds for every increasing sequence

$$\mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{O}_\Omega$$

of coherent ideal sheaves.

PROOF. We follow the argument of [De4, Chapter II, (3.22)], spelling out the reductions. We first prove that the sequence is stationary on one neighborhood of each fixed point. Compactness will then give the assertion near K.

We begin with two reductions in a fixed dimension. Locally, there is a surjective morphism $\alpha\colon \mathcal{O}_W^{\oplus q} \longrightarrow \mathcal{F}|_W$ for some $q \in \mathbb{Z}_{\geq 0}$. The inverse images $\widetilde{\mathcal{G}}_\nu := \alpha^{-1}(\mathcal{G}_\nu|_W)$ form an increasing sequence of coherent subsheaves of $\mathcal{O}_W^{\oplus q}$. Indeed, $\operatorname{Ker}\alpha$ is coherent by the Three Lemma for coherent sheaves (Proposition III.1.39), and the same proposition applies to

$$0 \longrightarrow \operatorname{Ker}\alpha \longrightarrow \widetilde{\mathcal{G}}_\nu \longrightarrow \mathcal{G}_\nu|_W \longrightarrow 0.$$

Stationarity of the inverse images implies stationarity of the original sequence, by surjectivity of α.

Next, for an increasing sequence $\mathcal{E}_\nu \subset \mathcal{O}_W^{\oplus q}$ with $q \geq 2$, let $\mathcal{J}_\nu$ be its image under projection to the last coordinate, and let $\mathcal{E}'_\nu := \mathcal{E}_\nu \cap \mathcal{O}_W^{\oplus(q-1)}$, where $\mathcal{O}_W^{\oplus(q-1)}$ is embedded by adjoining a zero last coordinate. The image $\mathcal{J}_\nu$ is locally finitely generated and hence coherent by Proposition III.1.43. The Three Lemma for coherent sheaves (Proposition III.1.39) then shows that $\mathcal{E}'_\nu$ is coherent as well. If both sequences are stationary on a common neighborhood, so is $\mathcal{E}_\nu$: at each stalk, subtract an element of an earlier term with the same last coordinate, and the difference belongs to the stationary kernel. Induction on q, with $q = 0$ immediate, therefore reduces the problem to coherent ideal sheaves. These rank and quotient reductions apply separately in every dimension. Thus, once the ideal case is proved in a given dimension, the full module assertion is available in that dimension.

For $n = 0$, a nonempty open subset of $\mathbb{C}^0$ is a point, and the only ideals of $\mathbb{C}$ are 0 and $\mathbb{C}$. The assertion follows. Equivalently, an increasing sequence of subspaces of a finite-dimensional vector space is stationary.

For the positive-dimensional ideal case, we may work on a connected neighborhood of the chosen point. A nonzero coherent ideal sheaf on a connected open set has a nonzero stalk at every point. To see this, the locus of zero stalks is open by local finite generation. Its complement is also open: a nonzero local section on a connected neighborhood has a nonzero germ at every point of that neighborhood by the identity principle. Connectedness proves the assertion. If all terms of the sequence are zero, there is nothing to prove. Otherwise, we can choose a nonzero germ in some term $\mathcal{I}_{\nu_*}$ at the chosen point. If this germ is a unit, all terms with $\nu \geq \nu_*$ equal the structure sheaf on a smaller neighborhood, so we may exclude this case below.

For $n = 1$, translate the point to 0 and write the chosen germ as $f(w) = w^\mu u(w)$, where $\mu \geq 1$ and $u(0) \neq 0$. On a sufficiently small disc D, the function u is nowhere zero and $w^\mu \in \mathcal{I}_{\nu_*}(D)$. For every $\nu \geq \nu_*$, membership in $\mathcal{I}_{\nu,0}$ is therefore determined modulo w^μ. Taylor division identifies

$$\mathcal{O}_{\mathbb{C},0}/w^\mu \mathcal{O}_{\mathbb{C},0} \cong \bigoplus_{j=0}^{\mu-1} \mathbb{C} w^j$$

as vector spaces, so the increasing sequence of images in this finite-dimensional space is stationary. Hence the stalks at 0 are stationary. At every $x \in D \setminus \{0\}$, w^μ is a unit, and all the stalks $\mathcal{I}_{\nu,x}$ with $\nu \geq \nu_*$ equal $\mathcal{O}_{D,x}$. Consequently, the sequence is stationary on the same disc D. The reductions above now give the module assertion in dimension one.

For $n = 2$, write the coordinates as (z, w) and put the chosen point at $(0, 0)$. By Lemma III.1.30 and the Weierstraß preparation theorem (Theorem III.1.29), after a linear change of coordinates and a shrinking we can choose

$$P(z, w) = w^\mu + \sum_{j=0}^{\mu-1} a_j(z) w^j, \qquad a_j(0) = 0, \qquad \mu \geq 1,$$

with $P \in \mathcal{I}_{\nu_*}(W)$ on a product $W := \Delta \times D$ of open discs centered at 0. By continuity of the roots (Lemma I.1.3), we can shrink Δ so that all roots of $P(z, \bullet)$ remain in a smaller disc whose closure is contained in D. This product and P are fixed for all $\nu \geq \nu_*$.

We describe carefully the coefficient sheaves used for the one-dimensional reduction. For $\nu \geq \nu_*$ and an open subset $\Delta_1 \subset \Delta$, put

$$\mathcal{H}_\nu(\Delta_1) := \left\{ (r_0, r_1, \ldots, r_{\mu-1}) \in \mathcal{O}_\Delta(\Delta_1)^{\oplus\mu} \;\middle|\; \sum_{j=0}^{\mu-1} r_j(z) w^j \in \mathcal{I}_\nu(\Delta_1 \times D) \right\}.$$

These define an increasing sequence of subsheaves of $\mathcal{O}_\Delta^{\oplus\mu}$. We show that they are coherent and that they determine the original ideals, without assuming that a germ at one point of a fiber is defined near the other points of that fiber.

Fix $a \in \Delta$, let $b_1, b_2, \ldots, b_t$ be the distinct roots of $P(a, w)$, and put $A := \mathcal{O}_{\Delta,a}$ and $B_i := \mathcal{O}_{W,(a,b_i)}$. Preparation at these points gives monic factors $Q_i \in A[w]$ with $Q_i(a, w) = (w - b_i)^{\mu_i}$ and $P = \prod_{i=1}^t Q_i$, where $\sum_i \mu_i = \mu$. As in the proof of Lemma III.1.42, polynomial division and uniqueness of Weierstraß division identify the local factor with a polynomial; the other factors are units in B_i. Weierstraß division at each root and polynomial recombination give

$$A[w]/PA[w] \;\cong\; \bigoplus_{i=1}^t B_i/PB_i.$$

More explicitly, reduction modulo the Q_i sends a polynomial of degree less than μ to its remainders of degrees less than μ_i. This is a square linear map over A. At $z = a$ it is invertible because the polynomials $(w - b_i)^{\mu_i}$ are pairwise relatively prime. Its determinant is therefore a unit in A, which proves the asserted recombination.

Under this isomorphism, the coefficient module $\mathcal{H}_{\nu,a}$ corresponds exactly to

$$\bigoplus_{i=1}^t \mathcal{I}_{\nu,(a,b_i)}/PB_i.$$

Indeed, membership at the finitely many roots holds on their neighborhoods after shrinking the base. All nearby roots stay in these neighborhoods, and away from the zeros of P the ideal is the whole structure sheaf. This is precisely the condition in the definition of $\mathcal{H}_\nu$.

For completeness, fix ν and choose finite local generators of $\mathcal{I}_\nu$ near each (a, b_i). Using the preceding isomorphism, represent each of their classes by a polynomial of degree less than μ, with zero class at the other roots. After shrinking the base, these give finitely many polynomials $R_1, R_2, \ldots, R_e$ belonging to $\mathcal{I}_\nu$ on the whole product. Together with P, they generate $\mathcal{I}_\nu$ on this smaller product: near each (a, b_i) this follows from the chosen local generators and their congruences modulo P, and elsewhere P is a unit. The coefficients of the polynomial remainders of

$$w^k R_j \quad \text{modulo } P \qquad (0 \leq k < \mu,\ 1 \leq j \leq e)$$

then generate $\mathcal{H}_\nu$ over $\mathcal{O}_\Delta$ on a smaller base neighborhood. To check this at a nearby base point, use the same direct-sum description to represent the coefficients multiplying the R_j by polynomials of degree less than μ, then expand those polynomials in powers of w. Thus

$\mathcal{H}_\nu$ is locally finitely generated, and is coherent by Proposition III.1.43. The neighborhoods used to prove this may depend on ν; the sheaves $\mathcal{H}_\nu$ themselves are all defined on the fixed Δ.

Apply the already proved one-dimensional module assertion to $\mathcal{H}_{\nu_*} \subset \mathcal{H}_{\nu_*+1} \subset \cdots \subset \mathcal{O}_\Delta^{\oplus\mu}$. They are stationary on one neighborhood Δ_0 of 0. The direct-sum description at every $a \in \Delta_0$ then gives stationarity of the ideals at every root of $P(a, w)$. Outside those roots P is a unit, so the original sequence is stationary on $\Delta_0 \times D$. The rank and quotient reductions give the module assertion in dimension two.

For $n \geq 3$, assume the module assertion in dimension $n-1$. Use $z' = (z^1, z^2, \ldots, z^{n-1})$ in place of z and a polydisc $\Delta' \subset \mathbb{C}^{n-1}$ in place of Δ. Preparation again gives a fixed monic polynomial $P(z', w)$. The same construction, including the factorization at the finitely many roots over any base point and the finite list of remainders, gives coherent sheaves $\mathcal{H}_\nu \subset \mathcal{O}_{\Delta'}^{\oplus\mu}$. By the induction hypothesis they are stationary on one smaller base neighborhood. The preceding argument then gives stationarity of the ideals on its product with D. Applying the rank and quotient reductions completes the induction.

Finally, cover K by finitely many open neighborhoods $U_1, U_2, \ldots, U_m$ on which the original sequence is stationary, with respective indices $\nu_1, \nu_2, \ldots, \nu_m$. For $U := \bigcup_{j=1}^m U_j$ and $\nu_0 := \max\{\nu_1, \nu_2, \ldots, \nu_m\}$, the claimed equality holds on U for every $\nu \geq \nu_0$. When K is empty, take $U = \emptyset$ and $\nu_0 = 1$. □

The conclusion is stronger than stabilization of the submodules $\mathcal{G}_{\nu,x}$ separately for each x: both the neighborhood U and the index ν_0 are common to all $\nu \geq \nu_0$. It does not assert stabilization on all of a non-compact Ω.

Here is a useful consequence for families which are not initially given by a finite list of functions.

Corollary III.1.45. *Let $\Omega \subset \mathbb{C}^n$ be open, let $S \subset \mathcal{O}_\Omega(\Omega)$ be any set, and let $\mathcal{J} := \sum_{f\in S} \mathcal{O}_\Omega f$ be the ideal sheaf generated by this family. Then $\mathcal{J}$ is coherent. More precisely, for every relatively compact open subset V of Ω, there exist finitely many $f_1, f_2, \ldots, f_r \in S$ such that*

$$\mathcal{J}|_V = \sum_{j=1}^{r} (\mathcal{O}_\Omega|_V)\, f_j|_V.$$

Here $r = 0$ is allowed when $\mathcal{J}|_V = 0$.

Proof. For a finite subset $E \subset S$, let $\mathcal{J}_E := \sum_{f\in E} \mathcal{O}_\Omega f$. These ideal sheaves are coherent by Proposition III.1.43. Fix V as in the statement. The family of subsheaves $\mathcal{J}_E|_V$ has a maximal element under inclusion. Indeed, otherwise one could choose finite subsets $E_1 \subset E_2 \subset \cdots$ of S such that $\mathcal{J}_{E_\nu}|_V$ increases strictly at every step; finite unions let us make the subsets themselves increasing. This contradicts the strong Noetherian property (Theorem III.1.44), applied to this sequence on Ω and the compact set $\overline{V}$.

Choose a finite E_0 attaining such a maximal element. For every finite $E \subset S$, the ideal $\mathcal{J}_{E_0\cup E}|_V$ contains $\mathcal{J}_{E_0}|_V$, so maximality makes them equal. It follows that $\mathcal{J}_E|_V \subset \mathcal{J}_{E_0}|_V$ for every E. Every germ in $\mathcal{J}$ is a finite sum of multiples of germs of functions in S, so

$\mathcal{J}|_V = \mathcal{J}_{E_0}|_V$. Every point of Ω admits such a neighborhood V; hence $\mathcal{J}$ is locally finitely generated and is coherent by the same proposition. □

2 Complex analytic geometry on global objects

2.1 Complex manifolds

The preceding section was written on open subsets of complex Euclidean spaces. We now use these as local models for geometric objects. The coordinate invariance established in §1.3 will allow us to carry the constructions over to these objects without repeating the coordinate calculations. We begin with complex manifolds. For the general language of locally ringed spaces, refer to Definitions II.3.89 and II.3.92 in Chapter II.

Complex manifolds and holomorphic maps

Definition III.2.1. A *complex manifold* is an object $(X, \mathcal{O}_X)$ of $(\mathsf{LocRingSp}_{\mathbb{C}})$ such that X is Hausdorff and second countable and the following condition holds: for every $x \in X$, there exist an open neighborhood $U \subset X$ of x, an integer $n \geq 0$, an open subset $V \subset \mathbb{C}^n$, and an isomorphism

$$(z, z^\sharp)\colon (U, \mathcal{O}_X|_U) \longrightarrow (V, \mathcal{O}_V) \tag{138}$$

in $(\mathsf{LocRingSp}_{\mathbb{C}})$. The underlying homeomorphism $z\colon U \to V$ is called a *holomorphic coordinate chart*, and its components $(z^1, z^2, \ldots, z^n)$ are called *holomorphic local coordinates*. We usually write X for $(X, \mathcal{O}_X)$ when the structure sheaf is understood. □

Let us relate this definition to the usual description by a holomorphic atlas (see also [De4, Chapter I, §3.C]). The argument is the complex-analytic counterpart of Example II.3.97. Let $V \subset \mathbb{C}^m$ and $W \subset \mathbb{C}^n$ be open subsets, and consider a morphism $(F, F^\sharp)$ between the local models $(V, \mathcal{O}_V)$ and $(W, \mathcal{O}_W)$. Remark II.3.95 gives

$$(F^\sharp(h))(x) = h(F(x))$$

for every local holomorphic function h on W and every point x where this expression is defined. Thus $F^\sharp(h) = h \circ F$ as functions. Applying this to the coordinate functions of W, we see that all components of F are holomorphic. Conversely, a holomorphic map $F\colon V \to W$ defines $F^\sharp$ by composition; the induced homomorphisms on the local rings are local, since a function vanishing at $F(x)$ pulls back to one vanishing at x. In particular, an isomorphism between local models has a biholomorphic underlying map.

It follows that the transition maps between the charts in (138) are biholomorphic. Conversely, suppose that a Hausdorff second-countable space X has an open covering by charts $z_\alpha\colon U_\alpha \to V_\alpha$, where each V_α is open in a complex Euclidean space, and that all transition maps are biholomorphic. On an open subset $W \subset X$, set

$$\mathcal{O}_X(W) := \left\{ f\colon W \to \mathbb{C} \;\middle|\; f \circ z_\alpha^{-1} \text{ is holomorphic on } z_\alpha(W \cap U_\alpha) \text{ for every } \alpha \right\}.$$

Composition with the transition maps shows that this condition is independent of the chart. Restrictions and gluing of functions make $\mathcal{O}_X$ a sheaf, and each chart identifies its restriction with $\mathcal{O}_{V_\alpha}$. This gives the local isomorphisms required by Definition III.2.1. Starting instead with that definition, evaluation in the residue fields identifies $\mathcal{O}_X$ with this sheaf of functions: this assertion can be checked in each local model. The two descriptions therefore encode the same structure.

The integer n in a chart containing x is independent of that chart: the differentials of a transition map and its inverse are mutually inverse complex-linear maps, as in §1.3. We write $\dim_x X = n$ and call this the *dimension of X at x*. As agreed in §I.8, dimension for a complex manifold means complex dimension unless otherwise specified. It is locally constant, hence constant on each connected component, but X need not be connected. We say that X is *of pure dimension n* when this value is n at every point. The global dimension notation for complex spaces, including manifolds, is fixed in Definition III.2.40 and Remark III.2.41.

By Hartogs' separate analyticity theorem (Theorem III.1.18), holomorphic functions are locally represented by convergent power series and hence are of class C^∞. Thus the real and imaginary parts of holomorphic coordinates give X an underlying C^∞ manifold structure without boundary, with $\dim_{\mathbb{R},x} X = 2\dim_x X$. For a biholomorphic transition map F, its holomorphic Jacobian is invertible, so (124) gives $\det(DF) = |\det(D^{\mathrm{hol}}F)|^2 > 0$. The complex charts therefore determine an orientation, which we use when integrating forms on X; in particular, X is orientable.

One may consequently regard a complex manifold as a C^∞ manifold equipped with a *complex structure*, namely a maximal holomorphic atlas inducing the given C^∞ structure. After forgetting this additional structure, the results about C^∞ manifolds in Chapters I–II apply under their stated hypotheses, and we use the notation introduced there in this sense. For example, $C^\infty(X)$ denotes the complex-valued C^∞ functions on the underlying manifold, not just its holomorphic functions; similarly, TX and $\mathcal{A}^q(X)$ have their previous meanings. Moreover, the coordinate invariance proved in §1.3 makes the previously introduced notions of type, ∂ and $\overline{\partial}$, semi-positivity, and Levi forms well-defined on X; their notation is recorded below.

Definition III.2.2. Let X and Y be complex manifolds. A continuous map $F\colon X \to Y$ is called *holomorphic* if, near every point of X, its expression in holomorphic coordinates on X and Y has holomorphic components. A bijective map is called *biholomorphic* if it and its inverse are holomorphic. A holomorphic function on X is a holomorphic map $X \to \mathbb{C}$; these are precisely the elements of $\Gamma(X, \mathcal{O}_X)$. □

The condition on coordinate expressions is independent of the charts, since the transition maps are biholomorphic. The calculation for local models above also shows that a holomorphic map $F\colon X \to Y$ determines a unique morphism $(F, F^\sharp)$ in $(\mathsf{LocRingSp}_{\mathbb{C}})$, with

$$F^\sharp\colon \mathcal{O}_Y \longrightarrow F_*\mathcal{O}_X, \qquad F^\sharp_W(h) = h \circ F \in \mathcal{O}_X(F^{-1}(W)) \quad (h \in \mathcal{O}_Y(W)).$$

Conversely, every such morphism is obtained from a holomorphic map in this way. Thus biholomorphisms are exactly the isomorphisms of these locally ringed spaces. Notice that

the target of $F^\sharp$ is the direct image sheaf on Y, as in Chapter II; no new convention for morphisms is being introduced.

EXAMPLE III.2.3 (Basic examples of complex manifolds). The following examples illustrate the local model description.

(*i*) The space $\mathbb{C}^n$, every open subset $U \subset \mathbb{C}^n$, and in particular every domain $\Omega \subset \mathbb{C}^n$ are complex manifolds with their usual sheaves of holomorphic functions. More generally, an open subset U of a complex manifold X carries the restricted structure sheaf $\mathcal{O}_X|_U$.

(*ii*) Let $\pi\colon \widetilde{X} \to X$ be a covering in the topological sense of Chapter I, §7.1, where X is a complex manifold, and assume that $\widetilde{X}$ is second countable. For an evenly covered coordinate neighborhood $U \subset X$ and each connected component $\widetilde{U}$ of $\pi^{-1}(U)$, use the coordinates $z \circ \pi|_{\widetilde{U}}$, where z is a holomorphic chart on U. Their transition maps are restrictions of the transition maps on X. Thus $\widetilde{X}$ carries a unique complex structure for which π is locally biholomorphic. Its Hausdorff property follows from that of X and the disjoint components over evenly covered neighborhoods. In the terminology already noted in Chapter I, these are *unramified covering spaces*; branched coverings are not intended here. In particular, this construction applies to second-countable covering spaces of a domain in $\mathbb{C}^n$, without requiring the covering space to be connected.

(*iii*) Recall the space of complex lines $\mathbf{P}^n = \mathbf{P}(\mathbb{C}^{n+1})$ from Mini-Exercise I.7.16. For $n \geq 1$, write a line using homogeneous coordinates $[Z^0 : Z^1 : \cdots : Z^n]$. On the open subset $U_j := \{[Z] \mid Z^j \neq 0\}$, the ratios

$$\left(\frac{Z^k}{Z^j}\right)_{\substack{0\leq k\leq n\\ k\neq j}}$$

give a chart $U_j \to \mathbb{C}^n$. On overlaps the transition functions are ratios with nonvanishing denominators, so these charts define a complex manifold of dimension n. The quotient topology is Hausdorff, the finite atlas makes it second countable, and it is compact since every line has a unit vector representative. For $n = 0$, the space $\mathbf{P}^0$ is a point. See also [De4, Chapter V, §15.A, p. 277] for this construction, noting that the notation for projectivization there differs from ours.

The same construction defines $\mathbf{P}(V)$ for a complex vector space V of dimension $n+1$; it is independent of a chosen basis, since invertible linear changes induce biholomorphic changes of charts. With the convention of this note, $\mathbb{P}(V)$ denotes instead hyperplanes in V. It has the complex structure transported by the canonical bijection

$$\mathbf{P}(V^*) \longrightarrow \mathbb{P}(V), \qquad [\lambda] \longmapsto \operatorname{Ker} \lambda.$$

In particular, $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ is also a compact complex manifold of dimension n.

(*iv*) Finite products of complex manifolds carry product charts; for example, $\mathbb{C}^n \times \mathbb{P}^m$ is a complex manifold of dimension $n+m$. Likewise, consider a fiber bundle as in Chapter I, §7.6, whose base and fiber are complex manifolds and whose total space is Hausdorff

and second countable. If its changes of local product trivialization are biholomorphic, the product charts define a complex structure on the total space.

□

REMARK III.2.4 (Riemann surfaces, curves, and surfaces). A *Riemann surface* is a connected complex manifold of dimension one. Thus domains in $\mathbb{C}$, their connected covering spaces as in Example III.2.3 (*ii*), and the Riemann sphere $\mathbb{P}^1$ are examples; recall the description of $\mathbb{P}^1$ in Chapter I, §I.9. We do not assume connectedness for complex manifolds in general, but we do assume it when using the term Riemann surface. In the manifold setting, one of pure dimension one is also called a *complex curve*, whereas a *complex surface* has pure complex dimension two and real dimension four. The word "surface" in the traditional name "Riemann surface" refers instead to its underlying real dimension two; we retain this customary name. These names for dimensions do not restrict the terms curve and surface to non-singular objects; their use for complex spaces is specified in Remark III.2.50. □

Here is a first application of the local model description to the results already proved.

COROLLARY III.2.5. *Let X be a complex manifold. The structure sheaf $\mathcal{O}_X$ is coherent. For every $x \in X$, the ring $\mathcal{O}_{X,x}$ is Noetherian and a UFD.*

PROOF. A coordinate chart identifies $\mathcal{O}_X|_U$ with the sheaf of holomorphic functions on an open subset of $\mathbb{C}^n$. Oka's first coherence theorem (Theorem III.1.41) applies there. Local generators and relations are transported by this isomorphism, so Definition III.1.36 gives the coherence of $\mathcal{O}_X$. The same chart identifies $\mathcal{O}_{X,x}$ with $\mathcal{O}_{\mathbb{C}^n,z(x)}$, to which Theorems III.1.32 and III.1.33 apply. For $n = 0$, this local ring is simply $\mathbb{C}$. □

The local theory on a complex manifold

We now fix a complex manifold X of pure dimension n. For a manifold with components of different dimensions, the following constructions are made on each connected component separately. The notation TX denotes the real tangent bundle of the underlying C^∞ manifold, and T_xX denotes its fiber at x. The endomorphisms J defined in holomorphic charts by (92) agree on overlaps, because the real differentials of holomorphic transition maps are complex-linear. They therefore define $J\colon TX \to TX$ with $J^2 = -\mathrm{id}$. The corresponding eigenspaces in the complexification give

$$T^{\mathbb{C}}X := TX \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}X \oplus T^{0,1}X, \qquad T_X := T^{1,0}X. \tag{139}$$

Both summands have complex rank n. We write $T_{X,x}$ for the fiber of T_X, distinguishing it from the real tangent space T_xX. The identification of (TX, J) with $T^{1,0}X$ is the one explained in Remark III.1.1; it is not equality of two subbundles of $T^{\mathbb{C}}X$.

In a chart $(z^1, z^2, \ldots, z^n)$, a vector in T_X has the expression $\sum_{j=1}^n v^j \partial/\partial z^j$. If $w = F(z)$ is a change of holomorphic coordinates, its new fiber coordinates are $D^{\mathrm{hol}}F(z)\, v$, by the chain rule from the preceding section. Thus the induced charts on the total space of T_X are

related by the holomorphic maps $(z, v) \mapsto (F(z), D^{\mathrm{hol}}F(z)\, v)$. The rank-$n$ complex vector bundle with these holomorphic local trivializations is called the *holomorphic tangent bundle*. In particular, a vector field of type $(1,0)$ is holomorphic precisely when its coefficients in these frames are holomorphic. The general terminology of holomorphic vector bundles will be treated in the next subsection.

The decomposition (139) also defines the bundles $\Lambda^{p,q}X$ of forms of type (p,q):

$$\Lambda^{p,q}X := \left(\bigwedge\nolimits^{p}(T^{1,0}X)^*\right) \wedge \left(\bigwedge\nolimits^{q}(T^{0,1}X)^*\right).$$

We use the same notation as on complex domains:

$$\mathcal{A}^{p,q}(U) := C^\infty(U, \Lambda^{p,q}X|_U), \qquad \mathcal{K}^{p,q}(U) := \left(\mathcal{A}_c^{n-p,n-q}(U)\right)^\vee \quad (U \subset X \text{ open}).$$

Here $0 \le p, q \le n$, the subscript c means compact support, and the continuous dual is taken for the test-form topology fixed in Chapter I. Thus $\mathcal{K}^{p,q}(U)$ is the space of currents of bidegree (p,q), or total degree $p+q$, in the convention of this note. A smooth form α defines the current $\eta \mapsto \int_U \alpha \wedge \eta$. These descriptions agree in each chart with those in §1.1. Restrictions give the sheaves $\mathcal{A}_X^{p,q}$ and $\mathcal{K}_X^{p,q}$, whose sections on U are the respective spaces just displayed. This is the type decomposition of the sheaves of forms and currents already used in Example II.3.65.

The operators ∂ and $\overline{\partial}$ defined in charts agree on overlaps by (125). They consequently define operators on these spaces and sheaves, of bidegrees $(1,0)$ and $(0,1)$, respectively. The identities $d = \partial + \overline{\partial}$, $\partial^2 = \overline{\partial}^2 = 0$, and $\partial\overline{\partial} + \overline{\partial}\partial = 0$ hold both for forms and for currents, since they hold in every chart. The normalization of d^c is unchanged:

$$d^c = \frac{1}{4\pi\sqrt{-1}}(\partial - \overline{\partial}), \qquad dd^c = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}.$$

The transport of currents between charts uses only biholomorphic maps. It does not provide a pullback of an arbitrary current by an arbitrary holomorphic map.

Reality and semi-positivity of (p,p)-forms and currents, for $1 \le p \le n$, also make sense on X: we require the conditions of §1.1 in holomorphic charts. Their invariance was checked in §1.3. For example, the coefficient matrices of a real $(1,1)$-form in two charts are related by (126). Hence positivity and semi-positivity can be checked in either chart. For a real C^2 function ρ on X, the Levi form is defined on T_X by the complex Hessian in a holomorphic chart; its independence of that chart follows from (127). This describes the same form as $\sqrt{-1}\partial\overline{\partial}\rho$, with the conventions for the associated Hermitian form fixed in §1.1.

Likewise, let $M \subset X$ be a C^∞ real hypersurface and let ρ be a local real defining function for M, with $d\rho \neq 0$ on M. Its complex tangent spaces and real complex tangent bundle are given locally by

$$T_x^{1,0}M = \{\xi \in T_{X,x} \mid (\partial\rho)_x(\xi) = 0\}, \qquad HM = TM \cap J(TM).$$

The hypersurface Levi form is the restriction of the Levi form of ρ to $T_x^{1,0}M$. Formula (128) and the change-of-defining-function calculation in Mini-Exercise III.1.12 show that this definition is independent of coordinates, with the same qualification concerning the sign of a

defining function as before. In particular, the condition that this Levi form vanish is intrinsic. We therefore call M *Levi-flat* when its Levi form vanishes at every point.

REMARK III.2.6 (Hermitian metrics and fundamental forms). Recall that $T_X = T^{1,0}X$ is the holomorphic tangent bundle, of rank n on an n-dimensional component of X. A C^∞ Hermitian metric h on this bundle is also called an *Hermitian metric on* X. The correspondence of Remark III.1.3 is compatible with holomorphic coordinate changes by (126). It therefore gives a global bijection between these metrics and positive real $(1,1)$-forms on X. The form corresponding to h is called its *fundamental form*:

$$\omega_h = \sqrt{-1}\sum_{j,k=1}^{n} h_{j\overline{k}}\, dz^j \wedge d\overline{z^k}, \qquad h_{j\overline{k}} = h\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right). \tag{140}$$

In particular, $h(v,w) = -\sqrt{-1}\,\omega_h(v,\overline{w})$ for $v, w \in T_X$ recovers the metric without choosing coordinates. The induced real Riemannian metric satisfies $g_h(u,v) = \omega_h(u, Jv)$ by Mini-Exercise III.1.4; its volume form, with the complex orientation, is $\omega_h^n/n!$. The latter identity can be checked in a frame orthonormal for h, so it does not require $d\omega_h = 0$.

If $d\omega_h = 0$, we call h a *Kähler metric* and ω_h a *Kähler form*. The positive-form normalizations of §1.3 apply in charts: at any point we have (97), and in the Kähler case Remark III.1.25 gives (129). On a one-dimensional complex manifold every Hermitian metric is Kähler, since its fundamental form has top real degree. A real closed positive $(1,1)$-form is also often called a Kähler form, even when the corresponding Kähler metric is not specified separately. □

Complex submanifolds, holomorphic immersions, and embeddings

DEFINITION III.2.7. A *complex submanifold* Y of a complex manifold X is a closed subset such that, for every $y \in Y$, there are holomorphic coordinates $z\colon U \to V \subset \mathbb{C}^n$ centered at y (i.e., $z(y) = 0$) and an integer m with $0 \le m \le n$ for which

$$z(Y \cap U) = \{z \in V \mid z^j = 0 \text{ for } m+1 \le j \le n\}. \tag{141}$$

It is given the subspace topology and the complex manifold structure with local coordinates $(z^1, z^2, \dots, z^m)|_{Y\cap U}$ (with the evident interpretation for $m = 0, 1$). □

The induced coordinate changes on Y are holomorphic. The inclusion $j\colon Y \to X$ is holomorphic as well: in these coordinates it is given by $(z^1, z^2, \dots, z^m) \mapsto (z^1, z^2, \dots, z^m, 0, \dots, 0)$. Moreover, Y is a properly embedded C^∞ submanifold, in accordance with the convention of §I.8. If closedness is intended only inside a specified open subset of X, we say that Y is a complex submanifold of that open subset.

Equivalently, near each point of Y, it is the common zero set of $n - m$ holomorphic functions whose differentials are complex-linearly independent there.

MINI-EXERCISE III.2.8. Let X be a complex manifold, let $Y \subset X$ be a complex submanifold, and let $j\colon Y \hookrightarrow X$ be the inclusion. For every open subset $U \subset X$, set

$$\mathcal{I}_Y(U) := \{f \in \mathcal{O}_X(U) \mid f|_{Y\cap U} = 0\}.$$

(i) Show that $\mathcal{I}_Y$ is coherent.
(ii) Show that restriction of holomorphic functions induces an isomorphism

$$j^{-1}(\mathcal{O}_X/\mathcal{I}_Y) \cong \mathcal{O}_Y.$$

DEFINITION III.2.9. A holomorphic map $F\colon Y \to X$ between complex manifolds is a *holomorphic immersion* if the induced complex-linear map $(F_*)_y\colon T_{Y,y} \to T_{X,F(y)}$ is injective for every $y \in Y$. Here $(F_*)_y$ denotes the restriction of the complexified tangent map to the $(1,0)$ tangent space. Equivalently, its coordinate expression has complex Jacobian of rank $\dim_y Y$ at every y. It is a *holomorphic embedding* if it is a holomorphic immersion and $F\colon Y \to F(Y)$ is a homeomorphism, where $F(Y)$ has the subspace topology induced from X. A holomorphic embedding that is also proper is called a *proper holomorphic embedding*. □

By the holomorphic inverse function theorem, every holomorphic immersion is locally, on the source, a holomorphic embedding. Closedness of the image is not part of the definition of an embedding. If the embedding is proper, its image is closed and is a complex submanifold in the convention of Definition III.2.7. Conversely, the inclusion of a complex submanifold in that convention is a proper holomorphic embedding.

We now return to Levi-flat real hypersurfaces. Let M be a C^∞ Levi-flat real hypersurface in a complex manifold X of pure dimension n. Its distribution $HM = TM \cap J(TM)$ is well-defined independently of coordinates by §1.3. In every holomorphic chart, Mini-Exercise III.1.16 (iv) shows that HM is involutive. It has real rank $2n-2$; hence the Frobenius theorem recalled in §8.10 gives a C^∞ foliation $\mathcal{F}$ on M with $T\mathcal{F} = HM$. This is the *Levi foliation* of M. The descriptions in different charts agree because they have the same integral leaves of this intrinsic distribution. For $n = 1$, the leaves are points, just as in Remark III.1.17.

A leaf carries its own C^∞ manifold topology, and its inclusion into X is an immersion; it need not be an embedding. For $n \geq 2$, the following exercise constructs its holomorphic coordinates using a projection in each ambient chart and checks compatibility when the ambient chart changes. This proves that the inclusion of the leaf is a holomorphic immersion.

MINI-EXERCISE III.2.10. Let X be a complex manifold of pure dimension $n \in \mathbb{Z}_{\geq 2}$, $M \subset X$ a C^∞ Levi-flat real hypersurface, and $\mathcal{F}$ its Levi foliation. Let $\mathcal{L}$ be a leaf and denote its natural C^∞ immersion by

$$\iota\colon \mathcal{L} \longrightarrow X.$$

For $p \in \mathcal{L}$, choose a holomorphic chart $z\colon U \to \Omega \subset \mathbb{C}^n$ around $\iota(p)$ and put $\iota_z := z \circ \iota$ on $\iota^{-1}(U)$. In the coordinate calculations below, J denotes the standard complex structure of the relevant complex Euclidean space, as in (92).
(i) Show that there exists a complex-linear map $P\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ such that

$$d(P \circ \iota_z)_p\colon T_p\mathcal{L} \longrightarrow T_{P(\iota_z(p))}\mathbb{C}^{n-1}$$

is an isomorphism of real vector spaces.
(*ii*) After shrinking an open neighborhood $V \subset \iota^{-1}(U)$ of p in $\mathcal{L}$, use the real inverse function theorem to regard $P \circ \iota_z|_V$ as a C^∞ coordinate map onto a domain $W \subset \mathbb{C}^{n-1}$. Set

$$\Phi := \iota_z \circ (P \circ \iota_z|_V)^{-1} \colon W \longrightarrow \Omega.$$

Show that

$$d\Phi_\zeta \circ J = J \circ d\Phi_\zeta \qquad (\zeta \in W),$$

and conclude that Φ is holomorphic.
(*iii*) Show that the coordinate maps obtained from all such choices of z and P form a complex atlas on $\mathcal{L}$. Conclude that $\mathcal{L}$ has a natural complex manifold structure of dimension $n-1$ for which

$$\iota \colon \mathcal{L} \longrightarrow X$$

is a holomorphic immersion.

2.2 Holomorphic vector bundles

We now put the vector bundles of Chapter I, §7.7 and the locally free modules of Chapter II into the holomorphic setting. Throughout this subsection, X is a complex manifold; no connectedness or constancy of its dimension is assumed. We first fix a rank $r \in \mathbb{Z}_{\geq 0}$.

Definition III.2.11. A *holomorphic vector bundle* of *rank* r on X is a complex vector bundle $\pi \colon E \to X$ whose total space is a complex manifold and which has local trivializations $\psi_j \colon \pi^{-1}(V_j) \to V_j \times \mathbb{C}^r$ that are biholomorphic, for an open covering $\{V_j\}_{j \in J}$ of X. Thus the diagram

$$\begin{array}{ccc} \pi^{-1}(V_j) & \xrightarrow{\ \psi_j\ } & V_j \times \mathbb{C}^r \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{Pr}_1} & \\ & V_j & \end{array}$$

commutes, and the changes of trivialization have the form

$$\psi_j \circ \psi_k^{-1}(x, \eta_k) = (x, g_{jk}(x)\eta_k), \qquad g_{jk} \colon V_j \cap V_k \to \mathrm{GL}_r(\mathbb{C}) \text{ holomorphic.} \tag{142}$$

A bundle of rank 1 is a *holomorphic line bundle*. A *holomorphic section* on an open subset $U \subset X$ is a holomorphic map $s \colon U \to E$ such that $\pi \circ s = \mathrm{id}_U$. A *local holomorphic frame* on U consists of holomorphic sections $e_1, e_2, \ldots, e_r$ whose values form a basis of $E_x := \pi^{-1}(x)$ for every $x \in U$. □

The projection π is holomorphic, and E is also a C^∞ complex vector bundle on the underlying C^∞ manifold X. The local frames associated with ψ_j are obtained by applying ψ_j^{-1} to the standard basis sections of $V_j \times \mathbb{C}^r$. Conversely, a holomorphic frame gives such a trivialization: its coefficient matrix in any given trivialization is holomorphic and invertible,

and its inverse is holomorphic as well. When no other category is specified in the holomorphic setting, we will use *vector bundle* and *line bundle* in this sense.

A *holomorphic vector-bundle morphism in the sense of Chapter I, §7.7* over X is a holomorphic, fiber-preserving map $F\colon E \to E'$ which is linear on every fiber and has locally constant fiber rank. A *holomorphic bundle isomorphism* is such a map which is an isomorphism on every fiber; its inverse is holomorphic by the local matrix description. We write $E \cong E'$ when the holomorphic category is understood, and retain $E \cong_{\text{top}} E'$ for a merely topological bundle isomorphism. A bundle is *holomorphically trivial* if it is holomorphically isomorphic to $X \times \mathbb{C}^r$. The operations E^*, $E \oplus E'$, $E \otimes E'$, $\bigwedge^k E$, and $\operatorname{Hom}(E, E')$ are defined as in Chapter I: the induced transition matrices are holomorphic, so these are again holomorphic bundles.

Holomorphic sections and locally free sheaves

For a holomorphic vector bundle $E \to X$, its sheaf of holomorphic sections is denoted by $\mathcal{O}_X(E)$:

$$\mathcal{O}_X(E)(U) := \{s\colon U \to E \mid s \text{ is holomorphic and } \pi \circ s = \mathrm{id}_U\}.$$

Addition of sections and multiplication by holomorphic functions make this an $\mathcal{O}_X$-module. Let $U \subset X$ be an open set on which E has a holomorphic frame $e_1, e_2, \dots, e_r$. Every section $s \in \mathcal{O}_X(E)(U)$ then has a unique expression $s = \sum_{\nu=1}^{r} f_\nu e_\nu$, with $f_\nu \in \mathcal{O}_X(U)$. The same statement holds after restricting to any open subset of U. Hence the coefficient map gives an isomorphism of sheaves $\mathcal{O}_X(E)|_U \cong (\mathcal{O}_X|_U)^{\oplus r}$. In the terminology of Definition II.3.103, $\mathcal{O}_X(E)$ is locally free of rank r.

Proposition III.2.12. *The assignment $E \mapsto \mathcal{O}_X(E)$ induces a bijection between holomorphic isomorphism classes of rank-r holomorphic vector bundles on X and isomorphism classes of locally free $\mathcal{O}_X$-modules of rank r.*

Proof. The construction after Definition II.3.103 associates to local trivializations of a locally free module $\mathcal{E}$ transition matrices $g_{jk} \in \mathrm{GL}_r(\mathcal{O}_X(V_j \cap V_k))$ with $g_{jk}g_{k\ell} = g_{j\ell}$. Glue $V_j \times \mathbb{C}^r$ by $(x, \eta_k) \mapsto (x, g_{jk}(x)\eta_k)$. These maps are biholomorphic, so the vector-bundle construction of Lemma I.7.35 gives a holomorphic bundle E. A countable trivializing cover may be chosen because X is second-countable; the resulting total space is Hausdorff and second-countable, as required for a complex manifold. The coefficient maps identify $\mathcal{O}_X(E)$ with $\mathcal{E}$. Changes of frames act by the same matrices on both constructions, so they are inverse on isomorphism classes. See also [De4, Chapter V, §8 and §11]. □

The fiber E_x is not the stalk $\mathcal{O}_X(E)_x$. Evaluation of a germ of a section at x instead gives

$$\mathcal{O}_X(E)_x \big/ \mathfrak{m}_{X,x}\mathcal{O}_X(E)_x \;\cong\; E_x, \qquad [s_x] \longmapsto s(x), \tag{143}$$

where $\mathfrak{m}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$. Indeed, in a local frame, evaluation is evaluation of the r coefficient germs; its kernel consists precisely of those for which all coefficients vanish at x. For a finite locally free module whose rank is not constant, the preceding construction is applied separately on each connected component, on which the locally constant rank is constant.

Remark III.2.13 (From vector-bundle morphisms to sheaf morphisms). The locally constant-rank condition in Chapter I, §7.7 allows us to begin with vector bundles as a relative version of linear algebra: kernels, images, and cokernels can be taken as vector bundles. For more general questions, however, this restriction on morphisms has two limitations.

First, it is not preserved by composition. On the base $\mathbb{C}$, consider the trivial holomorphic bundles $\mathbb{C}\times\mathbb{C}$ and $\mathbb{C}\times\mathbb{C}^2$ of ranks 1 and 2, respectively, each with its first projection to $\mathbb{C}$. Let a map the first bundle to the second and b the second to the first, both over the identity of the base. Their restrictions to the fibers over z, identified with maps $a_z\colon \mathbb{C}\to\mathbb{C}^2$ and $b_z\colon \mathbb{C}^2\to\mathbb{C}$, are given by

$$a_z(t) = (t, zt), \qquad b_z(u, v) = v.$$

Both maps depend holomorphically on z and have rank 1 on every fiber, so they are vector-bundle morphisms in the sense of Chapter I. Their composition is $(b\circ a)_z(t) = zt$, which has rank 0 at $z = 0$ and rank 1 elsewhere. Thus maps satisfying our rank condition are not closed under composition; they cannot by themselves serve as the morphisms of a category with the usual composition.

Second, even multiplication by a holomorphic function need not satisfy this rank condition. Regard $\mathbb{C}\times\mathbb{C}$ as the trivial holomorphic line bundle over $\mathbb{C}$ via the first projection, and consider the map over this base

$$\Phi\colon \mathbb{C}\times\mathbb{C}\longrightarrow\mathbb{C}\times\mathbb{C}, \qquad \Phi(z, t) = (z, zt).$$

The kernels of its fiber maps have dimension 1 at the origin and 0 elsewhere, so their union is not a vector subbundle. Nevertheless, multiplication of sections by z gives a well-defined morphism of $\mathcal{O}_{\mathbb{C}}$-modules

$$\varphi\colon \mathcal{O}_{\mathbb{C}}\longrightarrow\mathcal{O}_{\mathbb{C}}, \qquad f\longmapsto zf.$$

Its kernel, image, and cokernel as sheaves are

$$\begin{aligned}\operatorname{Ker}\varphi &= 0,\\ \operatorname{Image}\varphi &= z\mathcal{O}_{\mathbb{C}}\cong\mathcal{O}_{\mathbb{C}},\\ \operatorname{Coker}\varphi &= \mathcal{O}_{\mathbb{C}}/z\mathcal{O}_{\mathbb{C}}.\end{aligned}$$

Indeed, $zf = 0$ implies $f = 0$ off the origin and hence also at the origin by continuity. The same argument proves that multiplication by z identifies $\mathcal{O}_{\mathbb{C}}$ with its image sheaf. Although this image sheaf is locally free, its inclusion into $\mathcal{O}_{\mathbb{C}}$ induces the zero map on the fiber at 0, where fibers are obtained by (143). Thus it is not the inclusion of a line subbundle. The cokernel is zero off 0 and has stalk $\mathcal{O}_{\mathbb{C},0}/(z)\cong\mathbb{C}$ at 0; it is not locally free. In particular, taking the kernel of a sheaf morphism and then its fiber need not give the kernel of the corresponding fiber map. Kernels, images, and cokernels still exist in the category of sheaves of modules, as explained in Proposition II.3.58; they need not remain locally free.

For holomorphic bundles E and F on a general complex manifold X, an $\mathcal{O}_X$-linear morphism $\mathcal{O}_X(E)\to\mathcal{O}_X(F)$ is represented in local frames by a holomorphic matrix. The frame-change rule glues these matrices to a fiber-preserving, fiberwise linear holomorphic map $E\to F$, and conversely any such map induces a morphism of section sheaves. No locally

constant-rank condition is needed for this correspondence. Accordingly, in subsequent sheaf-theoretic discussions of holomorphic vector bundles, a morphism $E \to F$ is understood as an $\mathcal{O}_X$-linear morphism between their sheaves of holomorphic sections, unless stated otherwise. The phrase "vector-bundle morphism in the sense of Chapter I" retains the rank condition, which will be stated explicitly whenever it is needed to obtain subbundles or quotient bundles.

This distinction also clarifies the role of the Hom bundle. Put $\mathcal{E} := \mathcal{O}_X(E)$ and $\mathcal{F} := \mathcal{O}_X(F)$. The notation $\mathrm{Hom}(E,F)$ continues to denote the bundle with fiber $\mathrm{Hom}_{\mathbb{C}}(E_x, F_x)$, not a set of restricted bundle morphisms. In local frames, its holomorphic sections and morphisms of section sheaves are represented by the same arbitrary holomorphic matrices, with the same change-of-frame rule. Recall the notation of Chapter II, §3.4. The symbol $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ denotes the set of morphisms from $\mathcal{E}$ to $\mathcal{F}$ in $(\mathsf{Mod}(\mathcal{O}_X))$; the subscript specifies $\mathcal{O}_X$-linearity, with no condition on fiber rank. The different typeface in $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ indicates the *sheaf* of such morphisms on open subsets. More precisely, for every open subset $U \subset X$, its sections are

$$\Gamma\big(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})\big) = \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{E}|_U, \mathcal{F}|_U).$$

The right-hand side consists of morphisms of the restricted sheaves, not merely module homomorphisms between their spaces of sections on U. Thus the Hom bundle, the sheaf Hom, and the set of sheaf morphisms are distinguished. The local matrix description above gives

$$\mathcal{O}_X\big(\mathrm{Hom}(E,F)\big) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}).$$

Taking global sections gives

$$\Gamma\big(X, \mathcal{O}_X(\mathrm{Hom}(E,F))\big) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}),$$

where the right-hand side is the set of $\mathcal{O}_X$-linear sheaf morphisms. Neither identification imposes a fiber-rank condition. Indeed, multiplying the identity section of the Hom bundle of the trivial line bundle by z already gives the map considered above; restricting to locally constant-rank maps would not even be compatible with multiplication by holomorphic functions. □

In rank one, Proposition III.2.12 identifies holomorphic line bundles, up to holomorphic isomorphism, with $\mathrm{Pic}(X,\mathcal{O}_X)$ of Chapter II. We abbreviate this group to $\mathrm{Pic}(X)$ when the structure sheaf is understood. Tensor products and duals of line bundles correspond to the group law and inverses; in particular, $[L] := [\mathcal{O}_X(L)]$ is the class of L. When no confusion is possible, we also write L for this class. The identification with $\check{H}^1(X,\mathcal{O}_X^*)$ is the one in Remark II.3.104. For a holomorphic line bundle $L \to X$, we write $L^{-1} := L^*$ and, for $m \in \mathbb{Z}_{\geq 0}$, set $L^m := L^{\otimes m}$ and $L^{-m} := (L^{-1})^{\otimes m}$. Here L^0 is naturally interpreted as the holomorphically trivial line bundle on X.

Example III.2.14. For $r \in \mathbb{Z}_{\geq 0}$, the product $X \times \mathbb{C}^r$, together with the first projection $X \times \mathbb{C}^r \to X$, is a holomorphic vector bundle with sheaf of sections $\mathcal{O}_X^{\oplus r}$. The holomorphic tangent bundle T_X constructed in the preceding subsection and its dual T_X^* are further examples; their

ranks are $\dim_x X$ on each component. In holomorphic coordinates, $\partial/\partial z^1, \partial/\partial z^2, \dots, \partial/\partial z^n$ and $dz^1, dz^2, \dots, dz^n$ are their respective local frames.

For a complex vector space V of dimension $n+1$, where $n \in \mathbb{Z}_{\geq 0}$, the space

$$L_{\text{taut}} := \{(\ell, v) \in \mathbf{P}(V) \times V \mid v \in \ell\}$$

together with the projection $\pi\colon L_{\text{taut}} \to \mathbf{P}(V)$, $(\ell, v) \mapsto \ell$, induced by the first projection, is the *tautological line bundle* on $\mathbf{P}(V)$, the space of lines rather than hyperplanes in V. Choose a basis of V and write $Z = (Z^0, Z^1, \dots, Z^n)$ for a nonzero representative of ℓ. On the chart where $Z^j \neq 0$, the vector Z/Z^j is a nonvanishing holomorphic section. On an overlap these vectors differ by the nowhere-vanishing holomorphic factor Z^k/Z^j, giving the required line-bundle trivializations. Its sheaf of sections is usually written $\mathcal{O}_{\mathbf{P}(V)}(-1)$; compare [De4, Chapter V, §15.A], where the space of lines is written $P(V)$. See also Figure III.1. □

Remark III.2.15 (The tautological bundle and a blow-up). There is another useful map on the total space of L_{taut}. For $V = \mathbb{C}^{n+1}$, consider

$$\beta\colon L_{\text{taut}} \longrightarrow V, \qquad (\ell, v) \longmapsto v.$$

Over $V \setminus \{0\}$ its inverse is $v \mapsto (\mathbb{C}v, v)$, whereas $\beta^{-1}(0)$ is the zero section $\mathbf{P}(V)$. The map is proper: L_{taut} is closed in $\mathbf{P}(V) \times V$, and $\mathbf{P}(V)$ is compact. In the chart of $\mathbf{P}(V)$ where the i-th homogeneous coordinate is nonzero, put $u_j = Z^j/Z^i$ for $j \neq i$. The coordinates $(t, (u_j)_{j \neq i})$ on the total space give

$$v^i = t, \qquad v^j = t u_j \quad (j \neq i). \tag{144}$$

Thus the total space is again a complex manifold. The map β is called the *blow-up of V at the origin*; the origin is its *center*. For $n \geq 1$, the zero section is its *exceptional hypersurface*, where the map ceases to be a local biholomorphism. This construction is also called a *monoidal transformation*; see [De4, Chapter V, Lemma (15.4)]. See also Figure III.1.

For a point p of a complex manifold X, choose a coordinate neighborhood of p, perform this construction there, and glue to $X \setminus \{p\}$ using the biholomorphism away from the center. This gives a complex manifold $\mathrm{Bl}_p X$ and a proper surjective holomorphic map $\mathrm{Bl}_p X \to X$, independent of the coordinates up to isomorphism over X; see [BM3, §1.2, pp. 46–47]. For $\dim_p X \geq 2$, the exceptional hypersurface is $\mathbf{P}(T_{X,p})$. If $\dim_p X = 1$, the same construction is an isomorphism, so there is no exceptional locus in the sense of a failure of local biholomorphy. We do not use the point construction on zero-dimensional components. Blow-ups can also be defined with a coherent ideal sheaf as center, not merely with a point; in that generality the new space need not be non-singular. We return to smooth centers and the precise meaning of a proper modification in §2.5; for the analytic ideal-sheaf construction, see [GM, §3]. □

Corollary III.2.16. *Every finite locally free $\mathcal{O}_X$-module is coherent. In particular, $\mathcal{O}_X(E)$ is coherent for every holomorphic vector bundle $E \to X$.*

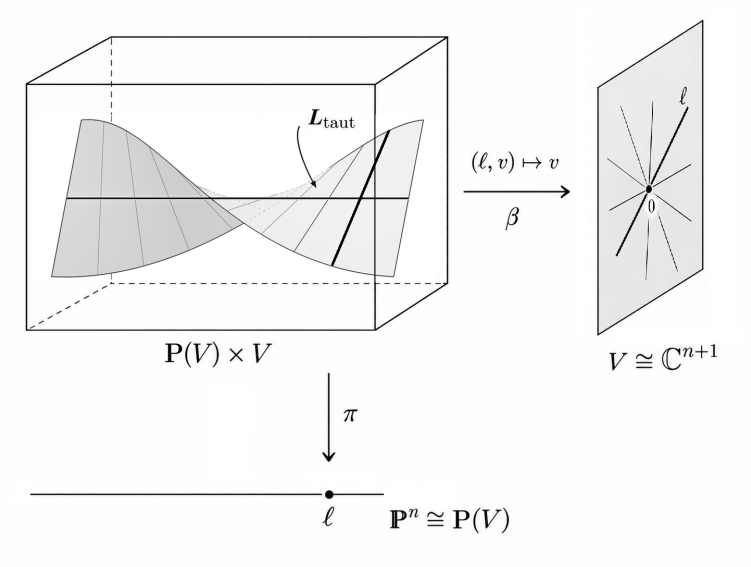


Figure III.1: Schematic picture of the tautological line bundle $L_{\text{taut}} \subset \mathbf{P}(V) \times V$, its projection π, and the blow-up map $\beta(\ell, v) = v$. The bold oblique line on the left represents the fiber $\pi^{-1}(\ell)$, which β maps onto the bold line $\ell \subset V$ on the right. The horizontal line in L_{taut} represents the image of the zero section of π, namely $\mathbf{P}(V) \times \{0\} = \beta^{-1}(0)$, which is the exceptional hypersurface of β for $n \geq 1$.

PROOF. By Corollary III.2.5, $\mathcal{O}_X$ is coherent. Its finite direct sums are coherent by the consequence of the Three Lemma for coherent sheaves (Proposition III.1.39). Locally a finite locally free module is such a direct sum, and coherence is local. □

Consequently, the kernel, image, and cokernel of any $\mathcal{O}_X$-linear morphism between finite locally free modules are coherent, although they need not be locally free. Indeed, the image is locally finitely generated and is a subsheaf of the coherent target, so it is coherent by the subsheaf criterion used in Lemma III.1.40. The Three Lemma for coherent sheaves (Proposition III.1.39) then applies to the kernel–image and image–cokernel short exact sequences; see also [GrR, Annex, §4.2, Consequence 2]. Thus the maps of nonconstant rank in Remark III.2.13 can be treated within coherent sheaves, rather than only locally free sheaves.

Bundle-valued forms and the delbar-operator

Let $E \to X$ be a holomorphic vector bundle of rank $r \in \mathbb{Z}_{\geq 0}$. An *$E$-valued (p, q)-form* on an open subset $U \subset X$ is a C^∞ section of $\Lambda^{p,q} X \otimes E$ over U. We write $\mathcal{A}^{p,q}_X(E)$ for the sheaf of these forms, so that

$$\mathcal{A}^{p,q}_X(E)(U) = C^\infty(U, (\Lambda^{p,q} X \otimes E)|_U).$$

The smooth bundle $\Lambda^{p,q} X$ is the one already defined from (139). The holomorphic bundle $\bigwedge^p T^*_X \otimes E$ has sheaf of sections $\Omega^p_X(E) := \mathcal{O}_X(\bigwedge^p T^*_X \otimes E)$, whose elements are called *E-valued holomorphic p-forms*. We write Ω^p_X when E is the trivial line bundle. In particular,

$\Omega^0_X(E) = \mathcal{O}_X(E)$. The holomorphic line bundle $K_X := \det T^*_X$ is called the *canonical bundle* of X.

In a local holomorphic frame $e_1, e_2, \ldots, e_r$, write $\eta = \sum_{\nu=1}^r \eta_\nu \otimes e_\nu$, with scalar forms η_ν of type (p,q), and set

$$\overline{\partial}_E \eta := \sum_{\nu=1}^r (\overline{\partial}\eta_\nu) \otimes e_\nu. \tag{145}$$

Mini-Exercise III.2.17. (i) Show that (145) is independent of the holomorphic coordinates and the local holomorphic frame. Thus it defines a $\mathbb{C}$-linear map

$$\overline{\partial}_E \colon \mathcal{A}^{p,q}_X(E)(U) \longrightarrow \mathcal{A}^{p,q+1}_X(E)(U).$$

Show also that $\overline{\partial}^2_E = 0$.
(ii) Does applying d to the scalar coefficient forms in a local frame, namely

$$d\eta := \sum_{\nu=1}^r (d\eta_\nu) \otimes e_\nu,$$

define an operator on E-valued forms independently of the chosen holomorphic frame? Explain the difference from (i).

In local coordinates, the resulting operator has the familiar scalar expression on each coefficient:

$$\eta = \sum_{\#I=p} \sum_{\#\overline{J}=q} \sum_{\nu=1}^r f_{I\overline{J},\nu}(z)\, dz^I \wedge d\overline{z^J} \otimes e_\nu,$$

$$\overline{\partial}_E \eta = \sum_{\#I=p} \sum_{\#\overline{J}=q} \sum_{\nu=1}^r \sum_{j=1}^n \frac{\partial f_{I\overline{J},\nu}}{\partial \overline{z^j}}\, d\overline{z^j} \wedge dz^I \wedge d\overline{z^J} \otimes e_\nu.$$

Here the increasing multi-indices and the wedge products have the same meaning as in §1.1. As usual, we may omit the subscript E when the coefficient bundle is fixed: $\overline{\partial} := \overline{\partial}_E$.

The kernel of $\overline{\partial}$ on $\mathcal{A}^{p,0}_X(E)$ is $\Omega^p_X(E)$, by the characterization of holomorphic functions in Theorem III.1.18 and the same coordinate calculation. In particular, on every open subset $U \subset X$, one has $\mathcal{O}_X(E)(U) = \{s \in \mathcal{A}^{0,0}_X(E)(U) \mid \overline{\partial} s = 0\}$. No metric has been chosen in this construction.

Remark III.2.18 (Sheaf notation and bundle notation). For ease of reference, we collect the sheaf notation used on a complex manifold X. In the descriptions below, $U \subset X$ is open, and the restriction maps are the usual restrictions.

- $\underline{G}_X$ is the sheaf of locally constant G-valued functions, where G is an abelian group, for example $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$; see Example II.3.54.
- $\mathcal{O}_X$ is the sheaf of holomorphic functions, and $\mathcal{O}^*_X$ is its sheaf of units: $\mathcal{O}^*_X(U)$ consists of the nowhere vanishing holomorphic functions on U, with multiplication as the group operation.

- $\mathcal{A}_X^{p,q}$ is the sheaf of C^∞ forms of bidegree (p,q), whereas $\mathcal{K}_X^{p,q}$ is the sheaf of currents of that bidegree. For a holomorphic vector bundle E, $\mathcal{A}_X^{p,q}(E)$ and $\mathcal{K}_X^{p,q}(E)$ denote the sheaves of E-valued C^∞ (p,q)-forms and (p,q)-currents, respectively.

- For a holomorphic vector bundle $E \to X$, $\mathcal{O}_X(E)$ is its sheaf of holomorphic sections. If E has rank r, this is a locally free $\mathcal{O}_X$-module of rank r, as in Proposition III.2.12.

- Ω_X^p is the sheaf of holomorphic p-forms:

$$\Omega_X^p = \mathcal{O}_X\big(\bigwedge^p T_X^*\big), \qquad \Omega_X^p(E) = \mathcal{O}_X\big(\bigwedge^p T_X^* \otimes E\big).$$

  Thus Ω_X^p is the kernel of $\overline{\partial}$ on $\mathcal{A}_X^{p,0}$; in particular, $\Omega_X^0 = \mathcal{O}_X$.

- $K_X := \det T_X^*$ is the canonical bundle, a holomorphic line bundle on X. If X has pure dimension n, then $\mathcal{O}_X(K_X) = \Omega_X^n$.

- For a closed subset $A \subset X$, $\mathcal{I}_A \subset \mathcal{O}_X$ is the ideal sheaf of functions vanishing on A: $\mathcal{I}_A(U) = \{f \in \mathcal{O}_X(U) \mid f|_{A\cap U} = 0\}$. For analytic subsets, this is the notation in (155).

- For a point $p \in X$, we write $\mathfrak{m}_p := \mathcal{I}_{\{p\}}$. Explicitly,

$$\mathfrak{m}_p(U) = \begin{cases} \{f \in \mathcal{O}_X(U) \mid f(p) = 0\}, & p \in U, \\ \mathcal{O}_X(U), & p \notin U. \end{cases}$$

  This is a sheaf on X, whereas $\mathfrak{m}_{X,p} = (\mathfrak{m}_p)_p$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$.

- On $\mathbb{P}^n$, using the homogeneous-coordinate line model and the tautological bundle of Example III.2.14, we write

$$\mathcal{O}_{\mathbb{P}^n}(m) := \mathcal{O}_{\mathbb{P}^n}(L_{\mathrm{taut}}^{-m}) \quad (m \in \mathbb{Z}).$$

  For $m \geq 0$, the bundle L_{taut}^{-m} means $(L_{\mathrm{taut}}^{-1})^{\otimes m}$, the m-fold tensor power of the dual tautological bundle; for $m < 0$, it means $L_{\mathrm{taut}}^{\otimes(-m)}$. The zeroth tensor power is the trivial line bundle, so $\mathcal{O}_{\mathbb{P}^n}(0) = \mathcal{O}_{\mathbb{P}^n}$.

It is customary to use the same notation for a holomorphic vector bundle and its locally free sheaf of holomorphic sections. We avoid this identification when possible, since we also use the total spaces of bundles as complex manifolds, as in Remark III.2.15. An exception is Ω_X^p, which may also denote the holomorphic cotangent exterior power itself. When the distinction matters, we write $\bigwedge^p T_X^*$ for the holomorphic vector bundle and $\Omega_X^p = \mathcal{O}_X(\bigwedge^p T_X^*)$ for its sheaf of sections. □

2.3 Hermitian metrics and Chern curvatures

Mini-Exercise III.2.17 shows that applying d to the coefficients in a local frame is not intrinsic. A connection supplies the correction needed under a change of frame. We recall the basic constructions, following [De3, §3, pp. 23–25]. Throughout this subsection, E is a holomorphic vector bundle of rank r on a complex manifold X. We use the spaces $\mathcal{A}^q(X,E)$ of Chapter I, §8.6.

Definition III.2.19. A *connection* on E is a $\mathbb{C}$-linear differential operator D from $\mathcal{A}^0(X,E)$ to $\mathcal{A}^1(X,E)$ satisfying $D(fs) = df \otimes s + fDs$ for $f \in C^\infty(X)$ and $s \in \mathcal{A}^0(X,E)$. □

Let D be a connection on the rank-r bundle E, and let $U \subset X$ be an open set with a smooth frame $e_1, e_2, \ldots, e_r$. Write

$$De_\nu = \sum_{\lambda=1}^{r} \Gamma_\nu^\lambda \otimes e_\lambda, \qquad \Gamma = (\Gamma_\nu^\lambda)_{1\le\lambda,\nu\le r},$$

where each Γ_ν^λ is a 1-form on U. We call Γ the *connection matrix* in this frame; λ is its row index and ν its column index. For a section $s = \sum_{\nu=1}^r u^\nu e_\nu$, the Leibniz rule gives

$$Ds = \sum_{\lambda=1}^{r} \left(du^\lambda + \sum_{\nu=1}^{r} \Gamma_\nu^\lambda u^\nu \right) \otimes e_\lambda.$$

Thus, if u is the column of coefficients of s, the coefficient column of Ds is $du + \Gamma u$.

Applying the same formula to columns of differential forms defines an extension of D to E-valued forms, with local expression $Du = du + \Gamma \wedge u$. Here and below an equality in a frame means equality of the coefficient columns in that frame. If $u_j = g_{jk} u_k$ is the change of coefficients between two smooth frames, the connection matrices on sections satisfy

$$\Gamma_k = g_{jk}^{-1} \Gamma_j g_{jk} + g_{jk}^{-1} dg_{jk}.$$

Consequently, $du_j + \Gamma_j \wedge u_j = g_{jk}(du_k + \Gamma_k \wedge u_k)$ also for columns of forms, so the local extensions agree on overlaps. The resulting operator satisfies

$$D(\alpha \wedge u) = d\alpha \wedge u + (-1)^p \alpha \wedge Du, \qquad \alpha \in \mathcal{A}^p(X), \quad u \in \mathcal{A}^q(X,E).$$

Matrix multiplication is combined with wedge product, so that $(\Gamma \wedge \Gamma)_\nu^\lambda = \sum_{\mu=1}^r \Gamma_\mu^\lambda \wedge \Gamma_\nu^\mu$. A direct calculation gives

$$D^2 u = (d\Gamma + \Gamma \wedge \Gamma) \wedge u. \tag{146}$$

Since D^2 is C^∞-linear, the local matrices in (146) define a global $\mathrm{Hom}(E,E)$-valued 2-form, the curvature of D. Unlike an $\mathcal{O}_X$-linear sheaf morphism, a connection satisfies a Leibniz rule with the extra term $df \otimes s$.

Recall that a C^∞ *Hermitian metric* h on E assigns a positive definite Hermitian form to each fiber, smoothly in the base point. Such metrics exist by the smooth version of

Lemma I.7.34, recalled in Chapter I, §8. Following Remark I.2.3, we write $\langle v, w\rangle_h$ for the Hermitian inner product of two vectors in the same fiber. In a local frame $e_1, e_2, \dots, e_r$, put $H = (\langle e_\lambda, e_\mu\rangle_h)_{1\le\lambda,\mu\le r}$. For bundle-valued forms we use the scalar-valued pairing

$$\{u, v\}_h := \sum_{\lambda,\mu=1}^{r} \langle e_\lambda, e_\mu\rangle_h \, u_\lambda \wedge \overline{v_\mu}, \qquad u = \sum_{\lambda=1}^{r} u_\lambda \otimes e_\lambda, \quad v = \sum_{\mu=1}^{r} v_\mu \otimes e_\mu.$$

This is independent of the frame. A connection is *compatible with* h, or *Hermitian*, if

$$d\{u, v\}_h = \{Du, v\}_h + (-1)^p \{u, Dv\}_h \qquad (u \in \mathcal{A}^p(X, E)).$$

It suffices to check this on sections. In matrix notation the condition is $dH = {}^t\Gamma\, H + H\overline{\Gamma}$.

Proposition III.2.20. *Let X be a complex manifold, let $E \to X$ be a holomorphic vector bundle of rank $r \in \mathbb{Z}_{\ge 0}$, and let h be a C^∞ Hermitian metric on E. There is a unique connection compatible with h whose $(0,1)$-part is the operator defined in (145). It is called the* Chern connection *and is denoted by*

$$D_h = \partial_h + \overline{\partial}.$$

In a local holomorphic frame, with metric matrix H as above,

$$\Gamma_h = (\overline{H})^{-1}\partial\overline{H}, \qquad D_h u = du + \Gamma_h \wedge u, \qquad \partial_h u = (\overline{H})^{-1}\partial(\overline{H}u), \tag{147}$$

where u is the column of scalar coefficient forms of an E-valued differential form in this frame.

Proof. In a holomorphic frame the prescribed $(0,1)$-part says that Γ_h has type $(1,0)$. The $(1,0)$-part of metric compatibility is $\partial H = {}^t\Gamma_h\, H$, which uniquely determines Γ_h as in (147); its conjugate gives the remaining part of the compatibility condition. Under the change of coefficients $u_j = g_{jk}u_k$ of (142), one has $H_k = {}^t g_{jk}\, H_j \overline{g_{jk}}$. The displayed formula then gives $\Gamma_{h,k} = g_{jk}^{-1}\Gamma_{h,j}g_{jk} + g_{jk}^{-1}dg_{jk}$. Thus the local operators patch, proving existence as well as uniqueness; compare [De3, §3, (3.5)–(3.6)]. □

For the Chern connection D_h of Proposition III.2.20, (147) gives $\partial_h^2 = 0$, while $\overline{\partial}^2 = 0$ by Mini-Exercise III.2.17. For a local coefficient column u of an E-valued differential form, we therefore obtain

$$D_h^2 u = (\partial_h\overline{\partial} + \overline{\partial}\partial_h)u = \overline{\partial}\big((\overline{H})^{-1}\partial\overline{H}\big) \wedge u. \tag{148}$$

The coefficient matrices define a global $\mathrm{Hom}(E,E)$-valued $(1,1)$-form, which we denote by Θ_h, so that $D_h^2 u = \Theta_h \wedge u$. We call Θ_h the *Chern curvature* of h; the same term is also used for $\sqrt{-1}\Theta_h$. Metric compatibility also implies that $(\sqrt{-1}\Theta_h)(\xi,\eta)$ is an h-Hermitian endomorphism for real tangent vectors ξ, η. Indeed, in a smooth orthonormal frame $\Gamma_h^\dagger = -\Gamma_h$ and $\Theta_h^\dagger = -\Theta_h$, where the adjoint also conjugates the differential forms; see [De3, §3, (3.2)–(3.7)].

For a line bundle L, these formulas become scalar. Let e_j be local holomorphic frames and use the fiber-coordinate transition functions of (142), so that $e_k = g_{jk}e_j$. Put $a_j = |e_j|_h^2$ and $\varphi_j = -\log a_j$; the real function φ_j is the *local weight* of h. Then $a_k = |g_{jk}|^2 a_j$ and $\varphi_k = \varphi_j - \log|g_{jk}|^2$. The rank-one case of (147) and (148) reads

$$\Gamma_{h,j} = -\partial\varphi_j, \qquad \sqrt{-1}\Theta_h|_{V_j} = \sqrt{-1}\partial\overline{\partial}\varphi_j, \qquad \frac{\sqrt{-1}}{2\pi}\Theta_h|_{V_j} = dd^c\varphi_j. \tag{149}$$

Since g_{jk} is holomorphic and nowhere zero, $\partial\overline{\partial}\log|g_{jk}|^2 = 0$, so these expressions patch. In particular, $\sqrt{-1}\Theta_h$ is a real closed $(1,1)$-form.

Mini-Exercise III.2.21. Let h and h' be C^∞ Hermitian metrics on the same holomorphic line bundle L. Show that $h' = e^{-\psi}h$ for a unique $\psi \in C^\infty(X,\mathbb{R})$ and that

$$\frac{\sqrt{-1}}{2\pi}(\Theta_{h'} - \Theta_h) = dd^c\psi. \tag{150}$$

Deduce that $\{(\sqrt{-1}/2\pi)\Theta_h\} \in H^2_{\mathrm{DR}}(X,\mathbb{R})$ is independent of h.

Proposition III.2.22. *Let X be a complex manifold, let L be a holomorphic line bundle on X, and let h be a C^∞ Hermitian metric on L. Then*

$$c_1^{\mathbb{R}}(L) = \left\{\frac{\sqrt{-1}}{2\pi}\Theta_h\right\} \in H^2(X,\mathbb{R}),$$

where $c_1^{\mathbb{R}}(L)$ denotes the image of the integral first Chern class of Example II.3.86 under $\mathbb{Z} \to \mathbb{R}$, and we identify de Rham cohomology with $H^2(X,\mathbb{R})$ by the de Rham isomorphism. This class is independent of h and is natural under holomorphic pullback.

Proof. Choose a locally finite open covering $\mathcal{U} = \{U_j\}_{j\in J}$ which trivializes L and whose nonempty finite intersections are contractible. To obtain such a covering, take a locally finite refinement of a holomorphic trivializing cover, subdivide a locally finite triangulation so that its open vertex stars refine this covering, and apply the construction of Lemma I.7.20. Choose a holomorphic frame e_j of L on U_j. Write $U_{jk} = U_j \cap U_k$ and $U_{jk\ell} = U_j \cap U_k \cap U_\ell$. Let $g_{jk} \in \mathcal{O}_X^*(U_{jk})$ be determined by $e_k = g_{jk}e_j$, and choose holomorphic functions $f_{jk} \in \mathcal{O}_X(U_{jk})$ such that $g_{jk} = \exp(2\pi\sqrt{-1}f_{jk})$. Finally, put $\varphi_j = -\log|e_j|_h^2$.

The family $\{(U_{jk}, f_{jk})\}$ is a Čech 1-cochain. On triple intersections, set

$$n_{jk\ell} := f_{k\ell} - f_{j\ell} + f_{jk}.$$

Since $g_{jk}g_{k\ell} = g_{j\ell}$, these are integer-valued locally constant functions, and the Čech 2-cocycle $\{(U_{jk\ell}, n_{jk\ell})\}$ represents $c_1(L)$ by Example II.3.86.

The relation between the local weights gives

$$\varphi_j - \varphi_k = \log|g_{jk}|^2 = 2\pi\sqrt{-1}\big(f_{jk} - \overline{f_{jk}}\big).$$

As f_{jk} is holomorphic,

$$\partial(\varphi_j - \varphi_k) = 2\pi\sqrt{-1}\, df_{jk}, \qquad \overline{\partial}(\varphi_j - \varphi_k) = -2\pi\sqrt{-1}\, d\overline{f_{jk}}.$$

Put $F = (\sqrt{-1}/2\pi)\Theta_h$. The definition of d^c and (149) therefore give

$$d^c\varphi_j - d^c\varphi_k = d\,\mathrm{Re}\, f_{jk}, \qquad dd^c\varphi_j = F|_{U_j}.$$

Moreover, $\mathrm{Re}\, f_{k\ell} - \mathrm{Re}\, f_{j\ell} + \mathrm{Re}\, f_{jk} = n_{jk\ell}$. These are the local primitive relations for the degree-two Čech–de Rham comparison of Example II.3.82, with the signs in Remark A.1.5. Thus the real image of $\big[\{(U_{jk\ell}, n_{jk\ell})\}\big]$ corresponds to $\{F\}$, proving the assertion. Metric independence also follows from Mini-Exercise III.2.21; naturality follows from the exponential connecting homomorphism and pullback. See also [De4, Chapter V, Theorem 9.5]. □

The tangent bundle supplies familiar examples of this calculation. We use the fundamental form and real metric of Remark III.2.6. In one variable, if $a = |\partial/\partial z|_h^2$, then $\omega_h = \sqrt{-1}a\, dz \wedge d\bar{z}$ and $g_h = 2a(dx^2 + dy^2)$, while (149) gives

$$\sqrt{-1}\Theta_h = -\sqrt{-1}\partial\overline{\partial}\log a. \tag{151}$$

Example III.2.23 (Three metrics on Riemann surfaces). (*i*) On $\mathbb{P}^1$, let $z = Z^0/Z^1$ and $w = Z^1/Z^0$ be the non-homogeneous coordinates, with $w = 1/z$ on the overlap. The *Fubini–Study metric* h_{FS} is specified by

$$\left|\frac{\partial}{\partial z}\right|^2_{h_{\mathrm{FS}}} = \frac{1}{2\pi(1+|z|^2)^2}, \qquad \sqrt{-1}\Theta_{h_{\mathrm{FS}}} = \frac{2\sqrt{-1}\, dz \wedge d\bar{z}}{(1+|z|^2)^2}. \tag{152}$$

Indeed, $\partial/\partial w = -z^2\partial/\partial z$, so the coefficient of the metric in the w-frame is again $1/(2\pi(1+|w|^2)^2)$. Thus the metric extends smoothly across infinity. Its fundamental form is $\omega_{\mathrm{FS}} = dd^c\log(1+|z|^2)$ on the z-chart. Our normalization is $\int_{\mathbb{P}^1}\omega_{\mathrm{FS}} = 1$; compare [De4, Chapter VI, (4.4)]. Formula (151) gives the curvature in (152) and hence $\int_{\mathbb{P}^1}(\sqrt{-1}/2\pi)\Theta_{h_{\mathrm{FS}}} = 2$.

(*ii*) The *Euclidean metric* $dx^2 + dy^2$ on $\mathbb{C}$ corresponds to $|\partial/\partial z|^2_{h_{\mathrm{Eucl}}} = 1/2$. Its coefficient is constant, so $\sqrt{-1}\Theta_{h_{\mathrm{Eucl}}} = 0$.

(*iii*) On $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ we use the *Poincaré metric* in the normalization $g_{h_{\mathrm{Poinc}}} = 4(dx^2 + dy^2)/(1-|z|^2)^2$. Here $|\partial/\partial z|^2_{h_{\mathrm{Poinc}}} = 2/(1-|z|^2)^2$ and

$$\sqrt{-1}\Theta_{h_{\mathrm{Poinc}}} = 2\sqrt{-1}\partial\overline{\partial}\log(1-|z|^2) = -\frac{2\sqrt{-1}\, dz \wedge d\bar{z}}{(1-|z|^2)^2}. \tag{153}$$

□

Remark III.2.24 (A special curvature property). The preceding computations give

$$\sqrt{-1}\Theta_{h_{\mathrm{FS}}} = 4\pi\omega_{\mathrm{FS}}, \qquad \sqrt{-1}\Theta_{h_{\mathrm{Eucl}}} = 0, \qquad \sqrt{-1}\Theta_{h_{\mathrm{Poinc}}} = -\omega_{h_{\mathrm{Poinc}}}.$$

Thus the curvature of each tangent line bundle is a constant multiple of the fundamental form. This is a special property, not one shared by every Hermitian metric: on a Riemann surface

the proportionality factor is generally a function of the point. One direction in differential geometry is to single out metrics by curvature conditions of this kind and to study their existence and uniqueness. To mention only the terminology, without introducing its general definition here, the three metrics above are examples of *Kähler–Einstein metrics*. □

REMARK III.2.25 (Checking the sign on the projective line). Use the coordinates of Example III.2.23 and the complex orientation on $\mathbb{P}^1$. Writing $e_w = g_{zw}e_z$, the tautological frames $(z, 1)$ and $(1, w)$ and the relation $\partial/\partial w = -z^2\partial/\partial z$ give

$$g_{zw} = z^{-1}, \quad z, \quad -z^2 \quad \text{for } L_{\text{taut}},\ L_{\text{taut}}^{-1},\ T_{\mathbb{P}^1},\ \text{respectively.}$$

Thus $T_{\mathbb{P}^1} \cong L_{\text{taut}}^{-2}$, since the constant factor -1 can be removed by changing one frame. Proposition III.2.22 and Example III.2.23 (i) give

$$\langle c_1(T_{\mathbb{P}^1}), [\mathbb{P}^1]\rangle = \int_{\mathbb{P}^1} \frac{\sqrt{-1}}{2\pi}\Theta_{h_{\text{FS}}} = 2.$$

Additivity of c_1 under tensor products then gives $\langle c_1(L_{\text{taut}}^{-1}), [\mathbb{P}^1]\rangle = 1$. Hence $c_1(L_{\text{taut}}^{-1})$ is the positive generator of $H^2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z}$, and $c_1(L_{\text{taut}})$ is its negative. □

MINI-EXERCISE III.2.26 (The Fubini–Study metric). Let $V = \mathbb{C}^{n+1}$ with its standard Hermitian inner product, where $n \geq 1$. Regard $\mathbb{P}(V)$ as the space of hyperplanes $H \subset V$. The quotient line V/H identifies naturally with the fiber of L_{taut}^{-1} on $\mathbb{P}(V) = \mathbf{P}(V^*)$. Give this line the quotient metric induced by the inner product on V. Compute its Chern curvature and show that it is positive definite at every point of $\mathbb{P}(V)$.

2.4 Analytic subsets and complex spaces

We now allow local holomorphic equations to define objects which need not be manifolds. There are two pieces of data to retain: their common zero set and the quotient of the sheaf of holomorphic functions by the chosen equations. The latter records information which the zero set alone may lose.

DEFINITION III.2.27. Let X be a complex manifold. A subset $A \subset X$ is called an *analytic subset* (or *analytic set*) if, for every $x \in X$, there exist an open neighborhood U of x and finitely many $f_1, f_2, \ldots, f_r \in \mathcal{O}_X(U)$ such that

$$A \cap U = \{y \in U \mid f_1(y) = f_2(y) = \cdots = f_r(y) = 0\}. \tag{154}$$

The topology on A is the subspace topology. □

Since the functions $f_1, f_2, \ldots, f_r$ are continuous, $A \cap U$ is closed in U for every neighborhood in (154); hence A is closed in X.

For a fixed analytic subset $A \subset X$, its *defining ideal sheaf* (also called its *vanishing ideal sheaf*) is defined on open subsets $U \subset X$ by

$$\mathcal{I}_A(U) := \{f \in \mathcal{O}_X(U) \mid f|_{A\cap U} = 0\}. \tag{155}$$

Restrictions and gluing are inherited from $\mathcal{O}_X$, as in Example III.1.38. Each ideal $(\mathcal{I}_A)_x$ is radical: if a power of a representative vanishes on A near x, then the representative itself vanishes there. The equations in (154) generate a coherent ideal sheaf $\mathcal{I} = (f_1, f_2, \ldots, f_r) \subset \mathcal{O}_X|_U$, by Proposition III.1.43 in coordinate charts, but only $\mathcal{I} \subset \mathcal{I}_A|_U$ follows immediately. For instance, z and z^2 have the same zero set in $\mathbb{C}$, whereas $(z^2) \neq (z) = \mathcal{I}_{\{0\}}$. The defining ideal sheaf $\mathcal{I}_A$ of a general analytic subset is also coherent, but this is a nontrivial theorem, which we record below (*Oka's second coherence theorem*). We first construct complex spaces using ideals already known to be coherent.

Definition III.2.28. Let X be a complex manifold and $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf. The *closed complex subspace defined by* $\mathcal{I}$ is the locally ringed space $(S, \mathcal{O}_S)$, where

$$S = \operatorname{Supp}(\mathcal{O}_X/\mathcal{I}), \qquad \mathcal{O}_S = j^{-1}(\mathcal{O}_X/\mathcal{I}), \quad j\colon S \hookrightarrow X. \tag{156}$$

The support notation is defined in Chapter II (p. 266). Such a subspace is also called a *closed analytic subspace*. Here S has the subspace topology, and j^{-1} is the inverse image of sheaves from Chapter II. No radical condition is imposed on $\mathcal{I}$. □

For a complex submanifold $Y \subset X$, take $\mathcal{I}_Y$ as defined in Mini-Exercise III.2.8. By part (ii) of that exercise, taking $\mathcal{I} = \mathcal{I}_Y$ in (156) recovers the usual structure sheaf of Y. Let us now check the ingredients of this definition in general. Since $\mathcal{I}$ is coherent, near each point of X we can choose an open neighborhood U and finitely many sections $f_1, f_2, \ldots, f_r \in \mathcal{I}(U)$ generating $\mathcal{I}|_U$. The same sections generate every stalk on U, as in the semi-stalkwise viewpoint of Remark III.1.37. A point $y \in U$ belongs to S precisely when all $f_j(y)$ vanish. Indeed, otherwise one generator is a unit in $\mathcal{O}_{X,y}$, while if all vanish then $\mathcal{I}_y \subset \mathfrak{m}_{X,y}$ is a proper ideal. Thus S is an analytic subset of X. For $y \in S$, taking stalks in (156) gives $\mathcal{O}_{S,y} \cong \mathcal{O}_{X,y}/\mathcal{I}_y$. The ideal $\mathfrak{m}_{X,y}/\mathcal{I}_y$ in this quotient is maximal, since the quotient by it is $\mathbb{C}$. Every element outside that ideal is the image of a unit of $\mathcal{O}_{X,y}$. Hence $\mathcal{O}_{S,y}$ is a local ring. We denote its unique maximal ideal by $\mathfrak{m}_{S,y}$. With the preceding identification, we have

$$\mathcal{O}_{S,y} \cong \mathcal{O}_{X,y}/\mathcal{I}_y, \qquad \mathfrak{m}_{S,y} = \mathfrak{m}_{X,y}/\mathcal{I}_y, \qquad \mathcal{O}_{S,y}/\mathfrak{m}_{S,y} \cong \mathbb{C}.$$

In particular, $(S, \mathcal{O}_S)$ is an object of $(\mathsf{LocRingSp}_{\mathbb{C}})$. The quotient map gives the inclusion a morphism of locally ringed spaces, with $j^\sharp\colon \mathcal{O}_X \to j_*\mathcal{O}_S$ and $j_*\mathcal{O}_S \cong \mathcal{O}_X/\mathcal{I}$. Here j_* is the direct image of Definition II.3.92 in Chapter II. The last isomorphism can be checked on stalks: on S it is the quotient above, and outside the closed set S both sheaves vanish.

Every section of $\mathcal{O}_S$ locally lifts to a holomorphic function on X, but this is a statement about a quotient sheaf; it need not have one lift over an entire prescribed open set.

Definition III.2.29. A *complex space* is an object $(X, \mathcal{O}_X)$ of $(\mathsf{LocRingSp}_{\mathbb{C}})$ such that X is Hausdorff and second countable and the following condition holds. For every $x \in X$, there exist an open neighborhood $U \subset X$ of x, an integer $N \in \mathbb{Z}_{\geq 0}$, an open subset $\Omega \subset \mathbb{C}^N$, and a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_\Omega$ with the following property: if $(S, \mathcal{O}_S)$ is the quotient model (156) defined by $\mathcal{I}$, then

$$(U, \mathcal{O}_X|_U) \cong (S, \mathcal{O}_S)$$

as objects of $(\mathsf{LocRingSp}_{\mathbb{C}})$.

For a complex space $(X, \mathcal{O}_X)$, a point $x \in X$ is *non-singular* if there exist an open neighborhood $U \subset X$ of x and an open subset V of a complex Euclidean space of finite dimension such that $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_V)$ in $(\mathsf{LocRingSp}_{\mathbb{C}})$; otherwise it is *singular*. The space is called *non-singular* if all its points are non-singular. It is called *reduced* if, for every $x \in X$, $\mathcal{O}_{X,x}$ is a reduced ring, that is, has no nonzero nilpotent element. □

It is customary to abbreviate a complex space $(X, \mathcal{O}_X)$ to X when its structure sheaf is clear from the context. In these notes, we prefer to retain the pair whenever confusion may arise. For an analytic subset with its specified reduced complex structure, or a reduced complex space with its fixed sheaf of holomorphic functions, we often use X for the complex space and also for its underlying topological space, according to context. This is analogous to regarding a complex manifold as a C^∞ manifold or a topological space by forgetting part of its structure. We use this abbreviation with particular care when the complex space is not assumed to be reduced, since its structure sheaf may contain information not detected by point values. When X is reduced, the elements of $\Gamma(U, \mathcal{O}_X)$ for an open subset $U \subset X$ are called *holomorphic functions on* U.

Topological properties of a complex space, such as compactness, refer to its underlying topological space. Ordinary real-valued or complex-valued functions and their continuity are likewise understood on the underlying topological space. Sections of the structure sheaf are different objects: on a non-reduced space they need not be determined by their point values.

Thus a non-singular complex space is a complex manifold in the sense of Definition III.2.1. In a quotient model, reducedness means that each $\mathcal{I}_y$ is radical, by the ring-theoretic definition recalled in Chapter I, §I.5.

Remark III.2.30. We use *complex space* for both reduced and non-reduced objects, as in [GrR, Chapter 4, §3.2–§3.3]. In contrast, the local models in [No1, Definition 6.9.1] are reduced. The same restriction occurs in [De4, Chapter II, §5], which uses *complex analytic scheme* for possibly non-reduced objects in Definition (9.5). The terms *complex analytic space* and *analytic space* also occur in the literature; here the shorter name does not impose reducedness. All topological statements refer to the topology of these local models, not to a Zariski topology. In particular, the spaces are locally compact; no global embedding in a complex manifold is required. □

On a complex space $(X, \mathcal{O}_X)$, a coherent sheaf of $\mathcal{O}_X$-modules is also called a *coherent analytic sheaf*, and a coherent ideal subsheaf of $\mathcal{O}_X$ is called a *coherent analytic ideal sheaf*. Here "analytic" specifies the structure sheaf over which coherence is considered; the condition is still Definition III.1.36.

Example III.2.31. (i) A complex manifold is a complex space, using $\mathcal{I} = 0$ in its coordinate charts. If a complex submanifold $Y \subset X$ is given in local holomorphic coordinates by $z^{m+1} = \cdots = z^n = 0$, its defining ideal sheaf is generated by these coordinate functions. Mini-Exercise III.2.8 (ii) shows that the corresponding closed complex subspace has the usual structure sheaf of Y.

(*ii*) Let $j\colon \{0\} \hookrightarrow \mathbb{C}$ be the inclusion. The ideals (z) and (z^2) define the complex spaces

$$\big(\{0\}, j^{-1}(\mathcal{O}_{\mathbb{C}}/(z))\big), \qquad \big(\{0\}, j^{-1}(\mathcal{O}_{\mathbb{C}}/(z^2))\big),$$

which have the same one-point underlying space but different local rings, respectively

$$\mathbb{C}\{z\}/(z) \cong \mathbb{C}, \qquad \mathbb{C}\{z\}/(z^2) \cong \mathbb{C}[\varepsilon]/(\varepsilon^2).$$

The first is a point as a zero-dimensional complex manifold, whereas the second is the *double point*. A germ is represented by $f(0) + f'(0)\varepsilon$, so its value at the point does not determine it. In particular, ε is nonzero although its value is zero, and the double point is not reduced.

(*iii*) In $\mathbb{C}^2$ with coordinates (z, w), the ideal (zw) gives a complex space whose underlying set is the union of the two coordinate axes. At the origin, $(zw) = (z) \cap (w)$ in $\mathbb{C}\{z, w\}$, an intersection of prime ideals, so this quotient ring is reduced. It has nonzero elements represented by z, w whose product is zero, and hence is not the local ring of a complex manifold. Away from the origin the space is a complex line locally. Thus it is reduced, but singular at the origin. □

Example III.2.32 (Closed subsets which are not analytic). The subsets $S_1 = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_{>0}\}$ and $S_2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ from Example III.1.1.38 are not analytic subsets of $\mathbb{C}$. If S_1 were defined by finitely many holomorphic functions on a disk about 0, each defining function would vanish identically by the identity theorem, since $1/n \to 0$. Their common zero set would be the whole disk, not S_1 in that disk. For S_2, take a disk about a point of $|z| = 1$. Any defining function would vanish on its nonempty open intersection with $|z| < 1$, and hence on the whole disk, again a contradiction. A proper analytic subset of a connected plane domain has no accumulation point in that domain.

Consequently neither S_1 nor S_2, with its given inclusion in $\mathbb{C}$, is the underlying subset of a closed analytic subspace of $\mathbb{C}$: the support in Definition III.2.28 is always analytic, as checked above. In contrast, the open unit disk is a complex manifold. Closedness alone does not make a subset analytic. □

We next specify the morphisms, keeping track of the structure sheaf as well as the underlying space.

Definition III.2.33. A *holomorphic map* between complex spaces $(S, \mathcal{O}_S)$ and $(T, \mathcal{O}_T)$ is a morphism $(f, f^\sharp)$ in $(\mathsf{LocRingSp}_{\mathbb{C}})$. Thus $f\colon S \to T$ is continuous, $f^\sharp\colon \mathcal{O}_T \to f_*\mathcal{O}_S$ is a morphism of sheaves of $\mathbb{C}$-algebras, and each induced homomorphism $\mathcal{O}_{T,f(x)} \to \mathcal{O}_{S,x}$ is local. An isomorphism is also called a *biholomorphism*. If such an isomorphism exists, the two spaces are called *biholomorphic*, and we write $(S, \mathcal{O}_S) \cong (T, \mathcal{O}_T)$. □

Let $U \subset \mathbb{C}^N$ and $V \subset \mathbb{C}^M$ be open subsets, where $N, M \in \mathbb{Z}_{\geq 0}$. In local models defined by $\mathcal{I} \subset \mathcal{O}_U$ and $\mathcal{J} \subset \mathcal{O}_V$, a holomorphic map $F\colon U \to V$ which pulls $\mathcal{J}$ into $\mathcal{I}$ induces such a morphism by composition modulo the ideals.

Conversely, let us verify that a morphism between these models is locally induced by a holomorphic map between the ambient open sets. Let $\mathcal{I}$ and $\mathcal{J}$ be coherent ideal sheaves,

and write their quotient models as $(S, \mathcal{O}_S)$ and $(T, \mathcal{O}_T)$, where

$$S = \operatorname{Supp}(\mathcal{O}_U/\mathcal{I}), \qquad \mathcal{O}_S = j^{-1}(\mathcal{O}_U/\mathcal{I}), \quad j\colon S \hookrightarrow U,$$
$$T = \operatorname{Supp}(\mathcal{O}_V/\mathcal{J}), \qquad \mathcal{O}_T = k^{-1}(\mathcal{O}_V/\mathcal{J}), \quad k\colon T \hookrightarrow V.$$

Suppose that $(f, f^\sharp)\colon (S, \mathcal{O}_S) \to (T, \mathcal{O}_T)$ is a morphism in $(\mathsf{LocRingSp}_{\mathbb{C}})$. Fix $a \in S$ and denote the coordinates on V by $w^1, w^2, \dots, w^M$. After shrinking around a and $f(a)$, lift the sections $f^\sharp([w^j])$ to holomorphic functions F^j on U. Here $[w^j]$ denotes the class of the coordinate function in $\mathcal{O}_T$. The induced maps on residue fields are the identity on $\mathbb{C}$, since the stalk homomorphisms are local $\mathbb{C}$-algebra homomorphisms. Thus $F^j(x) = w^j(f(x))$ for $x \in S$. After a further shrinking, $F = (F^1, F^2, \dots, F^M)$ maps U to V and satisfies $F|_S = f$.

This equality of underlying maps is not sufficient when the source is non-reduced: it does not determine the images of sections with nilpotent information. We must also show that the induced local homomorphism $f_x^\sharp\colon \mathcal{O}_{T,f(x)} \to \mathcal{O}_{S,x}$ sends the class of every holomorphic germ h to the class of $h \circ F$. The following exercise verifies this remaining point and the condition on the defining ideals.

Mini-Exercise III.2.34. In the preceding setting, show that, for every $x \in S$ and $h \in \mathcal{O}_{V,f(x)}$,

$$f_x^\sharp([h]) = [h \circ F] \quad \text{in } \mathcal{O}_{S,x},$$

where the classes are taken modulo $\mathcal{J}_{f(x)}$ and $\mathcal{I}_x$, respectively. Deduce that F pulls $\mathcal{J}$ into $\mathcal{I}$ and induces $(f, f^\sharp)$ on the quotient models.

Definition III.2.35. We denote by (CpxSp) the category whose objects are complex spaces and whose morphisms are holomorphic maps. □

With Definition III.2.33, this is a full subcategory of $(\mathsf{LocRingSp}_{\mathbb{C}})$. We follow Remark II.3.93 and write f for the entire morphism when no confusion is likely. This abbreviation does not mean that the continuous map determines $f^\sharp$. For the double point, each homomorphism $\mathbb{C}[\varepsilon]/(\varepsilon^2) \to \mathbb{C}[\varepsilon]/(\varepsilon^2)$ given by $\varepsilon \mapsto a\varepsilon$, $a \in \mathbb{C}$, is local and defines a holomorphic self-map. All have the same underlying map of the one-point space, but their morphisms of structure sheaves differ.

We next apply the coherence results already available, rather than making a new finiteness assumption on these structure sheaves.

Proposition III.2.36. *Let $(X, \mathcal{O}_X)$ be a complex space. Its structure sheaf $\mathcal{O}_X$ is coherent. If $\mathcal{J} \subset \mathcal{O}_X$ is a coherent ideal sheaf, then $(A, \mathcal{O}_A)$, where*

$$A = \operatorname{Supp}(\mathcal{O}_X/\mathcal{J}), \qquad \mathcal{O}_A = j^{-1}(\mathcal{O}_X/\mathcal{J}), \quad j\colon A \hookrightarrow X,$$

is again a complex space. It is called the closed complex subspace of $(X, \mathcal{O}_X)$ defined by $\mathcal{J}$.

Proof. For the first assertion, work with a model $(X, \mathcal{O}_X)$, where $\Omega \subset \mathbb{C}^N$ is open, $\mathcal{I} \subset \mathcal{O}_\Omega$ is coherent, and $\mathcal{O}_X = j^{-1}(\mathcal{O}_\Omega/\mathcal{I})$ for the inclusion $j\colon X \hookrightarrow \Omega$ of its support. The $\mathcal{O}_X$-module $\mathcal{O}_X$ is generated by 1. Let $s_1, s_2, \dots, s_q$ be local sections of $\mathcal{O}_X$ near $y \in X$.

After shrinking an ambient neighborhood $V \subset \Omega$ of y, choose lifts $f_1, f_2, \ldots, f_q \in \mathcal{O}_\Omega(V)$ and generators $g_1, g_2, \ldots, g_m \in \mathcal{I}(V)$ of $\mathcal{I}|_V$. By Oka's first coherence theorem (Theorem III.1.41), $\mathcal{O}_\Omega$ is coherent. Thus the sheaf of relations $\mathcal{R}(f_1, f_2, \ldots, f_q, g_1, g_2, \ldots, g_m)$ is locally finitely generated. Shrink V so that it has finitely many generators on V.

Project these generators to their first q components and reduce modulo $\mathcal{I}$. They give relations among $s_1, s_2, \ldots, s_q$. Conversely, at every $x \in X \cap V$, the coefficients of a relation among the germs $s_{i,x}$ can be lifted to $\mathcal{O}_{\Omega,x}$. The resulting sum with $f_{1,x}, f_{2,x}, \ldots, f_{q,x}$ belongs to $\mathcal{I}_x$, so it is a combination of $g_{1,x}, g_{2,x}, \ldots, g_{m,x}$. Completing the lifted tuple by the negatives of those coefficients gives a relation among the f_i and g_j. Hence the projected generators generate the relations among the s_i at every point of $X \cap V$. Definition III.1.36 proves the first assertion.

For the second assertion, work in such a model and lift finitely many local generators of $\mathcal{J}$ to holomorphic functions on the ambient open set, shrinking around the point in question. Adjoining these lifts to generators of $\mathcal{I}$ gives a locally finitely generated ideal $\widetilde{\mathcal{J}}$, which is coherent by Proposition III.1.43. Taking the quotient by $\widetilde{\mathcal{J}}$ is the same as first quotienting by $\mathcal{I}$ and then by $\mathcal{J}$. Thus the corresponding quotient model agrees locally with $(A, \mathcal{O}_A)$. Its underlying space is locally a closed zero set, and hence is closed in X; it inherits Hausdorffness and second countability. This proves that $(A, \mathcal{O}_A)$ is a complex space, whose structure sheaf is coherent by the first assertion. □

In particular, coherent ideals may be used again on a singular or non-reduced complex space, not just on a manifold.

Radicals and the reduced structure

For a complex space $(X, \mathcal{O}_X)$, the value $s(x)$ of a local section of $\mathcal{O}_X$ means its image in $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong \mathbb{C}$. Definition III.2.27 extends to this setting by using these values in local equations. In a quotient model, lifting the equations to the ambient open set gives the same notion. The defining ideal sheaf $\mathcal{I}_A$ of an analytic subset $A \subset X$ is likewise defined by vanishing of these values, as in (155); it need not be the ideal originally chosen to define a subspace with underlying set A.

Let A be an analytic subset of a complex space X and let $x \in X$. The *germ of A at x* identifies local representatives when they agree on some smaller neighborhood of x. We denote it by $\underline{A}_x$ (The notations A_x and (A, x) also occur in the literature). In particular, equations defining the germ are considered on a common neighborhood which may be shrunk.

We record two further analytic results, with proofs referred to the sources below. They supply the reduced structures used here; neither is being deduced merely from the Noetherianity of individual local rings. Recall radicals and reduced rings from Chapter I, Mini-Exercise I.5.1 and the following paragraph. For an ideal sheaf $\mathcal{J}$, its radical $\sqrt{\mathcal{J}}$ is the ideal sheaf with stalks $\sqrt{\mathcal{J}_x}$.

THEOREM III.2.37 (Rückert's Nullstellensatz). *Let $N \in \mathbb{Z}_{\geq 0}$, let $a \in \mathbb{C}^N$, and let $\mathfrak{a} \subset \mathcal{O}_{\mathbb{C}^N,a}$ be an ideal. Let $\underline{V(\mathfrak{a})}_a$ be the germ at a of the common zero set of representatives of*

finitely many generators of $\mathfrak{a}$. *If* $I(\underline{A}_a)$ *denotes the ideal of germs vanishing on an analytic set germ* $\underline{A}_a$, *then*

$$I\big(\underline{V(\mathfrak{a})}_a\big) = \sqrt{\mathfrak{a}}. \tag{157}$$

More generally, if $\mathcal{J}$ *is a coherent ideal sheaf on a complex space* $(X, \mathcal{O}_X)$ *and* A *is its zero set, then* $\mathcal{I}_A = \sqrt{\mathcal{J}}$.

Proof. The proof proceeds by induction on dimension and uses the Weierstraß preparation theorem and Oka's first coherence theorem. For details, see [No1, Theorem 6.4.20] for the germ statement and [GrR, Chapter 4, §1.5, pp. 81–82] for the sheaf statement on complex spaces. □

The germ of the zero set is independent of the chosen generators after shrinking their common domain. The conclusion means that a germ vanishing there has some power in the defining ideal; the exponent is not asserted to be uniform over an arbitrary open set.

We use the following standard fact, known as Oka's second coherence theorem: for every analytic subset A of a complex space $(X, \mathcal{O}_X)$, its defining ideal sheaf $\mathcal{I}_A \subset \mathcal{O}_X$ is coherent. We use it here as a quoted result to keep the basic constructions concerning analytic spaces together, including the consequences below concerning reduction, the singular locus, and hypersurface ideals. Theorem III.2.163 in §2.11 records its statement and proof references; the local proof method and its independence of these applications are explained in Remark III.2.164.

In the terminology of Remark III.1.37, Noetherianity gives a *stalkwise* assertion: the ideal $(\mathcal{I}_A)_a$ is finitely generated for each fixed $a \in X$. The chosen germs have representatives vanishing on A in a common open neighborhood of a. This alone does not say that those representatives generate the full vanishing ideal at every nearby point. Oka's second coherence theorem (Theorem III.2.163) gives one open neighborhood U and finitely many $h_1, h_2, \dots, h_r \in \mathcal{I}_A(U)$ such that

$$(\mathcal{I}_A)_x = \sum_{j=1}^{r} \mathcal{O}_{X,x} h_{j,x} \qquad \text{for every } x \in U.$$

The neighborhood and the sections are fixed before x varies; this is the *semi-stalkwise* viewpoint of that remark. It is still a statement about stalks, not surjectivity on sections over the whole prescribed U.

Definition III.2.38 (Reduction). Let $(X, \mathcal{O}_X)$ be a complex space. Its *nilradical sheaf* $\mathcal{N}_X := \sqrt{0}$ is the ideal sheaf of nilpotent germs. The *reduction* is the pair $(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$, where $X_{\mathrm{red}} = X$ as topological spaces and $\mathcal{O}_{X_{\mathrm{red}}} := \mathcal{O}_X/\mathcal{N}_X$. In particular,

$$\mathcal{O}_{X_{\mathrm{red}},x} = \mathcal{O}_{X,x}/\sqrt{0} \qquad (x \in X).$$

□

Let $(X, \mathcal{O}_X)$ be a complex space. Rückert's Nullstellensatz (Theorem III.2.37) identifies $\mathcal{N}_X$ with the ideal of sections whose values vanish everywhere. Oka's second coherence theorem (Theorem III.2.163) makes this ideal coherent. Proposition III.2.36 therefore shows that $(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$ is a reduced closed complex subspace of $(X, \mathcal{O}_X)$. Locally, if $(X, \mathcal{O}_X)$ is defined by $\mathcal{I}$ in an open set $U \subset \mathbb{C}^N$, its reduction is defined by $\sqrt{\mathcal{I}}$, with structure sheaf $j^{-1}(\mathcal{O}_U/\sqrt{\mathcal{I}})$. In particular, an analytic subset A carries the canonical reduced structure $j^{-1}(\mathcal{O}_X/\mathcal{I}_A)$. The quotient defines a canonical morphism $(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}) \to (X, \mathcal{O}_X)$; a holomorphic map $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ induces $(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}}) \to (Y_{\mathrm{red}}, \mathcal{O}_{Y_{\mathrm{red}}})$, since ring homomorphisms send nilpotents to nilpotents. Reduction discards nilpotent information, not singularities in general: the double point reduces to a point, whereas the crossing in Example III.2.31 (*iii*) is already reduced.

For a complex space $(X, \mathcal{O}_X)$, we write X_{reg} for the set of non-singular points and $X_{\mathrm{sing}} := X \setminus X_{\mathrm{reg}}$ for its *singular locus*, using Definition III.2.29. The regular locus is open by definition. For an analytic subset A, the notation A_{reg} and A_{sing} refers to its canonical reduced structure. This is not the same as the singular locus of every complex subspace having support A; the double point already illustrates the distinction. We will explain the analyticity of the singular locus below, after introducing dimension.

Irreducibility and dimension

Definition III.2.39. A nonempty complex space $(X, \mathcal{O}_X)$ is *irreducible* if its underlying analytic set is not the union of two proper closed analytic subsets. It is *irreducible at* x if the germ $\underline{X_{\mathrm{red}}}_x$ cannot be written as the union of two proper analytic subgerms. □

Let $(X, \mathcal{O}_X)$ be a complex space, let $x \in X$, and put $R = \mathcal{O}_{X,x}$. This ring is Noetherian, since it is a quotient of a holomorphic local ring (Theorem III.1.32). As briefly recalled in Chapter I, §5.1, the following facts are standard commutative algebra: a Noetherian ring has finitely many minimal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$, and its nilradical is their intersection; see [Mat, §6] and [St, Tag 00FR, Tag 00E0]. By Rückert's Nullstellensatz (Theorem III.2.37), radical ideals correspond to analytic subgerms, with inclusions reversed, and prime ideals correspond to irreducible subgerms. Thus the minimal primes give a finite decomposition of $\underline{X_{\mathrm{red}}}_x$ into irreducible germs. If no member is contained in another, the members are unique up to order and are called its *irreducible components*, or *local branches*. In particular, the germ is irreducible if and only if $R/\sqrt{0} = \mathcal{O}_{X_{\mathrm{red}},x}$ is an integral domain. The analytic input in this translation is the quoted Nullstellensatz; the finite decomposition of the nilradical is a ring-theoretic fact. For the analytic formulation, see [No1, Theorem 6.2.9] and the Local Decomposition Lemma in [GrR, Chapter 4, §1.3–§1.4, pp. 78–81]. Thus the two axes of Example III.2.31 (*iii*) are the two components at their crossing. Global irreducibility does not imply irreducibility of every germ; Example III.2.47 below gives a nodal example of this distinction. Nor does irreducibility imply reducedness: the double point is irreducible, although its local ring is not an integral domain.

Definition III.2.40 (Local and global dimension). For a complex space $(X, \mathcal{O}_X)$ and $x \in X$, its *dimension at* x is

$$\dim_x X := \dim \mathcal{O}_{X,x}, \tag{158}$$

where the right-hand side is the Krull dimension recalled in Chapter I, §5.7. Its *dimension* is $\dim X := \sup_{x\in X} \dim_x X$, with $\dim \emptyset = -\infty$. □

This is intrinsic, since an isomorphism of locally ringed spaces induces isomorphisms of local rings. Finiteness at each point follows from the dimension theorem for Noetherian local rings recalled in Chapter I, §5.7, although the global supremum can be infinite. For a complex manifold, local coordinates identify the associated graded ring of $\mathcal{O}_{\mathbb{C}^n,a}$, with respect to its maximal ideal, with a polynomial ring in n variables. The Hilbert–Poincaré series computation in Chapter I, §5.7, therefore gives $\dim \mathcal{O}_{\mathbb{C}^n,a} = n$, in agreement with the previously defined manifold dimension (This is the same associated graded ring calculation as in Example I.5.18). For a fixed $N \in \mathbb{Z}_{\geq 0}$ and a nonempty analytic subset of an open subset of $\mathbb{C}^N$, all local rings are quotients of such rings, so their dimensions lie in $\{0, 1, \ldots, N\}$ and the supremum is a maximum.

The effect of reduction and of local decomposition is also ring-theoretic. Fix $x \in X$, put $R = \mathcal{O}_{X,x}$, and let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$ be its minimal prime ideals. Every prime ideal contains the nilradical, so passing to $R/\sqrt{0}$ preserves chains of prime ideals. Moreover, every prime ideal contains a minimal prime, by [St, Tag 00E0]. Hence any chain of primes lies above some $\mathfrak{p}_i$ and gives a chain in $R/\mathfrak{p}_i$; conversely, chains in that quotient lift to R. Consequently,

$$\dim R = \dim(R/\sqrt{0}) = \max_{1\leq i\leq s} \dim(R/\mathfrak{p}_i). \tag{159}$$

Thus $\dim_x X = \dim_x X_{\mathrm{red}}$, and the local dimension is the maximum of the dimensions of the irreducible components of the germ.

The dimension theorem also characterizes the dimension of a Noetherian local ring R as the least number of elements of its maximal ideal $\mathfrak{m}$ which generate an ideal with radical $\mathfrak{m}$; see [St, Proposition 10.60.9, Tag 00KQ]. For $R = \mathcal{O}_{X,x}$, Rückert's Nullstellensatz identifies that radical condition with the condition that x be an isolated common zero of local representatives of those elements. Thus $\dim_x X$ is also the least number of local holomorphic functions having x as an isolated common zero; in dimension zero, the empty family is allowed. This is the definition used in [GrR, Chapter 5, §1.1–§1.2, pp. 93–96].

Remark III.2.41 (Pure dimension). For $n \in \mathbb{Z}_{\geq 0}$, a nonempty complex space $(X, \mathcal{O}_X)$ is *of pure dimension* n if $\dim_x X = n$ for every $x \in X$.

In the author's experience, differential-geometric discussions of an n-dimensional C^∞ or complex manifold usually keep the dimension fixed, even when connectedness is not assumed. This motivated the fixed-dimension convention in Chapter I. For general complex spaces, however, connectedness does not force the local dimension to be constant, and the supremum used above is the usual global dimension. Since complex manifolds are particular complex spaces, from this point onward both $\dim X = n$ and the phrase "of dimension n" refer only to this supremum, also when X is a complex manifold. When all local dimensions are to equal n, we explicitly say "of pure dimension n". Statements from Chapter I that use a fixed real dimension are applied to a pure-dimensional complex manifold or separately on its connected components. In particular, the definition of manifold itself is unchanged. □

Finite projections and local dimension

A holomorphic map between complex spaces is called *finite* if its underlying continuous map is proper and every fiber is a finite set; surjectivity is not part of this definition.

Recall covering maps from Chapter I, §7.1, where the terms ramified and unramified were also mentioned. An *unramified covering* in the present holomorphic setting is a covering map with the following property: for each sufficiently small connected open set U in the base, its inverse image is a disjoint union of open sets, each mapped biholomorphically onto U. Each of these open sets is called a *sheet over* U. A covering with q sheets has q such sets, or equivalently q points in each fiber; the sheets over U need not extend to distinct global connected components.

A finite surjective holomorphic map between analytic sets is called a *branched* (or *ramified*) *covering* if, after removing a proper analytic subset of the target and its inverse image, it becomes an unramified covering and both remaining sets are dense in their respective spaces. The exceptional set is allowed to be empty; choosing such a set does not assert ramification at every point over it. These terms concern a map, whereas a local branch of an analytic set is an irreducible component of its germ.

EXAMPLE III.2.42. The curve $\{w^2 = z\}$ in $\mathbb{C}^2$ with coordinates (z, w) is non-singular, with coordinate w, although its projection to z has two sheets for $z \neq 0$ and only one point over 0. Away from $z = 0$ the projection is an unramified covering, but at $w = 0$ it is not locally biholomorphic. Thus this projection can branch even where the curve is non-singular. The two local choices of a square root do not give two global components of the covering over a punctured disk. □

We now record the local structure theorem for an irreducible analytic germ, and then illustrate its mechanism in two variables.

THEOREM III.2.43 (Local parametrization theorem). *Let A be an analytic subset of a neighborhood of 0 in $\mathbb{C}^N$, and suppose that A is irreducible at 0 and $\dim_0 A = d$, where $1 \leq d < N$. After an invertible complex-linear change of coordinates $z = (z', w) \in \mathbb{C}^d \times \mathbb{C}^{N-d}$, there are arbitrarily small polydiscs $\Delta' \subset \mathbb{C}^d$ and $\Delta'' \subset \mathbb{C}^{N-d}$ centered at 0 such that*

$$\pi \colon A_\Delta := A \cap (\Delta' \times \Delta'') \longrightarrow \Delta', \qquad (z', w) \longmapsto z' \tag{160}$$

is finite and surjective, with $\pi^{-1}(0) = \{0\}$. There are a proper analytic subset $S \subset \Delta'$ and an integer $q \geq 1$ such that $A_\Delta \setminus \pi^{-1}(S)$ is a connected complex manifold of pure dimension d, dense in A_Δ, and

$$\pi \colon A_\Delta \setminus \pi^{-1}(S) \longrightarrow \Delta' \setminus S$$

is an unramified covering with q sheets. Every fiber over S has at most q points.

PROOF. For the general case, see [De4, Chapter II, Theorem (4.19), pp. 95–97]; the finite projection and its covering description are also given in [No1, Theorems 6.4.14 and 6.4.16].

For an illustration in two variables, consider the zero set of a nonzero holomorphic germ f at $0 \in \mathbb{C}^2$, with $f(0) = 0$. To describe the reduced zero set, replace f by the product of its

distinct irreducible factors, using the UFD property (Theorem III.1.33). By Lemma III.1.30, linear coordinates (z, w) can be chosen so that $f(0, w) \not\equiv 0$. The Weierstraß preparation theorem (Theorem III.1.29) then gives $f = uP$, where u is a unit and

$$P(z, w) = w^m + \sum_{j=1}^{m} a_j(z) w^{m-j}, \qquad a_j(0) = 0.$$

Here $m \geq 1$. Choose $r > 0$ small. Since the coefficients vanish at 0, there is $\varepsilon > 0$ such that, for $|z| < \varepsilon$ and $|w| = r/2$,

$$|P(z, w) - w^m| < |w^m|$$

(Compare the estimate leading to (1)). Rouché's theorem, recalled in Chapter I, §I.9, shows that $P(z, \cdot)$ has exactly m roots counted with multiplicity in $|w| < r/2$. Thus the projection of $\{P = 0\} \cap \{|z| < \varepsilon,\ |w| < r\}$ onto $|z| < \varepsilon$ is surjective with finite fibers. It is proper: over a compact subset K of the base, the zero set is closed in the compact set $K \times \{|w| \leq r/2\}$.

As in the proof of the UFD property (Theorem III.1.33), uniqueness in preparation and Gauß' lemma make P square-free over the fraction field of $\mathbb{C}\{z\}$; compare [No1, Theorem 2.2.1 and Lemma 2.2.14]. Its discriminant $\delta(z)$, a polynomial in its coefficients which vanishes precisely for a repeated root, is therefore not identically zero. Away from $\{\delta = 0\}$, the holomorphic implicit function theorem recalled in §1.2 gives m local holomorphic graphs. They form the sheets of the unramified covering. The fiber over 0 is $\{0\}$ because $P(0, w) = w^m$, and no fiber has more than m points. The continuity of roots (Lemma I.1.3) shows that the part away from the discriminant is dense in the whole zero set.

It remains to explain connectedness when the reduced germ is irreducible. Shrink ε so that δ has no zero in $0 < |z| < \varepsilon$. If the covering over this punctured disk were not connected, choose one connected component, and form the monic polynomial whose roots over each z are the w-coordinates on that component. Form a second polynomial from the remaining roots. Their coefficients are single-valued holomorphic functions of z: locally they are symmetric functions of the holomorphic sheets. The bound $|w| < r/2$ makes these coefficient functions of z bounded on $0 < |z| < \varepsilon$, so the one-variable removable singularity theorem extends them holomorphically to $z = 0$. The two polynomials have positive degrees, their product is P, and their values at $z = 0$ are powers of w. They are therefore nonunit holomorphic factors of P at the origin, contrary to the irreducibility of the reduced germ and the UFD property. This proves connectedness in the plane-curve case. □

Corollary III.2.44. *Let A be an analytic subset of a complex manifold, and suppose that its germ at $x \in A$ is irreducible. There are arbitrarily small open neighborhoods U of x in the ambient manifold such that $A_{\mathrm{reg}} \cap U$ is connected and dense in $A \cap U$.*

Proof. Work in a coordinate neighborhood in $\mathbb{C}^N$ and put $d = \dim_x A$. If $0 < d < N$, Theorem III.2.43 gives a connected dense subset C of $A \cap U$ contained in $A_{\mathrm{reg}} \cap U$. Any set between a connected set and its closure is connected: a separation would induce a separation of the dense connected set. Thus $A_{\mathrm{reg}} \cap U$ is connected and dense. If $d = 0$ or $d = N$, then, after shrinking U, the set $A \cap U$ is respectively $\{x\}$ or U, so the conclusion holds in these cases as well. □

We will also use the fact that the complement of a proper analytic subset A of a connected complex manifold X is connected. Indeed, A has no interior by the identity theorem. In a sufficiently small coordinate ball, the complex line through two points outside A meets A in a discrete set: one of its local defining functions is nonzero at the first point, so its restriction to the line is not identically zero. The two points can therefore be joined while avoiding that discrete set. A chain of coordinate balls gives the assertion on X, since each nonempty overlap contains a point outside A. For a zero-dimensional connected manifold the assertion is immediate. A different proof using pluripotential theory will be given in Remark III.2.95.

THEOREM III.2.45. *Let $(X, \mathcal{O}_X)$ be a complex space, and put $W = (X_{\mathrm{red}})_{\mathrm{reg}}$. Then W is open and dense in X. Let $W = \coprod_{\alpha \in A} W_\alpha$ be its decomposition into connected components, and let X_α be the closure of W_α in X. The sets X_α form a locally finite family of closed irreducible analytic subsets covering X. They are precisely the maximal irreducible analytic subsets of X, called its* global irreducible components.

PROOF. Work with X_{red}. We outline the local-to-global argument of [De4, Chapter II, Theorem (5.3), pp. 102–103]. Near x, choose representatives $A_1, A_2, \ldots, A_s$ of the local branches. For each j, Corollary III.2.44 gives a neighborhood on which $(A_j)_{\mathrm{reg}}$ is connected and dense in A_j. Remove the other branches from this regular part. Their intersections are proper analytic subsets, so the preceding connectedness argument leaves a connected dense open part of A_j contained in W. Each such part therefore lies in a single component W_α. On a smaller common neighborhood, X_α is exactly the union of the local branches assigned to W_α. This finite local description proves analyticity and local finiteness of the closures, and also shows that they cover X and that W is dense. Openness of W follows from the definition of a non-singular point.

So far we have shown that the sets X_α are closed analytic subsets forming a locally finite cover of X, with W_α dense in X_α. It remains to identify them as the global irreducible components. We first prove that each X_α is irreducible. A connected complex manifold cannot be the union of two proper analytic subsets, since each has empty interior. Applying this to W_α and taking closures proves the claim.

Next, we show that every irreducible analytic subset $Z \subset X$ is contained in some X_α. Choose $x \in Z$ and an open neighborhood U meeting only finitely many X_α. Let Y be their union and Y' the union of all remaining X_α. Both are closed analytic subsets by local finiteness. Since $Z = (Z \cap Y) \cup (Z \cap Y')$ and $x \notin Y'$, irreducibility gives $Z \subset Y$. As Y is a finite union, irreducibility again places Z in one of its members.

Finally, no X_α is contained in a distinct X_β: indeed, their dense regular parts are disjoint and open in X, so such a containment is impossible. Together with the preceding steps, this proves the stated maximality. For the details of the local description of the closures, see the cited proof. □

COROLLARY III.2.46. *Every nonempty irreducible complex space is pure dimensional.*

PROOF. Let $(X, \mathcal{O}_X)$ be such a space. Reduction preserves irreducibility and local dimension. By Theorem III.2.45, $W = (X_{\mathrm{red}})_{\mathrm{reg}}$ is connected; let d be its manifold dimension. At $x \in X$, let d_j be the dimension of its j-th local branch. The Local Parametrization

Theorem (Theorem III.2.43) gives that branch a dense smooth open part of pure dimension d_j. After removing the other branches, a nonempty open part remains, and there the branch agrees locally with the whole reduced space. It is consequently also an open part of W, so $d_j = d$. The zero-dimensional and full ambient-dimensional cases are handled by the same local models as in the proof of Corollary III.2.44. Now (159) gives $\dim_x X = \max_j d_j = d$. □

More generally, for $n \in \mathbb{Z}_{\geq 0}$, a nonempty complex space $(X, \mathcal{O}_X)$ is of pure dimension n if and only if every local irreducible component at every point has dimension n; compare [GrR, Chapter 5, §4.2, p. 106]. If all branches have dimension n, (159) gives $\dim_x X = n$ at every point. Conversely, on a dense regular part of any local branch away from the other branches, the branch and the whole reduced space agree. The Local Parametrization Theorem (Theorem III.2.43) identifies the dimension of this part with that of the branch germ; if $\dim_x X = n$ everywhere, this dimension must be n.

Example III.2.47 (Global irreducibility and two local branches). Consider the analytic subset

$$C = \{[X : Y : Z] \in \mathbb{P}^2 \mid Y^2 Z = X^2(X + Z)\}.$$

The homogeneous equation has degree three, so its zero set is unchanged by rescaling the homogeneous coordinates; in each projective chart it is a holomorphic equation. In the chart $Z \neq 0$, put $x = X/Z$ and $y = Y/Z$. Near $p = [0 : 0 : 1]$, choose the holomorphic square root $h(x)$ of $1 + x$ with $h(0) = 1$. Then

$$y^2 - x^2(1 + x) = (y - xh(x))(y + xh(x)).$$

The two factors define distinct smooth branches, with tangent lines $y = x$ and $y = -x$ at p. Thus C is not irreducible at p. This is the local crossing referred to by the word *node*.

Nevertheless, C is globally irreducible. To see this, consider

$$\nu \colon \mathbb{P}^1 \longrightarrow C, \qquad [s : t] \longmapsto [t(s^2 - t^2) : s(s^2 - t^2) : t^3].$$

The three coordinates never vanish simultaneously, and substitution verifies the equation of C, so ν is holomorphic. For $t \neq 0$, writing $u = s/t$ gives $(x, y) = (u^2 - 1, u(u^2 - 1))$. Every affine point with $x \neq 0$ is obtained uniquely by taking $u = y/x$; the remaining affine point p has the two preimages $u = 1, -1$. The only point of C with $Z = 0$ is $[0 : 1 : 0]$, the image of $[1 : 0]$. Hence ν is surjective and is one-to-one away from p.

If C were a union of two proper closed analytic subsets, their inverse images would be proper closed analytic subsets of $\mathbb{P}^1$. Each would be finite, by the one-variable identity theorem and compactness, and their union could not cover $\mathbb{P}^1$. This proves global irreducibility. The parametrization by $\mathbb{P}^1$, which has the holomorphic inverse $u = y/x$ away from the indicated points, also explains the adjective *rational* in the name *nodal rational cubic*. Thus the curve is globally irreducible and of pure dimension one, although its germ at the node has two distinct one-dimensional branches. □

COROLLARY III.2.48 (Upper semicontinuity of the local dimension). *Let X be a complex space. For every $a \in X$ there exists an open neighborhood U of a such that*

$$\dim_x X \leq \dim_a X \qquad (x \in U). \tag{161}$$

Equivalently, $\{x \in X \mid \dim_x X \geq k\}$ is closed in X for every integer k.

PROOF. Work in a reduced local model near a. Theorem III.2.45 gives a locally finite family of closed irreducible components. After shrinking a neighborhood U of a, the only components meeting U are the finitely many $Z_1, Z_2, \ldots, Z_s$ which contain a. By Corollary III.2.46, each Z_i has a constant local dimension, say d_i. The local components of their union are obtained from those of the Z_i by discarding contained components, which does not change the maximum of their dimensions. Thus (159) gives

$$\dim_x X = \max_{i:\, x \in Z_i} d_i \leq \max_{1 \leq i \leq s} d_i = \dim_a X \qquad (x \in U).$$

The same formula shows that reduction does not change the dimensions, so this proves the assertion for non-reduced spaces as well. See also [GrR, Chapter 5, §1.1, p. 94] and [No1, Proposition 6.4.29]. □

EXAMPLE III.2.49. The point and the double point in Example III.2.31 (*ii*) both have dimension zero, by the reduction invariance in (159). The space defined by (w^2) in $\mathbb{C}^2$ has pure dimension one, with reduction the line $w = 0$; it is not non-singular at any point, although its reduction is a manifold.

The reduced crossing $Y = \{zw = 0\} \subset \mathbb{C}^2$ of Example III.2.31 (*iii*) also has pure dimension one. At the origin its two branches each have dimension one, so (159) gives $\dim_0 Y = \max\{1, 1\} = 1$. Its local ring $\mathbb{C}\{z, w\}/(zw)$ has nonzero zero divisors, unlike a manifold's local ring. Thus the origin is singular, but the local dimension does not jump there.

In contrast, in $\mathbb{C}^3$ with coordinates (u, v, w) the reduced union $X = \{w = 0\} \cup \{u = v = 0\}$ is connected with $\dim X = 2$, but is not pure dimensional. Its local dimension is two on the plane (including the origin) and one on the line away from the origin. In particular, $\dim_{(0,0,t)} X = 1$ for $t \neq 0$, whereas (159) gives $\dim_0 X = \max\{2, 1\} = 2$. The jump at the limit point has exactly the direction permitted by (161). Its germ at the origin has components of dimensions two and one. □

REMARK III.2.50 (Curves and surfaces). A complex space of pure dimension one is called a *complex curve*, and one of pure dimension two a *complex surface*. Reducedness and irreducibility are separate conditions. If such a space is a complex manifold, we say *non-singular complex curve* or *non-singular complex surface*. Thus a connected non-singular complex curve is precisely a Riemann surface, as in Remark III.2.4. In particular, the calculations above involving metrics and the first Chern class were on Riemann surfaces, not on arbitrary complex curves. □

Definition III.2.51 (Codimension and complex hypersurfaces). Let A be an analytic subset of a complex manifold X and let $x \in A$. Its *codimension in X at x* is

$$\operatorname{codim}_{X,x} A := \dim_x X - \dim_x A. \tag{162}$$

For nonempty A, we put $\operatorname{codim}_X A := \inf_{x\in A} \operatorname{codim}_{X,x} A$. We say that A has *pure codimension c* if the local codimension is c at every point of A. An analytic subset of pure codimension one is called a *complex hypersurface*, or simply a *hypersurface* when there is no risk of confusion with a real hypersurface. □

For a closed complex subspace of a complex manifold X, codimension refers to its support, so it does not change on reduction. The pure-codimension condition is imposed at every point, not just at points on the largest-dimensional components. For instance, the plane and transverse line in Example III.2.49 have union of codimension one in $\mathbb{C}^3$, but this union is not a complex hypersurface. A complex hypersurface may be singular; $\{zw = 0\} \subset \mathbb{C}^2$ is an example.

The singular locus and the implicit function theorem

Theorem III.2.52 (Analyticity of the singular locus). *For every complex space X, the subset X_{sing} is a closed analytic subset in the sense explained above (p. 420). If X is reduced, then X_{reg} is dense in X.*

Proof. For the full argument, see [GrR, Chapter 6, §2, pp. 116–118]. We recall the local description, following also [De4, Chapter II, Theorem (4.31), pp. 100–101]. Let A be a reduced analytic subset of pure dimension $d < N$ in an open subset $U \subset \mathbb{C}^N$. By Oka's second coherence theorem (Theorem III.2.163), after shrinking around the point under consideration we can choose $h_1, h_2, \dots, h_r \in \mathcal{I}_A(U)$ generating the full ideal sheaf $\mathcal{I}_A|_U$. The holomorphic implicit function theorem recalled in §1.2 gives

$$x \in A_{\text{reg}} \quad \Longleftrightarrow \quad \operatorname{rank} \left(\frac{\partial h_j}{\partial z^k}(x) \right)_{1\le j\le r,\, 1\le k\le N} = N - d. \tag{163}$$

A nonzero minor of order $N - d$ cuts out a smooth d-dimensional germ containing $\underline{A}_x$; equality follows because a proper analytic subgerm of a smooth d-dimensional germ has smaller dimension. Conversely, at a regular point the transverse coordinate functions belong to the full ideal and have independent differentials, giving the required rank. Thus A_{sing} is defined inside A by the vanishing of these minors. For a general reduced germ, take its finitely many local irreducible components: the singular locus is the union of their singular loci and their pairwise intersections. This also handles components of different dimensions; reduced zero-dimensional germs and full ambient-dimensional germs are non-singular. The density assertion is already contained in Theorem III.2.45. The cited proofs give the details of this local description.

For a non-reduced space, on the common underlying space one has

$$X_{\text{sing}} = (X_{\text{red}})_{\text{sing}} \cup \operatorname{Supp} \mathcal{N}_X.$$

Indeed, the germ agrees with its reduction exactly where the nilradical stalk vanishes. The nilradical sheaf is coherent, as explained after Definition III.2.38. In a local finite presentation of this sheaf, its support is the locus where the presenting matrix is not surjective, described by its maximal minors. It is therefore analytic. This proves the remaining case. □

Example III.2.53 (Checking a Fermat cubic by differentiation). Consider the analytic subset

$$F = \{[X : Y : Z] \in \mathbb{P}^2 \mid X^3 + Y^3 + Z^3 = 0\}.$$

In the chart $Z \neq 0$, with $x = X/Z$ and $y = Y/Z$, its equation is $x^3 + y^3 + 1 = 0$. The two partial derivatives are $3x^2$ and $3y^2$. Their simultaneous vanishing would give $x = y = 0$, which does not satisfy the equation. The implicit function theorem therefore gives a complex one-dimensional submanifold near every point in this chart. The same calculation in the other two charts covers F, proving $F_{\text{sing}} = \emptyset$. Thus non-singularity here can be checked by differentiating the local equations, without finding a global parametrization. □

As an application of Theorem III.2.52, we prove the following dimension comparison for finite holomorphic maps.

Lemma III.2.54 (Dimension and finite maps). *Let $f\colon X \to Y$ be a finite holomorphic map of complex spaces. Then $\dim X \leq \dim Y$. If f is also surjective, then $\dim X = \dim Y$.*

Before proving this lemma, let us describe the decomposition into smooth pieces used in its proof. The underlying set of a complex space can be decomposed into at most countably many smooth locally closed complex manifolds, whose dimensions do not exceed that of the space. For example, suppose that X is reduced of pure dimension three, that $S := X_{\text{sing}}$ is a surface, that $C := S_{\text{sing}}$ is a curve, and that $C_{\text{sing}} = \{p\}$, giving S and C their reduced structures. Then

$$X = (X \setminus S) \sqcup (S \setminus C) \sqcup (C \setminus \{p\}) \sqcup \{p\}.$$

These pieces have dimensions $3, 2, 1, 0$, respectively. The points of $S \setminus C$ are singular in X, but form the regular part of S. In general, start with X_{red}, remove its regular part, give the remaining singular locus its reduced structure, and repeat. The singular loci are analytic by Theorem III.2.52. Their dimensions strictly decrease in each finite-dimensional local model: a proper analytic subgerm of an irreducible germ corresponds to a nonzero ideal in its local domain, and a chain of primes above that ideal can be lengthened by prepending the zero prime. Thus every local iteration terminates. Second countability, with connected components separated when necessary, gives the asserted countable disjoint decomposition (Compare [De4, Chapter II, Proposition (5.6), p. 104]).

Proof of Lemma III.2.54. We prove this lemma following [De4, Chapter II, Lemma (8.1), p. 116]. Reduction changes neither the underlying map nor the dimensions, by (159), so we may work with reduced spaces. The empty cases are immediate.

Fix $x \in X$, put $y = f(x)$, and choose a local branch of X at x of dimension m. Its image lies in one local branch of Y at y, of dimension p: the inverse images of the finitely many target branches are analytic, and their union contains the irreducible source germ. The Local

Parametrization Theorem (Theorem III.2.43) gives a finite projection of a small representative of that target branch to a polydisc in $\mathbb{C}^p$. The zero-dimensional and full ambient-dimensional cases use a point and a coordinate neighborhood, respectively. Compose with f and restrict to a small source representative; the resulting map g still has finite fibers. On its dense smooth part of dimension m, take a point where the complex rank r of the differential of g is maximal. If $r > 0$, a nonzero $r \times r$ minor makes this rank constant nearby. Using the corresponding r independent components as part of a coordinate system, the inverse function theorem shows that the remaining components depend only on these r coordinates. Thus a fiber is locally an $(m - r)$-dimensional manifold (For $r = 0$, the map is locally constant). Finiteness forces $m-r = 0$, and hence $m = r \leq p$. Taking the maximum over source branches and then the supremum over x gives $\dim X \leq \dim Y$.

Suppose now that f is surjective and, to obtain a contradiction, that $\dim X < \dim Y$. Theorem III.2.45 and Corollary III.2.46 give a nonempty coordinate open set $U \subset Y_{\text{reg}}$ of some pure dimension $p > \dim X$. Decompose $f^{-1}(U)$ into at most countably many smooth locally closed pieces M_j as described above. Each has complex dimension less than p, so every point is critical for the underlying C^∞ map $f|_{M_j} \colon M_j \to U$. Sard's theorem, recalled in Chapter I, §8.1, makes each image $f(M_j)$ a set of Lebesgue measure zero in U. Their countable union cannot equal the nonempty open set U, contrary to surjectivity. It is necessary here to cover the singular part of the source too; using only X_{reg} would not account for its image. This proves the reverse inequality. □

Two real-analytic tools

We record two results of Łojasiewicz. The first connects analytic subsets with the topology of Chapter I; the second gives a quantitative estimate for real-analytic functions near a point.

THEOREM III.2.55 (Triangulation compatible with analytic subsets). *Let X be a complex manifold of pure dimension n, let $A \subset X$ be a nonempty analytic subset, and let $B_1, B_2, \ldots, B_s$ be finitely many analytic subsets of A. There is a locally finite cell complex structure $(X, \mathfrak{E})$ admitting a simplicial structure in the sense of Chapter I, §7.4, such that A and each B_j are subcomplexes. In particular, the induced cell decomposition on A makes it triangulable and makes each B_j a subcomplex of A. If A is compact, its induced cell decomposition is finite.*

PROOF. See [Ło1, §3, Theorem 2, p. 464]. Holomorphic coordinate changes are real analytic. Moreover, locally A and each B_j are given in X by holomorphic equations, which can be replaced by their real and imaginary parts. They therefore satisfy the hypotheses of the cited theorem, which applies more generally to sets described by real-analytic equalities and inequalities. It supplies a triangulation of X whose open simplices are each contained in, or disjoint from, every prescribed subset. Since these subsets are closed, all faces of a simplex contained in one of them are contained in it as well. The last assertion follows from compactness and local finiteness. □

REMARK III.2.56 (The underlying topological spaces). This gives an actual CW structure with the given topology, not merely a CW homotopy type. Thus, for a compact analytic

subset $B \subset A$, the subcomplex neighborhood results of Chapter I, §7.3, apply: B has an open neighborhood in A that deformation retracts onto B. The pair (A, B) also meets the topological hypotheses of Example II.3.117 in Chapter II. If A is compact, its singular homology groups are finitely generated, by the finite CW-complex results in Chapter I. □

For the next result, recall that a function is *real analytic*, or of class C^ω, if it is locally given by a convergent power series in real coordinates.

THEOREM III.2.57 (Łojasiewicz gradient inequality). *Let $\Omega \subset \mathbb{R}^m$ be open, let $f\colon \Omega \to \mathbb{R}$ be real analytic, and let $a \in \Omega$. There are an open neighborhood $U \subset \Omega$ of a, a constant $c > 0$, and a number θ with $0 < \theta < 1$ such that*

$$|(df)_x| \geq c\,|f(x) - f(a)|^\theta \qquad (x \in U), \tag{164}$$

where the norm of the real covector $(df)_x$ is induced by the Euclidean metric on $\mathbb{R}^m$. The constants and the neighborhood may depend on f and a.

PROOF. See [Ło2, no. 18, Proposition 1], applied to $f - f(a)$. □

For example, if $g_1, g_2, \ldots, g_s$ are holomorphic near $a \in \mathbb{C}^n$ and vanish at a, then $f := \sum_{j=1}^s |g_j|^2$ is real analytic in the underlying real coordinates, and its zero set is their common zero set. The inequality applies to this ambient function even when the zero set is singular.

Holomorphic vector bundles and invertible sheaves on reduced spaces

Let $r \in \mathbb{Z}_{\geq 0}$ and let X be a reduced complex space, for example an analytic subset. A holomorphic vector bundle of rank r on X is a topological complex vector bundle on X, as in Chapter I, §7.7, with local trivializations over an open covering $\{U_j\}$ of X whose changes of coordinates are $(x, \eta_k) \mapsto (x, g_{jk}(x)\eta_k)$, where $g_{jk} \in \mathrm{GL}_r(\mathcal{O}_X(U_j \cap U_k))$. Here the entries are holomorphic functions on the reduced space and the determinant is an invertible holomorphic function. A section is holomorphic if its coefficient functions in these trivializations are holomorphic. This defines its sheaf $\mathcal{O}_X(E)$. Rank one gives holomorphic line bundles.

PROPOSITION III.2.58. *On a reduced complex space X, the assignment $E \mapsto \mathcal{O}_X(E)$ gives a bijection between holomorphic isomorphism classes of vector bundles of rank r and isomorphism classes of locally free $\mathcal{O}_X$-modules of rank r. In particular, line bundles correspond to invertible sheaves.*

PROOF. The proof of Proposition III.2.12 uses only local frames and their transition matrices, and applies here unchanged. More explicitly, the gluing construction following Definition II.3.103 gives matrices $g_{jk}g_{k\ell} = g_{j\ell}$, and Lemma I.7.35 glues the products $U_j \times \mathbb{C}^r$. Their coefficient sheaves recover the given locally free module. Conversely, holomorphic frames identify the section sheaf with $\mathcal{O}_X^{\oplus r}$ locally. □

Via $L \mapsto \mathcal{O}_X(L)$, we identify the holomorphic isomorphism classes of line bundles with the isomorphism classes of invertible sheaves. Remark II.3.104 gives

$$\mathrm{Pic}(X) \cong \check{H}^1(X, \mathcal{O}_X^*).$$

For a non-reduced complex space $(X, \mathcal{O}_X)$, the same sheaf-theoretic gluing uses matrices in $\mathcal{O}_X$, retaining their nilpotent entries; their pointwise values on X_{red} alone do not specify that data.

2.5 Divisors and log resolutions

Let X be a complex manifold. A *meromorphic function* on an open subset $U \subset X$ is given locally by quotients g/h of holomorphic functions, where h is not identically zero on any connected component of the neighborhood. Two such quotients represent the same meromorphic function if they agree on their common domain wherever both denominators are nonzero. These functions form a sheaf, denoted by $\mathcal{M}_X$. Its unit sheaf is denoted by $\mathcal{M}_X^*$: a section $f \in \mathcal{M}_X(U)$ is a unit if $fg = 1$ for some $g \in \mathcal{M}_X(U)$. On the complex manifold under consideration, this means that f is not identically zero on any connected component of U. Although we follow the usual convention of calling them meromorphic *functions*, they need not be $\mathbb{C}$-valued functions on all of U.

As in one variable, we are interested in the zeros of holomorphic functions and the zeros and poles of meromorphic functions, counted with their orders, and in whether prescribed data of this kind can be realized by such functions. Divisors record these data. We first study their local equations and associated line bundles on manifolds, then distinguish Cartier and Weil divisors on complex spaces and discuss log resolutions.

In the preceding local arguments, we often used the same symbol for a function and its germ to keep the notation light. From this point onward, we distinguish them whenever possible: for a holomorphic or meromorphic function f defined near a point $x \in X$, we write $\underline{f}_x$ for its germ at x.

Definition III.2.59 (Weil divisors). On a complex manifold X, a *Weil divisor* is a locally finite formal sum $D = \sum_{\nu \in \Lambda} m_\nu H_\nu$, where the H_ν are distinct irreducible complex hypersurfaces and $m_\nu \in \mathbb{Z}$. Local finiteness means that every point has a neighborhood meeting only finitely many H_ν with $m_\nu \neq 0$. Its *support* is $\operatorname{Supp} D := \bigcup_{m_\nu \neq 0} H_\nu$. The divisor is *effective*, written $D \geq 0$, if all $m_\nu \geq 0$. Addition is addition of coefficients. More generally, on a complex space $(X, \mathcal{O}_X)$ of pure dimension n, the same definition uses irreducible analytic subsets of dimension $n-1$. □

The hypersurfaces in this definition need not be non-singular. On restriction to an open set, each H_ν is decomposed into its irreducible components, with the same coefficient on every branch. We use the following consequence of the algebra already prepared.

Proposition III.2.60 (Local equations of hypersurfaces). *Let H be a complex hypersurface in a complex manifold X. Every point has an open neighborhood U on which $\mathcal{I}_H|_U = f\mathcal{O}_U$ for a holomorphic function f on U. At a point $a \in H$, the germ $\underline{f}_a$ of f is, up to a unit, the product of pairwise nonassociated irreducible germs defining the local branches of H, each occurring once.*

Proof. At $a \in H$, the prime ideals of the branches have quotient dimension $\dim_a X - 1$. Theorem I.5.19 in Chapter I, applied to the UFD $\mathcal{O}_{X,a}$ (Theorem III.1.33 in coordinates), makes them principal, say $(\underline{p_1}_a), (\underline{p_2}_a), \ldots, (\underline{p_r}_a)$, where the p_j are holomorphic near

a. Rückert's Nullstellensatz (Theorem III.2.37) gives $\mathcal{I}_{H,a} = \bigcap_{j=1}^{r} (\underline{p_j}_a) = (\underline{p_1}_a \underline{p_2}_a \cdots \underline{p_r}_a)$. Choose a representative f vanishing on H near a. Oka's second coherence theorem (Theorem III.2.163) makes $\mathcal{I}_H / f\mathcal{O}_X$ locally finitely generated. Its stalk at a is zero, so finitely many local generators vanish on a common smaller neighborhood. Thus the equality holds there as an equality of sheaves, not only at a. Outside H use $f = 1$. This also proves the asserted description. □

Before associating a divisor to a function, we distinguish its order at a point from its orders along hypersurfaces.

Definition III.2.61 (Order of vanishing at a point). Let X be a complex manifold, $a \in X$, and let f be holomorphic near a, with germ $\underline{f}_a \in \mathcal{O}_{X,a}$. In holomorphic coordinates $z = (z^1, z^2, \ldots, z^n)$ centered at a, write

$$f(z) = \sum_{k \in (\mathbb{Z}_{\geq 0})^n} c_k z^k,$$

as permitted by Hartogs' separate analyticity theorem (Theorem III.1.18). For $\underline{f}_a \neq 0$, its *order of vanishing*, or *multiplicity*, at a is

$$\operatorname{ord}_a(f) := \min\{\, |k| \mid c_k \neq 0 \,\}.$$

We set $\operatorname{ord}_a(0) := \infty$. □

If $\mathfrak{m}_{X,a}$ is the maximal ideal of $\mathcal{O}_{X,a}$, then

$$\operatorname{ord}_a(f) = \sup\{\, r \in \mathbb{Z}_{\geq 0} \mid \underline{f}_a \in \mathfrak{m}_{X,a}^r \,\}.$$

Indeed, in the chosen coordinates, $\mathfrak{m}_{X,a}$ is generated by the germs of $z^1, z^2, \ldots, z^n$, and membership in its r-th power means that all Taylor terms of total degree less than r vanish. This also proves independence of the coordinates. For $n = 1$ the definition agrees with the order recalled in Chapter I, §I.9.

Recall now that $\mathcal{O}_{X,a}$ is a UFD by Theorem III.1.33, applied in coordinates, or equivalently by Corollary III.2.5. For $\underline{f}_a \neq 0$ with $f(a) = 0$, write

$$\underline{f}_a = \underline{u}_a \, \underline{p_1}_a^{m_1} \underline{p_2}_a^{m_2} \cdots \underline{p_r}_a^{m_r},$$

where $\underline{u}_a$ is a unit, the $\underline{p_\nu}_a$ are pairwise nonassociated irreducible germs, and $m_\nu \geq 1$. Each $(\underline{p_\nu}_a)$ is prime, and Theorem I.5.19 in Chapter I gives $\dim \mathcal{O}_{X,a}/(\underline{p_\nu}_a) = \dim_a X - 1$. Together with Rückert's Nullstellensatz (Theorem III.2.37), this says that the zero set of f, after choosing representatives on a sufficiently small open neighborhood of a, is the union of the hypersurfaces $\{p_\nu = 0\}$, each of pure codimension one. Thus, although an order is defined at each point, it is especially natural to record the order along each hypersurface on which f vanishes. At a non-singular point of that hypersurface, one can choose a transverse coordinate t so that the hypersurface is $\{t = 0\}$; counting powers of t recovers the one-variable picture in the normal direction.

DEFINITION III.2.62 (Order along a hypersurface). Let U be a connected open subset of a complex manifold X, let $H \subset U$ be an irreducible hypersurface, and let $f \in \mathcal{O}_X(U)$ be nonzero. At a point of H_{reg}, choose coordinates (z', t) on an open neighborhood $V \subset U$ so that $H \cap V = \{t = 0\}$ is connected. Write

$$f|_V = t^m h, \qquad h|_{H\cap V} \not\equiv 0,$$

with h holomorphic on V. The integer $m \geq 0$ is independent of the chosen regular point and coordinates, as explained below. It is the *order of vanishing of f along H*, denoted by $\mathrm{ord}_H(f)$. We set $\mathrm{ord}_H(0) := \infty$. □

The exponent is the first index for which the Taylor coefficient in the transverse variable t is not identically zero on $H \cap V$. The identity theorem on this connected manifold shows that the exponent is unchanged when V is restricted near any point of $H \cap V$. Two reduced local defining functions of H differ by a nowhere vanishing holomorphic factor, by Proposition III.2.60. Thus the exponents agree on overlapping coordinate neighborhoods. Since H_{reg} is connected when H is irreducible (Theorem III.2.45), they have a common value on H. The same value is the order on every local irreducible branch of H, since its regular points away from the other branches belong to H_{reg}.

The condition $h|_{H\cap V} \not\equiv 0$ does not require $h(b) \neq 0$ at every $b \in H \cap V$: another zero component may pass through b. At a point where $h(b) \neq 0$, one has $\mathrm{ord}_b(f) = m$, and restriction to a transverse disc also has one-variable order m. The excluded points form a proper analytic subset of $H \cap V$. This is the sense in which $\mathrm{ord}_H(f)$ is the order at *general* regular points of H, rather than necessarily the order at every point of H.

In particular, represent a nonzero holomorphic germ $\underline{f}_0$ by $f(z) = \sum_k c_k z^k$ on a sufficiently small polydisc centered at 0. For the coordinate hyperplane $H = \{z^j = 0\}$ in this polydisc,

$$\mathrm{ord}_{\{z^j=0\}}(f) = \min\{\, k_j \mid c_k \neq 0 \,\}.$$

This value is unchanged on a smaller polydisc. For example, the function $f\colon \mathbb{C}^2 \to \mathbb{C}$ defined by $f(z, w) = z^2 w^3$ has point order $\mathrm{ord}_0(f) = 5$, but orders 2 and 3 along $\{z = 0\}$ and $\{w = 0\}$, respectively.

MINI-EXERCISE III.2.63. Let $n \geq 1$, let $\Omega \subset \mathbb{C}^n$ be an open neighborhood of the origin, and let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be positive real numbers. Let $\varphi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be measurable and suppose that

$$\varphi(z) - \sum_{j=1}^{n} \gamma_j \log |z^j|^2$$

is bounded near the origin off the coordinate hyperplanes. Let $f\colon \Omega \to \mathbb{C}$ be holomorphic. For $\varepsilon > 0$, put $\Delta_\varepsilon := \{w \in \mathbb{C} \mid |w| < \varepsilon\}$. Show that, for every sufficiently small $\varepsilon > 0$,

$$\int_{\Delta_\varepsilon \times \cdots \times \Delta_\varepsilon} |f(z)|^2 e^{-\varphi(z)}\, d\lambda(z) < \infty$$

if and only if

$$\operatorname{ord}_{\{z^j=0\}}(f) \geq \lfloor \gamma_j \rfloor \qquad (j \in \{1, 2, \ldots, n\}),$$

where $d\lambda$ is Lebesgue measure on $\mathbb{C}^n$ and $\lfloor \gamma_j \rfloor$ is the greatest integer not exceeding γ_j.

In addition, let $\alpha \geq 0$ and let $\psi \colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be measurable, with

$$\psi(z) - \alpha \log \sum_{j=1}^{n} |z^j|^2$$

bounded near the origin away from the origin itself. For the ball $B_\varepsilon := \{z \in \mathbb{C}^n \mid |z| < \varepsilon\}$, show that, for every sufficiently small $\varepsilon > 0$,

$$\int_{B_\varepsilon} |f(z)|^2 e^{-\psi(z)}\, d\lambda(z) < \infty \quad \Longleftrightarrow \quad \underline{f}_0 \in \mathfrak{m}_{\mathbb{C}^n,0}^{\max\{0,\lfloor \alpha \rfloor - n + 1\}},$$

where $\underline{f}_0$ is the germ of f at the origin and the zeroth power is understood to be the whole local ring.

We now record the hypersurface orders as coefficients of a divisor.

Definition III.2.64 (Meromorphic functions and principal divisors). Let X be a complex manifold and $U \subset X$ open. For a holomorphic function f on U which is not identically zero on any connected component, its *zero divisor* is

$$\operatorname{div}(f) := \sum_{\nu \in \Lambda} \operatorname{ord}_{H_\nu}(f)\, H_\nu,$$

where the H_ν are the irreducible hypersurface components of $f^{-1}(0)$.

For $f \in \mathcal{M}_X^*(U)$, represented locally as g/h with g, h having nonzero germs, set

$$\operatorname{div}(f) := \operatorname{div}(g) - \operatorname{div}(h)$$

on that neighborhood. If the coefficient of a hypersurface in $\operatorname{div}(f)$ is $-m < 0$, we say that f has a *pole of order* m along that hypersurface. The divisor of a meromorphic unit is called a *principal divisor*. □

The family of zero components is locally finite, and the preceding orders agree on overlaps, including when a component splits into local branches. Unique factorization makes the difference for g/h independent of the quotient used. In particular, $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$. For every open subset $U \subset X$, the homomorphism $f \longmapsto \operatorname{div}(f)$ on $\mathcal{M}_X^*(U)$ has kernel $\mathcal{O}_X^*(U)$: after cancelling irreducible factors, a meromorphic germ of divisor zero is a unit.

For the preceding monomial example, $\operatorname{div}(z^2 w^3) = 2\{z = 0\} + 3\{w = 0\}$. For the cusp $C = \{y^2 = x^3\}$ in a sufficiently small connected open neighborhood of the origin, the germ $\underline{F}_0$ of $F(x, y) = y^2 - x^3$ is irreducible: by Weierstraß preparation and its uniqueness, a nontrivial factorization would give a holomorphic root $y = a(x)$ with $a(x)^2 = x^3$, impossible since $2 \operatorname{ord}_0(a) = 3$. Thus $\operatorname{ord}_0(F) = 2$, whereas $\operatorname{ord}_C(F) = 1$ and $\operatorname{div}(F) = C$ with coefficient one near the origin.

By Proposition III.2.60, a Weil divisor D on a complex manifold X admits an open covering $\{U_j\}_{j\in J}$ of X and meromorphic equations $f_j \in \mathcal{M}_X^*(U_j)$: $\operatorname{div}(f_j) = D|_{U_j}$, with $f_j/f_k \in \mathcal{O}_X^*(U_j \cap U_k)$. Indeed, near a point only finitely many branches occur, and the product of their equations raised to the prescribed integer powers has the required divisor. For effective D, the f_j may be chosen holomorphic. This is where non-singularity of the ambient space is being used. The local modules $f_j^{-1}\mathcal{O}_{U_j}$ agree on overlaps. They define an invertible subsheaf of $\mathcal{M}_X$, denoted by $\mathcal{O}_X(D)$.

Proposition III.2.65 (The line bundle of a divisor). *A divisor D on a complex manifold X determines a holomorphic line bundle $[D]$ with a canonical meromorphic section σ_D. For local equations f_j as above, its local frames e_j satisfy*

$$g_{jk} := \frac{f_j}{f_k}, \qquad e_k = g_{jk}e_j, \qquad \sigma_D|_{U_j} = f_j e_j. \tag{165}$$

The section is holomorphic if D is effective, and its zero divisor is then D, including the multiplicities. Its sheaf of holomorphic sections is canonically identified with $\mathcal{O}_X(D)$.

Proof. The units g_{jk} satisfy $g_{jk}g_{k\ell} = g_{j\ell}$, so they are multiplicative Cousin data in the convention of (142). The gluing construction of Proposition III.2.12 gives $[D]$, and $f_k e_k = f_j e_j$ gives σ_D. Mapping e_j to $1/f_j$ identifies its sections with $\mathcal{O}_X(D)$ and maps σ_D to the meromorphic section 1. Replacing the equations by unit multiples therefore changes neither this sheaf nor its distinguished meromorphic section. For effective D, the local coefficient f_j is holomorphic and has precisely the prescribed zeros. □

We call σ_D the *canonical section associated with D*. Division by σ_D gives the identification

$$\mathcal{O}_X([D]) \longrightarrow \mathcal{O}_X(D), \qquad s \longmapsto s/\sigma_D,$$

whose inverse is $f \longmapsto f\sigma_D$. Locally, $s = a_j e_j$ corresponds to a_j/f_j. Consequently, for every connected open subset $V \subset X$,

$$\Gamma(V, \mathcal{O}_X(D)) = \{0\} \cup \{f \in \mathcal{M}_X^*(V) \mid \operatorname{div}(f) + D|_V \geq 0\}.$$

Indeed, $f\sigma_D$ is holomorphic exactly when each ff_j is holomorphic. By unique factorization in the local rings of X, this is equivalent, for $f \neq 0$, to the displayed divisor condition. For an arbitrary open set, the description is understood on each connected component. No effectiveness assumption on D is needed for this description; σ_D itself is holomorphic when D is effective.

Multiplication of local equations gives $[D + D'] \cong [D] \otimes [D']$ and $[-D] \cong [D]^{-1}$. For $D \geq 0$, the inverse bundle has sheaf of sections $\mathcal{O}_X(-D) = f_j\mathcal{O}_{U_j} \subset \mathcal{O}_X$. Thus D also defines a closed complex subspace with structure sheaf $j^{-1}(\mathcal{O}_X/\mathcal{O}_X(-D))$, where $j\colon \operatorname{Supp} D \hookrightarrow X$. This ideal retains the multiplicities: for $D = 2\{z = 0\}$ in $\mathbb{C}$, it is (z^2), not the full vanishing ideal $\mathcal{I}_{\operatorname{Supp} D} = (z)$.

Example III.2.66. On $\mathbb{P}^1$, let $p = [0 : 1]$, with coordinates $z \mapsto [z : 1]$ and $w \mapsto [1 : w]$. The effective divisor $D = p$ has equations $f_0 = z$ and $f_\infty = 1$. Consequently $g_{0\infty} = z$, so $[p] \cong L_{\text{taut}}^{-1}$, the dual of the tautological bundle. Its sheaf of holomorphic sections is $\mathcal{O}_{\mathbb{P}^1}(1)$, and σ_p has a simple zero at p. □

Cartier divisors and the distinction on a singular space

On a general complex space $(X, \mathcal{O}_X)$, define $\mathcal{M}_X$ by localizing $\mathcal{O}_X$ at its non-zero-divisors and taking the associated sheaf. In other words, denominators must be non-zero-divisors in the local rings. This agrees with the preceding definition on a manifold. The localization definition also retains nilpotents when X is non-reduced. Here $\mathcal{M}_X^*$ again denotes the unit sheaf, using the definition by multiplicative inverses.

Definition III.2.67 (Cartier divisors). A *Cartier divisor* on a complex space $(X, \mathcal{O}_X)$ is represented on some open covering $\{U_j\}_{j\in J}$ of X by meromorphic units $f_j \in \mathcal{M}_X^*(U_j)$ with $f_j/f_k \in \mathcal{O}_X^*(U_j \cap U_k)$, modulo restriction to common refinements and multiplication by local holomorphic units. The f_j are called *local defining functions* of the Cartier divisor. Equivalently, it is a global section of $\mathcal{M}_X^*/\mathcal{O}_X^*$. It is *effective* if its equations can be chosen holomorphic; these equations are necessarily non-zero-divisors. They define an invertible ideal $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ and a closed complex subspace called an *effective Cartier divisor*. □

These definitions use the local-ring and sheaf operations of Chapter II. The construction (165) works for Cartier divisors as well, using a finite locally free $\mathcal{O}_X$-module of rank one for the associated line bundle on a complex space.

On a complex manifold X, the local divisors $\operatorname{div}(f_j)$ associated with a Cartier divisor agree on overlaps, since f_j/f_k is a holomorphic unit. They therefore define a Weil divisor. Conversely, Proposition III.2.60 gives local defining functions for every Weil divisor, and the kernel computation above shows that two such functions with the same divisor differ by a holomorphic unit. Thus Weil and Cartier divisors are identified on a complex manifold. In this setting, we usually say simply *divisor*.

For a reduced complex space X of pure dimension $n \geq 1$, Weil divisors are locally finite integral sums of irreducible analytic subsets of dimension $n-1$. The principal divisors $\operatorname{div}(f)$ of meromorphic units f are defined using the multiplicities of their local zero and pole divisors, and the $\operatorname{div}(f_j)$ again glue to a Weil divisor associated with the Cartier divisor (See [AS, Example 6.1] for the zero divisors). When $\dim X_{\text{sing}} \leq n-2$, these multiplicities can be computed by transverse orders on X_{reg} (See [De4, Chapter II, Definition (6.7)(c) and (6.12), pp. 107–109]). In general, this correspondence need be neither injective nor surjective. In particular, a Weil divisor need not admit local meromorphic defining functions with the prescribed multiplicities and holomorphic unit ratios, as the following example shows.

Example III.2.68 (A Weil divisor which is not Cartier). Let (x, y, z) be the standard coordinates on $\mathbb{C}^3$, let $X = \{xy = z^2\} \subset \mathbb{C}^3$ have its reduced structure, and let $D = \{x = z = 0\} \subset X$, with coefficient one. The only singular point of this surface is the origin, as follows by differentiating its equation. Thus the singular locus has codimension two, and the preceding description of Weil coefficients on X_{reg} applies.

We show that D does not come from a Cartier divisor near the origin. Consider the surjective holomorphic map

$$\Phi\colon \mathbb{C}^2 \longrightarrow X, \qquad (s,t) \longmapsto (s^2, t^2, st).$$

It is locally biholomorphic away from the origin: where $s \neq 0$ use a local square root of x, and where $t \neq 0$ use one of y. If a local meromorphic unit f on X had $\operatorname{div}(f) = D$, then $F = f \circ \Phi$ would be a meromorphic function near $0 \in \mathbb{C}^2$ with divisor $\{s = 0\}$, with coefficient one. Every hypersurface germ in $\mathbb{C}^2$ meets the punctured neighborhood, so no further divisor component can be hidden at the origin. The UFD property (Theorem III.1.33) gives $F = su$ for a holomorphic unit u near the origin. But $F(-s,-t) = F(s,t)$, so $u(-s,-t) = -u(s,t)$, contradicting $u(0,0) \neq 0$.

Nevertheless, $y \neq 0$ at every nonzero point of D, where $x = z^2/y$. Hence $\operatorname{div}(x) = 2D$. Thus twice D does come from a Cartier divisor. A Weil divisor with a positive integer multiple which comes from a Cartier divisor is called $\mathbb{Q}$-*Cartier*; this example is $\mathbb{Q}$-Cartier but not Cartier. □

Normal crossings and modifications

Definition III.2.69 (Normal crossings). A reduced hypersurface H in a complex manifold has *normal crossings* if, near every point of H, there are holomorphic coordinates $z^1, z^2, \ldots, z^n$ centered at that point in which $H = \{z^1 z^2 \cdots z^r = 0\}$ for some $1 \leq r \leq n$. It has *simple normal crossings*, abbreviated *SNC*, if in addition every global irreducible component is non-singular. For an effective divisor D, saying that D has SNC support refers to $\operatorname{Supp} D$ with its reduced structure; the coefficients of D need not be one. We allow the empty support. □

For example, the coordinate axes in $\mathbb{C}^2$ have SNC. The distinction between normal crossings and SNC concerns global components: two local branches may belong to a single singular irreducible component. These branches must not be treated as two global non-singular components. Locally, an effective divisor with normal-crossing support has a defining equation $u(z)(z^1)^{m_1}(z^2)^{m_2}\cdots(z^r)^{m_r}$, where u is a unit and the m_j are positive integers.

A *proper modification* $\pi\colon X' \to X$ will mean a proper surjective holomorphic map of reduced complex spaces for which there is a nowhere dense analytic subset $A \subset X$ such that $\pi^{-1}(A)$ is nowhere dense and $\pi|_{X' \setminus \pi^{-1}(A)}\colon X' \setminus \pi^{-1}(A) \to X \setminus A$ is biholomorphic. The point blow-up of Remark III.2.15 is such a map. More generally, if Z is a closed complex submanifold of X of positive codimension c, choose coordinates $z^1, z^2, \ldots, z^n$ with $Z = \{z^1 = z^2 = \cdots = z^c = 0\}$. Its blow-up is obtained by blowing up the normal coordinate factor $\mathbb{C}^c$ at zero and leaving the remaining coordinates fixed. In the chart indexed by $i \in \{1, 2, \ldots, c\}$, the map is

$$z^i = t, \qquad z^j = t u_j \ (1 \leq j \leq c,\ j \neq i), \qquad z^k = v^k \ (k > c).$$

These local constructions glue to a complex manifold $\operatorname{Bl}_Z X$ and a proper modification of X; see [BM3, §1.2]. For $c \geq 2$, the inverse image of Z is its exceptional hypersurface. For $c = 1$, the blow-up is an isomorphism. The *strict transform* of an analytic subset not contained in Z is the closure of its inverse image away from Z, with reduced structure. For the blow-up of $0 \in \mathbb{C}^2$, the strict transform of a line through 0 is the closure of the inverse image of the punctured line; the full inverse image also contains the exceptional curve.

DEFINITION III.2.70 (Pullback of a Cartier divisor). Let $f\colon (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a holomorphic map of complex spaces, and let D be an effective Cartier divisor represented by holomorphic equations h_j on an open covering $\{U_j\}_{j\in J}$ of X. Assume that $f^\sharp(h_j)$ is a non-zero-divisor at every point of $f^{-1}(U_j)$. The equations $f^\sharp(h_j)$ then define an effective Cartier divisor on X', called the *pullback* of D and denoted by f^*D. Here $f^\sharp$ is the structure-sheaf morphism of Definition II.3.92; on reduced spaces, $f^\sharp(h_j) = h_j \circ f$.

For a Cartier divisor not necessarily effective, the same construction applies whenever its local meromorphic equations can be pulled back as meromorphic units. More explicitly, after refining the covering, write an equation as a_j/b_j with a_j, b_j holomorphic non-zero-divisors. If both $f^\sharp(a_j)$ and $f^\sharp(b_j)$ are non-zero-divisors, use the quotient $f^\sharp(a_j)/f^\sharp(b_j)$ as the pulled-back equation. □

This construction is independent of the permitted representations: equalities between fractions remain equalities after cross multiplication and application of $f^\sharp$, and a holomorphic unit pulls back to a unit. Thus the equations have unit ratios on overlaps and are unchanged as Cartier data under refinement. The condition on non-zero-divisors is essential. For example, the effective Cartier divisor $\{x = 0\}$ in $\mathbb{C}^2$ cannot be pulled back in this sense by the inclusion of $\{x = 0\}$ itself: the pulled-back equation is zero. The corresponding line bundle can still be pulled back, but its pulled-back canonical section does not define a Cartier divisor.

For a composition $\pi\colon X' \to X$ of point blow-ups of a complex manifold of pure dimension two, the local biholomorphism away from the exceptional curves shows that these pullbacks are defined for every divisor D on X. We call π^*D the *total transform* of D. If $D = \sum_\nu m_\nu H_\nu$, its strict transform is $D' := \sum_\nu m_\nu H'_\nu$, where H'_ν is the strict transform of H_ν under the successive blow-ups. The difference $\pi^*D - D'$ is supported on the exceptional curves, but its coefficients need not be one. The transition functions in (165) give $[\pi^*D] \cong \pi^*[D]$, with the canonical sections corresponding. For $D \geq 0$, one also has $\mathcal{O}_{X'}(-\pi^*D) = \mathcal{O}_X(-D)\mathcal{O}_{X'}$, where the right-hand side is the ideal generated by pulled-back local equations, not the inverse-image sheaf.

In particular, let $\sigma\colon X' \to X$ be one point blow-up, with center p and exceptional curve E, and let h be a local holomorphic equation of an effective divisor D at p. If $m := \operatorname{ord}_p(h)$ is its point order from Definition III.2.61, equivalently the smallest total degree of a nonzero Taylor term at p, then

$$\sigma^*D = D' + mE. \tag{166}$$

To see the coefficient, use local coordinates (x, y) centered at p and write $h = h_m + h_{m+1} + \cdots$ in homogeneous parts. In the chart $(x, y) = (t, tu)$, $h(t, tu) = t^m(h_m(1, u) + th_{m+1}(1, u) + \cdots)$. Since $h_m(1, u)$ is not identically zero, the order along E is m. Away from E the map is biholomorphic, so the remaining divisor is D'. For successive blow-ups one must apply this to the *whole* total transform at the new center, not just to the strict transform of the original curve.

We also record the *pushforward* of Weil divisors in the setting needed below. For the above composition π, let H'_ν be strict transforms of distinct irreducible curves $H_\nu \subset X$, and let E_ℓ be its exceptional curves. For a locally finite divisor of the form $G' = \sum_\nu m_\nu H'_\nu + \sum_\ell b_\ell E_\ell$,

set

$$\pi_* G' := \sum_\nu m_\nu H_\nu. \tag{167}$$

The strict transform maps with one sheet away from the finitely many points blown up on X; each exceptional curve is contracted to a point, which has codimension two and contributes no divisor on X. Properness ensures local finiteness of the image family: only finitely many components of G' meet the inverse image of a sufficiently small compact neighborhood of any point of X. In particular, $\pi_*(\pi^* D) = D$. This divisor pushforward is not the direct image of a sheaf. For more general proper maps one also has to count the number of sheets over a general point of the image curve; that construction is not needed here.

In contrast, taking the set-theoretic inverse image of the support of a Weil divisor neither guarantees codimension one nor supplies multiplicities. The inclusion example above illustrates the first problem; the map $\mathbb{C} \to \mathbb{C}$, $w \mapsto w^m$, with $m \geq 2$, illustrates the second, since the divisor $\{0\}$ pulls back as $m\{0\}$, not $\{0\}$. On manifolds, Weil divisors are Cartier, so their pullback is available under the stated conditions; the distinction is between these operations, not a claim that Weil divisors can never be pulled back.

A coherent ideal $\mathcal{I}$ also defines a blow-up, with the ideal rather than only its zero set as center; the general analytic construction and its universal property are in [GM, §3, Lemmata 3.1, 3.5, and 3.6]. For an ideal nonzero on every connected component of a manifold, this makes the ideal locally principal on the new space, but does not by itself guarantee that the new space is non-singular or that the resulting support has SNC. The following deeper result obtains all these properties using smooth centers.

Theorem III.2.71 (Log resolution of a coherent ideal near a compact set). *Let X be a complex manifold and let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf which is not identically zero on any connected component. For every compact subset $C \subset X$, there are an open neighborhood U of C and a finite sequence of blow-ups with smooth closed centers, whose composite is a proper modification $\pi\colon \widetilde{U} \to U$, such that*

$$\mathcal{I}\mathcal{O}_{\widetilde{U}} = \mathcal{O}_{\widetilde{U}}(-D) \tag{168}$$

for an effective divisor D with SNC support. Here $\widetilde{U}$ is non-singular, $\mathcal{I}\mathcal{O}_{\widetilde{U}}$ is the ideal generated by pullbacks of local sections of $\mathcal{I}$, and π is biholomorphic over $U \setminus V(\mathcal{I})$. The support of D and the exceptional hypersurfaces of π together have SNC.

Proof. This is the form of Hironaka's resolution theorem for coherent ideals. See [BM1, Remark 1.8 and Theorem 1.10], with the relatively compact and locally finite versions explained after Theorem 1.6. Applying that construction near the compact set C and shrinking its neighborhood gives the finite sequence above. The total transform is monomial with respect to a collection of smooth hypersurfaces, including the exceptional hypersurfaces, which have normal crossings. We use this theorem without reproducing its proof. For a complete proof in the analytic category, see also [Wł, Theorem 2.0.3, §4.1, and §§5–7]. □

Such a π is called a *log resolution of the ideal $\mathcal{I}|_U$*. No radical assumption is imposed on $\mathcal{I}$: (168) concerns the original ideal and its multiplicities, not just its zero set. In coordinates

adapted to the SNC support it says $\mathcal{I}\mathcal{O}_{\widetilde{U}} = ((z^1)^{m_1}(z^2)^{m_2}\cdots(z^r)^{m_r})$. This is an equality of ideals generated after pullback, not an identification with the inverse-image sheaf $\pi^{-1}\mathcal{I}$. The unit ideal is allowed and gives $D = 0$; the zero ideal is excluded because it cannot become invertible under a modification. If X is compact, one may take $U = X$; for general non-compact X we do not assert that one finite sequence suffices on all of X.

Remark III.2.72 (Desingularization of reduced complex spaces). A reduced complex space X, without a prescribed global embedding in a manifold, also admits a proper modification $\rho\colon \widehat{X} \to X$ such that $\widehat{X}$ is non-singular and ρ is biholomorphic over X_{reg}. It can be constructed by blow-ups with smooth closed centers lying over X_{sing}, using finitely many stages when X is compact and a locally finite construction in general. Here local finiteness means that only finitely many stages are nontrivial over each relatively compact open subset of X. See [BM1, Theorem 1.6 and the discussion following it, pp. 4–6, and the discussion "Universal and canonical desingularization", p. 11]; the complete article is [BM2]. For a complete analytic proof of the existence of ρ, even with $\rho^{-1}(X_{\mathrm{sing}})_{\mathrm{red}}$ an SNC hypersurface, see [Wł, Theorem 2.0.1, Proposition 4.0.2, §4.3, and §§5–7]. The passage from local embeddings to abstract spaces is proved in Proposition 4.0.2 and §4.3 of that paper; its Theorems 2.0.2–2.0.3 state the finite factorizations near compact sets in local ambient manifolds.

In particular, if $S := X_{\mathrm{sing}}$ is compact, one may obtain a finite composite $\pi\colon \widetilde{X} \to X$ of blow-ups with smooth closed centers such that $\widetilde{X}$ is non-singular, π is biholomorphic over $X \setminus S$, and $\pi^{-1}(S)$, with its reduced structure, is an SNC hypersurface, possibly empty. For finiteness, restrict the locally finite construction to a relatively compact neighborhood of S and glue it to the identity over $X \setminus S$; all centers lie over S. To ensure the last assertion, apply Theorem III.2.71 to $\mathcal{I}_S\mathcal{O}_{\widehat{X}}$ near the compact set $\rho^{-1}(S)$, where $\mathcal{I}_S$ is the defining ideal sheaf of S. This ideal is nonzero on every connected component of $\widehat{X}$ and is the unit ideal away from $\rho^{-1}(S)$. The additional blow-ups therefore glue to the identity outside that neighborhood. Their total transform is invertible with SNC support, and its zero set is exactly $\pi^{-1}(S)$. □

Example III.2.73 (A log resolution of a cusp). Let $X_0 = \mathbb{C}^2$ with coordinates (x, y), put $F(x, y) := y^2 - x^3$, and let $D_0 := \mathrm{div}(F)$. Its support $C_0 = \{y^2 = x^3\}$ is the cusp, with coefficient one. We compute three point blow-ups $\sigma_k\colon X_k \to X_{k-1}$, for $k = 1, 2, 3$. Write C_k for the strict transform of C_0 on X_k, and $E_i^{(k)}$ for the strict transform on X_k of the exceptional curve created by σ_i, for $i \le k$. Thus $E_i^{(k)}$ denotes a reduced curve, not its total transform. Set $\pi_k := \sigma_1 \circ \cdots \circ \sigma_k$ and $D_k := \pi_k^* D_0$. In each chart we substitute into the *original* function F to retain all multiplicities. Figure III.2 summarizes the successive configurations.

(*i*) The first blow-up. Blow up the origin of X_0. The two charts of (144) give

$$\begin{aligned} (x, y) = (u, uv) &: \qquad F \circ \sigma_1 = u^2(v^2 - u), \\ (x, y) = (st, t) &: \qquad F \circ \sigma_1 = t^2(1 - s^3 t). \end{aligned}$$

In the first chart, $E_1^{(1)} = \{u = 0\}$ and $C_1 = \{v^2 - u = 0\}$. The strict transform is non-singular, since the partial derivative of $v^2 - u$ with respect to u is -1, but its tangent line at $(0, 0)$ is $u = 0$. Thus C_1 is tangent to $E_1^{(1)}$ there. In the other chart, the strict transform

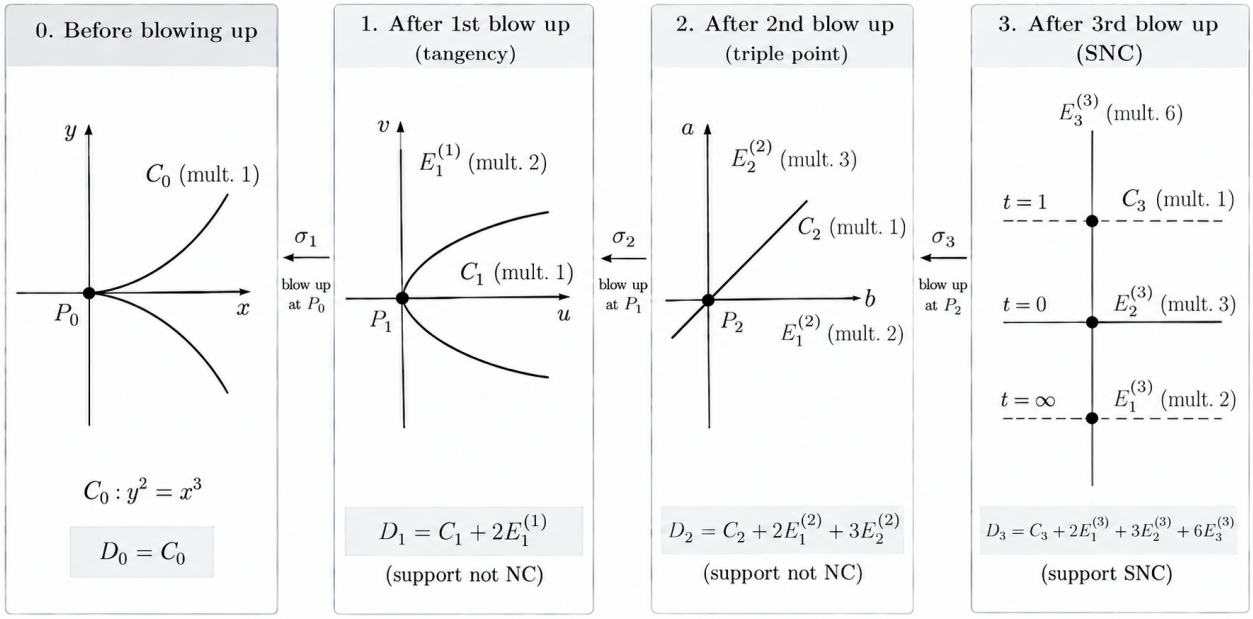


Figure III.2: Schematic supports and multiplicities in the three point blow-ups of the cusp. In the last panel, $t = 0, 1, \infty$ are distinct points on $E_3^{(3)}$.

$1 - s^3 t = 0$ does not meet the exceptional curve $t = 0$. Consequently $D_1 = C_1 + 2E_1^{(1)}$, whose support does not have normal crossings.

(*ii*) The second blow-up. Blow up the tangency point $(u, v) = (0, 0)$ on X_1. The two new charts give

$$\begin{aligned} (u, v) = (ab, b) : &\qquad F \circ \pi_2 = a^2 b^3 (b - a), \\ (u, v) = (c, cd) : &\qquad F \circ \pi_2 = c^3 (cd^2 - 1). \end{aligned}$$

In the first chart the three components are $E_1^{(2)} = \{a = 0\}$, $E_2^{(2)} = \{b = 0\}$, and $C_2 = \{b - a = 0\}$. In the other chart, $E_2^{(2)} = \{c = 0\}$, $E_1^{(2)}$ is absent, and $cd^2 - 1 = 0$ does not meet $c = 0$. We obtain $D_2 = C_2 + 2E_1^{(2)} + 3E_2^{(2)}$. Although the three curves at $(a, b) = (0, 0)$ are non-singular with pairwise distinct tangent lines, their union does not have normal crossings: in a complex surface, the coordinate model in Definition III.2.69 allows at most two branches through one point. Thus a third blow-up is needed.

(*iii*) The third blow-up. Blow up this triple point $(a, b) = (0, 0)$ on X_2. Now

$$\begin{aligned} (a, b) = (r, rt) : &\qquad F \circ \pi_3 = r^6 t^3 (t - 1), \\ (a, b) = (\alpha\beta, \beta) : &\qquad F \circ \pi_3 = \alpha^2 \beta^6 (1 - \alpha). \end{aligned}$$

In the first chart $E_3^{(3)} = \{r = 0\}$, $E_2^{(3)} = \{t = 0\}$, and $C_3 = \{t = 1\}$. In the second chart $E_3^{(3)} = \{\beta = 0\}$, $E_1^{(3)} = \{\alpha = 0\}$, and $C_3 = \{\alpha = 1\}$. On the overlap, $\alpha = 1/t$ and $\beta = rt$. Hence the intersections of $E_3^{(3)} \cong \mathbb{P}^1$ with $E_2^{(3)}$, C_3, and $E_1^{(3)}$ occur at the three distinct points $t = 0, 1, \infty$, respectively. Each intersection is transverse; there is no triple point.

Putting $\pi := \pi_3$, the complete total transform is

$$\pi^* D_0 = C_3 + 2E_1^{(3)} + 3E_2^{(3)} + 6E_3^{(3)}. \tag{169}$$

The curve C_0 is non-singular away from the origin, as its two partial derivatives $-3x^2$ and $2y$ do not vanish simultaneously there. Away from the successive centers, each blow-up is biholomorphic. The chart computations therefore check every possible failure of normal crossings. All the final components are non-singular, and the support in (169) has SNC. Equivalently, $(F)\mathcal{O}_{X_3} = \mathcal{O}_{X_3}(-\pi^* D_0)$, so π is a log resolution of the ideal (F).

The new exceptional coefficients are 2, 3, and 6. The second is $2+1$, since the first total transform has local equation $u^2(v^2-u)$ of order three at the tangency point. The third is $2+3+1$, from $a^2b^3(b-a)$ at the triple point. These agree with (166). Moreover, (167) gives $\pi_* C_3 = C_0$ and $\pi_* E_i^{(3)} = 0$ for $i = 1, 2, 3$, so $\pi_*(\pi^* D_0) = D_0$. □

Example III.2.74 (Keeping the multiplicities). Consider $\mathcal{I} = (x, y^2)$ on $\mathbb{C}^2$. In the two charts of the blow-up of the origin, substitution gives

$$(x,y) = (u, uv): \quad \mathcal{I}\mathcal{O} = (u), \qquad (x,y) = (st, t): \quad \mathcal{I}\mathcal{O} = t(s,t).$$

The second ideal is not principal at $(s,t) = (0,0)$. Blow up that point of the first blow-up. In the new charts, $(s,t) = (a, ab)$ and $(s,t) = (ab, b)$, the total-transform ideals are $(a^2 b)$ and (b^2), respectively. Elsewhere the already principal ideal is unchanged. Thus two point blow-ups give a log resolution of $\mathcal{I}$, with SNC support and with the indicated multiplicities. Its radical (x, y) would instead become principal after the first blow-up; discarding nilpotent information at the start would therefore change this calculation. □

2.6 Pluriharmonic functions

We now turn to functions whose complex Hessian vanishes. Throughout this subsection, X is a complex manifold.

Definition III.2.75. A function $h\colon X \to \mathbb{C}$ is called *pluriharmonic* if it is of class C^2 and $\partial\overline{\partial} h \equiv 0$. A function $g\colon X \to \mathbb{C}$ is called *antiholomorphic* if $\overline{g}$ is holomorphic. □

The condition $\partial\overline{\partial} h = 0$ is independent of holomorphic coordinates, by (125); it is also equivalent to $dd^c h = 0$ with our normalization. Holomorphic and antiholomorphic functions are pluriharmonic. Moreover, $\overline{\partial\overline{\partial} h} = -\partial\overline{\partial h}$, so a complex-valued function is pluriharmonic if and only if its real and imaginary parts are pluriharmonic. In particular, $\operatorname{Re} f$ and $\operatorname{Im} f$ are pluriharmonic whenever f is holomorphic.

Example III.2.76. On $\mathbb{C}^2$, let (z^1, z^2) be the standard coordinates, and write $z^j = x^j + \sqrt{-1}\, y^j$, where $x^j = \operatorname{Re} z^j$ and $y^j = \operatorname{Im} z^j$ for $j = 1, 2$. Let g_0 be the Euclidean metric. With the sign convention of Mini-Exercise I.8.59, its Laplace–Beltrami operator on functions is

$$\Delta_{g_0} = -\sum_{j=1}^{2} \left(\frac{\partial^2}{\partial (x^j)^2} + \frac{\partial^2}{\partial (y^j)^2} \right).$$

The function $h(z^1, z^2) = |z^1|^2 - |z^2|^2$ satisfies $\Delta_{g_0} h = 0$, so it is harmonic for g_0. It is not pluriharmonic: its complex Hessian has diagonal entries $1, -1$ and is not zero. This is consistent with $\Delta_{g_0} h = -4 \operatorname{tr} H(h)$ from Remark III.1.11.

Conversely, keep the same complex structure and set $g = e^{2x^1} g_0$. The function $u = x^1 = \operatorname{Re} z^1$ is pluriharmonic. However, $dV_g = e^{4x^1} dV_{g_0}$, and computing $\Delta_g = d_g^\dagger d$ by integration by parts gives

$$\Delta_g u = -e^{-4x^1} \frac{\partial}{\partial x^1} \left(e^{2x^1} \frac{\partial u}{\partial x^1} \right) = -2e^{-2x^1} \neq 0.$$

Thus u is not harmonic for this metric. □

PROPOSITION III.2.77. *Let $h\colon X \to \mathbb{R}$ be of class C^2. Then h is pluriharmonic if and only if every point has a connected open neighborhood V on which $h = \operatorname{Re} F$ for some $F \in \mathcal{O}_X(V)$. Such an F is unique up to a purely imaginary constant on V. More generally, a complex-valued C^2 function is pluriharmonic if and only if it is locally the sum of a holomorphic function and an antiholomorphic function. Every pluriharmonic function is of class C^∞.*

PROOF. Suppose first that h is real-valued and $\partial\overline{\partial} h = 0$. In a chart $(z^1, z^2, \dots, z^n)$, each coefficient $\partial h / \partial z^j$ is C^1 and is annihilated by $\overline{\partial}$. It is therefore holomorphic, hence smooth, by Theorem III.1.18. Thus ∂h is a smooth holomorphic $(1,0)$-form, and $d(\partial h) = -\partial\overline{\partial} h = 0$. Choose V whose coordinate image is a ball. The Poincaré lemma (Remark I.8.45) gives a smooth function f on V such that $df = \partial h$. Comparing types gives $\overline{\partial} f = 0$, so f is holomorphic. As h is real-valued,

$$d(f + \overline{f}) = \partial h + \overline{\partial} h = dh.$$

Consequently $h - 2 \operatorname{Re} f$ is a real constant on V; adding this constant to $2f$ gives the required F. The converse follows from the observations above. If F_1 and F_2 have the same real part, their difference is holomorphic with values in $\sqrt{-1}\mathbb{R}$. The open mapping theorem, applied on coordinate discs, makes it locally constant, hence a purely imaginary constant on the connected set V.

For complex-valued h, apply the real-valued assertion to $\operatorname{Re} h$ and $\operatorname{Im} h$ on a common neighborhood and use $\operatorname{Re} F = (F + \overline{F})/2$. This writes h as a holomorphic function plus an antiholomorphic one. Conversely, any such sum is pluriharmonic. These local representations also prove the asserted smoothness. □

The word "locally" in this proposition cannot in general be omitted. The next exercise makes the global obstruction visible using de Rham cohomology.

MINI-EXERCISE III.2.78. Let X be a connected complex manifold and let $h\colon X \to \mathbb{R}$ be pluriharmonic.
(*i*) Show that $h = \operatorname{Re} F$ for some $F \in \mathcal{O}_X(X)$ if and only if the closed form ∂h is exact. Deduce that such an F exists if $H^1(X, \mathbb{C}) = 0$. In particular, it exists if X is contractible.
(*ii*) On $X = \mathbb{C}^*$, show that $h(z) = \log|z|$ is pluriharmonic but is not the real part of any globally defined holomorphic function.

Finally, let $\mathcal{H}_X$ denote the sheaf of complex-valued pluriharmonic functions, and let $\overline{\mathcal{O}}_X$ denote the sheaf of antiholomorphic functions, with the ordinary pointwise addition and scalar multiplication. Their sections and restriction maps are those of functions on open subsets of X. The local description gives an exact sequence of sheaves of complex vector spaces

$$0 \longrightarrow \underline{\mathbb{C}}_X \xrightarrow{\iota} \mathcal{O}_X \oplus \overline{\mathcal{O}}_X \xrightarrow{q} \mathcal{H}_X \longrightarrow 0, \tag{170}$$

where $\iota(c) = (c, -c)$ for a locally constant function c, and $q(f, g) = f + g$ on every open set. Indeed, $f + g = 0$ implies that $f = -g$ is both holomorphic and antiholomorphic, hence locally constant; the surjectivity of q is local by Proposition III.2.77. It is not a sequence of $\mathcal{O}_X$-modules; multiplication by a holomorphic function need not preserve pluriharmonicity.

The next exercise applies the long exact cohomology sequence to measure the obstruction to a global decomposition.

Mini-Exercise III.2.79. Show that every pluriharmonic function $h\colon \mathbb{C}^* \to \mathbb{C}$ can be written as $h(z) = f(z) + g(z) + \lambda \log|z|$ for $z \in \mathbb{C}^*$, where $f\colon \mathbb{C}^* \to \mathbb{C}$ is holomorphic, $g\colon \mathbb{C}^* \to \mathbb{C}$ is antiholomorphic, and $\lambda \in \mathbb{C}$ is uniquely determined.

2.7 Plurisubharmonic functions and basic pluripotential theory

We studied subharmonic functions in Chapter I, §I.9. Their higher-dimensional counterparts, plurisubharmonic functions, play a central role in complex analysis and geometry.

Definition III.2.80. Let $\Omega \subset \mathbb{C}^n$ be open, with $n \geq 1$. An upper semi-continuous function $\varphi\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic* if, for every $a \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^n \setminus \{0\}$, the function $w \mapsto \varphi(a + w\xi)$ is subharmonic on $\{w \in \mathbb{C} \mid a + w\xi \in \Omega\}$. We write $\mathrm{Psh}(\Omega)$ for the set of these functions. Identically $-\infty$ functions are allowed, also on individual connected components. □

For an upper semi-continuous φ, the defining condition is equivalently the submean inequality

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + re^{\sqrt{-1}\theta}\xi)\, d\theta \tag{171}$$

whenever the closed affine disk under consideration is contained in Ω. Finite maxima and sums of plurisubharmonic functions are plurisubharmonic, as are their positive scalar multiples. A decreasing limit of plurisubharmonic functions is plurisubharmonic: upper semi-continuity is preserved, and the submean inequalities pass to the limit by monotone convergence after subtracting a local upper bound.

Proposition III.2.81. *Let $n \in \mathbb{Z}_{\geq 1}$ and let $\Omega \subset \mathbb{C}^n$ be open. A real-valued C^2 function φ on Ω is plurisubharmonic if and only if its complex Hessian $H(\varphi)$ is positive semi-definite everywhere, or equivalently $\sqrt{-1}\partial\overline{\partial}\varphi \geq 0$.*

PROOF. Fix $a \in \Omega$ and a non-zero vector $\xi = (\xi^1, \xi^2, \dots, \xi^n) \in \mathbb{C}^n$. For $v(w) = \varphi(a + w\xi)$ near $w = 0$, the chain rule gives

$$\frac{\partial^2 v}{\partial w \partial \overline{w}}(0) = \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z^j \partial \overline{z^k}}(a) \xi^j \overline{\xi^k}.$$

Apply the one-variable criterion in Mini-Exercise I.9.6 to each affine line. The equivalence with semi-positivity of the $(1,1)$-form is Lemma III.1.7, applied to its continuous coefficients pointwise. □

The same positivity has a meaning without differentiability, but then it describes an almost-everywhere equivalence class.

PROPOSITION III.2.82 (Local integrability and the current criterion). *Let $n \in \mathbb{Z}_{\geq 1}$ and let $\Omega \subset \mathbb{C}^n$ be a domain. If $\varphi \in \mathrm{Psh}(\Omega)$ and $\varphi \not\equiv -\infty$, then $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ and $\sqrt{-1}\partial\overline{\partial}\varphi$ is a semi-positive $(1,1)$-current. Conversely, a real L^1_{loc} equivalence class u has a plurisubharmonic representative if and only if $\sqrt{-1}\partial\overline{\partial}u \geq 0$ as a current. This representative is unique.*

PROOF. Since spherical measure is invariant under $\xi \mapsto e^{\sqrt{-1}\theta}\xi$, averaging (171) over unit vectors ξ gives the spherical mean inequality. Integrating for $0 < \rho < r$ with the polar-coordinate weight $\rho^{2n-1}d\rho$ and normalizing gives the Euclidean ball inequality

$$\varphi(a) \leq \frac{1}{\lambda(B(a,r))} \int_{B(a,r)} \varphi \, d\lambda, \qquad \overline{B(a,r)} \subset \Omega,$$

where λ is Lebesgue measure on $\mathbb{C}^n$. The averaging is legitimate for extended integrals by applying Tonelli's theorem after subtracting a local upper bound. At a point where $\varphi(a) > -\infty$, this proves integrability on such balls. The propagation by overlapping balls in the proof of Lemma I.9.4 works in exactly the same way in $\mathbb{R}^{2n}$ and gives local integrability on Ω.

We continue by regularization, as in Mini-Exercise I.9.6. Choose a non-negative, radial $\rho \in C^\infty_c(\mathbb{C}^n)$ supported in the unit ball with $\int \rho \, d\lambda = 1$, and set $\rho_\varepsilon(z) = \varepsilon^{-2n}\rho(z/\varepsilon)$. On $\Omega_\varepsilon := \{z \in \Omega \mid \mathrm{dist}(z, \mathbb{C}^n \setminus \Omega) > \varepsilon\}$, the convolution $\varphi_\varepsilon = \varphi * \rho_\varepsilon$ is smooth and plurisubharmonic, by integrating the line submean inequalities for translates of φ. The one-variable monotonicity of circular means, averaged over complex directions, shows that φ_ε decreases as ε decreases. Upper semi-continuity and the submean inequalities give $\varphi_\varepsilon \searrow \varphi$ pointwise. Convolution also converges in L^1_{loc}. The smooth criterion and passage to the distributional limit therefore give $\sqrt{-1}\partial\overline{\partial}\varphi \geq 0$.

Conversely, suppose that $\sqrt{-1}\partial\overline{\partial}u \geq 0$. Convolution with ρ_ε preserves this positivity, so $u_\varepsilon = u * \rho_\varepsilon$ is smooth and plurisubharmonic on Ω_ε. Applying the preceding monotonicity to u_δ and then letting $\delta \searrow 0$ shows that u_ε decreases as $\varepsilon \searrow 0$. Its pointwise limit is plurisubharmonic by the decreasing-limit property, and represents u because $u_\varepsilon \to u$ in L^1_{loc}. Finally, the submean inequality and upper semi-continuity imply that a plurisubharmonic function is the limit of its small-ball averages at every point. These averages depend only on its L^1_{loc} class, proving uniqueness. □

PROPOSITION III.2.83. *Let $F\colon \Omega' \to \Omega$ be holomorphic between open subsets of finite-dimensional complex Euclidean spaces. If $\varphi \in \mathrm{Psh}(\Omega)$, then $\varphi \circ F \in \mathrm{Psh}(\Omega')$.*

PROOF. In the smooth case the chain rule expresses the complex Hessian of $\varphi \circ F$ as ${}^t\!A\, H(\varphi)(F)\, \overline{A}$, where $A = D^{\mathrm{hol}}F$ is the holomorphic Jacobian. This matrix is positive semi-definite; A need not be invertible or square. Locally apply this to the decreasing smooth regularizations above and pass to the limit. If the target component has $\varphi \equiv -\infty$, the assertion is immediate. □

Consequently plurisubharmonicity is invariant under holomorphic changes of coordinates. On a complex manifold X, we define $\mathrm{Psh}(X)$ by requiring plurisubharmonicity in every chart; one covering by charts suffices. At an isolated point the affine-line condition is vacuous. The preceding propositions apply locally, and $\varphi \in \mathrm{Psh}(X)$ is locally integrable if it is not identically $-\infty$ on any connected component. For a real-valued C^2 function φ, being pluriharmonic is equivalent to both φ and $-\varphi$ being plurisubharmonic. More generally, the same characterization holds without the C^2 assumption:

LEMMA III.2.84. *Let X be a complex manifold and let $\varphi\colon X \to \mathbb{R}$ be a function. Then φ is pluriharmonic if and only if both φ and $-\varphi$ are plurisubharmonic.*

PROOF. The forward implication follows from Proposition III.2.81. Conversely, upper semi-continuity of both signs makes φ continuous, hence locally bounded. In a coordinate neighborhood, use the radial regularizations ρ_ε from the proof of Proposition III.2.82. The submean inequalities for φ and $-\varphi$ give $\varphi * \rho_\varepsilon \geq \varphi$ and $-\varphi * \rho_\varepsilon \geq -\varphi$ on smaller neighborhoods. Thus $\varphi = \varphi * \rho_\varepsilon$ there, so φ is smooth. Applying Proposition III.2.81 to both signs now gives $\partial\overline{\partial}\varphi = 0$. On a zero-dimensional component the assertion is immediate. □

DEFINITION III.2.85 (Strict plurisubharmonicity). Let X be a complex manifold. A function $\varphi \in \mathrm{Psh}(X)$ is called *strictly plurisubharmonic* if, near each point, there is $\varepsilon > 0$ such that $\varphi - \varepsilon|z|^2$ is plurisubharmonic, where z denotes local holomorphic coordinates. □

Equivalently, for every real $\psi \in C_c^\infty(X)$, $\varphi + t\psi$ is plurisubharmonic for real t sufficiently close to 0. Indeed, local lower bounds by $\varepsilon|z|^2$ dominate the complex Hessian of $t\psi$ on its compact support. Conversely, use a smooth cutoff times $|z|^2$ and a negative t. This also proves coordinate independence. This is the complex counterpart of strict convexity in Chapter I, §4.2. For a real-valued C^2 function φ, strict plurisubharmonicity is equivalent to $H(\varphi) > 0$ at every point, or equivalently $\sqrt{-1}\partial\overline{\partial}\varphi > 0$. Indeed, the definition and the smooth criterion give $H(\varphi) \geq \varepsilon\mathbf{I}$ locally, where $\mathbf{I}$ denotes the identity matrix. Conversely, positivity at a point gives such a lower bound on a sufficiently small neighborhood by Mini-Exercise I.3.3; then $\varphi - \varepsilon|z|^2$ is plurisubharmonic. This is the same argument as in Proposition I.4.7 (*ii*). In particular, $|z|^2$ is strictly plurisubharmonic on $\mathbb{C}^n$.

We now apply the convex functions prepared in Chapter I, §4.2. Let $\chi\colon \mathbb{R}^r \to \mathbb{R}$ be convex and non-decreasing in each variable. If $\varphi_1, \varphi_2, \ldots, \varphi_r$ are finite-valued plurisubharmonic functions, then $\chi(\varphi_1, \varphi_2, \ldots, \varphi_r)$ is plurisubharmonic. The submean inequality and Jensen's inequality prove this on complex lines, first after replacing each φ_j by $\max\{\varphi_j, -M\}$ and

then by decreasing limits. For $r = 1$ we may also allow the value $-\infty$, putting $\chi(-\infty) = \lim_{t\to-\infty}\chi(t)$.

For a real-valued C^2 function φ with values in an open interval I and $\chi \in C^2(I)$, the chain rule gives

$$H(\chi\circ\varphi) = (\chi'\circ\varphi)H(\varphi) + (\chi''\circ\varphi)\left(\frac{\partial\varphi}{\partial z^j}\frac{\partial\varphi}{\partial\overline{z^k}}\right)_{j,k}.$$

The last matrix is positive semi-definite. Thus, if $\chi' > 0$ and $\chi'' \geq 0$, composition preserves C^2 strict plurisubharmonicity. In particular, this applies to a non-decreasing locally strictly convex C^2 function on I: Proposition I.4.7 (*ii*) gives $\chi'' > 0$, and then $\chi' > 0$ throughout the open interval. The functions in Example I.4.8 give concrete choices on their indicated intervals. Strict convexity of the outer function alone does not make an arbitrary plurisubharmonic function strictly plurisubharmonic; a constant inner function already shows this.

For several real-valued C^2 plurisubharmonic functions, put $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_r)$ and $A = (\partial\varphi_\ell/\partial z^j)_{\ell,j}$ in local coordinates. If χ is C^2, convex, and non-decreasing in each variable on a convex open neighborhood of the range of φ, then

$$H(\chi\circ\varphi) = \sum_{\ell=1}^{r}(\chi_\ell\circ\varphi)H(\varphi_\ell) + {}^tA\,\mathrm{Hess}_\varphi(\chi)\,\overline{A}, \qquad \chi_\ell := \frac{\partial\chi}{\partial t^\ell}.$$

The real Hessian is positive semi-definite by Proposition I.4.7 (*i*), and $\chi_\ell \geq 0$, so both terms are positive semi-definite. If all the φ_ℓ are strictly plurisubharmonic and $\sum_\ell \chi_\ell > 0$ on their range, the first term is positive definite. For example, both functions in Mini-Exercise I.4.10 preserve C^2 strict plurisubharmonicity under these compositions, since all their first partial derivatives are positive. So does the regularized maximum of Example I.4.9: differentiating its translation identity gives $\chi_1 + \chi_2 = 1$.

The second term can also supply strict positivity. On $\mathbb{C}^n$, take $\varphi_j = \mathrm{Re}\,z^j$. Then $H(\chi\circ\varphi) = \frac{1}{4}\mathrm{Hess}_\varphi(\chi)$. Thus the locally strictly convex function in Mini-Exercise I.4.10 gives the smooth strictly plurisubharmonic function $\log\bigl(1+\sum_{j=1}^n e^{\mathrm{Re}\,z^j}\bigr)$, even though each inner function is pluriharmonic.

More generally, let $\Omega \subset \mathbb{R}^n$ be convex and open, and let $v\colon \Omega \to \mathbb{R}$ be of class C^2. The function $u(z) := v(\mathrm{Re}\,z)$ on $\Omega + \sqrt{-1}\mathbb{R}^n$ satisfies $H(u) = \frac{1}{4}\mathrm{Hess}_{\mathrm{Re}\,z}(v)$. Thus u is plurisubharmonic exactly when v is convex, and is strictly plurisubharmonic exactly when v is locally strictly convex in the sense of Chapter I, §4.2. This follows from Proposition I.4.7.

Proposition III.2.86 (Upper envelopes on a manifold). *Let $\{\varphi_\alpha\}_{\alpha\in I}$ be a non-empty family in* $\mathrm{Psh}(X)$ *which is locally uniformly bounded above, where X is a complex manifold. Put $v = \sup_\alpha \varphi_\alpha$. Then its upper semi-continuous regularization*

$$v^*(x) := \inf_{U\in\mathcal{N}_X(\{x\})}\sup_{y\in U} v(y),$$

where U ranges over open neighborhoods of x, belongs to $\mathrm{Psh}(X)$.

PROOF. Work in a chart. For $v^*(a) > -\infty$, choose $a_\nu \to a$ and indices α_ν such that $\varphi_{\alpha_\nu}(a_\nu) \to v^*(a)$. Apply (171) on affine disks with centers a_ν and fixed radius and direction. Local uniform upper bounds and the reverse Fatou inequality give

$$\begin{aligned} v^*(a) &= \lim_{\nu\to\infty} \varphi_{\alpha_\nu}(a_\nu) \\ &\le \limsup_{\nu\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \varphi_{\alpha_\nu}(a_\nu + re^{\sqrt{-1}\theta}\xi)\, d\theta \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \limsup_{\nu\to\infty} \varphi_{\alpha_\nu}(a_\nu + re^{\sqrt{-1}\theta}\xi)\, d\theta \\ &\le \frac{1}{2\pi} \int_0^{2\pi} v^*(a + re^{\sqrt{-1}\theta}\xi)\, d\theta. \end{aligned}$$

If $v^*(a) = -\infty$, this inequality is automatic. Together with upper semi-continuity, it proves the assertion. □

We now give the definition on a complex space. For a possibly non-reduced complex space $(X, \mathcal{O}_X)$, the functions in question are defined on the underlying set X. We use its reduced structure for local embeddings; the original structure sheaf $\mathcal{O}_X$ is retained in the other constructions.

DEFINITION III.2.87 (Plurisubharmonic functions on a complex space). Let $(X, \mathcal{O}_X)$ be a complex space. A function $\varphi\colon X \to \mathbb{R} \cup \{-\infty\}$ is *plurisubharmonic* if, for each $a \in X$, there are an open neighborhood U of a, an embedding $j\colon U \hookrightarrow \Omega \subset \mathbb{C}^N$ of U with its reduced structure as a closed analytic subset of an open set Ω, and $\widetilde{\varphi} \in \mathrm{Psh}(\Omega)$ such that $\varphi|_U = \widetilde{\varphi} \circ j$. It is *strictly plurisubharmonic* if these extensions $\widetilde{\varphi}$ can be chosen strictly plurisubharmonic. A real-valued function on X is of class C^k if it locally has real-valued C^k extensions in such embeddings, $k \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$. When we say C^k *(strictly) plurisubharmonic*, we require local extensions having both properties simultaneously. □

We next give a criterion using holomorphic disks rather than local ambient extensions.

THEOREM III.2.88 (Fornæss–Narasimhan disk criterion). *Let $(X, \mathcal{O}_X)$ be a complex space and let $\varphi\colon X \to \mathbb{R} \cup \{-\infty\}$ be upper semi-continuous. Then φ is plurisubharmonic if and only if $\varphi \circ f$ is subharmonic for every holomorphic map $f\colon \Delta \to X$, where $\Delta = \{w \in \mathbb{C} \mid |w| < 1\}$.*

PROOF. For reduced spaces, see the original paper [FN]; see also [GPR, Chapter V, Theorem 1.5] for an exposition. For a non-reduced X, every map from the reduced disk factors through X_{red}, since nilpotent holomorphic functions pull back to zero. Thus the statement is unchanged under our convention. □

The definitions are independent of the chosen local embedding: after shrinking, the required extensions exist in any such embedding (See [Gr3, §1, assertion (3)] for the smooth setting and [GPR, Chapter V, Definition 1.4 and the following remarks] for the general plurisubharmonic setting). Richberg's extension theorem also gives local extensions that are simultaneously continuous and plurisubharmonic when the given function is continuous and

plurisubharmonic. For a closed complex subspace $(Y, \mathcal{O}_Y)$ of $(X, \mathcal{O}_X)$, a continuous strictly plurisubharmonic function $\varphi\colon Y \to \mathbb{R}$ extends to a continuous strictly plurisubharmonic function on one open neighborhood of the whole of Y in X; if φ is smooth strictly plurisubharmonic, the extension can be chosen smooth strictly plurisubharmonic (See [Ri] and [GPR, Chapter V, Theorem 1.10]).

Mini-Exercise III.2.89. Let $(X, \mathcal{O}_X)$ be a complex space and let $f_1, f_2, \ldots, f_r \in \mathcal{O}_X(X)$, with $r \geq 1$. Show that $\log \sum_{j=1}^r |f_j|^2$ is plurisubharmonic, with value $-\infty$ at the common zeros.

Lemma III.2.90 (Maximum principle). *Let $(X, \mathcal{O}_X)$ be a complex space and let $u \in \mathrm{Psh}(X)$. If u has a local maximum at $a \in X$, then u is constant on some open neighborhood of a.*

Proof. Put $M = u(a)$. If $M = -\infty$, the assertion is immediate. Otherwise, work on the reduction of an open neighborhood V of a on which $u \leq M$. By Remark III.2.72, choose a proper surjective holomorphic map $\pi\colon \widehat{V} \to V$ with $\widehat{V}$ non-singular. The pullback $u \circ \pi$ is plurisubharmonic by Theorem III.2.88. At each point of $\pi^{-1}(a)$ it attains the local maximum M, so the one-variable maximum principle on complex lines makes it constant near that point. Thus $u \circ \pi = M$ on an open set W containing $\pi^{-1}(a)$. Properness makes π closed (Mini-Exercise I.3.16 (i)), so $V_0 := V \setminus \pi(\widehat{V} \setminus W)$ is an open neighborhood of a, and surjectivity gives $u = M$ on V_0. See also [GPR, Chapter V, §1.1, property (5), p. 226]. □

Lemma III.2.91. *Let $(X, \mathcal{O}_X)$ be a connected compact complex space. Every plurisubharmonic function on X is constant, possibly equal to $-\infty$.*

Proof. Let $u \in \mathrm{Psh}(X)$. Upper semi-continuity and compactness give a maximum M. If $M = -\infty$, there is nothing to prove. Otherwise, the non-empty set $S := \{x \in X \mid u(x) = M\}$ is closed, since its complement $\{u < M\}$ is open. Every point of S is a local maximum, so Lemma III.2.90 shows that S is also open. Connectedness gives $S = X$. □

The author would appreciate information or references concerning the following question about pluriharmonic functions on complex spaces.

Question III.2.92. *Let $(X, \mathcal{O}_X)$ be a complex space and let $u\colon X \to \mathbb{R}$ be a function such that both u and $-u$ are plurisubharmonic. For every $a \in X$, does there exist a local embedding $j\colon U \hookrightarrow \Omega \subset \mathbb{C}^N$ of X_{red} near a and a real-valued pluriharmonic function h on Ω such that $u|_U = h \circ j$?* □

We next discuss sets on which plurisubharmonic functions can take the value $-\infty$, and removal of singularities across them.

Definition III.2.93 (Pluripolar sets). Let X be a complex manifold. A subset $E \subset X$ is called *pluripolar* if every point of X has a connected open neighborhood U and a function $\varphi \in \mathrm{Psh}(U)$, $\varphi \not\equiv -\infty$, such that $E \cap U \subset \{x \in U \mid \varphi(x) = -\infty\}$. It is *locally complete pluripolar* if the functions can be chosen so that equality holds. We call E *complete pluripolar in X* if $E = \{x \in X \mid \varphi(x) = -\infty\}$ for a single $\varphi \in \mathrm{Psh}(X)$ which is not identically $-\infty$ on any connected component of X. □

A pluripolar set has Lebesgue measure zero in every chart, by Proposition III.2.82 applied to its local defining functions.

An analytic subset A of a complex manifold which contains no connected component is locally complete pluripolar. Indeed, local defining equations $f_1, f_2, \dots, f_r$ give the function $\log \sum_j |f_j|^2$ of Mini-Exercise III.2.89, whose $-\infty$ set is exactly A locally.

LEMMA III.2.94. *Let X be a complex manifold and let $E \subset X$ be a closed pluripolar set. If $u \in \mathrm{Psh}(X \setminus E)$ is locally bounded above near every point of E, then it has a unique extension $\widetilde{u} \in \mathrm{Psh}(X)$. Its values on E are*

$$\widetilde{u}(a) = \limsup_{X \setminus E \ni x \to a} u(x), \qquad a \in E.$$

If $h\colon X \setminus E \to \mathbb{C}$ is pluriharmonic and locally bounded near E, then it has a unique pluriharmonic extension to X.

PROOF. Locally near E, choose a coordinate neighborhood U and $\varphi \in \mathrm{Psh}(U)$, $\varphi \not\equiv -\infty$, with $E \cap U \subset \{\varphi = -\infty\}$. After shrinking and subtracting a constant from φ, we may assume $\varphi \le 0$ and $u \le C$ on $U \setminus E$. For $\varepsilon > 0$, put

$$v_\varepsilon(x) := \begin{cases} u(x) + \varepsilon\varphi(x), & x \in U \setminus E, \\ -\infty, & x \in U \cap E. \end{cases}$$

This function is upper semi-continuous at points of E, since u is bounded above there and φ tends to $-\infty$. On every complex line its local submean inequalities hold away from E and are automatic at points of E. Thus $v_\varepsilon \in \mathrm{Psh}(U)$.

As $m \to \infty$, the functions $v_{1/m}$ increase and are bounded above by C. Proposition III.2.86 therefore gives a plurisubharmonic function $w := (\sup_{m \ge 1} v_{1/m})^*$. On $U \setminus E$ we have $w \le u$ by upper semi-continuity of u, and equality holds wherever φ is finite. The set $\{\varphi = -\infty\}$ has measure zero, so uniqueness of plurisubharmonic representatives gives $w = u$ throughout $U \setminus E$. On a component where $u \equiv -\infty$, this also follows directly from $w \le u$.

Any two local extensions agree almost everywhere, since E has measure zero, and hence agree everywhere by Proposition III.2.82; the identically $-\infty$ case is immediate as well. They therefore glue. The ball submean inequality, with E omitted from the integral, gives $\widetilde{u}(a) \le \sup_{B(a,r) \setminus E} u$ for every sufficiently small r. Letting $r \searrow 0$ and using upper semi-continuity proves the displayed formula.

For a real-valued pluriharmonic h, apply the first assertion to both h and $-h$. This uses the two-sided local bound on h. The resulting extensions H and K are locally bounded and satisfy $H + K = 0$ off E. Their sum is plurisubharmonic and zero almost everywhere, so Proposition III.2.82 gives $K = -H$ everywhere. Lemma III.2.84 therefore makes H pluriharmonic. Apply this to the real and imaginary parts for complex-valued h. Uniqueness follows from continuity and the density of $X \setminus E$. □

In particular, a continuous function on X which is plurisubharmonic, or pluriharmonic, off E has the same property on X.

The manifold hypothesis in Lemma III.2.94 cannot simply be dropped. On the crossing $A = \{zw = 0\} \subset \mathbb{C}^2$, the function $\log(|z|^2 + |w|^2)|_A$ is plurisubharmonic and has $-\infty$ set $\{0\}$. But the bounded function equal to 0 on $\{w = 0, z \neq 0\}$ and to 1 on $\{z = 0, w \neq 0\}$ has no plurisubharmonic extension to A. Upper semi-continuity would force its value at 0 to be at least 1, while the submean inequality on the disk $\zeta \mapsto (\zeta, 0)$ forces that value to be at most 0. This is a separate issue from the definition of plurisubharmonic functions on complex spaces.

Remark III.2.95. Lemma III.2.94 also shows that $X \setminus E$ is connected whenever X is a connected complex manifold and $E \subset X$ is a closed pluripolar set. Otherwise, the function equal to 0 on one connected component of $X \setminus E$ and to 1 on all the others is bounded and locally constant, hence pluriharmonic. By the lemma, it extends to a pluriharmonic function h on X. Since $X \setminus E$ is dense, continuity gives $h(X) = \{0, 1\}$, contrary to the connectedness of X. In particular, this applies when E is a proper analytic subset of X, giving another proof of the fact used before Theorem III.2.45. □

Integration currents and the Lelong–Poincaré formula

The logarithmic plurisubharmonic functions above also connect analytic subsets with currents. Recall from Example I.8.21 in Chapter I that integration over an oriented submanifold Y defines a current T_Y. For an analytic subset, the additional issue is integrability near its singular points.

Theorem III.2.96. *Let X be a complex manifold of pure dimension n, and let $A \subset X$ be a closed analytic subset of pure dimension p, where $0 \leq p \leq n$. Integration with the complex orientation,*

$$T_A(\eta) := \int_{A_{\mathrm{reg}}} \eta, \qquad \eta \in \mathcal{A}_c^{p,p}(X), \tag{172}$$

defines a d-closed semi-positive current $T_A \in \mathcal{K}^{n-p,n-p}(X)$. In particular, the integral is absolutely convergent even when the support of η meets A_{sing}. If A is compact, then T_A has compact support.

Proof. The case $p = 0$ is the locally finite sum of Dirac currents at the points of A. For $p > 0$, work near a point of A in a coordinate open set of X. By the local parametrization theorem (Theorem III.2.43), generic linear projections to $\mathbb{C}^p$ are finite on sufficiently small representatives of the local components of A. Choose linear coordinates for which all the finitely many coordinate projections $\pi_I(z) = (z^{i_1}, z^{i_2}, \ldots, z^{i_p})$, $|I| = p$, have this property, and shrink to a common neighborhood V. For each I, the number of sheets is bounded by an integer N_I, and $\pi_I(A \cap V)$ is contained in a bounded polydisk $B_I \subset \mathbb{C}^p$. With $\beta = \frac{\sqrt{-1}}{2} \sum_j dz^j \wedge d\overline{z^j}$, the Euclidean volume form on A_{reg} is the restriction of $\beta^p/p!$. Expanding this form and integrating each projection, counted with its sheets, gives

$$\int_{A_{\mathrm{reg}} \cap V} \frac{\beta^p}{p!} = \sum_{|I|=p} \int_{A_{\mathrm{reg}} \cap V} \pi_I^* dV_{\mathbb{C}^p} \leq \sum_{|I|=p} N_I \operatorname{Vol}(B_I) < \infty.$$

The change-of-variables formula is applied off the analytic branch loci, which have measure zero. Thus A_{reg} has locally finite volume, including near A_{sing}. On a fixed compact set K, this yields $|T_A(\eta)| \le C_K \|\eta\|_{C^0(K)}$. As in Chapter I, this proves continuity on compactly supported test forms and shows that the coefficients are measures. Forms of other bidegrees restrict to zero in top degree, giving the asserted bidegree. Semi-positivity follows because every test form $\bigwedge_{j=1}^{p}(\sqrt{-1}a_j \wedge \overline{a_j})$ restricts non-negatively to the complex tangent spaces of A_{reg}: its top-degree coefficient is a squared absolute value of a determinant.

Off A_{sing}, closedness follows from Stokes' theorem (Remark I.8.22). Closedness across the singular set is Lelong's result; it also follows from the extension theorem of Skoda, El Mir, and Sibony in [De3, §1.C, Theorem (1.18), pp. 10–12]. Indeed, A_{sing} is locally complete pluripolar, the coefficient measures just constructed have locally finite mass near it, and (172) is exactly the extension by zero of those measures from $X \setminus A_{\mathrm{sing}}$. □

Remark III.2.97. In the setting of Theorem III.2.96, we also write $\int_A \eta := \int_{A_{\mathrm{reg}}} \eta$ for $\eta \in \mathcal{A}^{p,p}(X)$ when $\operatorname{Supp}\eta \cap A$ is compact. This includes the cases where A is compact or η has compact support. The integral is absolutely convergent. It equals $T_A(\chi\eta)$, where $\chi \in C_c^\infty(X)$ is one near $\operatorname{Supp}\eta \cap A$, and is independent of χ. In particular, for compactly supported η, this is simply $T_A(\eta)$. □

For an analytic set with components of different dimensions, integration is performed separately in each dimension, rather than assigning the whole set a single bidegree.

Remark III.2.98 (Divisor bundles and integration currents). Let $D = \sum_\nu m_\nu H_\nu$ be a divisor on a complex manifold X. We write $T_D := \sum_\nu m_\nu T_{H_\nu}$ for its integration current. The sum is locally finite and is semi-positive when D is effective. Some references denote this current by $[D]$. In these notes, however, $[D]$ denotes the holomorphic line bundle associated with D; we always use T_D, not $[D]$, for the integration current. □

Theorem III.2.99 (Lelong–Poincaré formula). *Let X be a complex manifold and let $f \in \mathcal{O}_X(X)$ be not identically zero on any connected component. Then*

$$dd^c \log|f|^2 = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|f|^2 = T_{\mathrm{div}(f)}. \tag{173}$$

Proof. The function $\log|f|^2$ is plurisubharmonic by Mini-Exercise III.2.89 and locally integrable by Proposition III.2.82. The same proposition shows that the left-hand side is a d-closed semi-positive current. Outside the zeros of f, it is zero because $\log|f|^2$ is locally the real part of a holomorphic function. At a non-singular point of the reduced zero set, choose a transverse coordinate t and write $f = t^m g$, where g is holomorphic and nowhere zero and m is the order along the hypersurface. The one-variable formula (58) in Chapter I, followed by integration in the remaining coordinates, gives $dd^c \log|f|^2 = mT_{\{t=0\}}$ there.

The two sides have measure coefficients: this follows for $T_{\mathrm{div}(f)}$ from the construction in Theorem III.2.96, and for the left-hand side from an elementary positivity argument (See [De4, Chapter III, Proposition (1.14)]). Their difference S is therefore a d-closed current of degree two with measure coefficients, supported in the singular set E of the reduced zero set.

Every component of E has complex codimension at least two. Near a smooth point of E, choose real coordinates in which $E = \{x^1 = x^2 = \cdots = x^k = 0\}$, with $k \geq 4$. The support condition and the measure coefficients give $x^\ell S = 0$ for $1 \leq \ell \leq k$, and $dS = 0$ then gives $dx^\ell \wedge S = 0$. Write $S = \sum_{a<b} S_{ab}\, dx^a \wedge dx^b$, with measure coefficients S_{ab}. For each $a < b$, choose $\ell \leq k$ distinct from both a and b. The coefficient of $dx^\ell \wedge dx^a \wedge dx^b$ in $dx^\ell \wedge S = 0$ then gives $S_{ab} = 0$. Hence $S = 0$ there, and its support is contained in E_{sing}. Repeating on the successive reduced singular loci, as in the smooth decomposition on p. 430, proves $S = 0$ everywhere. □

Remark III.2.100 (Pullback and restriction through local potentials). Currents can be pushed forward by a smooth map which is proper on their support, using the pairing with pulled-back test forms from Chapter I. An arbitrary current, however, has no pullback under an arbitrary map. We will specify admissible operations when they are needed; the following case is particularly useful.

Let T be a d-closed semi-positive $(1,1)$-current on a complex manifold X, locally written as $T = \sqrt{-1}\partial\overline{\partial}u$ with u plurisubharmonic (such local potentials exist by Lemma III.2.118 in the next subsection). For a holomorphic map $F\colon Y \to X$ between complex manifolds, assume that these local compositions $u \circ F$ are in L^1_{loc}. Then

$$F^*T := \sqrt{-1}\partial\overline{\partial}(u \circ F)$$

defines a d-closed semi-positive $(1,1)$-current on Y. The condition means that no such composition is identically $-\infty$ on a connected component of its domain. Two local potentials for T differ by a smooth pluriharmonic function. Indeed, their L^1_{loc} difference has zero complex Hessian, so both signs have plurisubharmonic representatives by Proposition III.2.82. These are negatives of each other by uniqueness. Lemma III.2.84 then proves smoothness and pluriharmonicity. The holomorphic pullback of their difference is again pluriharmonic. Consequently the construction is independent of the potentials and glues on overlaps. It includes restriction to a complex submanifold when the same integrability condition holds.

There is also a concrete way to express restriction along a possibly singular closed analytic subset $A \subset X$ of pure dimension $p \geq 1$. Require $u|_{A_{\mathrm{reg}}}$ to be locally integrable with respect to its induced volume, including near A_{sing}: for each compact set K in a potential chart, $\int_{A_{\mathrm{reg}} \cap K} |u|\, dV_{A_{\mathrm{reg}}} < \infty$. The weighted current uT_A is then defined by integration, and

$$T \wedge T_A := \sqrt{-1}\partial\overline{\partial}(uT_A)$$

is a d-closed semi-positive $(n-p+1, n-p+1)$-current on X representing this restriction, where $n = \dim X$ on the chart. For locally bounded potentials, this is the Bedford–Taylor construction; see [De3, §2.A, Proposition (2.1)]. Under the present integrability assumption, every decreasing sequence of smooth plurisubharmonic functions $u_j \searrow u$ converges locally in L^1 for the induced volume on A_{reg}, including near A_{sing}. Since $dT_A = 0$, the semi-positive currents $\sqrt{-1}\partial\overline{\partial}u_j \wedge T_A$ converge to $\sqrt{-1}\partial\overline{\partial}(uT_A)$, independently of the sequence. Independence of potentials again follows from pluriharmonicity of their difference and $dT_A = 0$. On A_{reg} this is the usual restriction followed by integration. For singular A we keep

the stated integrability hypothesis; connectedness alone does not exclude a potential which is $-\infty$ on one branch: on the connected crossing $\{zw = 0\} \subset \mathbb{C}^2$, the plurisubharmonic function $\log|z|^2$ restricts to $-\infty$ on the whole branch $\{z = 0\}$. □

EXAMPLE III.2.101. On $\mathbb{C}^2$, with coordinates (z, w), let $A = \{z = 0\}$ and $B = \{w = 0\}$ be the w-axis and the z-axis, respectively. The Lelong–Poincaré formula (Theorem III.2.99) gives

$$T_A = dd^c \log|z|^2, \qquad T_B = dd^c \log|w|^2.$$

The restriction of $\log|z|^2$ to B is locally integrable, including at the origin. Thus Remark III.2.100 defines $T_A \wedge T_B = dd^c(\log|z|^2 T_B)$. For every $\psi \in C_c^\infty(\mathbb{C}^2)$, we have

$$(T_A \wedge T_B)(\psi) = \int_B \log|z|^2 \, dd^c(\psi|_B) = \psi(0,0),$$

where the last equality is the one-variable formula (58) in Chapter I, applied on $B \cong \mathbb{C}$. Consequently

$$T_A \wedge T_B = T_{\{(0,0)\}}, \qquad \int_{\mathbb{C}^2} T_A \wedge T_B = 1.$$

Here the integral is the total mass of the resulting compactly supported measure. Interchanging z and w gives the same product in the opposite order. Thus the transverse intersection of these two axes contributes one. □

Lelong numbers and the hypersurface part of a current

We measure a plurisubharmonic singularity by comparison with a logarithmic pole. This will also recover the orders of vanishing introduced in §2.5.

DEFINITION III.2.102 (Lelong numbers). Let X be a complex manifold of pure dimension $n \geq 1$. Let u be plurisubharmonic on a neighborhood of $x \in X$, and assume that it is not identically $-\infty$ on the component containing x. Its *Lelong number* is

$$\nu(u, x) := \sup\{\gamma \geq 0 \mid u(z) \leq \gamma \log|z - x|^2 + O(1) \text{ near } x\}, \tag{174}$$

where coordinates centered at x are used and $O(1)$ means a uniform upper bound on a punctured neighborhood. For a $(1,1)$-current T admitting local representations $T = \sqrt{-1}\partial\overline{\partial} v$ with v plurisubharmonic, define its Lelong number by $\nu(T, x) := 2\pi\nu(v, x)$. □

Although the preceding definition uses local plurisubharmonic potentials, it in fact applies to every d-closed semi-positive $(1,1)$-current: such potentials always exist, as will be proved in Lemma III.2.118.

Changing coordinates changes the logarithm by a bounded function; adding a bounded function to a potential does not change the supremum. Thus the definitions are independent of these choices, since two potentials for T differ by a smooth pluriharmonic function. Let X be a complex manifold of pure dimension $n \geq 1$, let $x \in X$, and let u be plurisubharmonic

near x, not identically $-\infty$ on the component containing x. For a holomorphic function f near x with non-zero germ $\underline{f}_x$, our normalization of d^c gives

$$\nu(dd^c u, x) = \nu(u, x), \qquad \nu(T_{\mathrm{div}(f)}, x) = \mathrm{ord}_x(f).$$

In particular, a reduced hypersurface has generic Lelong number one, but its Lelong number at a singular point can be larger.

The following theorem on Lelong numbers plays an important role in applications of pluripotential theory to complex geometry.

THEOREM III.2.103 (Siu decomposition). *Let T be a d-closed semi-positive $(1,1)$-current on a complex manifold X of pure dimension $n \geq 1$. For every $c > 0$, the set $E_c(T) := \{x \in X \mid \nu(T, x) \geq c\}$ is a closed analytic subset of codimension at least one. There is a unique decomposition*

$$T = \sum_j \lambda_j T_{H_j} + R, \tag{175}$$

where the H_j are distinct irreducible hypersurfaces, $\lambda_j > 0$, the at most countable sum converges in the weak topology of currents, and R is d-closed and semi-positive with $\dim E_c(R) \leq n - 2$ for every $c > 0$. The hypersurfaces are precisely the codimension-one components of the sets $E_c(T)$, and $\lambda_j = \inf_{x \in H_j} \nu(T, x)$ is the generic Lelong number along H_j*: it equals $\nu(T, x)$ outside a countable union of proper analytic subsets of H_j. The expression (175) is called the* Siu decomposition *of T.*

PROOF. See [De3, Theorem (2.10), Lemma (2.17), and Theorem (2.18)]. The results there hold more generally for closed semi-positive currents of bidegree (q, q), with integration over codimension q analytic subsets and residual upper-level sets of codimension greater than q. □

REMARK III.2.104. Let X be a complex manifold of pure dimension $n \geq 1$, let $x \in X$, and let $f_1, f_2, \ldots, f_r$, $r \geq 1$, be holomorphic functions on an open neighborhood of x, not all with zero germ at x. Put $u = \log \sum_j |f_j|^2$ and $m = \min_j \mathrm{ord}_x(f_j)$. Then

$$\nu(u, x) = \nu(dd^c u, x) = m.$$

Indeed, in local coordinates near x, Taylor expansion gives $\sum_j |f_j(z)|^2 \leq C|z - x|^{2m}$ for some constant $C > 0$ on a sufficiently small neighborhood of x. Choose a direction on which a non-zero degree-m homogeneous term does not vanish. Along that line, $u(x + t\xi) = m \log |t|^2 + O(1)$, excluding every larger logarithmic coefficient. Along an irreducible hypersurface H, the generic number of $dd^c u$ is likewise $\min_j \mathrm{ord}_H(f_j)$. Thus the rate of divergence to $-\infty$ is analytic data associated with the ideal $(f_1, f_2, \ldots, f_r)$, and its logarithmic coefficients recover vanishing multiplicities. Changing generators changes u by a locally bounded function off the common zero set, since each generating system is a holomorphic matrix times the other and both matrices are bounded on smaller neighborhoods; this does not assert that distinct ideals always give distinct singularities modulo bounded terms.

The residual current in a Siu decomposition need not be smooth: on $\mathbb{C}^2$, $dd^c \log(|z|^2+|w|^2)$ has Lelong number one at 0 and zero at every other point, but has no hypersurface term to subtract.

This separation of hypersurface contributions from the remaining analytic singularities is an important tool in analytic studies of the divisor decompositions known as *Zariski decompositions*. Related directions include Tsuji's *analytic Zariski decompositions* [Ts2, §2] and Boucksom's *divisorial Zariski decompositions* [Bouc, §1 and §2.2.1]. The Siu decomposition concerns a specified current; relating it to a decomposition of a divisor or its class involves the choice of that current. □

Finally, on a complex manifold the smooth local weights of a Hermitian line bundle in (149) are plurisubharmonic exactly when its curvature $\sqrt{-1}\Theta_h$ is semi-positive, and strictly plurisubharmonic exactly when that curvature is positive. A change of holomorphic frame changes the weight by $-\log|g_{jk}|^2$, where g_{jk} is the holomorphic transition function between the two frames. This difference is pluriharmonic. This is the connection between the function-theoretic definition and the previously introduced curvature convention.

2.8 The $\overline{\partial}$-equation and L^2 methods

Integral kernels are useful for local solvability of the $\overline{\partial}$-equation. In Exercise II.1 (i)–(ii), the Cauchy kernel gives a solution for compactly supported data in one variable, and a cutoff then gives local solutions on plane domains. Higher-dimensional counterparts are provided by the Bochner–Martinelli kernel and the Koppelman formula; see [De4, Chapter I, §3.D, especially (3.24)–(3.27)].

On a complex manifold, however, local solutions need not yield a global primitive. Their compatibility can be studied using Čech cohomology. Another description interprets local solvability as exactness of sheaves: the fine sheaves of C^∞ forms and currents then give resolutions, and the cohomology of their global sections records the obstruction to global solvability. This is the passage already made in Exercise II.1 (ii)–(iii) in one variable. Below, kernel formulas supply local exactness, while L^2 estimates and the functional analysis of Chapter I provide global solutions under suitable hypotheses and describe the obstruction when a solution need not exist.

The basic L^2 existence theory is due to Hörmander and Andreotti–Vesentini, following fundamental work of Kohn; the complete Kähler and singular-weight formulations used below are developed in Demailly's work. See [De3, §5.A] and [De1, §§4–5]. For a systematic treatment and applications to Oka–Cartan theory, see also [Oh5, Chapters 2–3].

Kähler forms and Nakano's identity

Recall from Remark III.2.6 that an Hermitian metric on a complex manifold is a *Kähler metric* if its fundamental form ω is d-closed. A complex manifold equipped with such a metric is called a *Kähler manifold*, and ω is its *Kähler form*. We usually specify the form and write (X,ω); the corresponding Hermitian and Riemannian metrics are understood through (140). Throughout the arguments involving (n,q)-forms, X has pure complex dimension n.

Its volume form is $dV_\omega = \omega^n/n!$, as computed from the normal form in Remark III.1.25. Completeness refers to the associated Riemannian distance, on each connected component.

Let (E,h) be a holomorphic vector bundle with a C^∞ Hermitian metric. The metrics h and ω induce a fiberwise Hermitian inner product on $\Lambda^{p,q}X \otimes E$, denoted by $\langle u,v\rangle_{h,\omega}$. For measurable sections u,v of finite L^2 norm, the corresponding L^2 inner product is

$$\langle\!\langle u,v\rangle\!\rangle_{h,\omega} := \int_X \langle u,v\rangle_{h,\omega}\, dV_\omega.$$

We denote the induced norm by $\|u\|_{h,\omega}$, so that $\|u\|^2_{h,\omega} = \langle\!\langle u,u\rangle\!\rangle_{h,\omega}$. The Hilbert space $L^{p,q}_{(2)}(X,E;h,\omega)$ consists of E-valued (p,q)-currents with measurable coefficients of finite L^2 norm; these coefficients are taken modulo equality almost everywhere, as in Chapter I, §8.6. Following Chapter I, §8.9, write $\overline\partial^\dagger_{h,\omega,\mathrm{formal}}$ and $\partial^\dagger_{\omega,\mathrm{formal}}$ for the formal adjoints on C^∞ forms with respect to this inner product. The latter is the formal adjoint of ∂_h; its differential expression is independent of h, as the next exercise shows. This does not assert that its L^2 domain is independent of h. The subscript "formal" distinguishes these expressions from Hilbert-space adjoints with specified domains. We use the notation

$$\Box_{h,\omega} = \overline\partial\overline\partial^\dagger_{h,\omega,\mathrm{formal}} + \overline\partial^\dagger_{h,\omega,\mathrm{formal}}\overline\partial, \qquad \Box'_{h,\omega} = \partial_h\partial^\dagger_{\omega,\mathrm{formal}} + \partial^\dagger_{\omega,\mathrm{formal}}\partial_h. \tag{176}$$

These are differential operators; their Hilbert-space domains will be specified when they enter an L^2 argument. Let $L_\omega u = \omega\wedge u$ and let $\Lambda_\omega = L^\dagger_\omega$ be its pointwise adjoint. For operators of total degrees a,b, the bracket means the graded commutator $[A,B] = AB - (-1)^{ab}BA$.

Mini-Exercise III.2.105. Use the complex-linear Hodge star $* = *_\omega$ of Chapter I, §8.9, extended to E-valued forms by the same pointwise convention $\{*u,v\}_h = \langle u,v\rangle_{h,\omega}dV_\omega$. It acts coefficientwise as the scalar star and is independent of h. Show that

$$\overline\partial^\dagger_{h,\omega,\mathrm{formal}} = *\partial_h *, \qquad \partial^\dagger_{\omega,\mathrm{formal}} = *\overline\partial *, \qquad \Lambda_\omega = *^{-1}L_\omega *\,.$$

Theorem III.2.106 (Nakano's identity). *On a Kähler manifold (X,ω), the operators on (E,h)-valued forms satisfy*

$$\Box_{h,\omega} = \Box'_{h,\omega} + [\sqrt{-1}\Theta_h\wedge\bullet,\Lambda_\omega]. \tag{177}$$

In particular, put $A^{p,q}_{h,\omega} := [\sqrt{-1}\Theta_h\wedge\bullet,\Lambda_\omega]$ on (p,q)-forms. For every $u\in\mathcal{A}^{p,q}_c(X,E)$,

$$\|\overline\partial u\|^2_{h,\omega} + \|\overline\partial^\dagger_{h,\omega,\mathrm{formal}}u\|^2_{h,\omega} \geq \int_X \langle A^{p,q}_{h,\omega}u,u\rangle_{h,\omega}\, dV_\omega. \tag{178}$$

Thus $\Box_{h,\omega}\geq A^{p,q}_{h,\omega}$ in this quadratic-form sense, not as a pointwise comparison of the values of differential operators.

Proof. We recall the computation underlying the identity, also called the Bochner–Kodaira–Nakano identity in [De3, §4.C, (4.5)–(4.7)]. For constant metrics and a constant unit frame on $\mathbb{C}^n$, with $\omega = \sqrt{-1}\sum_j dz^j\wedge d\overline{z^j}$, let e_j and $\overline e_j$ denote exterior multiplication by dz^j

and $d\overline{z^j}$, respectively, and let $\bar\iota_j = \bar e_j^\dagger$. The exterior-algebra relations give $[\bar\iota_j, L_\omega] = -\sqrt{-1}e_j$. Write $L = L_\omega$ and $\Lambda = L^\dagger$, and omit the metric subscripts from the Euclidean formal adjoints. Since $\overline{\partial}^\dagger = -\sum_j \bar\iota_j\, \partial/\partial z^j$, this gives the first identity below; taking adjoints and complex conjugates gives the other three:

$$\begin{aligned} [\overline{\partial}^\dagger, L] &= \sqrt{-1}\partial, \qquad [\Lambda, \overline{\partial}] = -\sqrt{-1}\partial^\dagger, \\ [\partial^\dagger, L] &= -\sqrt{-1}\overline{\partial}, \quad [\Lambda, \partial] = \sqrt{-1}\overline{\partial}^\dagger. \end{aligned}$$

For a variable Kähler metric, use Lemma III.1.9 and (129). At any prescribed point the metric coefficients are $\delta_{jk} + O(|z|^2)$ in suitable holomorphic coordinates. A holomorphic frame of E can likewise be chosen with metric matrix $\mathbf{I} + O(|z|^2)$ there, where $\mathbf{I}$ is the identity matrix: a constant change normalizes the value, and a holomorphic change linear in z cancels its first derivatives. The first-order operators therefore have the same expressions at that point as in the constant-metric calculation. We obtain

$$\begin{aligned} [\overline{\partial}^\dagger_{h,\omega,\mathrm{formal}}, L_\omega] &= \sqrt{-1}\partial_h, \qquad [\Lambda_\omega, \overline{\partial}] = -\sqrt{-1}\partial^\dagger_{\omega,\mathrm{formal}}, \\ [\partial^\dagger_{\omega,\mathrm{formal}}, L_\omega] &= -\sqrt{-1}\overline{\partial}, \quad [\Lambda_\omega, \partial_h] = \sqrt{-1}\overline{\partial}^\dagger_{h,\omega,\mathrm{formal}}. \end{aligned} \tag{179}$$

As the point was arbitrary, these are operator identities on X. The graded Jacobi identity and $[\partial_h, \overline{\partial}] = \Theta_h \wedge \bullet$ now give

$$\Box_{h,\omega} = -\sqrt{-1}[\overline{\partial}, [\Lambda_\omega, \partial_h]] = \Box'_{h,\omega} + [\sqrt{-1}\Theta_h \wedge \bullet, \Lambda_\omega].$$

Finally, integration by parts on compactly supported forms gives $\langle\langle \Box'_{h,\omega} u, u\rangle\rangle = \|\partial_h u\|^2 + \|\partial^\dagger_{\omega,\mathrm{formal}} u\|^2 \geq 0$, proving the inequality. □

In the terminology of Remark III.1.37, $\Box_{h,\omega}$ is local but not pointwise in its argument u: its value involves derivatives of u. Once the metrics are fixed, $A^{p,q}_{h,\omega}$ is instead a pointwise endomorphism of the form bundle, computed from the curvature and the metric at the point. The usefulness of (178) is that this fiberwise linear algebra gives a lower bound for the quadratic form of the differential operator after integration.

The normal form is used only for the first-order commutation relations; the second derivatives of the bundle metric remain in the curvature term of (177).

The complete-metric estimate

In the Hilbert spaces above, we use the maximal distributional realization of $\overline{\partial}$: its domain consists of those u for which $\overline{\partial}u$ is again L^2. This operator is closed and densely defined. Its Hilbert-space adjoint, in the sense of Definition I.6.17, is denoted by $\overline{\partial}^\dagger_{h,\omega}$. It agrees with the formal adjoint on compactly supported C^∞ forms, but the equality of their full domains is ensured by completeness (Recall also the incomplete-interval example, Example I.8.62).

Indeed, on a complete Hermitian manifold, Hopf–Rinow (Chapter I, §8.4) and the distance function give cutoffs χ_R with compact support, $\chi_R \to 1$, and $|d\chi_R| \leq C/R$. The commutators

of these cutoffs with $\overline{\partial}$ and its formal adjoint are multiplication or contraction by their first derivatives. Their L^2 errors tend to zero. After this cutoff, a partition of unity and local convolution approximate in the joint graph norm. For the variable coefficients of the formal adjoint, the required commutator estimate is Friedrichs' lemma; see [De4, Chapter VIII, Lemma (2.4), Theorem (3.2), and Lemma (3.3)]. This is the same density argument as in Theorem I.8.63, now for the Chern connection rather than only a unitary flat bundle. If there are several components, first restrict to finitely many of them and then apply the cutoffs.

Consequently, compactly supported C^∞ forms are dense in $\operatorname{Dom}\overline{\partial}\cap\operatorname{Dom}\overline{\partial}^\dagger_{h,\omega}$ for $\|u\|+\|\overline{\partial}u\|+\|\overline{\partial}^\dagger_{h,\omega}u\|$. The maximal distributional realization of the formal adjoint equals the Hilbert-space adjoint. The analogous assertion holds for ∂_h. In this complete, C^∞-metric setting we therefore omit "formal" and write $\overline{\partial}^\dagger_{h,\omega}$ and $\partial^\dagger_\omega$, retaining the dependence of their domains on h. On a complete Kähler manifold, the density and Fatou's lemma extend (178) to the joint domain whenever $A^{p,q}_{h,\omega}\ge 0$.

THEOREM III.2.107 (L^2 existence on a complete Kähler manifold). *Let (X,ω) be a complete Kähler manifold of pure dimension n, and let (E,h) be a holomorphic vector bundle with a C^∞ Hermitian metric. Fix $0\le p\le n$ and $1\le q\le n$. Assume that $A=A^{p,q}_{h,\omega}\ge c\mathbf{I}$ on X for a constant $c>0$, where $\mathbf{I}$ is the identity on the form bundle. For every $g\in L^{p,q}_{(2)}(X,E;h,\omega)$ with $\overline{\partial}g=0$, there is $f\in L^{p,q-1}_{(2)}(X,E;h,\omega)$ such that*

$$\overline{\partial}f=g,\qquad \|f\|^2_{h,\omega}\le c^{-1}\|g\|^2_{h,\omega}. \tag{180}$$

PROOF. Let H_1 be the L^2 space in degree $(p,q-1)$, and let H_2 be the closed kernel of $\overline{\partial}$ in degree (p,q), with the metrics (h,ω) fixed. Apply Theorem I.6.19 with $H_3=0$, $T=\overline{\partial}$, and $S=0$. The maximal realization $T\colon H_1\xrightarrow{\mathrm{op}}H_2$ is closed and densely defined. Its adjoint is the restriction of $\overline{\partial}^\dagger_{h,\omega}$ to H_2. The extended Nakano inequality gives $c\|v\|^2\le\|T^\dagger v\|^2$ for $v\in\operatorname{Dom}T^\dagger$. Thus $\operatorname{Ker}T^\dagger=0$, and the cited theorem gives $\operatorname{Ran}T=H_2$. The norm bound is the quantitative conclusion of its proof, (10), with $C_g=c^{-1/2}\|g\|$. □

REMARK III.2.108. The constant c may be replaced by a continuous function $c\colon X\to\mathbb{R}_{>0}$, possibly tending to zero at infinity, provided that g also satisfies $C_g^2:=\int_X|g|^2_{h,\omega}/c\,dV_\omega<\infty$. Then $\overline{\partial}f=g$ has a solution with $\|f\|^2_{h,\omega}\le C_g^2$. Indeed, weighted Cauchy–Schwarz and the extended Nakano inequality on $H_2=\operatorname{Ker}\overline{\partial}$ give

$$|\langle\!\langle v,g\rangle\!\rangle_{h,\omega}|\le C_g\|T^\dagger v\|_{h,\omega}\qquad(v\in\operatorname{Dom}T^\dagger). \tag{181}$$

One uses the range criterion (10) directly, rather than a uniform closed-range estimate. More precisely, for positive definite A, the same argument allows $C_g^2=\int_X\langle A^{-1}g,g\rangle_{h,\omega}dV_\omega$ whenever this integral is finite. See [De4, Chapter VIII, Theorem (4.5) and Remarks (4.6)–(4.8)] for this general estimate and the minimal-norm solution. □

Top holomorphic degree and changing the base metric

Let X be a complex manifold of pure dimension $n\ge1$. Recall that its canonical line bundle is $K_X=\bigwedge^n T^*_X$. For holomorphic coordinates $z_j=(z_j^1,z_j^2,\dots,z_j^n)$ on U_j, we abbreviate

its local frame by

$$dz_j^* := dz_j^1 \wedge dz_j^2 \wedge \cdots \wedge dz_j^n.$$

If η is an E-valued $(n,0)$-form, the scalar pairing of §2.3 gives

$$|\eta|^2_{h,\omega}\, dV_\omega = (\sqrt{-1})^{n^2}\{\eta,\eta\}_h. \tag{182}$$

The right-hand side is a non-negative top-degree form and is independent of ω. We therefore write the corresponding L^2 norm as $\|\eta\|_h$, omitting ω from the subscript. For an $(n,0)$-current η with locally L^2 coefficients, the same formula is an identity of locally integrable densities. In a line-bundle frame e_j with $|e_j|^2_h = e^{-\varphi_j}$ and $\eta = a_j dz_j^* \otimes e_j$, it is $(\sqrt{-1})^{n^2}|a_j|^2 e^{-\varphi_j} dz_j^* \wedge \overline{dz_j^*}$. Holomorphicity of η is not needed for this identity.

For a Hermitian line bundle (L,h), consider its Chern curvature $\sqrt{-1}\Theta_h$. By the Hermitian spectral theorem (2), at each point we can choose an ω-orthonormal coframe which diagonalizes $\sqrt{-1}\Theta_h$. Ordered with multiplicities as $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_n$, these eigenvalues are intrinsic and depend continuously on the point by Exercise I.2 (with its order reversed). The diagonalizing coframe is chosen only at the point. Write $u = dz^I \wedge d\overline{z^J} \otimes e$ in this coframe, where $\#I = p$, $\#J = q$, and $|e|_h = 1$ at the point. Exterior multiplication and contraction give

$$\langle A^{p,q}_{h,\omega} u, u\rangle_{h,\omega} = \left(\sum_{j\in I}\gamma_j + \sum_{j\in J}\gamma_j - \sum_{j=1}^n \gamma_j\right)|u|^2_{h,\omega}. \tag{183}$$

Mini-Exercise III.2.109. Let X be a complex manifold of pure dimension n, let ω be a C^∞ positive real $(1,1)$-form on X, and let (L,h) be a holomorphic line bundle with a C^∞ Hermitian metric. With the notation above, prove (183).

When $p = n$ only $\sum_{j\in J}\gamma_j$ remains. Thus, if $\sqrt{-1}\Theta_h > 0$,

$$A^{n,q}_{h,\omega} \ge (\gamma_1 + \cdots + \gamma_q)\mathbf{I}, \qquad (A^{n,q}_{h,\omega})^{-1} \le (\gamma_1 + \cdots + \gamma_q)^{-1}\mathbf{I}. \tag{184}$$

Here $\mathbf{I}$ denotes the identity on the fiber $(\Lambda^{n,q}X \otimes L)_x$ at the point x under consideration. This is the special feature of top holomorphic degree.

Lemma III.2.110 (Comparison of the L^2 densities). *Let $\omega' \ge \omega$ be Hermitian forms and keep h fixed. For every E-valued (n,q)-form u,*

$$|u|^2_{h,\omega'}\, dV_{\omega'} \le |u|^2_{h,\omega}\, dV_\omega. \tag{185}$$

For $q = 0$ there is equality. If $E = L$ is a line bundle with positive curvature $\sqrt{-1}\Theta_h$, then, for $q \ge 1$, the inverse-curvature density also decreases:

$$\langle (A^{n,q}_{h,\omega'})^{-1}u, u\rangle_{h,\omega'} dV_{\omega'} \le \langle (A^{n,q}_{h,\omega})^{-1}u, u\rangle_{h,\omega} dV_\omega. \tag{186}$$

For $q = 1$ these two inverse-curvature densities are equal.

PROOF. For the first assertion, diagonalize ω' relative to ω at the point, with eigenvalues $a_1, a_2, \dots, a_n \geq 1$. In an ω-unitary coframe and an h-unitary frame, write $u = \sum_{J,\lambda} u_{J,\lambda} dz^* \wedge d\overline{z^J} \otimes e_\lambda$. The determinant from the holomorphic top-degree factor cancels that from the volume. Hence the density for ω' is

$$\sum_{J,\lambda} \frac{|u_{J,\lambda}|^2}{\prod_{j\in J} a_j}\, dV_\omega,$$

whereas that for ω has no denominator. This proves (185) and the equality for $q = 0$. The inverse-curvature comparison is the linear-algebra statement of [De1, Lemma 3.2, (3.2)–(3.3)], applied to $\sqrt{-1}\Theta_h$; it is distinct from the norm comparison just proved. For $q = 1$ one can see the equality directly: if a is the column of the antiholomorphic coefficients, the inverse-curvature density is obtained by contracting a with the inverse of the Hermitian coefficient matrix of $\sqrt{-1}\Theta_h$, times the coordinate top-degree density. The base metric cancels altogether. □

THEOREM III.2.111 (L^2 estimate in top holomorphic degree). *Let X be a complex manifold of pure dimension n admitting a complete Kähler metric. Let ω be any Kähler form on X, not necessarily complete, and (L, h) a holomorphic line bundle with a C^∞ Hermitian metric such that $\sqrt{-1}\Theta_h > 0$. Let $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ be its curvature eigenvalues relative to ω. For $1 \leq q \leq n$ and a $\overline{\partial}$-closed $g \in L^{n,q}_{(2)}(X, L; h, \omega)$ with*

$$\int_X \frac{|g|^2_{h,\omega}}{\gamma_1 + \cdots + \gamma_q}\, dV_\omega < \infty,$$

there is $f \in L^{n,q-1}_{(2)}(X, L; h, \omega)$ satisfying

$$\overline{\partial} f = g, \qquad \|f\|^2_{h,\omega} \leq \int_X \frac{|g|^2_{h,\omega}}{\gamma_1 + \cdots + \gamma_q}\, dV_\omega.$$

PROOF. For complete ω, set $c = \gamma_1 + \cdots + \gamma_q$ and apply Remark III.2.108 with (184). In general, choose a complete Kähler form $\widehat{\omega}$ and put $\omega_\varepsilon = \omega + \varepsilon\widehat{\omega}$. This is complete for every $\varepsilon > 0$. Apply the sharper estimate of the same remark for ω_ε. Lemma III.2.110 bounds its inverse-curvature integral by the one for ω, which is at most the displayed gamma-sum integral by (184). This use of the inverse-curvature comparison keeps the bound uniform as the base metric changes.

For fixed $\delta > 0$ and $\varepsilon \leq \delta$, the norm-density comparison in degree $(n, q-1)$ bounds the solutions f_ε in the Hilbert space for ω_δ. Take weakly convergent subsequences, diagonally as $\delta \searrow 0$. Local equivalence of the C^∞ metrics identifies their limits as a single distributional solution f. Weak lower semi-continuity gives the common bound for each ω_δ, and monotone convergence of the norm density as $\delta \searrow 0$ gives the required estimate. □

Singular Hermitian metrics and weighted estimates

DEFINITION III.2.112. Let X be a complex manifold and let L be a holomorphic line bundle on X. A *singular Hermitian metric* on L is an almost-everywhere metric of the form

$$h = h_\infty e^{-\varphi}, \qquad \varphi \in L^1_{\mathrm{loc}}(X, \mathbb{R}),$$

where h_∞ is a C^∞ Hermitian metric. On an open covering $\{U_j\}$ of X with holomorphic frames e_j, write $|e_j|^2_{h_\infty} = e^{-\varphi_{\infty,j}}$. In these frames, the local weights of h are $\varphi_j = \varphi + \varphi_{\infty,j}$, so that $|e_j|^2_h = e^{-\varphi_j}$ almost everywhere and

$$\varphi_k = \varphi_j - \log|g_{jk}|^2 \quad \text{on } U_j \cap U_k, \qquad e_k = g_{jk} e_j.$$

Its curvature current is

$$\sqrt{-1}\Theta_h = \sqrt{-1}\Theta_{h_\infty} + \sqrt{-1}\partial\overline{\partial}\varphi = \sqrt{-1}\partial\overline{\partial}\varphi_j \quad \text{on } U_j.$$

If it is semi-positive, the local weights φ_j are taken to be their plurisubharmonic representatives by Proposition III.2.82. □

Changing the reference metric changes φ by a C^∞ function, so this definition does not depend on that choice. The global function φ need not itself be plurisubharmonic; its complex Hessian must be combined with the reference curvature. The metric is positive and finite almost everywhere, though a local plurisubharmonic weight may take the value $-\infty$. Norms and integrability refer to the almost-everywhere metric. Our weight convention remains $e^{-\varphi_j}$.

In a coordinate ball, convolution gives decreasing C^∞ regularizations of a plurisubharmonic weight on smaller balls, and the corresponding metrics increase to h. If the complex Hessian dominates a constant positive form, convolution preserves that bound. On a manifold, however, independent convolutions in different frames need not satisfy the transition rule above. The global construction must control both the gluing and the curvature error. Methods for doing so are developed in [De4, Chapter VIII] and [De1, §4 and Theorem 9.1].

For an open set $U \subset X$, we specify the set over which we integrate by writing

$$\langle\!\langle a, b \rangle\!\rangle_{h,\omega;U} := \int_U \langle a, b\rangle_{h,\omega}\, dV_\omega, \qquad \|a\|^2_{h,\omega;U} := \langle\!\langle a, a\rangle\!\rangle_{h,\omega;U}.$$

For $(n,0)$-forms we write $\langle\!\langle a, b\rangle\!\rangle_{h;U}$ and $\|a\|_{h;U}$, since the integrated pairing is independent of the base metric by (182) and polarization. Pointwise pairings retain both metric subscripts.

THEOREM III.2.113 (Hörmander estimate with a singular metric). *Let (X,ω) be a Kähler manifold of pure dimension n admitting a C^∞ plurisubharmonic exhaustion φ (i.e., $\varphi\colon X \to \mathbb{R}$ is C^∞ and plurisubharmonic, and $\{x \in X \mid \varphi(x) < c\}$ has compact closure in X for every $c \in \mathbb{R}$). Let L be a holomorphic line bundle on X and let h be a singular Hermitian metric on L such that*

$$\sqrt{-1}\Theta_h \geq \varepsilon\omega$$

in the sense of currents, for a continuous function $\varepsilon\colon X \to \mathbb{R}_{>0}$. *For* $1 \le q \le n$ *and a* $\overline{\partial}$*-closed* L*-valued* (n,q)*-current* g *with measurable coefficients satisfying*

$$\int_X |g|^2_{h,\omega} dV_\omega < \infty, \qquad \int_X \frac{|g|^2_{h,\omega}}{q\varepsilon}\, dV_\omega < \infty,$$

there is an L*-valued* $(n, q-1)$*-current* f *satisfying*

$$\overline{\partial} f = g, \qquad \int_X |f|^2_{h,\omega}\, dV_\omega \le \int_X \frac{|g|^2_{h,\omega}}{q\varepsilon}\, dV_\omega. \tag{187}$$

Proof. For the general statement, see [De1, Theorem 5.1, pp. 475–480, and §4]. Its inverse-curvature bound implies the stated estimate because the absolutely continuous part of $\sqrt{-1}\Theta_h$ is at least $\varepsilon\omega$, and its operator on (n,q)-forms is at least $q\varepsilon\mathbf{I}$ by (184), where $\mathbf{I}$ is the identity on $(\Lambda^{n,q}X \otimes L)_x$. We explain the case $q = 1$, first on sublevel sets and then on X. Put

$$C_g := \int_X \frac{|g|^2_{h,\omega}}{\varepsilon}\, dV_\omega.$$

If $C_g = 0$, take $f = 0$. Henceforth assume $C_g > 0$.

Write $h = h_\infty e^{-\psi}$, using ψ for the weight to distinguish it from the exhaustion φ. Apply [De1, Theorem 9.1, pp. 504–505] to ψ with the Kähler form ω and lower bound $\varepsilon\omega - \sqrt{-1}\Theta_{h_\infty}$. It gives C^∞ metrics $h_\nu \nearrow h$ and continuous functions $0 \le \lambda_\nu \le \lambda$, with λ continuous on X, such that

$$\sqrt{-1}\Theta_{h_\nu} \ge (\varepsilon - \lambda_\nu)\omega, \qquad \lambda_\nu \longrightarrow 0 \quad \text{almost everywhere.}$$

This convergence need not be uniform on compact sets, so the approximating curvatures need not be positive.

Adding a constant, assume $\varphi \ge 1$. For integers $m \ge 2$, put $\Omega_m := \{x \in X \mid \varphi(x) < m\}$, omitting empty sublevels. The calculation in Step 1, with φ in place of φ_m, shows that $\omega + \sqrt{-1}\partial\overline{\partial}\varphi^2$ is a complete Kähler form on X, so the cited general theorem applies under the present hypothesis.

We proceed in two steps.

Step 1. For each m, we show that there exists an L-valued $(n,0)$-current f_m on Ω_m satisfying

$$\overline{\partial} f_m = g|_{\Omega_m}, \qquad \|f_m\|^2_{h;\Omega_m} \le C_g.$$

Step 2. Choosing each f_m to have minimum $\|\cdot\|_{h;\Omega_m}$-norm among the solutions in Step 1, we show that the f_m converge in L^2 on each fixed sublevel to an L-valued $(n,0)$-current f on X satisfying (187).

Step 1. Fix m throughout this step. Set

$$\varphi_m := -\log(1 - \varphi/m), \qquad \omega_m := \omega + \sqrt{-1}\partial\overline{\partial}\varphi_m^2 \quad \text{on } \Omega_m.$$

The function φ_m is a positive C^∞ plurisubharmonic exhaustion of Ω_m. Moreover,

$$\omega_m = \omega + 2\varphi_m\sqrt{-1}\partial\overline{\partial}\varphi_m + 2\sqrt{-1}\partial\varphi_m \wedge \overline{\partial}\varphi_m \geq \omega.$$

The last term bounds $|d\varphi_m|_{\omega_m}$ by a constant. Thus φ_m is bounded on each distance ball; its exhaustion property and Hopf–Rinow show that ω_m is complete on each component.

We suppress restrictions of currents and metrics to Ω_m, but specify both metrics and the set of integration in all L^2 pairings, except that the base metric is omitted in type $(n,0)$ as above. Put

$$C_{g,m} := \int_{\Omega_m} \frac{|g|^2_{h,\omega}}{\varepsilon}\, dV_\omega \leq C_g, \qquad M_m := \sup_{\overline{\Omega_m}} \lambda < \infty.$$

In the Hilbert spaces $L^{n,j}_{(2)}(\Omega_m, L; h_\nu, \omega_m)$, let $T_{\nu,m}$ and $S_{\nu,m}$ be the maximal distributional $\overline{\partial}$ operators from degrees $j = 0$ to 1 and from $j = 1$ to 2, respectively (For $n = 1$, take $S_{\nu,m} = 0$). Their domains consist of the currents whose distributional $\overline{\partial}$ is in the next L^2 space. These operators are closed and densely defined. Let $P_{\nu,m}$ be the orthogonal projection onto $\operatorname{Ker} S_{\nu,m}$. Since $h_\nu \leq h$ and $\omega_m \geq \omega$, Lemma III.2.110 gives $g \in \operatorname{Ker} S_{\nu,m}$. For $u \in \operatorname{Dom} T^\dagger_{\nu,m}$, the inclusion $\operatorname{Image} T_{\nu,m} \subset \operatorname{Ker} S_{\nu,m}$ gives

$$P_{\nu,m}u \in \operatorname{Dom} T^\dagger_{\nu,m}, \qquad T^\dagger_{\nu,m}P_{\nu,m}u = T^\dagger_{\nu,m}u.$$

Indeed, pairing $T_{\nu,m}a$ with $P_{\nu,m}u$ gives the same functional of a as pairing it with u.

On $(n,1)$-forms set $A_m := [\varepsilon\omega \wedge \bullet, \Lambda_{\omega_m}] > 0$. The curvature bound and $\omega \leq \omega_m$ imply $A^{n,1}_{h_\nu,\omega_m} \geq A_m - \lambda_\nu \mathbf{I}$, where $\mathbf{I}$ is the identity on $(\Lambda^{n,1}X \otimes L)_x$. Consequently Nakano's inequality gives

$$\int_{\Omega_m} \langle A_m v, v\rangle_{h_\nu,\omega_m} dV_{\omega_m} \leq \|T^\dagger_{\nu,m}v\|^2_{h_\nu;\Omega_m} + \|S_{\nu,m}v\|^2_{h_\nu,\omega_m;\Omega_m} + \|\lambda_\nu^{1/2}v\|^2_{h_\nu,\omega_m;\Omega_m}$$

on the joint domain. To justify this use of the inequality, apply the complete-metric adjoint and joint graph-norm density statements preceding Theorem III.2.107. After moving the curvature error to the right, that term is continuous in L^2, since $\lambda_\nu \leq M_m$ on Ω_m; Fatou's lemma applies to the non-negative term on the left. Thus the already established density result extends the inequality from compactly supported forms without a new cutoff in this proof.

Apply the inverse-density comparison of Lemma III.2.110 to the positive form $\varepsilon\omega$, not to the possibly non-positive curvature of h_ν. Since $h_\nu \leq h$, it gives

$$\int_{\Omega_m} \langle A_m^{-1} g, g\rangle_{h_\nu,\omega_m} dV_{\omega_m} \leq C_{g,m}.$$

Weighted Cauchy–Schwarz and the preceding inequality, with $v = P_{\nu,m}u$, yield, for $u \in \operatorname{Dom} T^\dagger_{\nu,m}$,

$$|\langle\!\langle u, g\rangle\!\rangle_{h_\nu,\omega_m;\Omega_m}|^2 \leq C_{g,m}\big(\|T^\dagger_{\nu,m}u\|^2_{h_\nu;\Omega_m} + \|\lambda_\nu^{1/2}P_{\nu,m}u\|^2_{h_\nu,\omega_m;\Omega_m}\big).$$

As in Chapter I, (10), this defines a bounded linear functional on the image of

$$u \longmapsto (T^{\dagger}_{\nu,m}u, \lambda_\nu^{1/2} P_{\nu,m} u)$$

in the Hilbert direct sum of degrees $(n,0)$ and $(n,1)$. Hahn–Banach and the Riesz representation theorem give currents $u_{\nu,m}$ and $v_{\nu,m}$ in these degrees such that

$$g = \overline{\partial} u_{\nu,m} + e_{\nu,m}, \qquad e_{\nu,m} := P_{\nu,m}(\lambda_\nu^{1/2} v_{\nu,m}),$$
$$\|u_{\nu,m}\|^2_{h_\nu;\Omega_m} + \|v_{\nu,m}\|^2_{h_\nu,\omega_m;\Omega_m} \le C_{g,m}.$$

The adjoint of $u \mapsto \lambda_\nu^{1/2} P_{\nu,m} u$ is $v \mapsto P_{\nu,m}(\lambda_\nu^{1/2} v)$. The representation identity also gives $u_{\nu,m} \in \operatorname{Dom} T^{\dagger\dagger}_{\nu,m} = \operatorname{Dom} T_{\nu,m}$, so the displayed equation holds in distributions. At this stage there is still a curvature error.

We next let $\nu \to \infty$ with m fixed. Write $e_\nu = e_{\nu,m}$. Its norm satisfies $\|e_\nu\|^2_{h_\nu,\omega_m;\Omega_m} \le M_m C_{g,m}$. Set $r_\nu = (h_\nu/h_1)^{1/2}$ and $r = (h/h_1)^{1/2}$, so that $1 \le r_\nu \nearrow r < \infty$ almost everywhere. The sequence $r_\nu e_\nu$ is bounded in the fixed Hilbert space $L^{n,1}_{(2)}(\Omega_m, L; h_1, \omega_m)$. From any subsequence take a weakly convergent further subsequence, and write its limit as re. Then $e \in L^{n,1}_{(2)}(\Omega_m, L; h, \omega_m)$. Dominated convergence for r_ν^{-1} against compactly supported test forms shows that $e_\nu \to e$ in distributions along this subsequence. Each e_ν is $\overline{\partial}$-closed, hence so is e. Since $h_\nu \le h$, this implies $P_{\nu,m} e = e$ for every ν. Also $r_\nu e \to re$ strongly for (h_1, ω_m) on Ω_m. Thus

$$\|e\|^2_{h,\omega_m;\Omega_m} = \lim_\nu \langle\!\langle e, e_\nu \rangle\!\rangle_{h_\nu,\omega_m;\Omega_m}$$
$$= \lim_\nu \langle\!\langle e, \lambda_\nu^{1/2} v_{\nu,m} \rangle\!\rangle_{h_\nu,\omega_m;\Omega_m} = 0.$$

Indeed, the absolute value of the last pairing is at most

$$C_{g,m}^{1/2} \left(\int_{\Omega_m} \lambda_\nu |e|^2_{h,\omega_m} dV_{\omega_m} \right)^{1/2},$$

which tends to zero by dominated convergence, using $\lambda_\nu \le M_m$. Every such weak limit is zero, so $e_{\nu,m} \to 0$ as a current. No convergence of the projections $P_{\nu,m}$ is used (Compare [De1, pp. 479–480]).

For each fixed k, the bound for $u_{\nu,m}$ also holds with h_k in place of h_ν when $\nu \ge k$. Weak compactness and diagonal extraction in these fixed Hilbert spaces give a single current s_m with $\overline{\partial} s_m = g$ on Ω_m. The limits agree as currents because the smooth metric h_1 controls ordinary local L^2 norms. Weak lower semi-continuity gives $\|s_m\|^2_{h_k;\Omega_m} \le C_{g,m}$. Letting $k \to \infty$ and using $h_k \nearrow h$ and monotone convergence, we obtain $\|s_m\|^2_{h;\Omega_m} \le C_{g,m}$.

Step 2. In $L^{n,0}_{(2)}(\Omega_m, L; h, \omega)$, the solutions of $\overline{\partial} f = g$ form a non-empty closed affine subspace: convergence in this norm implies convergence in distributions locally, since $h \ge h_1$. Choose its minimum-norm element f_m by the Hilbert-space projection theorem recalled in Chapter I, §6.2. It is orthogonal to the closed subspace of $\overline{\partial}$-closed elements. Put $E_m := \|f_m\|^2_{h;\Omega_m} \le C_{g,m}$. For $j > m$, the difference $f_j|_{\Omega_m} - f_m$ is $\overline{\partial}$-closed; in local frames its coefficients are holomorphic by Theorem III.1.18 (iv). Orthogonality therefore gives

$$\|f_j|_{\Omega_m} - f_m\|^2_{h;\Omega_m} = \|f_j|_{\Omega_m}\|^2_{h;\Omega_m} - E_m$$
$$\le E_j - E_m, \qquad E_m \le E_j \le C_g.$$

The increasing sequence E_m is bounded, so this inequality makes $f_m|_{\Omega_k}$ a Cauchy sequence in L^2 for each fixed k. Their limits agree on overlaps and define an $(n,0)$-current f on X with $\overline{\partial} f = g$. For each k one has $\|f\|^2_{h;\Omega_k} \leq C_g$; monotone convergence as $k \to \infty$ gives $\|f\|^2_{h;X} \leq C_g$. This is (187) for $q = 1$.

The independence of the $(n,0)$-norm from the base metric is essential here: solutions constructed using ω_m can be compared on nested sublevels using the same metric h. For $q \geq 2$ the solution has type $(n, q-1)$, so the base metric must remain in its norm. To recover the bound for the prescribed ω, one can use the complete forms $\omega + \delta\omega_m$ on Ω_m and keep track of the estimates as $\delta \searrow 0$. These degrees are covered by the general result cited at the start (See also [De1, pp. 474–476]). □

Local exactness and Dolbeault cohomology

LEMMA III.2.114 (Dolbeault–Grothendieck lemma). *Let E be a holomorphic vector bundle on a complex manifold X. Let g be an E-valued (p,q)-current on an open subset $U \subset X$, where $q \geq 1$, and assume $\overline{\partial} g = 0$. For every $a \in U$, there are an open neighborhood $V \subset U$ of a and an E-valued $(p, q-1)$-current f on V such that $\overline{\partial} f = g|_V$. If g is a C^∞ form, f can be chosen as a C^∞ form.*

PROOF. In local holomorphic coordinates and a holomorphic frame of E, the assertion reduces coefficientwise to scalar $(0,q)$-currents. A standard proof uses the Bochner–Martinelli kernel and the Koppelman formula; see [De4, Chapter I, §3.D and Lemma (3.29)(b)] for both the C^∞ and current cases. For C^∞ forms in one variable with trivial coefficients, this is the local construction in Exercise II.1 (i)–(ii). □

Let X be a complex manifold and let E be a holomorphic vector bundle on X. Recall $\Omega^p_X(E) = \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$ from §2.2, and write $\mathcal{K}^{p,q}_X(E)$ for the sheaf of E-valued (p,q)-currents. In degree zero, each coefficient of a $(p,0)$-current annihilated by $\overline{\partial}$, in holomorphic coordinates and a holomorphic frame, is annihilated by the Euclidean Laplacian. Local elliptic regularity (Remark I.8.40) makes these coefficients smooth, and the Cauchy–Riemann equations then make them holomorphic. Together with the lemma, this gives the exact sequences (Dolbeault's lemma)

$$0 \longrightarrow \Omega^p_X(E) \longrightarrow \mathcal{A}^{p,0}_X(E) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1}_X(E) \xrightarrow{\overline{\partial}} \cdots,$$
$$0 \longrightarrow \Omega^p_X(E) \longrightarrow \mathcal{K}^{p,0}_X(E) \xrightarrow{\overline{\partial}} \mathcal{K}^{p,1}_X(E) \xrightarrow{\overline{\partial}} \cdots.$$

The sheaves of forms and currents are $\mathcal{C}^\infty_X$-modules, hence fine by Example II.3.64. They are therefore fine resolutions of $\Omega^p_X(E)$; for $p = 0$, the resolved sheaf is $\mathcal{O}_X(E)$.

DEFINITION III.2.115. For a holomorphic vector bundle E on a complex manifold X, its *Dolbeault cohomology* (also called *$\overline{\partial}$-cohomology*) and its compactly supported Dolbeault

cohomology are

$$H^{p,q}(X,E) := \frac{\operatorname{Ker}(\overline{\partial}\colon \mathcal{A}^{p,q}(X,E) \to \mathcal{A}^{p,q+1}(X,E))}{\operatorname{Image}(\overline{\partial}\colon \mathcal{A}^{p,q-1}(X,E) \to \mathcal{A}^{p,q}(X,E))},$$
$$H_c^{p,q}(X,E) := \frac{\operatorname{Ker}(\overline{\partial}\colon \mathcal{A}_c^{p,q}(X,E) \to \mathcal{A}_c^{p,q+1}(X,E))}{\operatorname{Image}(\overline{\partial}\colon \mathcal{A}_c^{p,q-1}(X,E) \to \mathcal{A}_c^{p,q}(X,E))}.$$

For $q = 0$ the denominators are zero. In particular, $H^{p,0}(X,E) = H^0(X,\Omega^p_X(E))$ and $H_c^{p,0}(X,E) = \Gamma_c(X,\Omega^p_X(E))$. For the trivial line bundle we write $H^{p,q}(X)$ and $H_c^{p,q}(X)$.

On $\mathcal{A}^{p,q}(X,E)$ use the Fréchet topology, and on $\mathcal{A}_c^{p,q}(X,E)$ the strict countable LF topology, as in Chapter I, §8.6. Each kernel has the subspace topology, and each cohomology group has the quotient topology. The images are not replaced by their closures, so these quotient spaces need not be Hausdorff. □

Theorem III.2.116 (Dolbeault isomorphism). *Let E be a holomorphic vector bundle on a complex manifold X. There are natural isomorphisms of complex vector spaces*

$$H^{p,q}(X,E) \cong H^q\big(\mathcal{K}^{p,\bullet}(X,E),\overline{\partial}\big) \cong H^q(X,\Omega^p_X(E)),$$
$$H_c^{p,q}(X,E) \cong H^q\big(\mathcal{K}_c^{p,\bullet}(X,E),\overline{\partial}\big) \cong H_c^q(X,\Omega^p_X(E)).$$

The comparisons with current cohomology are induced by the inclusion of C^∞ forms into currents.

Proof. Apply Propositions II.3.63 and II.3.49 to the two fine resolutions above. For compact supports use Proposition II.3.70: X is locally compact, paracompact, and Hausdorff, and each compact subset is contained in the interior of another compact subset. Inclusion is a map of the resolutions inducing the identity on $\Omega^p_X(E)$, which proves the asserted compatibility. For $p = 0$ and trivial coefficients on a plane domain, the first isomorphism with sheaf cohomology recovers Exercise II.1 (*iii*) in degree one. □

Example III.2.117 (Dolbeault cohomology of a ball). Let $B = \{z \in \mathbb{C}^n \mid |z| < R\}$, where $0 < R < \infty$. Then $H^{p,q}(B) = 0$ for $0 \le p \le n$ and $q \ge 1$. Unlike the local assertion of Lemma III.2.114, this statement concerns the whole ball, without shrinking it.

Put $s(z) = \sum_{j=1}^n |z^j|^2$ and $\omega_0 = \sqrt{-1}\sum_j dz^j \wedge d\overline{z^j}$. The ball admits a complete Kähler metric. Indeed, with $\rho = 1 - \log(1 - s/R^2)$, the form $\widehat{\omega} = \omega_0 + \sqrt{-1}\partial\overline{\partial}\rho^2$ bounds $|d\rho|_{\widehat{\omega}}$ by a constant. Since ρ tends to infinity at the boundary, its distance balls are relatively compact, and Hopf–Rinow (Chapter I, §8.4) gives completeness. The estimates below will still use ω_0.

Take $g \in \mathcal{A}^{p,q}(B)$ with $\overline{\partial} g = 0$ and $1 \le q \le n$. The weight must also control the growth of this particular g. For $t \ge R^{-2}$, set

$$M(t) := \max_{s(z) \le R^2 - 1/t} \log(1 + |g(z)|^2_{\omega_0}),$$

and extend M constantly to $0 \le t \le R^{-2}$. The maximum is finite and varies continuously with the radius, so M is continuous. By Mini-Exercise I.4.11 (*ii*), choose $\theta \in C^\infty(\mathbb{R}_{>0},\mathbb{R})$ with $\theta > M$, $\theta' > 0$, and $\theta'' > 0$. Set

$$\chi(s) := s + \theta\big((R^2 - s)^{-1}\big), \qquad \varphi(z) := \chi(s(z)), \qquad |1|^2_h := e^{-\varphi}$$

on the trivial line bundle. Then $|g|^2_{\omega_0} e^{-\varphi} \le 1$, so g has finite weighted L^2 norm. Also $\chi' \ge 1$, $\chi'' > 0$, and

$$\sqrt{-1}\Theta_h = \chi'(s)\omega_0 + \sqrt{-1}\chi''(s)\partial s \wedge \overline{\partial} s \ge \omega_0.$$

Thus h is a C^∞ metric on B, and the sum of its smallest q curvature eigenvalues relative to ω_0 is at least q.

Write $g = \sum_I a_I \wedge dz^I$ with scalar $(0,q)$-forms a_I. The finitely many $a_I \wedge dz^*$ all have finite weighted norms, and right multiplication by dz^* commutes with $\overline{\partial}$. Apply Theorem III.2.111 to these (n,q)-forms and restore the dz^I factors. We obtain a $(p, q-1)$-current f with weighted L^2 coefficients such that $\overline{\partial} f = g$ on B. On compact subsets the weight is bounded above and below, so f has ordinary locally L^2 coefficients. The class of g is therefore zero in the current complex. By the injectivity of the comparison in Theorem III.2.116, its class in $H^{p,q}(B)$ is zero as well. The case $q > n$ is immediate.

For $q = 1$, the solution f itself is C^∞ by Theorem I.8.37, applied to its coefficients. For $q \ge 2$, one may choose the coefficientwise solutions of minimal L^2 norm; their distributional co-closedness and local elliptic regularity likewise give C^∞ solutions, as in [De4, Chapter VIII, Remark (4.6)]. The cohomological argument above does not use this additional regularity of the chosen L^2 primitive. □

We can now justify the local potentials used in Remark III.2.100. The same argument also gives a global statement under a cohomological hypothesis, without assuming compactness or a Kähler metric.

LEMMA III.2.118 ($\partial\overline{\partial}$-lemma in bidegree $(1,1)$). *Let X be a complex manifold. Assume $H^1(X, \mathcal{O}_X) = 0$. Then every smooth complex-valued d-exact $(1,1)$-form α on X can be written as $\alpha = \partial\overline{\partial}\psi$ with $\psi \in C^\infty(X)$. The analogous assertion holds for d-exact $(1,1)$-currents, with ψ a distribution. If α is real, the expression may be written $\alpha = \sqrt{-1}\partial\overline{\partial} u$ with u real.*

In particular, on an arbitrary complex manifold, every real d-closed $(1,1)$-current has the form $\sqrt{-1}\partial\overline{\partial} u$ near each point. The local potential u can be chosen smooth if the current is smooth, and plurisubharmonic if it is semi-positive. Two plurisubharmonic potentials for the same current differ by a smooth real-valued pluriharmonic function on their common domain.

PROOF. Write $\alpha = d\beta$ and decompose the smooth 1-form as $\beta = \beta^{1,0} + \beta^{0,1}$. Comparing types gives

$$\partial\beta^{1,0} = 0, \qquad \overline{\partial}\beta^{0,1} = 0, \qquad \alpha = \overline{\partial}\beta^{1,0} + \partial\beta^{0,1}.$$

Both $\overline{\beta^{1,0}}$ and $\beta^{0,1}$ are $\overline{\partial}$-closed $(0,1)$-forms. By the hypothesis and the Dolbeault isomorphism (Theorem III.2.116), there are smooth functions f_1, f_2 with $\beta^{1,0} = \partial\overline{f_1}$ and $\beta^{0,1} = \overline{\partial} f_2$. Hence

$$\alpha = \partial\overline{\partial}(f_2 - \overline{f_1}).$$

The current Dolbeault resolution in the same theorem proves the identical assertion for currents. If α is real and $\alpha = \partial\overline{\partial}\psi$, complex conjugation gives $\alpha = -\partial\overline{\partial}\overline{\psi}$; averaging yields $\alpha = \sqrt{-1}\partial\overline{\partial}(\operatorname{Im}\psi)$.

For the local assertion, the Poincaré lemma for currents (Lemma I.8.52) first gives $T = dS$ near the point, with S real. Then $\overline{\partial} S^{0,1} = 0$, and Lemma III.2.114 gives $S^{0,1} = \overline{\partial} f$ after

shrinking the neighborhood. As $S^{1,0} = \partial\overline{f}$, we obtain $T = \sqrt{-1}\partial\overline{\partial}(2\,\mathrm{Im}\, f)$. Use the smooth Poincaré and Dolbeault lemmata when T is smooth.

If $T \geq 0$, the distribution u has semi-positive complex Hessian. The distributional criterion in [De4, Chapter I, Theorem (5.8)] gives its locally integrable plurisubharmonic representative. Finally, the difference of two potentials has vanishing complex Hessian, hence vanishing Laplacian in local coordinates. Local elliptic regularity makes it smooth; it is therefore pluriharmonic. □

THEOREM III.2.119 (Kodaira vanishing). *Let X be a compact complex manifold and L a holomorphic line bundle admitting a C^∞ Hermitian metric h with $\sqrt{-1}\Theta_h > 0$. Then*

$$H^q(X, \mathcal{O}_X(K_X \otimes L)) = 0 \qquad (q \geq 1),$$

where the canonical bundle is defined componentwise if the dimensions differ.

PROOF. The curvature $\omega = \sqrt{-1}\Theta_h$ is itself a Kähler form. On a component of dimension n, its curvature eigenvalues are all one, so $A^{n,q}_{h,\omega} = q\mathbf{I}$. Compactness makes the metric complete and all smooth data L^2. Theorem III.2.107 solves $\overline{\partial} f = g$ for each closed smooth L-valued (n,q)-form with $q \geq 1$. The L^2 solution is a current, so the current version of Theorem III.2.116 proves the claim. No additional regularity of this particular solution is needed. □

Harmonic representatives on compact complex manifolds

On a compact complex manifold, the elliptic decomposition prepared in Chapter I gives another way to represent Dolbeault classes and to prove vanishing. Here ω may be the fundamental form of any C^∞ Hermitian metric; it need not be Kähler. We keep the notation $\Box_{h,\omega}$ of (176). Compactness implies completeness, so the adjoints below are also Hilbert-space adjoints, by the domain identification above.

PROPOSITION III.2.120 (Hodge decomposition for the Dolbeault complex). *Let X be a compact complex manifold of pure dimension n, endowed with a C^∞ Hermitian metric whose fundamental form is ω. Let (E,h) be a holomorphic vector bundle with a C^∞ Hermitian metric. For $0 \leq p, q \leq n$, put*

$$\mathbb{H}^{p,q}_{h,\omega}(X,E) := \{u \in \mathcal{A}^{p,q}(X,E) \mid \Box_{h,\omega} u = 0\}.$$

This space is finite-dimensional, and there is an L^2-orthogonal decomposition of smooth forms

$$\mathcal{A}^{p,q}(X,E) = \mathbb{H}^{p,q}_{h,\omega}(X,E) \oplus \overline{\partial}\mathcal{A}^{p,q-1}(X,E) \oplus \overline{\partial}^\dagger_{h,\omega}\mathcal{A}^{p,q+1}(X,E),$$

where terms outside $0 \leq q \leq n$ are zero. Every Dolbeault class has a unique harmonic representative; the map $u \mapsto \{u\}$ gives

$$\mathbb{H}^{p,q}_{h,\omega}(X,E) \cong H^{p,q}(X,E) \cong H^q(X, \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)).$$

PROOF. The case $n = 0$ is immediate. For $n \geq 1$, the exterior multiplication and contraction calculation of Lemma I.8.60 applies to the symbol of $\overline{\partial}$ and gives

$$\sigma_{\Box_{h,\omega}}(x,\xi) = -\frac{1}{2}|\xi|^2_{g_\omega}\mathbf{I}$$

for real cotangent vectors ξ, where g_ω is the associated Riemannian metric and $\mathbf{I}$ is the identity on $(\Lambda^{p,q}X \otimes E)_x$. Thus $\Box_{h,\omega}$ is elliptic. This uses only the values of the metrics, not the Kähler condition. Integration by parts gives

$$\langle\langle \Box_{h,\omega}u, u\rangle\rangle_{h,\omega} = \|\overline{\partial}u\|^2_{h,\omega} + \|\overline{\partial}^\dagger_{h,\omega}u\|^2_{h,\omega},$$

so it is self-adjoint and its kernel is $\operatorname{Ker}\overline{\partial} \cap \operatorname{Ker}\overline{\partial}^\dagger_{h,\omega}$. Apply Theorems I.8.31 and I.8.32, with Remark I.8.35, to $P = \Box_{h,\omega}$ on $\Lambda^{p,q}X \otimes E$. The underlying real manifold has its complex orientation. We obtain finite-dimensionality and $\mathcal{A}^{p,q}(X,E) = \operatorname{Ker}\Box_{h,\omega} \oplus \operatorname{Image}\Box_{h,\omega}$ on smooth forms. Expanding $\Box_{h,\omega}$ gives the asserted three summands. They are pairwise orthogonal because $\overline{\partial}^2 = 0$ and by the kernel identity above, exactly as in Chapter I, §8.9. If u is $\overline{\partial}$-closed, its component $\overline{\partial}^\dagger_{h,\omega}b$ satisfies $\overline{\partial}\overline{\partial}^\dagger_{h,\omega}b = 0$, hence $\|\overline{\partial}^\dagger_{h,\omega}b\|^2 = 0$ by integration by parts. Thus u differs from its harmonic component by a $\overline{\partial}$-exact form. Orthogonality gives uniqueness, and Theorem III.2.116 gives the last isomorphism. □

This yields another proof of Kodaira vanishing (Theorem III.2.119). On each component take $\omega = \sqrt{-1}\Theta_h$ and represent a Dolbeault class by a harmonic L-valued (n,q)-form u. For $q \geq 1$, Nakano's inequality (178) and $A^{n,q}_{h,\omega} = q\mathbf{I}$ give

$$0 = \|\overline{\partial}u\|^2_{h,\omega} + \|\overline{\partial}^\dagger_{h,\omega}u\|^2_{h,\omega} \geq q\|u\|^2_{h,\omega}.$$

Hence $u = 0$. The Hodge decomposition itself only required a Hermitian metric; the Kähler condition is used here for Nakano's inequality.

Finiteness and duality on compact spaces

We record the compact consequences separately, also to compare harmonic representatives with a Čech argument. For manifolds, Example III.2.117 gives vanishing on a ball. The same weighted L^2 technique gives vanishing with locally free coefficients on finite intersections of coordinate balls, making such coverings Leray; the exhaustion and vanishing argument is given in Corollary III.2.156 below. For a finite locally free sheaf $\mathcal{F}$ on a reduced complex space, give $\mathcal{F}(U)$ the topology of uniform convergence of its holomorphic coefficients on compact sets in local frames. On a reduced complex space, too, a locally uniform limit of holomorphic functions is holomorphic. Applied to coefficients in local frames, this gives completeness and makes $\mathcal{F}(U)$ a Fréchet space. The topology is independent of the frames. On a finite covering, use the finite-product topology on Čech cochains and the kernel-quotient topology on their cohomology, as in Chapter II. Via a Leray covering this gives the *natural topology* on $H^q(X,\mathcal{F})$; the resulting topology is independent of the Leray covering and, on a

manifold, agrees with the Dolbeault topology. For these topological facts and compactness of restrictions to relatively compact open subsets, see [De4, Chapter IX, §4.C, especially Proposition (4.7)].

LEMMA III.2.121 (Compact finiteness and Serre duality). *Let X be a compact complex manifold and $\mathcal{F}$ a finite locally free $\mathcal{O}_X$-module.*

(*i*) *For every $q \geq 0$, $H^q(X, \mathcal{F})$ is a finite-dimensional Hausdorff TVS with the natural topology above.*

(*ii*) *If X has pure dimension n, then for $0 \leq q \leq n$ integration induces*

$$H^q(X, \mathcal{F}) \cong H^{n-q}\big(X, \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{F}^*\big)^\vee.$$

Here $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, and $\cong$ denotes the canonical TVS isomorphism induced by this pairing. For $\mathcal{F} = \mathcal{O}_X(E)$, it sends $\{\alpha\}$ to $\{\beta\} \mapsto \int_X \beta \wedge \alpha$, where $\alpha \in \mathcal{A}^{0,q}(X, E)$ and $\beta \in \mathcal{A}^{n,n-q}(X, E^*)$ are closed, and the coefficients are paired by $E^* \otimes E \to \mathbb{C}$.

PROOF. For (*i*), apply Proposition III.2.120 componentwise. The harmonic projection is continuous in the C^∞ topology, since it is a finite sum of L^2 pairings with fixed smooth forms. In each degree a closed form is exact precisely when its harmonic projection vanishes. Thus the image of $\overline{\partial}$ is closed, and the quotient is the finite-dimensional space of harmonic forms.

For a Čech proof, choose finite coverings $\mathcal{U} = \{U_j\}_{j \in J}$ and $\mathcal{U}^* = \{U_j^*\}_{j \in J}$ by concentric coordinate balls, with $\overline{U_j^*}$ compact in U_j. Such pairs exist by compactness, and both coverings are Leray for $\mathcal{F}$ by Corollary III.2.156. Leray's theorem (Theorem II.3.79) makes the restriction $\check{H}^q(\mathcal{U}, \mathcal{F}) \to \check{H}^q(\mathcal{U}^*, \mathcal{F})$ an isomorphism, both sides representing $H^q(X, \mathcal{F})$. Lemma III.2.122 below now proves (*i*) by Schwartz's theorem (Theorem I.6.56), without using harmonic forms.

For (*ii*), use Proposition III.2.58 to write $\mathcal{F} = \mathcal{O}_X(E)$. The Hodge star and the conjugate-linear metric dual give an antilinear map $u \mapsto \beta_u$ from E-valued $(0,q)$-forms to E^*-valued $(n, n-q)$-forms, normalized by $\beta_u \wedge v = \langle v, u \rangle_{h,\omega} dV_\omega$. The adjoint identities in Mini-Exercise III.2.105, applied also to the dual metric, show that this map carries harmonic forms bijectively to harmonic forms. Hence a non-zero harmonic u pairs with β_u to give $\|u\|^2_{h,\omega} > 0$. The same argument on the dual bundle proves non-degeneracy in the other variable. Finite-dimensionality makes the resulting isomorphism a TVS isomorphism by Theorem I.6.4. This is the harmonic proof of Serre duality; see also [De3, §4.B].

Alternatively, the Čech proof of (*i*) gives the Hausdorff conditions in every degree. Apply the general Serre duality statement in Theorem III.2.132 below with $p = 0$. Since X is compact, compact supports impose no restriction. Its duality map, together with the Dolbeault isomorphism, gives (*ii*) without harmonic integral theory. □

LEMMA III.2.122 (A compact shrinking criterion for Čech cohomology). *Let X be a reduced complex space and $\mathcal{F}$ a finite locally free $\mathcal{O}_X$-module. Let $\mathcal{U} = \{U_j\}_{j \in J}$ be a finite open covering of X, and let $\mathcal{U}^* = \{U_j^*\}_{j \in J}$ be a family of open subsets with $\overline{U_j^*}$ compact and*

contained in U_j for every j. The family $\mathcal{U}^*$ need not cover X; its Čech complex is formed on its union. Fix $q \geq 0$. If the natural restriction map

$$\check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^q(\mathcal{U}^*, \mathcal{F})$$

is surjective, then its target is finite-dimensional and Hausdorff for the kernel-quotient topology above.

PROOF. Write $U_I = \bigcap_{j \in I} U_j$ and $U_I^* = \bigcap_{j \in I} U_j^*$ for the finite intersections appearing in the Čech complex. Then $\overline{U_I^*}$ is compact in U_I. The restriction $\mathcal{F}(U_I) \to \mathcal{F}(U_I^*)$ is compact by the Montel restriction property recalled above. There are only finitely many factors in each degree, so restriction is a compact map on the degree-q cochain spaces, and is a continuous morphism of Fréchet complexes. Apply Mini-Exercise I.6.57 with $C_+^\bullet = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$, $C_-^\bullet = \check{C}^\bullet(\mathcal{U}^*, \mathcal{F})$ and $\flat, \natural, \sharp$ equal to degrees $q-1, q, q+1$. For $q = 0$, take the degree-(-1) spaces to be zero. □

The Kähler case has further consequences, especially for coefficients whose flat structure is compatible with the holomorphic structure. We make this compatibility explicit, also for the complex spaces used elsewhere in this note.

DEFINITION III.2.123 (Unitary flatness). Let $(X, \mathcal{O}_X)$ be a complex space and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $r \geq 1$. As in the discussion following Definition II.3.103, choose an open covering $\{U_j\}_{j \in J}$ of X and $\mathcal{O}_X|_{U_j}$-linear isomorphisms

$$\tau_j \colon \mathcal{E}|_{U_j} \xrightarrow{\sim} (\mathcal{O}_X|_{U_j})^{\oplus r}.$$

For every open subset $V \subset U_j$, this identifies sections of $\mathcal{E}$ on V with columns of r sections of $\mathcal{O}_X$ on V. On $U_j \cap U_k$, let $g_{jk} \in \mathrm{GL}_r(\mathcal{O}_X(U_j \cap U_k))$ be the matrix representing $\tau_j \circ \tau_k^{-1}$. We call $\mathcal{E}$ *unitary flat* if the covering and these isomorphisms can be chosen so that every g_{jk} is locally constant with values in $\mathrm{U}(r)$. For rank one these are locally constant scalars of absolute value one. Here locally constant complex functions are viewed as sections of $\mathcal{O}_X$ through the constant functions. In particular, when X is non-reduced, the transition matrices must equal these constant matrices in $\mathcal{O}_X$, not merely after passage to X_{red}. □

When X is reduced, this is a compatible unitary flat structure on the holomorphic vector bundle corresponding to $\mathcal{E}$ by Proposition III.2.58. It is the $\mathrm{U}(r)$-flat structure of Chapter I, §7.7, required to be given by frames of that holomorphic bundle. On a complex manifold, declaring these frames orthonormal defines a C^∞ Hermitian metric h, which we call the *flat metric* of this structure. The connection acting coefficientwise as d in these frames is metric-compatible and has $(0,1)$-part $\overline{\partial}$. By uniqueness of the Chern connection it is D_h, so $\Theta_h = 0$. Conversely, for a holomorphic line bundle on a complex manifold, a C^∞ Hermitian metric of zero curvature yields unitary flat frames: a local weight φ_j is pluriharmonic, so $\varphi_j = 2\,\mathrm{Re}\, f_j$ locally by Proposition III.2.77. Replacing e_j by $e^{f_j} e_j$ makes its squared norm one. The transition functions then have absolute value one and are locally constant. We fix such a compatible flat structure and metric in the next statements, and use $D_h = \partial_h + \overline{\partial}$.

PROPOSITION III.2.124 (Kähler identities with unitary flat coefficients). *Let (X, ω) be a compact Kähler manifold of pure dimension n and let E be a holomorphic vector bundle on X with a fixed compatible unitary flat structure and its flat metric h. The Hodge Laplacian Δ_{D_h} for the flat connection D_h satisfies*

$$\Delta_{D_h} = 2\Box_{h,\omega} = 2\Box'_{h,\omega}. \tag{188}$$

Here the Hodge Laplacian has the convention of Chapter I, §8.9. Consequently its harmonic space in total degree k decomposes as

$$\mathbb{H}^k_{h,\omega}(X, E) = \bigoplus_{p+q=k} \mathbb{H}^{p,q}_{h,\omega}(X, E).$$

For scalar-valued forms, complex conjugation sends the harmonic space of type (p, q) onto that of type (q, p). For E-valued forms it instead gives a conjugate-linear isomorphism with the corresponding harmonic space for the dual flat bundle E^*.

PROOF. Nakano's identity and $\Theta_h = 0$ give $\Box_{h,\omega} = \Box'_{h,\omega}$. The commutation relations (179) and $\partial_h^2 = \overline{\partial}^2 = 0$ also give

$$[\partial_h, \overline{\partial}^\dagger_{h,\omega}] = 0, \qquad [\overline{\partial}, \partial^\dagger_\omega] = 0.$$

For example, substitute $\overline{\partial}^\dagger_{h,\omega} = -\sqrt{-1}[\Lambda_\omega, \partial_h]$ and use the graded Jacobi identity and $\partial_h^2 = 0$. Expanding Δ_{D_h} now gives $\Delta_{D_h} = \Box'_{h,\omega} + \Box_{h,\omega}$, proving (188). Since $\Box_{h,\omega}$ preserves type, the decomposition of harmonic spaces follows from the type decomposition of forms.

For scalar forms, Δ commutes with complex conjugation, as it is defined by the real Riemannian metric. Thus conjugation preserves harmonicity and exchanges the two bidegrees. In unitary flat frames the same assertion applies coefficientwise. The conjugated transition matrices are $\overline{g_{jk}} = {}^t(g_{jk}^{-1})$, which are those of E^*. This identifies the conjugate flat bundle with E^* and proves the last assertion; it does not identify E^* with E. □

LEMMA III.2.125 ($\partial\overline{\partial}$-lemma on compact Kähler manifolds). *In the setting of Proposition III.2.124, let α be a smooth E-valued (p, q)-form with $D_h\alpha = 0$. The following conditions are equivalent: α is D_h-exact; it is ∂_h-exact; it is $\overline{\partial}$-exact; and $\alpha = \partial_h\overline{\partial}\beta$ for a smooth E-valued $(p-1, q-1)$-form β. A space of forms with a negative index is understood to be zero. In particular, for scalar-valued forms these are the usual d, ∂, $\overline{\partial}$, and $\partial\overline{\partial}$ exactness conditions. Every real d-exact $(1, 1)$-form is therefore $\sqrt{-1}\partial\overline{\partial}u$ for a smooth real-valued function u.*

PROOF. An exact form for any of D_h, ∂_h, or $\overline{\partial}$ is orthogonal to the common harmonic space, by integration by parts and (188). Suppose that α is orthogonal to that space. Since $\overline{\partial}\alpha = 0$, the Hodge decomposition gives $\alpha = \overline{\partial}v$. Apply the analogous decomposition for ∂_h to v:

$$v = v_0 + \partial_h a + \partial^\dagger_\omega b,$$

where v_0 is harmonic, hence also $\overline{\partial}$-closed. The mixed commutation relation in the preceding proof gives

$$\alpha = -\partial_h \overline{\partial} a - \partial_\omega^\dagger \overline{\partial} b.$$

Here α and the first term are ∂_h-closed. The second term is both in $\operatorname{Ker} \partial_h$ and in the orthogonal space $\operatorname{Image} \partial_\omega^\dagger$, and therefore vanishes. Thus $\alpha = \partial_h \overline{\partial}(-a)$. The decomposition for ∂_h follows from the same elliptic argument as Proposition III.2.120, since $\square'_{h,\omega} = \square_{h,\omega}$. Conversely, $\partial_h \overline{\partial}$-exactness implies all three other conditions. For the real scalar assertion, take the imaginary part of the potential as in Lemma III.2.118. □

Corollary III.2.126 (Hodge decomposition in cohomology). *Let X be a compact Kähler manifold and let E be a holomorphic vector bundle on X with a fixed compatible unitary flat structure. There is a canonical decomposition*

$$H^k_{\mathrm{DR}}(X, E) \cong \bigoplus_{p+q=k} H^{p,q}(X, E),$$

independent of the choice of Kähler metric. The left-hand side also computes $H^k(X, \mathbb{C}_X(E))$, where $\mathbb{C}_X(E)$ is the sheaf of local flat sections from Chapter II, Example II.3.83. There are conjugate-linear isomorphisms $H^{p,q}(X, E) \to H^{q,p}(X, E^*)$. For scalar coefficients, with the bundle omitted from the notation, this is the Hodge symmetry

$$\overline{H^{p,q}(X)} \cong H^{q,p}(X).$$

Proof. Combine the harmonic decomposition in Proposition III.2.124 with Chapter I, §8.9, and Proposition III.2.120. To see independence of the metric, represent a Dolbeault class by a D_h-closed form of pure type, which is possible by choosing its harmonic representative. If two such forms represent the same Dolbeault class, their difference is D_h-closed and $\overline{\partial}$-exact. Lemma III.2.125 makes it D_h-exact, so its de Rham class is independent of the choice. The same lemma proves injectivity. The images are exactly the de Rham classes admitting D_h-closed representatives of the specified types, a description involving no Kähler metric. The conjugation statement follows from the preceding proposition. The harmonic representatives themselves, unlike this decomposition of cohomology, do depend on the metric. □

For a compact complex manifold of pure dimension n, put $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$; these finite integers are called the *Hodge numbers*. Their arrangement in a diamond, with rows indexed by $p+q = 0, 1, \ldots, 2n$, is called the *Hodge diamond*. For compact Kähler manifolds, Corollary III.2.126 and the de Rham comparison with singular cohomology give

$$b_k(X) = \sum_{p+q=k} h^{p,q}(X), \qquad h^{p,q}(X) = h^{q,p}(X).$$

In particular, $b_1(X) = 2h^{0,1}(X)$ is even, and the same holds for every odd Betti number. If X is connected and $n \geq 1$, its Kähler class is non-zero: otherwise $\omega = d\eta$ would imply $\int_X \omega^{\wedge n} = \int_X d(\eta \wedge \omega^{\wedge(n-1)}) = 0$, contrary to $\omega^{\wedge n} = n!\, dV_\omega > 0$ (Remark III.1.25). Hence $H^{1,1}(X) \neq 0$. More generally the same argument shows $\{\omega^{\wedge j}\} \neq 0$ for $0 \leq j \leq n$.

Example III.2.127 (Compact Riemann surfaces). Let C be a compact Riemann surface of genus g. Here g is the genus of its underlying compact real surface, with the orientation given by the complex structure, as in the surface classification recalled in Chapter I, §8.4. Every Hermitian metric on C is Kähler, since its fundamental form is of top degree and is therefore closed. The calculation from the handle decomposition in Chapter I, (25), gives $b_0(C) = b_2(C) = 1$ and $b_1(C) = 2g$. Hodge decomposition and symmetry therefore give the diamond

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

In particular, the space of holomorphic 1-forms has dimension g. □

Example III.2.128 (Complex projective space). On $\mathbb{P}^n$ the Fubini–Study form is locally $\omega_{\mathrm{FS}} = dd^c \log(1 + \sum_{j=1}^n |z^j|^2)$. As in Example III.2.23, these expressions agree on overlaps; their complex Hessians are positive definite, so they define a Kähler form. The CW calculation in Chapter I, Mini-Exercise I.7.16 and (18), gives one-dimensional cohomology in each even degree $0, 2, \ldots, 2n$, and zero in odd degrees. Since $\{\omega_{\mathrm{FS}}^j\}$ is a non-zero class of type (j, j), Hodge decomposition yields

$$h^{p,q}(\mathbb{P}^n) = \begin{cases} 1 & \text{if } p = q \text{ and } 0 \le p \le n, \\ 0 & \text{otherwise.} \end{cases}$$

For instance the Hodge diamond of $\mathbb{P}^2$ is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

□

Example III.2.129 (A primary Hopf surface). Fix a real number a with $0 < a < 1$, and let $F(z, w) = (az, aw)$ on $\mathbb{C}^2 \setminus \{(0, 0)\}$. Its iterates act freely and properly discontinuously, so

$$X = (\mathbb{C}^2 \setminus \{(0, 0)\})/\langle F \rangle$$

is a complex surface, a *primary Hopf surface* with diagonal contraction. A closed fundamental region is the shell $\{a \le (|z|^2 + |w|^2)^{1/2} \le 1\}$, whose boundary spheres are identified by F. Indeed, the polar-coordinate diffeomorphism

$$S^3 \times \mathbb{R} \longrightarrow \mathbb{C}^2 \setminus \{(0, 0)\}, \qquad (v, t) \longmapsto e^t v,$$

identifies F with $(v, t) \mapsto (v, t + \log a)$. Thus X is compact and diffeomorphic to $S^3 \times S^1$. The sphere calculation (19) and the Künneth formula recalled in Chapter I, §7.5, give $b_1(X) = 1$. By Corollary III.2.126, X admits no Kähler metric. □

Unitary flat line bundles and the $\partial\overline{\partial}$-lemma

For a complex space $(X, \mathcal{O}_X)$, the inclusion of locally constant unit-modulus functions into $\mathcal{O}_X^*$ induces

$$I_X \colon H^1(X, \underline{\mathrm{U}(1)}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X).$$

Here $\mathrm{Pic}(X)$ is the group of isomorphism classes of invertible $\mathcal{O}_X$-modules. The image of I_X consists of the classes of unitary flat invertible $\mathcal{O}_X$-modules.

LEMMA III.2.130 (Uniqueness of the unitary flat structure). *If $(X, \mathcal{O}_X)$ is a compact complex space, the map I_X is injective. In particular, on a reduced compact complex space, a holomorphic line bundle admits at most one unitary flat isomorphism class of compatible flat structures.*

PROOF. First suppose that X is reduced. Let a locally constant $\mathrm{U}(1)$-valued cocycle g_{jk} define a holomorphically trivial line bundle. A nowhere vanishing holomorphic section has local coefficient functions s_j satisfying $s_j = g_{jk} s_k$. The functions $\log |s_j|^2$ therefore glue to a real-valued plurisubharmonic function on X. By Lemma III.2.91 it is constant on each connected component. A holomorphic function of constant modulus on a reduced complex space is locally constant: apply the one-variable maximum principle on the regular loci of its local irreducible components and then use continuity. Thus the s_j are locally constant. Dividing them by their common positive modulus on each component expresses g_{jk} as a coboundary of locally constant unit-modulus functions. This proves injectivity.

For non-reduced X, restrict a holomorphic trivialization to X_{red}. The underlying topological space and its constant sheaf are unchanged, so the reduced case already implies that the original class in $H^1(X, \underline{\mathrm{U}(1)}_X)$ is zero. For the reduced setting, see also [Koi13, §2.3, Lemma 2.3]. □

PROPOSITION III.2.131 (Unitary flatness from the $\partial\overline{\partial}$-lemma). *Let X be a complex manifold on which every d-exact C^∞ $(1,1)$-form is $\partial\overline{\partial}$ of a C^∞ function. A holomorphic line bundle L on X is unitary flat if and only if $c_1^{\mathbb{R}}(L) = 0$, where $c_1^{\mathbb{R}}(L)$ is the image of the integral first Chern class in $H^2(X, \mathbb{R})$. In particular, this applies to a compact Kähler manifold.*

PROOF. For a flat metric, the Chern curvature vanishes. Proposition III.2.22 therefore proves one implication. Conversely, choose a C^∞ Hermitian metric h_0 on L. Its real closed form $\theta = \sqrt{-1}\Theta_{h_0}/(2\pi)$ represents $c_1^{\mathbb{R}}(L)$ and is therefore d-exact. By the hypothesis, $\theta = \sqrt{-1}\partial\overline{\partial} f$ for a real-valued C^∞ function f; the reality follows by conjugating and averaging, as in Lemma III.2.118. Changing the metric to $h = h_0 e^{2\pi f}$ gives

$$\sqrt{-1}\Theta_h = \sqrt{-1}\Theta_{h_0} - 2\pi\sqrt{-1}\partial\overline{\partial} f = 0.$$

The flat-metric characterization above proves unitary flatness. The compact Kähler assertion follows from Lemma III.2.125. See also [Koi13, §2.3, Proposition 2.4]. □

Serre duality

Let X be a complex manifold of pure dimension n and let E be a holomorphic vector bundle on X. The conventions of Chapter I, §8.6, give

$$\big(\mathcal{A}^{p,q}(X,E)\big)^\vee = \mathcal{K}_c^{n-p,n-q}(X,E^*), \qquad \big(\mathcal{A}_c^{p,q}(X,E)\big)^\vee = \mathcal{K}^{n-p,n-q}(X,E^*).$$

These duals carry the strong topology, as already agreed in Chapter I. In the first pairing, $S \in \mathcal{K}_c^{n-p,n-q}(X,E^*)$ acts on $\alpha \in \mathcal{A}^{p,q}(X,E)$ by evaluation $S(\alpha)$. In the second, $S \in \mathcal{K}^{n-p,n-q}(X,E^*)$ and $\alpha \in \mathcal{A}_c^{p,q}(X,E)$. If S is represented by a smooth E^*-valued $(n-p,n-q)$-form β, this pairing is $\int_X \beta\wedge\alpha$, using $E^*\otimes E\to\mathbb{C}$ on the coefficients. No Hermitian metric is involved. The smooth/current comparisons in Theorem III.2.116 are induced by inclusion, both with and without compact supports. We use these vector-space isomorphisms first; their topological refinement will enter only at the end of the proof below.

THEOREM III.2.132 (Serre duality). *Let X be a complex manifold of pure dimension n and E a holomorphic vector bundle. The bilinear pairing above induces natural continuous surjections*

$$\iota^{p,q}\colon H_c^{n-p,n-q}(X,E^*) \longrightarrow H^{p,q}(X,E)^\vee,$$
$$\iota_c^{p,q}\colon H^{p,q}(X,E) \longrightarrow H_c^{n-p,n-q}(X,E^*)^\vee.$$

All cohomology spaces have the quotient topologies of Definition III.2.115, and their duals have the strong topology. Spaces and maps with antiholomorphic degrees outside $0,1,\dots,n$ are interpreted as zero. For $0\le p,q\le n$, the following conditions are equivalent.

(*i*) *$\iota^{p,q}$ is a TVS isomorphism.*

(*ii*) *$\iota_c^{p,q+1}$ is a TVS isomorphism.*

(*iii*) *$H^{p,q+1}(X,E)$ is Hausdorff.*

(*iv*) *$H_c^{n-p,n-q}(X,E^*)$ is Hausdorff.*

(*v*) *The image of $\overline\partial\colon \mathcal{K}^{p,q}(X,E)\to\mathcal{K}^{p,q+1}(X,E)$ consists exactly of the currents T such that $T(\eta)=0$ for every $\overline\partial$-closed $\eta\in\mathcal{A}_c^{n-p,n-q-1}(X,E^*)$.*

(*vi*) *The image of $\overline\partial\colon \mathcal{A}_c^{n-p,n-q-1}(X,E^*)\to\mathcal{A}_c^{n-p,n-q}(X,E^*)$ consists exactly of the forms η such that $T(\eta)=0$ for every $\overline\partial$-closed $T\in\mathcal{K}^{p,q}(X,E)$.*

PROOF. For Serre's original duality theorem, see [Se1, §3]; for an exposition and the formulation used here, see [Oh5, §1.4.3, Theorem 1.33, pp. 39–40]. We first separate bijectivity from its topological refinement. Let (i') and (ii') denote (i) and (ii), respectively, with "TVS isomorphism" replaced by "bijection". After establishing surjectivity, we prove

$$(i') \iff (vi) \iff (iv),$$
$$(ii') \iff (v) \iff (iii).$$

We then connect the two chains by proving $(i') \iff (iii)$.

Put $B^j = \mathcal{A}^{p,j}(X,E)$ and $d_j = \overline{\partial}\colon B^j \to B^{j+1}$. Apply the Hahn–Banach construction in Lemma I.6.52 (i) to the Fréchet complex

$$B^{q-1} \xrightarrow{d_{q-1}} B^q \xrightarrow{d_q} B^{q+1}.$$

Its dual middle cohomology is $Q^q = \operatorname{Ker} d_{q-1}^\vee / \operatorname{Image} d_q^\vee$. The current differential convention makes the two transposes $(-1)^{p+q+1}\overline{\partial}$ and $(-1)^{p+q}\overline{\partial}$, respectively; these signs do not affect kernels or images. Thus Q^q is the compactly supported current cohomology, identified above with $H_c^{n-p,n-q}(X,E^*)$, and the construction gives the surjection $\iota^{p,q}$. Applying the same construction to the compactly supported smooth complex, followed by the smooth/current comparison, gives the surjections $\iota_c^{p,j}$ in every degree j. Consequently, only injectivity remains in (i') and (ii').

$(i') \iff (vi)$. A closed form $\eta \in \mathcal{A}_c^{n-p,n-q}(X,E^*)$ represents an element of $\operatorname{Ker} \iota^{p,q}$ exactly when it pairs to zero with every closed smooth (p,q)-form. Every closed (p,q)-current differs from a closed smooth form by a $\overline{\partial}$-exact current, by the smooth/current Dolbeault comparison. Since η is closed and compactly supported, its pairing with this difference is zero. Thus the kernel condition is equivalent to annihilating every closed (p,q)-current. Moreover, a form in the annihilator appearing in (vi) is necessarily closed: test against exact currents and use the transpose identity. Therefore (vi) says precisely that this kernel is zero, which is (i') by surjectivity.

$(vi) \iff (iv)$. Consider

$$D := \overline{\partial}\colon \mathcal{A}_c^{n-p,n-q-1}(X,E^*) \longrightarrow \mathcal{A}_c^{n-p,n-q}(X,E^*).$$

Its transpose is, up to sign, $\overline{\partial}$ on (p,q)-currents. The Hahn–Banach annihilator identity gives $(\operatorname{Ker} D^\vee)^\perp = \overline{\operatorname{Image} D}$, where the closure is taken in $\mathcal{A}_c^{n-p,n-q}(X,E^*)$. Thus (vi) is equivalent to closedness of $\operatorname{Image} D$. Since the next kernel is closed, this is exactly the Hausdorff property of $H_c^{n-p,n-q}(X,E^*)$, namely (iv).

$(ii') \iff (v)$. A closed smooth $(p,q+1)$-form represents an element of $\operatorname{Ker} \iota_c^{p,q+1}$ exactly when it annihilates all closed compactly supported smooth $(n-p,n-q-1)$-forms. The smooth/current comparison preserves this pairing and exactness, so the same kernel condition may be expressed with a closed $(p,q+1)$-current. An arbitrary current in the annihilator in (v) is already closed, by testing against exact compactly supported forms. Hence (v) is precisely the injectivity in (ii'). The zero-space convention covers the endpoint degrees as well.

$(v) \iff (iii)$. Put $\mathcal{R} = \overline{\partial}\mathcal{A}^{p,q}(X,E)$, with closure in the C^∞ topology of $\mathcal{A}^{p,q+1}(X,E)$. For a closed smooth form α of bidegree $(p,q+1)$, the Hahn–Banach theorem shows that $\alpha \in \overline{\mathcal{R}}$ exactly when every closed compactly supported current of complementary bidegree vanishes on α. By the compact-support comparison in Theorem III.2.116, each such current is $T_\eta + \overline{\partial}Q$, where η is smooth, closed, and compactly supported, and Q is also compactly supported. Since α is closed, Stokes' formula gives $(\overline{\partial}Q)(\alpha) = 0$. Thus $\alpha \in \overline{\mathcal{R}}$ if and only if $\int_X \eta \wedge \alpha = 0$ for every closed $\eta \in \mathcal{A}_c^{n-p,n-q-1}(X,E^*)$.

Assume (iii) and let T belong to the annihilator in (v). Testing against exact compactly supported forms shows that $\overline{\partial}T = 0$. The ordinary smooth/current comparison gives $T =$

$T_\alpha + \overline{\partial} S$ with α smooth and closed. Its pairings with closed compactly supported forms agree with those of T, so $\alpha \in \overline{\mathcal{R}}$ by the preceding argument. Condition (iii) makes $\mathcal{R}$ closed, hence $\alpha = \overline{\partial}\beta$ for a smooth β. It follows that $T = \overline{\partial}(T_\beta + S)$. The reverse inclusion in (v) follows from Stokes' formula. Conversely, assume (v) and take $\alpha \in \overline{\mathcal{R}}$; this form is closed because $\overline{\partial}$ is continuous and $\overline{\partial}^2 = 0$. The same annihilation property makes T_α exact by (v). Injectivity of the smooth/current comparison then makes α exact as a smooth form, so $\alpha \in \mathcal{R}$. Thus $\mathcal{R}$ is closed, proving (iii).

$(i') \iff (iii)$. By Lemma I.6.2, (iii) means that $\operatorname{Image} d_q$ is closed in $\operatorname{Ker} d_{q+1}$, hence in B^{q+1}. Lemma I.6.52 (ii) then makes $\iota^{p,q}$ bijective. Conversely, put $Z^q = \operatorname{Ker} d_q$ and let $(Z^q)^\perp \subset (B^q)^\vee$ be its annihilator. The definition of the natural map gives

$$\operatorname{Ker} \iota^{p,q} \cong (Z^q)^\perp / \operatorname{Image} d_q^\vee .$$

Under (i'), $\operatorname{Image} d_q^\vee = (Z^q)^\perp$ is weak-$*$ closed. The Fréchet closed-range theorem (Remark I.6.48) gives the closedness of $\operatorname{Image} d_q$ and hence (iii). Thus injectivity of the duality map in degree q is governed by the *outgoing* map d_q: its image is the denominator of $H^{p,q+1}(X,E)$, not of $H^{p,q}(X,E)$.

This proves the equivalence of $(i'), (ii'), (iii)$–(vi). For the topological refinement, we use [Oh5, Theorem 1.33, pp. 39–40] and [Cas, §3, Theorems 4–5, pp. 309–310, and §5, Theorem 1, pp. 316–317], together with the topological smooth/current comparisons in [Lau, Theorem 2.1, pp. 416–424, and the vector-bundle case on p. 427] (See also [LL, p. 306, footnote 3]). The canonical duality maps are continuous for the strong dual topologies, and under the equivalent conditions their inverses are continuous as well. Thus (i') and (ii') give (i) and (ii); the reverse implications are immediate. Here strong dual topology means uniform convergence on TVS bounded subsets of the cohomology space itself, not merely on images of TVS bounded sets of cocycles. This is why the topological refinement is invoked separately from the preceding bijectivity argument. □

Multiplier ideal sheaves and Nadel vanishing

The local integrability condition attached to a singular metric is itself an ideal-theoretic object. We use the metric weights already fixed above, so there is no factor two in the exponent below.

Definition III.2.133. Let X be a complex manifold, let $U \subset X$ be open, and let φ be a plurisubharmonic function on U. Its *multiplier ideal sheaf* $\mathcal{I}(\varphi) \subset \mathcal{O}_U$ is defined by

$$\mathcal{I}(\varphi)_a := \{ \underline{f}_a \in \mathcal{O}_{U,a} \mid |f|^2 e^{-\varphi} \text{ is integrable near } a \}, \qquad a \in U.$$

Here f is a holomorphic representative of $\underline{f}_a$. Integrability is measured in any smooth positive local volume form. The same definition applies when φ is locally a plurisubharmonic function plus a smooth function. Let L be a holomorphic line bundle on X and let h be a singular Hermitian metric on L whose local weights are of this type. The multiplier ideals of these local weights glue to an ideal sheaf on X, denoted by $\mathcal{I}(h)$. □

On a component where $\varphi \equiv -\infty$, we set $\mathcal{I}(\varphi) = 0$. If φ is locally bounded on U, then $\mathcal{I}(\varphi) = \mathcal{O}_U$. Changing coordinates or adding a smooth function multiplies the integrand locally by factors bounded above and below by positive constants. Changing a holomorphic frame has the same effect. This proves the asserted independence and gluing. For example, let $N \in \mathbb{Z}_{\geq 1}$ and give $\mathbb{C}^N$ coordinates $z = (z_1, z_2, \dots, z_N)$. For $1 \leq r \leq N$ and $\varphi = \sum_{j=1}^{r} \alpha_j \log |z_j|^2$ with $\alpha_j \geq 0$, Mini-Exercise III.2.63 gives

$$\mathcal{I}(\varphi) = \left(\prod_{j=1}^{r} (z_j)^{\lfloor \alpha_j \rfloor} \right).$$

Thus the earlier relation between weights and vanishing orders already computes multiplier ideals in the normal-crossing case. The additional assertion of Mini-Exercise III.2.63 gives a second useful model. On a coordinate ball centered at a in a manifold of pure dimension n, for $\alpha \geq 0$,

$$\mathcal{I}\big(\alpha \log |z-a|^2\big) = \mathfrak{m}_a^{\max\{0, \lfloor \alpha \rfloor - n + 1\}}. \tag{189}$$

Here $\mathfrak{m}_a$ is the ideal sheaf of the point a, and $|z-a|^2 = \sum_{j=1}^{n} |z^j - z^j(a)|^2$. The same formula holds after adding a bounded function to the weight. In particular,

$$\mathcal{I}\big(n \log |z-a|^2\big) = \mathfrak{m}_a, \qquad \mathcal{I}\big((n+1) \log |z-a|^2\big) = \mathfrak{m}_a^2. \tag{190}$$

THEOREM III.2.134 (Nadel's coherence theorem). *Let X be a complex manifold. The multiplier ideal sheaf on X of a plurisubharmonic function, or of a singular Hermitian metric on a holomorphic line bundle whose weights are locally plurisubharmonic plus smooth, is coherent.*

PROOF. The assertion is immediate on zero-dimensional components and on components with weight identically $-\infty$. Otherwise coherence is local, and bounded changes of weight do not change the ideal. Work on a bounded coordinate ball $B \subset \mathbb{C}^n$, where $n \in \mathbb{Z}_{\geq 1}$, with a plurisubharmonic weight φ. Let $\mathcal{H}$ be the space of holomorphic functions F on B with $\int_B |F|^2 e^{-\varphi} d\lambda < \infty$, where $d\lambda$ is Lebesgue measure. By Corollary III.1.45, the ideal sheaf $\mathcal{J}$ generated by all $F \in \mathcal{H}$ is coherent. Clearly $\mathcal{J} \subset \mathcal{I}(\varphi)$.

Fix $a \in B$, a germ $\underline{G}_a \in \mathcal{I}(\varphi)_a$, and $s \in \mathbb{Z}_{\geq 0}$. Choose a holomorphic representative G of $\underline{G}_a$ near a and a cutoff χ supported there, equal to one near a. Consider the equation $\overline{\partial} V = \overline{\partial}(\chi G)$ with the singular weight

$$\Phi_s(z) = \varphi(z) + (n+s) \log |z-a|^2 + |z|^2.$$

Its curvature is bounded below by the constant Kähler form. The right-hand side is supported away from a and has finite weighted norm. Theorem III.2.113, using $\alpha \mapsto \alpha \wedge dz^*$ as above, therefore gives a solution V satisfying

$$\int_B |V|^2 e^{-\varphi} |z-a|^{-2(n+s)} d\lambda < \infty.$$

Here the omitted factor $e^{-|z|^2}$ is bounded above and below on B. Near a, V is holomorphic by Theorem III.1.18 (iv). Since φ is bounded above there, (189) gives $\underline{V}_a \in \mathfrak{m}_{B,a}^{s+1}$. Also V is $L^2(e^{-\varphi})$ on B, since $|z-a|^{2(n+s)}$ is bounded. Thus $F = \chi G - V$ belongs to $\mathcal{H}$ and $\underline{G}_a - \underline{F}_a = \underline{V}_a \in \mathfrak{m}_{B,a}^{s+1}$.

We have proved $\underline{G}_a \in \mathcal{J}_a + \mathfrak{m}_{B,a}^{s+1}$ for every s. The Krull intersection theorem (Theorem I.5.15) applied to the finite module $\mathcal{O}_{B,a}/\mathcal{J}_a$ makes the intersection of these submodules zero in the quotient. Therefore $\underline{G}_a \in \mathcal{J}_a$ and $\mathcal{I}(\varphi) = \mathcal{J}$. □

THEOREM III.2.135 (Nadel vanishing). *Let (X,ω) be a Kähler manifold of pure dimension n admitting a C^∞ plurisubharmonic exhaustion ρ, meaning that $\{\rho < c\}$ has compact closure in X for every $c \in \mathbb{R}$. Suppose a holomorphic line bundle L on X has a singular Hermitian metric h with $\sqrt{-1}\Theta_h \geq \varepsilon\omega$ for a continuous positive function ε on X. Then*

$$H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes_{\mathcal{O}_X} \mathcal{I}(h)\big) = 0 \qquad (q \geq 1).$$

PROOF. Let $\mathcal{L}_h^{n,q}$ be the sheaf of L-valued (n,q)-currents u such that both u and $\overline{\partial}u$ have locally L^2 coefficients for h and a C^∞ base metric. Consider

$$0 \longrightarrow \mathcal{O}_X(K_X \otimes L) \otimes_{\mathcal{O}_X} \mathcal{I}(h) \longrightarrow \mathcal{L}_h^{n,0} \xrightarrow{\overline{\partial}} \mathcal{L}_h^{n,1} \xrightarrow{\overline{\partial}} \mathcal{L}_h^{n,2} \xrightarrow{\overline{\partial}} \cdots .$$

In degree zero the kernel consists of holomorphic top forms with the multiplier-ideal integrability condition. Indeed, the local upper bound on a weight makes these coefficients ordinary L^2_{loc}, and distributional $\overline{\partial}$-closedness implies holomorphicity. In positive degrees the complex is locally exact by Theorem III.2.113 on small coordinate balls. Multiplication by a smooth cutoff preserves both integrability conditions, so these sheaves are fine. They therefore give an acyclic resolution of $\mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)$.

Let g be a closed global section in positive degree. Adding a constant, take $\rho \geq 1$. The form $\omega + \sqrt{-1}\partial\overline{\partial}\rho^2$ is complete Kähler: the bound on $|d\rho|$ and the exhaustion property give the same Hopf–Rinow argument as in Example III.2.117. There is a smooth convex increasing function χ growing so fast that g satisfies both global integrability conditions of Theorem III.2.113 for $h_\chi = he^{-\chi\circ\rho}$. Such functions are available by the convex-function construction in Mini-Exercise I.4.11; the growth on successive compact sublevels can be chosen to dominate each of the two integrals. The curvature only increases and $\mathcal{I}(h_\chi) = \mathcal{I}(h)$. The theorem gives a global solution f for h_χ. On every relatively compact open set the smooth extra factor is bounded above and below, so $f \in \Gamma(X, \mathcal{L}_h^{n,q-1})$. The cohomology of the global resolution is zero in positive degrees, proving the theorem. □

From weighted solutions to projective embeddings

For a compact complex manifold X, call a holomorphic line bundle L *positive* if it admits a smooth Hermitian metric h with $\sqrt{-1}\Theta_h > 0$. We prove the following classical characterization of positivity on a compact complex manifold.

THEOREM III.2.136 (Kodaira embedding theorem). *Let X be a compact complex manifold and L a positive holomorphic line bundle. For $m \in \mathbb{Z}_{\geq 0}$, put $V_m := H^0(X, \mathcal{O}_X(L^m))$. There*

is an integer m_0 such that, for every $m \geq m_0$, the sections of L^m define a holomorphic embedding

$$\Phi_m \colon X \longrightarrow \mathbb{P}(V_m), \qquad \Phi_m^*(L_{\mathrm{taut}}^{-1}) \cong L^m.$$

Here $\mathbb{P}(V_m)$ is the space of hyperplanes in V_m, and L_{taut} is the tautological line bundle on the line model $\mathbf{P}(V_m^*) = \mathbb{P}(V_m)$. In a basis $s_0, s_1, \ldots, s_N$, the map is $x \mapsto [s_0(x) : s_1(x) : \cdots : s_N(x)]$, computed in any local frame. Conversely, if such an embedding is defined by sections of L^m for some integer $m > 0$, then L is positive.

Proof. First suppose that the embedding in the converse assertion is given. Pull back the quotient Fubini–Study metric of L_{taut}^{-1} from Mini-Exercise III.2.26. The resulting metric on L^m has positive curvature. Its positive m-th root in local frames defines a C^∞ metric on L whose curvature is $1/m$ times that curvature.

We use Nadel's vanishing theorem (Theorem III.2.135) with logarithmic weights. Work first on a connected component of pure dimension $n \geq 1$, and write $\mathcal{L}_m := \mathcal{O}_X(L^m)$. The space $V_m = \Gamma(X, \mathcal{L}_m)$ is finite-dimensional by Lemma III.2.121. We begin by identifying the interpolation that constructs an embedding.

At a point x, evaluation $V_m \to L_x^m$ must be non-zero: then its kernel is a hyperplane in V_m, defining $\Phi_m(x)$. If values can be prescribed independently at two distinct points, a section vanishing at one but not at the other distinguishes their hyperplanes. At one point a, prescribing a value and all first derivatives gives injectivity of the differential of Φ_m. Indeed, in a projective chart supplied by a section s_0 with $s_0(a) \neq 0$, the functions representing the map are ratios s/s_0. Arbitrary first jets of s give arbitrary differentials of these ratios at a.

Thus it suffices to make the following two restriction maps surjective:

$$\begin{aligned} V_m &\longrightarrow L_a^m \oplus L_b^m \qquad (a \neq b), \\ V_m &\longrightarrow \mathcal{L}_{m,a}/\mathfrak{m}_{X,a}^2 \mathcal{L}_{m,a} \qquad (a \in X). \end{aligned} \tag{191}$$

For $\mathcal{I} = \mathfrak{m}_a \cap \mathfrak{m}_b$ or $\mathcal{I} = \mathfrak{m}_a^2$, respectively, use the short exact sequence

$$0 \longrightarrow \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}_m \longrightarrow \mathcal{L}_m \longrightarrow (\mathcal{O}_X/\mathcal{I}) \otimes_{\mathcal{O}_X} \mathcal{L}_m \longrightarrow 0.$$

It is exact because $\mathcal{L}_m$ is locally free, hence flat. The last sheaf is supported at the specified point or points, and its global sections give the corresponding target in (191). The induced long exact sequence shows that $H^1(X, \mathcal{I} \otimes \mathcal{L}_m) = 0$ suffices.

Fix a C^∞ Hermitian metric h_0 on L with $\sqrt{-1}\Theta_{h_0} > 0$, and put $\omega := \sqrt{-1}\Theta_{h_0}$, a Kähler form. To apply Nadel's vanishing theorem (Theorem III.2.135), the bundle to equip with a singular metric is $F_m := L^m \otimes K_X^{-1}$, so that $K_X \otimes F_m = L^m$. By (190), it remains to construct singular Hermitian metrics on F_m with positive curvature and the logarithmic local models in that formula, such that their local weights have Lelong number n at each of two distinct points a, b, or Lelong number $n+1$ at a, with one lower bound on m for all choices of the points.

Choose finitely many coordinate sets U_j and open sets $U_j^* \subset U_j$ that still cover X, with $\overline{U_j^*}$ compact and contained in U_j. Fix a small radius $R_j > 0$ so that the translated coordinate balls of radius R_j centered at points of U_j^* have closure in U_j, and fix a C^∞ cutoff $\chi_j(t)$ that

equals one near $t = 0$ and is supported in $t < R_j^2$. Fix $a \in X$, choose j such that $a \in U_j^*$, and put

$$\eta_{j,a}(x) := \chi_j\big(|z_j(x) - z_j(a)|^2\big) \log |z_j(x) - z_j(a)|^2$$

in that chart, and extend it by zero outside. Near its pole the logarithm is plurisubharmonic. Every term involving a derivative of the cutoff is supported in an annulus-like domain with fixed inner and outer radii about a. Its derivatives are uniformly bounded as a varies in U_j^*. Finiteness of the covering therefore gives a constant C such that $\sqrt{-1}\partial\overline{\partial}\eta_{j,a} \geq -C\omega$ for every a and j with $a \in U_j^*$.

Give K_X^{-1} a fixed C^∞ Hermitian metric $h_{K^{-1}}$ with curvature bounded below by $-C_K\omega$. For distinct a, b, choose j, k with $a \in U_j^*$ and $b \in U_k^*$, and multiply $h_0^m \otimes h_{K^{-1}}$ by $e^{-n\eta_{j,a} - n\eta_{k,b}}$; for a single point use $e^{-(n+1)\eta_{j,a}}$. Both curvatures are bounded below by

$$\big(m - C_K - C\max\{2n, n+1\}\big)\omega.$$

Choose m_0 so that this coefficient is at least one for $m \geq m_0$. This choice is independent of a, b, even when they approach one another. Near each fixed pole, the other terms in the weight are bounded, so (190) identifies the multiplier ideals as $\mathfrak{m}_a \cap \mathfrak{m}_b$ and $\mathfrak{m}_a^2$, respectively. Nadel's vanishing theorem (Theorem III.2.135) now gives the required H^1 vanishings, and hence (191).

Consequently Φ_m is everywhere defined, injective, and immersive. Compactness makes it an embedding. The evaluation quotient identifies $\Phi_m^*(L_{\text{taut}}^{-1})$ with L^m. Finally, a compact manifold has finitely many components. Take the maximum of their bounds and use sections supported on one component to separate different components. This proves the theorem in the stated generality. □

2.9 Topological invariants of line bundles and divisors

For a reduced complex space X, recall the correspondence between holomorphic line bundles and invertible sheaves in Proposition III.2.58. The first Chern class of Example II.3.86 is a group homomorphism

$$c_1 \colon \operatorname{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}).$$

It classifies the underlying topological line bundle, not its holomorphic structure: $c_1(L) = c_1(M)$ if and only if $L \cong_{\text{top}} M$. In particular, $c_1(L) = 0$ means that L is *topologically trivial*.

For a complex space $(X, \mathcal{O}_X)$, the holomorphic exponential sequence is

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow \mathcal{O}_X \xrightarrow{\exp(2\pi\sqrt{-1}\,\bullet)} \mathcal{O}_X^* \longrightarrow 1. \tag{192}$$

Local holomorphic logarithms give exactness at the last term. Point values give a homomorphism $\mathcal{O}_X^* \to \mathcal{C}_X^*$, where $\mathcal{C}_X$ is the sheaf of complex-valued continuous functions. Comparison with the continuous exponential sequence in Example II.3.86 defines c_1 on $\operatorname{Pic}(X, \mathcal{O}_X)$, abbreviated to $\operatorname{Pic}(X)$, and agrees with the preceding first Chern class when X is reduced, with the same sign. Consequently it gives the exact sequence

$$H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \operatorname{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X). \tag{193}$$

DEFINITION III.2.137. For a complex space $(X, \mathcal{O}_X)$, set

$$\mathrm{Pic}^0(X) := \mathrm{Ker}\big(c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})\big).$$

When X is reduced, $\mathrm{Pic}^0(X)$ consists of the holomorphic isomorphism classes of topologically trivial holomorphic line bundles. □

The exact sequence above identifies this group with

$$\mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/\Lambda_X, \qquad \Lambda_X := \mathrm{Image}\big(H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X)\big). \tag{194}$$

The quotient of a finite-dimensional complex vector space by a lattice generated by a real basis is called a *compact complex torus*: translations give holomorphic charts, and a closed fundamental parallelepiped is compact.

PROPOSITION III.2.138. *For a compact Kähler manifold X, the quotient (194) equips $\mathrm{Pic}^0(X)$ with a canonical compact complex torus structure of complex dimension $h^{0,1}(X)$. Every member of $\mathrm{Pic}^0(X)$ is unitary flat.*

PROOF. If X is disconnected, apply the argument to its finitely many connected components and take the product. We may therefore assume that X is connected. Consider the map induced by the inclusion of real constants,

$$H^1(X, \mathbb{R}) \longrightarrow H^1(X, \mathcal{O}_X) \cong H^{0,1}(X).$$

Under the de Rham and Dolbeault comparisons it sends a real closed 1-form to its $(0,1)$-component. Indeed, projection onto type $(0, \bullet)$ is a morphism from the de Rham resolution to the Dolbeault resolution, extending the inclusion of constants. By the Hodge decomposition in Corollary III.2.126 this map is a real-linear isomorphism: a real class has conjugate $(1,0)$- and $(0,1)$-components, and the inverse sends $\beta \in H^{0,1}(X)$ to $\beta + \overline{\beta}$.

By Chapter I, §8.4, a compact manifold has finitely generated integral homology. The universal coefficient theorem in degree one and Mini-Exercise I.7.22 identify $H^1(X, \mathbb{Z})$ with a full lattice in $H^1(X, \mathbb{R})$, of rank $b_1(X) = 2h^{0,1}(X)$. Thus Λ_X is a full lattice in $H^{0,1}(X)$. This construction uses the canonical Hodge decomposition in cohomology, not a choice of harmonic representatives; it is independent of the Kähler metric. Unitary flatness follows from Proposition III.2.131. □

In this statement the kernel is taken in *integral* cohomology. In higher dimensions, unitary flatness only requires $c_1^{\mathbb{R}}(L) = 0$, and a non-zero torsion class $c_1(L)$ may remain. Thus we do not identify all unitary flat bundles with $\mathrm{Pic}^0(X)$.

LEMMA III.2.139. *Let D be a divisor on a complex manifold X, and let h be a C^∞ Hermitian metric on $[D]$. The canonical meromorphic section σ_D satisfies*

$$T_D = \frac{\sqrt{-1}}{2\pi}\Theta_h + dd^c \log |\sigma_D|_h^2 \tag{195}$$

in the sense of currents. In particular, T_D represents $c_1^{\mathbb{R}}([D])$.

PROOF. Use the local equations and frames of Proposition III.2.65: $\sigma_D = f_j e_j$ and $|e_j|_h^2 = e^{-\varphi_j}$. Then $\log|\sigma_D|_h^2 = \log|f_j|^2 - \varphi_j$ is locally integrable. For meromorphic f_j, this follows by writing it as a quotient of two non-zero holomorphic functions. The Lelong–Poincaré formula (Theorem III.2.99), applied to numerator and denominator, gives $dd^c \log|f_j|^2 = T_D$ locally. Also $dd^c\varphi_j = \sqrt{-1}\Theta_h/(2\pi)$ by (149). Subtraction gives the identity. The last term is d-exact, so the current comparison of Example II.3.82 and Proposition III.2.22 prove the final assertion. □

The same local calculation applies to any meromorphic section s of a line bundle which is non-zero on every connected component, with $D = \operatorname{div}(s)$; compare [De4, Chapter V, (13.2)–(13.6), p. 271].

EXAMPLE III.2.140. Let D be an effective divisor on a complex manifold X, with support $\operatorname{Supp} D$ as in Definition III.2.59. The canonical section σ_D defines a singular Hermitian metric $h_{\mathrm{sing},D}$ on $[D]$, also written $|\sigma_D|^{-2}$. In these notes, we call it the *canonically attached singular Hermitian metric* of D. To make this notation precise, take an open covering $\{U_j\}$ with holomorphic frames e_j of $[D]$ and write $\sigma_D|_{U_j} = \sigma_j e_j$. For $x \in U_j \setminus \operatorname{Supp} D$ and $\xi, \eta \in \mathbb{C}$, put

$$\big\langle \xi e_j(x), \eta e_j(x)\big\rangle_{h_{\mathrm{sing},D}} = \left(\frac{\xi e_j(x)}{\sigma_D(x)}\right)\overline{\left(\frac{\eta e_j(x)}{\sigma_D(x)}\right)} = \frac{\xi\overline{\eta}}{|\sigma_j(x)|^2}.$$

The quotients are the scalar coefficients relative to the basis $\sigma_D(x)$ of the fiber. Thus $|e_j|^2_{h_{\mathrm{sing},D}} = |\sigma_j|^{-2}$ almost everywhere, and the local weights are $\varphi_j = \log|\sigma_j|^2$. They are locally integrable and plurisubharmonic. On overlaps, $e_k = g_{jk}e_j$ and $\sigma_j = g_{jk}\sigma_k$ give $\varphi_k = \varphi_j - \log|g_{jk}|^2$. Hence they define a singular Hermitian metric in the sense of Definition III.2.112. The fiber formula is used off $\operatorname{Supp} D$; along $\operatorname{Supp} D$ the metric is understood in this almost-everywhere sense. By the Lelong–Poincaré formula (Theorem III.2.99),

$$\frac{\sqrt{-1}}{2\pi}\Theta_{h_{\mathrm{sing},D}} = dd^c \log|\sigma_j|^2 = T_D \quad \text{on } U_j.$$

Thus the normalized curvature current is T_D, including the multiplicities of D, and the curvature is semi-positive. □

On a compact Riemann surface

Let X be a compact Riemann surface. For a holomorphic line bundle L we put

$$\deg L := \int_X c_1(L) = \langle c_1(L), [X]\rangle = \int_X \frac{\sqrt{-1}}{2\pi}\Theta_h \in \mathbb{Z}, \tag{196}$$

where $[X] \in H_2(X,\mathbb{Z})$ is the fundamental class with the complex orientation, and h is any C^∞ Hermitian metric on L. The first integral abbreviates the pairing with $[X]$, as in Remark I.8.41. The last equality follows from Proposition III.2.22: the curvature form represents the image of $c_1(L)$ under $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}) \cong H^{1,1}(X)$. The last identification follows from Corollary III.2.126; every Hermitian metric on a Riemann surface is Kähler. Thus the integral is independent of h, and its integrality comes from the integral fundamental class.

PROPOSITION III.2.141. *On a compact Riemann surface X, the degree homomorphism $\mathrm{Pic}(X) \to \mathbb{Z}$ is surjective. On $\mathbb{P}^1$, it is an isomorphism. More explicitly, for every $m \in \mathbb{Z}$, the unique holomorphic isomorphism class of degree m is represented by L_{taut}^{-m}.*

PROOF. The Dolbeault isomorphism (Theorem III.2.116) gives $H^2(X, \mathcal{O}_X) \cong H^{0,2}(X) = 0$ by bidegree. Thus (193) makes c_1 surjective. Pairing with $[X]$ identifies $H^2(X, \mathbb{Z})$ with $\mathbb{Z}$, by (25) and the universal coefficient theorem (20) in Chapter I. For $X = \mathbb{P}^1$, Example III.2.128 also gives $H^1(X, \mathcal{O}_X) = H^{0,1}(X) = 0$, so c_1 is injective. Finally, $\deg L_{\mathrm{taut}}^{-1} = 1$ by Remark III.2.25. Tensor products add first Chern classes, proving the assertion. For $m \geq 0$ we have $L_{\mathrm{taut}}^{-m} = (L_{\mathrm{taut}}^{-1})^{\otimes m}$; negative tensor powers are interpreted by duality. Its sheaf of holomorphic sections is $\mathcal{O}_{\mathbb{P}^1}(m)$. □

If X has genus g, Proposition III.2.138 and $h^{0,1}(X) = g$ identify $\mathrm{Pic}^0(X)$ with a compact complex torus of dimension g; here Λ_X in (194) has rank $2g$ by (25) and (20). In particular, $\mathrm{Pic}^0(X)$ is non-trivial if $g > 0$, although all its members have the same underlying topological line bundle. All these degree-zero bundles are unitary flat. Conversely, a unitary flat line bundle on a compact Riemann surface has degree zero, since its flat metric has zero curvature.

On a compact Riemann surface, for $D = \sum_a m_a a$, (195) gives

$$\deg[D] = \sum_a m_a. \tag{197}$$

Indeed, integration of the exact current term is zero by Stokes, and the integration current of a point has mass one. Equivalently, the degree of a line bundle carrying such a section is the sum of its zero and pole orders, with their signs. No existence of a meromorphic section of an arbitrary line bundle is needed for the definitions below.

Curves and divisors on a complex surface

DEFINITION III.2.142. Let X be a complex manifold of pure dimension 2, let $C \subset X$ be a compact analytic subset of pure dimension 1, and let L be a holomorphic line bundle on X. We define

$$(L.C) := \int_C c_1(L) := \int_C \frac{\sqrt{-1}}{2\pi} \Theta_h, \tag{198}$$

where h is a C^∞ Hermitian metric on L. For a divisor D on X, set $(C.D) := ([D].C)$. □

These integrals use precisely the meaning of Remark III.2.97; equivalently, we evaluate the compactly supported current T_C on the curvature form, with a cutoff equal to one near C. Absolute convergence follows from Theorem III.2.96. If $h' = e^{-\psi} h$ is another C^∞ Hermitian metric on L, the difference of the normalized curvatures is $dd^c\psi$ by (150). Since $dT_C = 0$, its value on $dd^c\psi$ is zero, so (198) is independent of h. More generally, evaluation on T_C kills every exact 2-form. Thus $(L.C)$ depends only on $c_1(L)$ and is additive under tensor products of line bundles. In particular, it is unchanged by a continuous deformation of the underlying topological line bundle: the restrictions of a bundle on $X \times [0, 1]$ have the same Chern class, by naturality and homotopy invariance of cohomology.

If C is non-singular, $(L.C) = \deg(L|_C)$, interpreted as the sum over connected components. The number is an integer even for singular or reducible C. To see this without introducing a new notion of degree on a singular curve, write $C = \bigcup_{\nu=1}^{r} C_\nu$ for its global irreducible decomposition, as in Theorem III.2.45. Apply Theorem III.2.71 to the ideal of C near this compact set. The strict transforms $\widetilde{C}_\nu$ are non-singular compact curves. The induced maps $\pi_\nu \colon \widetilde{C}_\nu \to X$ are biholomorphic onto C_ν off finite sets: the non-trivial smooth centers on a surface are points. Properness gives compactness; the SNC support of the total transform gives non-singularity. Changing variables away from these finite sets, which have zero area, gives

$$(L.C) = \sum_{\nu=1}^{r} \int_{\widetilde{C}_\nu} \frac{\sqrt{-1}}{2\pi} \Theta_{\pi_\nu^* h} = \sum_{\nu=1}^{r} \deg(\pi_\nu^* L) \in \mathbb{Z}. \tag{199}$$

We call $(L.C)$ the *intersection number* of L and C. The next proposition explains this terminology.

Proposition III.2.143. *In the setting of Definition III.2.142, let D be a reduced effective divisor, meaning that all its non-zero coefficients are one. Suppose that at every point of $C \cap \operatorname{Supp} D$ both C and $\operatorname{Supp} D$ are non-singular and meet transversely. Then*

$$(C.D) = \#(C \cap \operatorname{Supp} D).$$

Proof. The intersection is discrete and compact, hence finite. Pull the canonical section σ_D back to each $\widetilde{C}_\nu$ in (199). Its zeros lie over $C \cap \operatorname{Supp} D$ and contribute $+1$ each by the local calculation in Example III.2.101. The degree formula (197), followed by (199), gives the result. □

With positive coefficients on D, the same argument counts those coefficients at transverse intersections. In the absence of transversality, the numerical definition still makes sense; it is not simply a count of distinct points.

Definition III.2.144. Let D_1, D_2 be divisors on a complex manifold X of pure dimension 2. If $\operatorname{Supp} D_1$ is compact, write $D_1 = \sum_{\nu=1}^{r} m_\nu C_\nu$ and set

$$(D_1.D_2) := \sum_{\nu=1}^{r} m_\nu([D_2].C_\nu) = \int_{D_1} c_1([D_2]).$$

Here integration over a divisor includes its coefficients. If only $\operatorname{Supp} D_2$ is compact, define $(D_1.D_2) := (D_2.D_1)$. □

Note that compactness and local finiteness make the displayed sum finite. The number is integral and additive in the divisors whenever the support condition is preserved.

Example III.2.145 (Self-intersection and the normal bundle). Let X be a complex manifold of pure dimension 2 and let $C \subset X$ be a holomorphically embedded compact Riemann surface, that is, a non-singular connected compact complex curve in X. We call $(C.C)$ the *self-intersection number* of C. The holomorphic tangent bundle T_C is a rank-one subbundle

of $T_X|_C$. Its quotient is the *holomorphic normal bundle* $N_{C/X}$, a holomorphic line bundle fitting into

$$0 \longrightarrow T_C \longrightarrow T_X|_C \longrightarrow N_{C/X} \longrightarrow 0.$$

Under the usual identification of real tangent vectors with their $(1,0)$-parts, its underlying real bundle is $TX|_C/TC$, the C^∞ normal bundle introduced after Theorem I.8.4.

In fact, $N_{C/X} \cong [C]|_C$. To see this, write the canonical section locally as $\sigma_C = f_j e_j$, where f_j is a reduced local defining function of C, so $(df_j)_x \neq 0$ for $x \in C$. The maps $v \mapsto (df_j)(v)e_j$ along C vanish on T_C and induce isomorphisms from the normal fibers to $[C]|_C$. They agree on overlaps because $f_j = g_{jk} f_k$ and $e_k = g_{jk} e_j$, while $f_k = 0$ along C. Consequently,

$$(C.C) = \deg([C]|_C) = \deg N_{C/X}.$$

□

LEMMA III.2.146. *Let D have compact support on a complex manifold X of pure dimension $n \geq 1$. There is a C^∞ Hermitian metric h_D on $[D]$ such that $|\sigma_D|_{h_D} = 1$ outside a compact set. Its normalized curvature $\alpha_D := \sqrt{-1}\Theta_{h_D}/(2\pi)$ is compactly supported, and*

$$\{\alpha_D\} = \{T_D\} \quad \text{in } H^2_{\mathrm{DR},c}(X, \mathbb{R}). \tag{200}$$

This class is independent of the metric satisfying the stated condition and maps to $c_1^{\mathbb{R}}([D])$ in ordinary cohomology. For every closed C^∞ $(2n-2)$-form η,

$$\int_D \eta = \int_X \alpha_D \wedge \eta. \tag{201}$$

PROOF. Start with any metric h_0. On $X \setminus \operatorname{Supp} D$, the section σ_D is a holomorphic trivialization. Choose $\chi \in C_c^\infty(X)$ equal to one near $\operatorname{Supp} D$ and put $\psi = (1-\chi)\log|\sigma_D|^2_{h_0}$, extended by zero near $\operatorname{Supp} D$. This is a global C^∞ function. The metric $h_D = e^{-\psi}h_0$ has the required property, so its curvature vanishes outside a compact set. The locally integrable function $u := \log|\sigma_D|^2_{h_D}$ is also zero there. By (195), $T_D - \alpha_D = dd^c u$, with compactly supported primitive $d^c u$. This proves (200) by the compact-support current comparison in Example II.3.73. Two such metrics have ratio e^{-v} with $v \in C_c^\infty(X, \mathbb{R})$, so their curvatures differ by $dd^c v$ and define the same class. Finally, pair the current identity with the closed form η. A cutoff equal to one near the compact support of u justifies Stokes' theorem and gives (201). □

In particular, if both divisors on a surface have compact support, choose the metrics of the lemma for each of them. Then

$$(D_1.D_2) = \int_X \alpha_{D_1} \wedge \alpha_{D_2} = (D_2.D_1).$$

For line bundles L_1, L_2 on a *compact* complex surface, we may similarly define

$$(L_1.L_2) := \int_X c_1(L_1) \wedge c_1(L_2). \tag{202}$$

The integral means evaluation of the integral cup product $c_1(L_1) \smile c_1(L_2)$ on $[X]$; the de Rham comparison sends cup product to wedge product (Chapter I, §7.5, and [De4, Chapter IV, §8]). Thus it is an integer, symmetric, and independent of the metrics.

Higher-dimensional intersections

If X is a compact complex manifold of pure dimension $n \geq 1$ and $L_1, L_2, \ldots, L_n$ are holomorphic line bundles, put

$$(L_1.L_2.\ldots.L_n) := \int_X c_1(L_1) \wedge c_1(L_2) \wedge \cdots \wedge c_1(L_n). \tag{203}$$

As above, this is the integral cup product evaluated on $[X]$, or the integral of the normalized curvature forms of arbitrary C^∞ Hermitian metrics. It is an integer and is symmetric and additive under tensor products in each variable.

More generally, let X be a possibly non-compact complex manifold of pure dimension n, and let $D_1, D_2, \ldots, D_n$ be divisors with $\operatorname{Supp} D_1$ compact. Define

$$(D_1.D_2.\ldots.D_n) := \int_{D_1} \frac{\sqrt{-1}}{2\pi}\Theta_{h_2} \wedge \cdots \wedge \frac{\sqrt{-1}}{2\pi}\Theta_{h_n}, \tag{204}$$

where h_j is a C^∞ Hermitian metric on $[D_j]$.

Similarly, let $Z \subset X$ be a compact analytic subset of pure dimension d, where $1 \leq d \leq n$, and let $L_1, L_2, \ldots, L_d$ be holomorphic line bundles on X. We write

$$\begin{aligned}(Z.L_1.L_2.\ldots.L_d) &:= \int_Z c_1(L_1) \wedge c_1(L_2) \wedge \cdots \wedge c_1(L_d) \\ &= T_Z\left(\frac{\sqrt{-1}}{2\pi}\Theta_{h_1} \wedge \frac{\sqrt{-1}}{2\pi}\Theta_{h_2} \wedge \cdots \wedge \frac{\sqrt{-1}}{2\pi}\Theta_{h_d}\right),\end{aligned}$$

where $h_1, h_2, \ldots, h_d$ are C^∞ Hermitian metrics on $L_1, L_2, \ldots, L_d$, respectively.

2.10 Pseudoconvexity, q-convexity, and Stein spaces

Pseudoconvexity is a central notion in several complex variables and may be viewed as a complex-analytic analogue of real convexity. We discuss it here together with the intermediate notions of q-convexity, concentrating mainly on the strongly pseudoconvex case. For a systematic treatment of generalized q-convexity, see [OP].

Hartogs extension and domains of holomorphy

Pseudoconvexity arises naturally in questions of analytic continuation. We first review the situation in complex Euclidean spaces. For a domain in $\mathbb{C}$ bounded by a smooth Jordan curve, one may ask whether every holomorphic function extends across each boundary point; the answer is no. In one complex variable, the function $(z-a)^{-1}$ prevents extension across a prescribed boundary point a. In several variables the geometry of an open set may instead force every holomorphic function on it to extend to a larger open set. The following elementary example is the *Hartogs phenomenon*.

EXAMPLE III.2.147 (A Hartogs figure). For $t > 0$, write $\Delta_t := \{\zeta \in \mathbb{C} \mid |\zeta| < t\}$ and $\Delta := \Delta_1$. For $0 < r, s < 1$, the *Hartogs figure*

$$H_{r,s} := (\Delta_r \times \Delta) \cup \big(\Delta \times (\Delta \setminus \overline{\Delta}_s)\big) \subset \mathbb{C}^2$$

has coordinates (z, w). Every $f \in \mathcal{O}_{H_{r,s}}(H_{r,s})$ extends uniquely to a holomorphic function on Δ^2. Indeed, for $(z, w) \in \Delta^2$ set

$$\widetilde{f}(z, w) := \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta|=\rho} \frac{f(z, \zeta)}{\zeta - w}\, d\zeta, \qquad \max\{s, |w|\} < \rho < 1.$$

Cauchy's theorem on annuli makes this independent of ρ. Locally one may fix ρ, so the integral is jointly holomorphic. As in Remark I.9.1, the existence of this Cauchy integral does not by itself say that it continues the original function. For $|z| < r$, however, the whole w-disc is available, and Cauchy's formula gives $\widetilde{f} = f$. The identity theorem on the connected set $H_{r,s}$ proves the required agreement everywhere; it also proves uniqueness. □

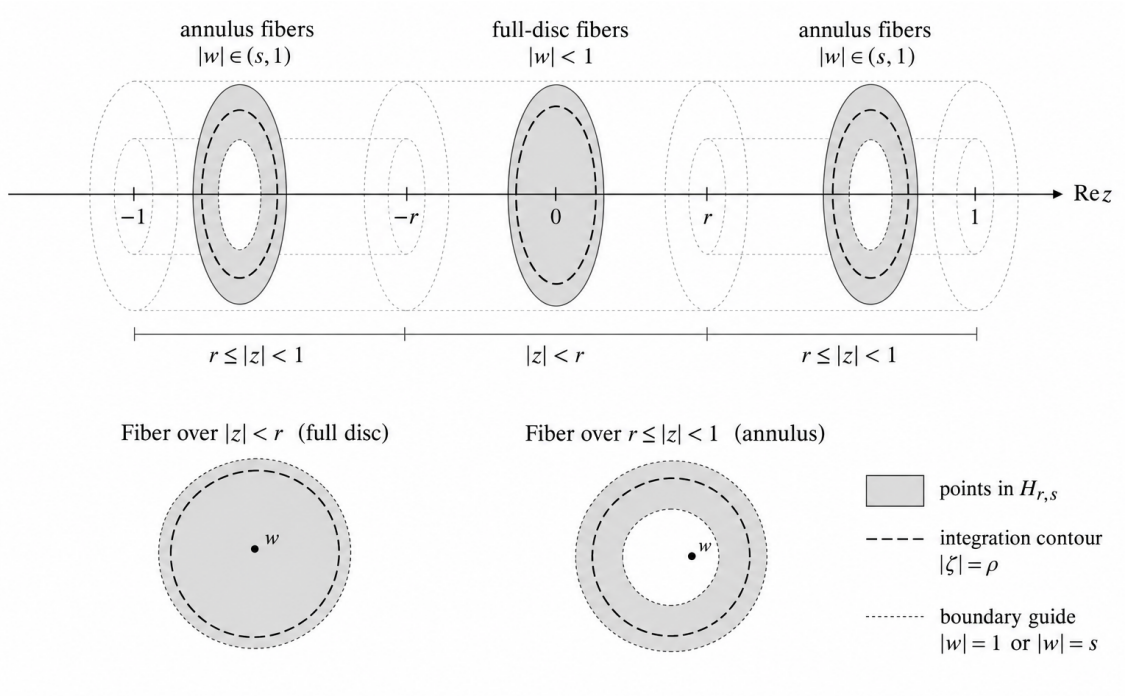


Figure III.3: The Hartogs figure $H_{r,s}$. Over $|z| < r$ the w-fiber is an entire disc, while over $r \le |z| < 1$ it is an annulus. The dashed circles indicate the contours $|\zeta| = \rho$ used in the fiberwise Cauchy integral.

A useful form of the same phenomenon is the *Kugelsatz*: if $B \subset \mathbb{C}^n$ is a ball, $n \ge 2$, and $K \subset B$ is compact with $B \setminus K$ connected, every holomorphic function on $B \setminus K$ extends to B. A proof using the Bochner–Martinelli kernel is given in [De4, Chapter I, Theorem (3.28)]. Here the results already prepared give a short cohomological proof. Example III.2.117 gives

$H^{n,n}(B) = H^{n,n-1}(B) = 0$. The first group is Hausdorff, so Theorem III.2.132 yields the TVS isomorphism

$$H_c^{0,1}(B) \cong H^{n,n-1}(B)^\vee = 0.$$

The Dolbeault comparison (Theorem III.2.116) and Proposition II.3.72 therefore make

$$\mathcal{O}_B(B) \longrightarrow \varinjlim_{L \subset B} \mathcal{O}_B(B \setminus L)$$

surjective, where L ranges over compact subsets of B. The extension obtained this way agrees with the given function outside a possibly larger compact subset. The identity theorem on $B \setminus K$ gives agreement on its entire original domain. Taking K to be a smaller closed ball fills a spherical shell; taking K to be a point shows that isolated points are removable for holomorphic functions in complex dimension at least two.

This leads to the question of which domains can be maximal domains of definition of holomorphic functions. To see the classical forms of the question, let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary (i.e., $\partial\Omega$ is a C^∞ real hypersurface). The following properties are known to be equivalent:

- *Domain of holomorphy.* For every connected open set U meeting $\partial\Omega$ and every connected component V of $U \cap \Omega$, some $f \in \mathcal{O}_\Omega(\Omega)$ has a restriction $f|_V$ which does not extend holomorphically to U.

- *Holomorphic convexity.* For every compact $\Gamma \subset \Omega$, the holomorphic hull

$$\widehat{\Gamma}_\Omega := \{x \in \Omega \mid |f(x)| \le \sup_\Gamma |f| \text{ for every } f \in \mathcal{O}_\Omega(\Omega)\}$$

  is compact.

- *Weak* 1*-completeness.* The domain Ω admits a C^∞ plurisubharmonic exhaustion function ρ, meaning that $\{\rho < c\}$ has compact closure in Ω for every $c \in \mathbb{R}$.

- *Strong* 1*-completeness.* The domain Ω admits a C^∞ strictly plurisubharmonic exhaustion function.

- *Levi pseudoconvexity of the boundary.* The Levi form of $\partial\Omega$, with the sign determined by the side Ω, is positive semi-definite at every boundary point.

This equivalence characterizes domains of holomorphy by complex-analytic convexity conditions. Accordingly, domains satisfying these equivalent conditions are called *pseudoconvex domains*. For proofs, see [De4, Chapter I, Theorems (6.11), (7.2), and (7.12), and Chapter VIII, Theorem (9.11)(a)]. A strictly plurisubharmonic exhaustion is smooth only on Ω and may tend to infinity at the boundary; its existence does not require the boundary Levi form to be positive definite. Without boundary regularity one can use the exhaustion condition. A domain is *locally pseudoconvex at its boundary* if every boundary point has a neighborhood U for which $U \cap \Omega$ is pseudoconvex. In $\mathbb{C}^n$ this characterizes pseudoconvexity;

see [De4, Chapter I, Theorem (7.7)]. For a C^2 boundary it is equivalent to the local Levi condition above.

The equivalence with a domain of holomorphy is the classical *Levi problem*. Kiyoshi Oka solved it for domains over $\mathbb{C}^n$ without interior ramification, including ordinary domains in $\mathbb{C}^n$; see [Ok]. Such domains arise when analytic continuation requires several sheets; the map need not be a topological covering of its image. On general complex manifolds and complex spaces, the Levi problem has further forms, and the different notions of boundary convexity, exhaustion, and holomorphic convexity must be distinguished.

Observation III.2.148 (The boundary Levi form and holomorphic functions). Assume $n \geq 2$. Fix a boundary point and the side $D = \{r < 0\}$. Lemma III.1.14 gives local holomorphic coordinates $(z^1, z^2, \dots, z^n)$ centered at that point. Set $z' := (z^1, \dots, z^{n-1})$ and $w := z^n$, and write these coordinates as (z', w). In these coordinates,

$$D = \left\{ \operatorname{Im} w > \sum_{j=1}^{n-1} \lambda_j |z^j|^2 + R(z', \operatorname{Re} w) \right\}, \qquad R(z', u) = o(|z'|^2 + |u|^2).$$

All statements here concern a sufficiently small coordinate neighborhood and this fixed side. If every λ_j is positive, the complex hyperplane $\{w = 0\}$ meets $\overline{D}$ near the origin only at the origin. Consequently $1/w$ is holomorphic on the local side D and its absolute value tends to infinity as a point of D tends to the origin. In the original coordinates the supporting set is a holomorphic hypersurface, not necessarily a linear hyperplane; this construction is local, not a claim about a global function.

If $\lambda_1 < 0$, choose $a > 0$ and $t_0 > 0$ sufficiently small, and set

$$\Delta := \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}.$$

For each $t \in (0, t_0]$, define a holomorphic map

$$\iota_t \colon \Delta \to D, \qquad \iota_t(\zeta) := (a\zeta, 0, \dots, 0, \sqrt{-1}t).$$

We call the image $\iota_t(\Delta)$ an analytic disc. Since this formula extends holomorphically to a neighborhood of $\overline{\Delta}$, the boundary circles, even at $t = 0$, form a fixed compact subset $\Gamma \subset D$. The maximum modulus principle gives

$$|F(\iota_t(0))| \leq \max_{|\zeta|=1} |F(\iota_t(\zeta))| \leq \sup_{\Gamma} |F|$$

for every holomorphic function F on D. Thus the centers belong to $\widehat{\Gamma}_D$ but tend to the boundary, contradicting holomorphic convexity. The conclusion is a uniform bound along the centers, not constancy on the discs. This explains directly why a negative Levi direction is incompatible with holomorphic convexity, while a positive definite Levi form permits a local holomorphic function with a boundary pole. □

Mini-Exercise III.2.149. Let X be a complex manifold and $\Omega \subset X$ a relatively compact domain (i.e., its closure in X is compact) with C^2 boundary. Assume that, with the sign determined by the side Ω, the Levi form of $\partial\Omega$ has at least one negative eigenvalue at every boundary point. Show that every holomorphic function on a connected open neighborhood of $\overline{\Omega}$ is constant.

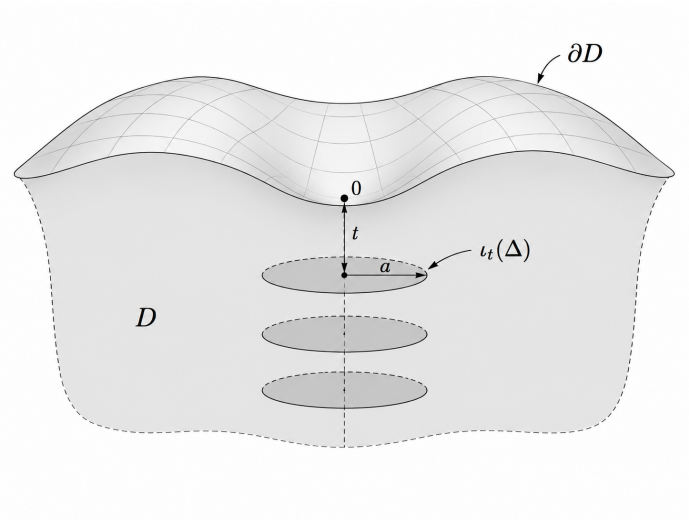


Figure III.4: Schematic picture of the analytic discs $\iota_t(\Delta) \subset D$ approaching the boundary point 0 in Observation III.2.148.

Exhaustions and strong or weak q-convexity

DEFINITION III.2.150. Let X be a complex manifold of pure dimension n, and let $q \in \mathbb{Z}_{\geq 1}$. A real-valued C^2 function φ is *strongly q-convex* at x if its complex Hessian has at least $n-q+1$ positive eigenvalues; it is *weakly q-convex* at x if it has at least $n-q+1$ non-negative eigenvalues. For $q \leq n$, equivalently its Levi form restricts to a positive definite, respectively semi-positive, form on a complex subspace of dimension $n-q+1$. If $q > n$, the condition is understood to be vacuous. On a complex space $(X, \mathcal{O}_X)$, require locally an embedding in an open set of $\mathbb{C}^N$ and a C^2 ambient extension which has at least $N-q+1$ positive, respectively non-negative, Levi eigenvalues on a neighborhood of the embedded patch; the condition is vacuous if $q > N$. This condition is independent of the local embedding; see [De4, Chapter IX, Definition (2.5) and Lemma (2.6)]. As with plurisubharmonic functions, pointwise functions on a non-reduced space are understood on its reduction. Strong and weak 1-convexity are, respectively, strict plurisubharmonicity and plurisubharmonicity in this regularity class. □

DEFINITION III.2.151. A complex space $(X, \mathcal{O}_X)$ is *strongly q-convex*, respectively *weakly q-convex*, if it admits a C^∞ exhaustion ρ which is strongly, respectively weakly, q-convex outside a compact subset of X. Here an exhaustion means that $\{\rho < c\}$ has compact closure in X for every real c. If the corresponding Levi condition holds on all of X, the space is called *strongly q-complete*, respectively *weakly q-complete*. □

With these definitions, weak q-convexity and weak q-completeness are equivalent. Indeed, choose c so that the exceptional compact set lies in $\{\rho < c\}$, and take a C^∞ convex function χ which is zero on $(-\infty, c]$, strictly increasing after c, and tends to infinity. The identity

$$\sqrt{-1}\partial\overline{\partial}(\chi \circ \rho) = \chi'(\rho)\sqrt{-1}\partial\overline{\partial}\rho + \chi''(\rho)\sqrt{-1}\partial\rho \wedge \overline{\partial}\rho$$

shows that $\chi\circ\rho$ is a weakly q-convex exhaustion everywhere. This flattening does not preserve strict positivity.

EXAMPLE III.2.152. For $R > 0$, the ball $\{|z| < R\} \subset \mathbb{C}^n$ is strongly 1-complete: $-\log(R^2 - |z|^2)$ is a strictly plurisubharmonic exhaustion. A compact complex space is weakly 1-complete, since a constant function is a plurisubharmonic exhaustion. A positive-dimensional compact complex space is not strongly 1-complete, by Lemma III.2.91 and strict positivity. A closed complex subspace of a strongly (respectively weakly) q-convex space is again strongly (respectively weakly) q-convex. If the ambient space is strongly (respectively weakly) q-complete, so is the closed subspace: restrict the corresponding exhaustion. See [De4, Chapter IX, Theorem (2.9)]. □

The boundary notions above are local, whereas an exhaustion is a global function. For example, weak 1-completeness on a general complex manifold does not assert that it is a domain in $\mathbb{C}^n$ or that it has a specified boundary.

LEMMA III.2.153. *Every weakly* 1*-complete Kähler manifold admits a complete Kähler metric.*

PROOF. We use the same construction as in Step 1 of the proof of Hörmander's estimate for singular metrics (Theorem III.2.113). Let X be a weakly 1-complete Kähler manifold and choose a C^∞ plurisubharmonic exhaustion φ. Adding a constant, assume $\varphi \geq 1$. For a given Kähler form ω, put

$$\widehat{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}\varphi^2 = \omega + 2\varphi\sqrt{-1}\partial\overline{\partial}\varphi + 2\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi.$$

This is Kähler, and the last term bounds the norm of $d\varphi$ for its Riemannian metric by a constant. Hence φ is bounded above on every distance ball. The exhaustion property makes closed distance balls compact, so Hopf–Rinow proves completeness. For example, on a coordinate ball $\{|z| < R\}$, a constant translate of $-\log(R^2 - |z|^2)$ is such an exhaustion. □

PROPOSITION III.2.154. *A connected non-compact complex manifold of pure dimension $n \geq 1$ is strongly n-complete. In particular, every open Riemann surface (that is, a non-compact Riemann surface) is strongly* 1*-complete.*

PROOF. For the general case, see [De4, Chapter IX, Theorem (3.5)], which gives an exhaustion with positive Levi trace for a C^∞ Hermitian metric. For simplicity, we explain the Kähler case. Suppose that X has a Kähler form ω and associated Riemannian metric g. For a real $(1,1)$-form η, $\mathrm{tr}_\omega\eta$ denotes the sum of its eigenvalues relative to ω. Greene–Wu's theorem (Theorem I.8.66) gives real-valued g-harmonic functions $h_1, h_2, \ldots, h_{4n+1}$ defining a proper C^∞ embedding into $\mathbb{R}^{4n+1}$. Thus $\rho := \sum_{j=1}^{4n+1} h_j^2$ is a nonnegative C^∞ exhaustion. The Kähler normal coordinates of Remark III.1.25 identify the Levi trace with $-\frac{1}{2}\Delta_g$. This is the use of the Kähler assumption; the product rule and harmonicity then give

$$\mathrm{tr}_\omega(\sqrt{-1}\partial\overline{\partial}\rho) = -\frac{1}{2}\Delta_g\rho = \sum_{j=1}^{4n+1}\left(|dh_j|_g^2 - h_j\Delta_g h_j\right) = \sum_{j=1}^{4n+1}|dh_j|_g^2 > 0,$$

where strict positivity follows because the embedding is an immersion. The Levi form therefore has at least one positive eigenvalue at each point, so ρ is strongly n-convex.

Taking $n = 1$ proves the last assertion. □

Vanishing and Stein spaces

THEOREM III.2.155. *Let X be a strongly 1-complete complex manifold of pure dimension $n \geq 1$, and let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank. Then $H^q(X, \mathcal{F}) = 0$ for $q \geq 1$. In particular, $H^{p,q}(X, E) = 0$ for every holomorphic vector bundle E, every p, and every $q \geq 1$.*

PROOF. Work on the open and closed constant-rank parts. By the correspondence with holomorphic vector bundles (Proposition III.2.58), write $\mathcal{F} = \mathcal{O}_X(E)$ for a holomorphic vector bundle E. Choose a strictly plurisubharmonic exhaustion $\psi \geq 1$. The form $\omega_0 := \sqrt{-1}\partial\overline{\partial}\psi$ is Kähler, and $\omega := \omega_0 + \sqrt{-1}\partial\overline{\partial}\psi^2$ is complete by Lemma III.2.153. Set $G := K_X^{-1} \otimes E$ and fix a C^∞ Hermitian metric h_0 on G. Identify E-valued $(0,q)$-forms with G-valued (n,q)-forms by wedging on the right with a local holomorphic volume form and tensoring with its dual. This identification is independent of the volume form and commutes with $\overline{\partial}$. Let $0 < \gamma_1(x) \leq \gamma_2(x) \leq \cdots \leq \gamma_n(x)$ be the eigenvalues of $\sqrt{-1}\partial\overline{\partial}\psi$ relative to ω. This is the curvature form of the trivial line bundle with metric $e^{-\psi}$. Thus (183) applies to its action on forms, coefficientwise on G. At a fixed point, choose an ω-orthonormal coframe diagonalizing this form. On a component with antiholomorphic multi-index J, where $\#J = q$, the endomorphism $[\sqrt{-1}\partial\overline{\partial}\psi \wedge \bullet, \Lambda_\omega]$ acts by multiplication by $\sum_{j\in J}\gamma_j$. The holomorphic degree n makes the other two sums in (183) cancel. Consequently its smallest eigenvalue is

$$s_q := \gamma_1 + \gamma_2 + \cdots + \gamma_q \geq q\gamma_1 > 0.$$

This is where the positive degree q enters the estimate.

Fix a smooth $\overline{\partial}$-closed representative g in the above description, with $1 \leq q \leq n$. We choose a weight for this g. Choose a continuous $M \geq 0$ such that $A^{n,q}_{h_0,\omega} \geq -M\mathbf{I}$, where $\mathbf{I}$ is the identity on $(\Lambda^{n,q}X \otimes G)_x$ at each $x \in X$. For $h := h_0 e^{-\chi(\psi)}$, where χ is a C^∞ convex increasing function, the curvature changes by

$$\sqrt{-1}\Theta_h = \sqrt{-1}\Theta_{h_0} + \sqrt{-1}\partial\overline{\partial}(\chi\circ\psi)\otimes \mathrm{id}_G.$$

Since $\sqrt{-1}\partial\overline{\partial}(\chi\circ\psi) = \chi'(\psi)\sqrt{-1}\partial\overline{\partial}\psi + \chi''(\psi)\sqrt{-1}\partial\psi\wedge\overline{\partial}\psi$, and the last semi-positive term induces a nonnegative commutator on (n,q)-forms, the preceding eigenvalue computation gives

$$A^{n,q}_{h,\omega} \geq \big(\chi'(\psi)s_q - M\big)\mathbf{I}.$$

On each compact sublevel set, s_q has a positive lower bound and M has a finite upper bound. Choose χ' sufficiently large on successive sublevels so that $\chi'(\psi)s_q - M \geq 1$, and choose χ sufficiently large that

$$\sum_{j\geq 1} e^{-\chi(j)} \int_{\{j\leq\psi\leq j+1\}} |g|^2_{h_0,\omega}\, dV_\omega < \infty.$$

Such simultaneous choices follow by taking increasing convex majorants of the sublevel bounds and integrating, as in Mini-Exercise I.4.11. Thus both the curvature and the growth of g have been taken into account. Nakano's inequality (178) now gives, for every $u \in \mathcal{A}_c^{n,q}(X,G)$,

$$\|\overline{\partial} u\|_{h,\omega}^2 + \|\overline{\partial}_{h,\omega}^{\dagger} u\|_{h,\omega}^2 \geq \int_X \langle A_{h,\omega}^{n,q} u, u\rangle_{h,\omega}\, dV_\omega \geq \|u\|_{h,\omega}^2.$$

The complete-metric density argument extends this estimate to the joint L^2 domain. Theorem III.2.107, with $c = 1$, then gives a weighted L^2 current f of type $(n, q-1)$ satisfying $\overline{\partial} f = g$. The weight is smooth and positive on every compact subset, so f is locally L^2. The smooth/current comparison in the Dolbeault isomorphism (Theorem III.2.116) makes the original cohomology class zero. Degrees $q > n$ vanish by the length of the Dolbeault resolution. Apply the same result to $\Omega_X^p(E)$ for the last assertion. □

Corollary III.2.156. *Let X be a complex manifold and let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank. Every open covering $\mathcal{U} = \{U_j\}_{j\in J}$ of X by strongly 1-complete open subsets is Leray for $\mathcal{F}$. In particular, X admits a locally finite covering by relatively compact coordinate balls whose coordinate maps are defined on neighborhoods of their closures, and every such covering is Leray for $\mathcal{F}$.*

Proof. Choose a nonnegative smooth strictly plurisubharmonic exhaustion ρ_j on each U_j. On a nonempty finite intersection $U_I := \bigcap_{j\in I} U_j$, put

$$\rho_I := \sum_{j\in I} \rho_j|_{U_I}.$$

This is strictly plurisubharmonic. Each sublevel set $\{\rho_I \leq c\}$ is a closed subset of the compact set $\bigcap_{j\in I}\{\rho_j \leq c\} \subset U_I$, so ρ_I is an exhaustion. Thus U_I is strongly 1-complete, and Theorem III.2.155, applied on its positive-dimensional connected components, gives $H^q(U_I, \mathcal{F}) = 0$ for $q \geq 1$. Zero-dimensional components are points and have no higher cohomology. This is the Leray property. Finally, a coordinate ball $\{|z| < R\}$ has the strictly plurisubharmonic exhaustion $1/(R^2 - |z|^2)$, and paracompactness supplies a locally finite covering by sufficiently small balls of the stated kind. □

Definition III.2.157. A complex space $(X, \mathcal{O}_X)$ is *holomorphically convex* if the hull

$$\widehat{\Gamma}_X := \{x \in X \mid |f(x)| \leq \sup_{\Gamma} |f| \text{ for every } f \in \mathcal{O}_X(X)\}$$

is compact for every compact $\Gamma \subset X$. It is a *Stein space* if it is holomorphically convex and its global holomorphic functions separate distinct points. A non-singular Stein space is called a *Stein manifold*. □

A complex space is Stein if and only if it is strongly 1-complete; see [GPR, Chapter V, Theorems 3.6–3.7, p. 239]. On a Stein manifold, global holomorphic functions produce a strictly plurisubharmonic exhaustion; see [De4, Chapter I, Theorem (6.18)]. Conversely, the

L^2 method just used, together with cutoff and interpolation arguments of the kind used for Kodaira's embedding theorem (Theorem III.2.136), produces functions separating points and controlling holomorphic hulls on successive sublevel sets. This is a route from an exhaustion to the Stein properties; the cited results provide the full argument. A Stein manifold of pure dimension n admits a proper holomorphic embedding in $\mathbb{C}^N$ for some finite N; one may take $N = 2n + 1$ in the classical embedding theorem of Remmert, Bishop, and Narasimhan. See [Forstn, Theorem 2.1(a) and §2.1] for this statement and the embedding and approximation arguments. Properness here is a global assertion, not just the existence of local holomorphic coordinates. In particular, open Riemann surfaces are Stein.

We next record the basic $q = 1$ finiteness statement in the Andreotti–Grauert theory, in the form used most often in these notes. It concerns a relatively compact domain with smooth strongly pseudoconvex boundary inside an arbitrary complex manifold, and its proof is based on the classical bumping method of Grauert [Gr1]. See also [OP, Theorem 4.1].

Let X be a complex manifold and $\Omega \subset X$ a relatively compact domain with C^∞ strongly pseudoconvex boundary (i.e., $\overline{\Omega}$ is compact in X). Put $M := \partial\Omega$. Here smooth boundary includes the one-sided condition: locally near M, there is a real-valued C^∞ function r such that $\Omega = \{r < 0\}$, $M = \{r = 0\}$, and $dr \neq 0$ on M. Strong pseudoconvexity means that the Levi form of r is positive definite on $T^{1,0}M$ along M.

Lemma III.2.158 (Grauert's bumping technique). *Let X be a complex manifold, let $\Omega \subset X$ be a relatively compact domain with C^∞ strongly pseudoconvex boundary, and put $M := \partial\Omega$. Assume $M \neq \emptyset$. There exist a tubular neighborhood $W \subset X$ of M with compact closure in X, finitely many coordinate open balls $B_1, B_2, \dots, B_N \subset W$ covering M, each with compact closure contained in W, and relatively compact domains*

$$\Omega_0 = \Omega \subset \Omega_1 \subset \cdots \subset \Omega_N \subset X$$

with the following properties.

1. *$\overline{\Omega} \subset \Omega_N$, and every Ω_j has C^∞ strongly pseudoconvex boundary.*
2. *For $j = 1, 2, \dots, N$,*

$$\Omega_j = \Omega_{j-1} \cup (\Omega_j \cap B_j), \qquad \overline{\Omega_j \setminus \Omega_{j-1}} \subset B_j. \tag{205}$$

3. *For $j \in \{1, 2, \dots, N\}$, both $\Omega_j \cap B_j$ and $\Omega_{j-1} \cap B_j$ are strongly 1-complete.*

The final domain Ω_N can be chosen relatively compact in any prescribed open neighborhood of $\overline{\Omega}$.

Proof. We use the cutoff construction of [De4, Chapter IX, Lemma (4.12)], in its boundary-local, strongly 1-complete form. We include the preparation of the defining function and the control of the intermediate boundaries. See also Figure III.5.

Step 1: a strictly plurisubharmonic defining function. Take finitely many local defining functions r_α, all negative on the side Ω, and a nonnegative subordinate C^∞ partition of unity

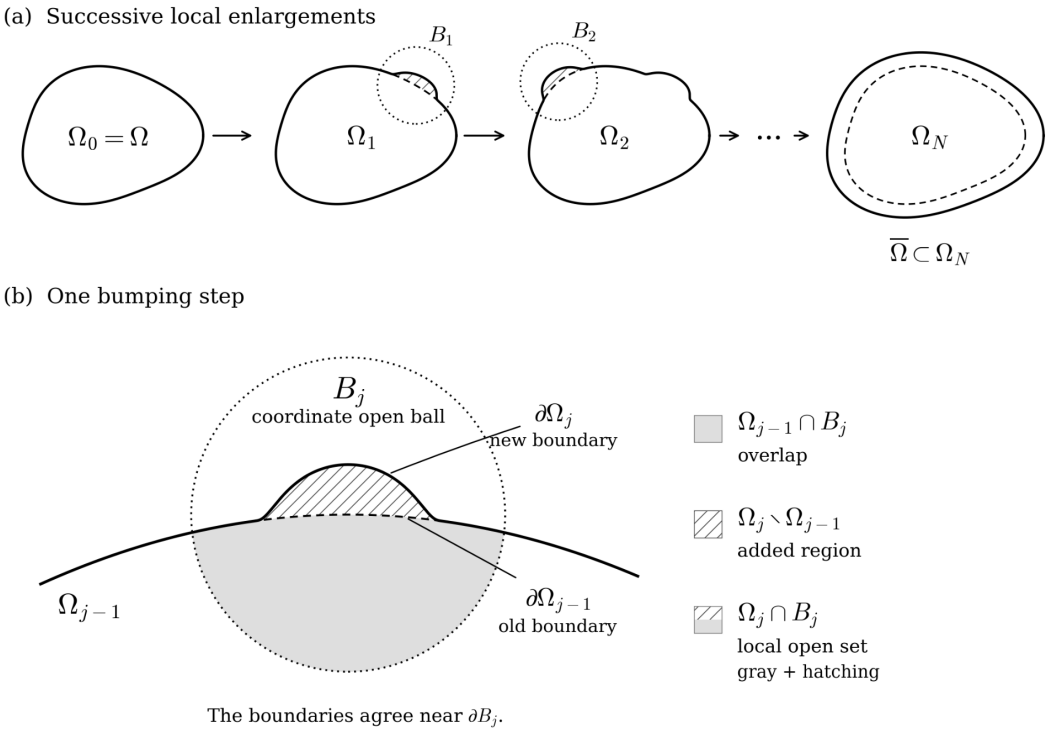


Figure III.5: Finite geometric bumping. The local open set $\Omega_j \cap B_j$ consists of the overlap $\Omega_{j-1} \cap B_j$ and the added set $\Omega_j \setminus \Omega_{j-1}$.

$\{\chi_\alpha\}$ near M. After restricting to a common neighborhood of M, set $r := \sum_\alpha \chi_\alpha r_\alpha$. At a point of M, all nonzero covectors dr_α are positive multiples of one another. Since $r_\alpha = 0$ on M,

$$dr|_M = \sum_\alpha \chi_\alpha \, dr_\alpha|_M \neq 0.$$

The sign of each local defining function also shows that $r < 0$ precisely on the side Ω near M. For complex tangent vectors, the terms involving derivatives of χ_α in $\partial\overline{\partial} r$ vanish on M. Thus the tangential Levi form of r is a positive weighted sum of the tangential Levi forms of the r_α, and is positive definite.

The covector $dr|_M$ trivializes the real normal line bundle $TX|_M / TM$. The tubular neighborhood theorem (Theorem I.8.4) therefore gives a parametrization of a neighborhood of M by $M \times (-a, a)$. The derivative of r in the normal direction is non-zero along M; reparametrizing each fiber by r and using compactness of M gives a common interval on which r is the normal coordinate.

Choose a positive constant A and put

$$\rho := \frac{e^{Ar} - 1}{A}.$$

This preserves the zero set and the sign of r, and

$$\sqrt{-1}\partial\overline{\partial}\rho = e^{Ar}\bigl(\sqrt{-1}\partial\overline{\partial}r + A\sqrt{-1}\partial r \wedge \overline{\partial}r\bigr). \tag{206}$$

By tangential positivity and compactness of M, a sufficiently large A makes ρ strictly plurisubharmonic near M. Since $d\rho = e^{Ar}dr$, a final reparametrization of the tubular fibers gives, after shrinking,

$$\kappa\colon M \times (-a, a) \longrightarrow W, \qquad \rho(\kappa(p,t)) = t,$$

where $a > 0$ and W has compact closure in X. We keep ρ strictly plurisubharmonic on W and have

$$M = \{x \in W \mid \rho(x) = 0\}, \qquad \Omega \cap W = \{x \in W \mid \rho(x) < 0\}. \tag{207}$$

The tubular neighborhood is chosen inside the prescribed neighborhood of $\overline{\Omega}$ when one has been specified.

Step 2: coordinate balls and cutoff functions. Choose coordinate open balls $B_1, B_2, \ldots, B_N \subset W$ covering M, with compact closures contained in W and with holomorphic coordinates z_j defined near $\overline{B}_j$ and radii $R_j > 0$ such that $z_j(B_j) = \{\zeta \in \mathbb{C}^n \mid |\zeta| < R_j\}$, where n is the dimension of the component containing Ω. Choose nonnegative real-valued functions $\theta_j \in C_c^\infty(B_j)$, extended by zero to W, such that

$$0 \le \sum_{j=1}^{N} \theta_j \le 1 \quad \text{on } W, \qquad \sum_{j=1}^{N} \theta_j = 1 \quad \text{near } M. \tag{208}$$

For instance, take a partition of unity subordinate to $B_1, B_2, \ldots, B_N, X \setminus M$ and discard the member associated with $X \setminus M$. For a positive parameter ε, define

$$\rho_0 := \rho, \qquad \rho_j := \rho - \varepsilon \sum_{k=1}^{j} \theta_k = \rho_{j-1} - \varepsilon\theta_j \quad (1 \le j \le N).$$

Choose $\varepsilon > 0$ sufficiently small so that all ρ_j remain strictly plurisubharmonic on W. This follows from C^2-stability of strict positivity on the compact union of the supports; outside that union nothing changes. Shrink ε further so that $\varepsilon < a/2$,

$$\frac{\partial}{\partial t}\bigl(\rho_j(\kappa(p,t))\bigr) > \frac{1}{2} \quad (p \in M,\ -a < t < a,\ 0 \le j \le N), \tag{209}$$

and the last equality in (208) holds whenever $|\rho| \le 2\varepsilon$ in W. The derivative bound is possible because the cutoff functions are fixed and compactly supported. The last requirement follows from compactness of M.

Step 3: the successive domains. Set

$$\Omega_j := \Omega \cup \{x \in W \mid \rho_j(x) < 0\} \quad (0 \le j \le N).$$

By (207) and $\rho_j \le \rho_{j-1} \le \rho$, these sets are open, $\Omega_0 = \Omega$, and $\Omega_{j-1} \subset \Omega_j$. They are relatively compact: every added point lies in the compact union of the supports of the cutoff functions, together with $\overline{\Omega}$.

For each fixed $p \in M$, (209) shows that $\rho_j(\kappa(p,t)) = 0$ has exactly one solution $t = h_j(p)$, with $0 \le h_j(p) \le \varepsilon$. The implicit function theorem gives $h_j \in C^\infty(M)$, and the part of Ω_j in the tubular neighborhood is exactly $t < h_j(p)$. Every added point can therefore be joined inside Ω_j to a point of Ω by decreasing t. Hence Ω_j is connected. The compact support of the modification produces no new boundary at the ends of the tubular neighborhood. Thus $\partial\Omega_j = \{\rho_j = 0\} \subset W$ is smooth, and it is strongly pseudoconvex because ρ_j is strictly plurisubharmonic there.

At the final step, (208) and the bound on the sum give

$$\Omega_N = \Omega \cup \{x \in W \mid \rho(x) < \varepsilon\}, \qquad \partial\Omega_N = \{x \in W \mid \rho(x) = \varepsilon\}.$$

In particular, $\overline{\Omega} \subset \Omega_N$.

Since $\rho_j - \rho_{j-1} = -\varepsilon\theta_j$ has support compactly contained in B_j, a point of $\Omega_j \setminus \Omega_{j-1}$ must satisfy $\theta_j > 0$. Consequently

$$\overline{\Omega_j \setminus \Omega_{j-1}} \subset \operatorname{Supp}\theta_j \subset B_j,$$

and (205) follows. Moreover,

$$\Omega_j \cap B_j = \{x \in B_j \mid \rho_j(x) < 0\},$$
$$\Omega_{j-1} \cap B_j = \{x \in B_j \mid \rho_{j-1}(x) < 0\}.$$

The overlap for the union in (205) is exactly

$$\Omega_{j-1} \cap (\Omega_j \cap B_j) = \Omega_{j-1} \cap B_j.$$

Step 4: strong 1-completeness of the local pieces. For each $\ell \in \{j-1, j\}$, consider on $\Omega_\ell \cap B_j$ the function

$$x \longmapsto -\frac{1}{\rho_\ell(x)} + \frac{1}{R_j^2 - |z_j(x)|^2}. \tag{210}$$

For $t < 0$, the function $f(t) = -1/t$ has $f'(t) = 1/t^2 > 0$ and $f''(t) = -2/t^3 > 0$. The first term in (210) is therefore strictly plurisubharmonic by the composition formula, and the coordinate-ball barrier in the second term is strictly plurisubharmonic as well. Both terms are positive. A bounded sublevel stays away from $\rho_\ell = 0$ and from ∂B_j; since $\overline{B}_j$ is compact, its closure is a compact subset of $\Omega_\ell \cap B_j$. Thus (210) is a C^∞ strictly plurisubharmonic exhaustion, proving the last assertion. □

We now apply the vanishing theorem for locally free sheaves of finite rank on strongly 1-complete complex manifolds (Theorem III.2.155) to the Mayer–Vietoris sequence for the covering $\Omega_j = \Omega_{j-1} \cup (\Omega_j \cap B_j)$.

COROLLARY III.2.159. *Let X be a complex manifold and let $\Omega \subset X$ be a relatively compact domain whose boundary $M := \partial\Omega$ is nonempty, C^∞, and strongly pseudoconvex. Let $\mathcal{F}$ be a locally free sheaf of finite rank on an open neighborhood of $\overline{\Omega}$. Choose the domains in Grauert's bumping technique (Lemma III.2.158) inside that neighborhood. Then the restriction maps*

$$H^q(\Omega_j, \mathcal{F}) \longrightarrow H^q(\Omega_{j-1}, \mathcal{F})$$

are surjective for $q \geq 1$ and are isomorphisms for $q \geq 2$. In particular, the same conclusions hold for

$$H^q(\Omega_N, \mathcal{F}) \longrightarrow H^q(\Omega, \mathcal{F}). \tag{211}$$

PROOF. By Grauert's bumping technique (Lemma III.2.158) and Theorem III.2.155,

$$H^q(\Omega_j \cap B_j, \mathcal{F}) = H^q(\Omega_{j-1} \cap B_j, \mathcal{F}) = 0 \quad (q \geq 1). \tag{212}$$

Apply the Mayer–Vietoris sequence (Proposition II.3.67) to the open covering $\Omega_j = \Omega_{j-1} \cup (\Omega_j \cap B_j)$. The relevant part is

$$\begin{aligned} H^{q-1}(\Omega_{j-1} \cap B_j, \mathcal{F}) &\longrightarrow H^q(\Omega_j, \mathcal{F}) \\ &\longrightarrow H^q(\Omega_{j-1}, \mathcal{F}) \oplus H^q(\Omega_j \cap B_j, \mathcal{F}) \\ &\longrightarrow H^q(\Omega_{j-1} \cap B_j, \mathcal{F}). \end{aligned}$$

The middle map consists of restrictions, and the last map is the first restriction minus the second. For $q \geq 1$, the last group and the second summand vanish, giving surjectivity. For $q \geq 2$, the first group also vanishes, giving injectivity. Composing the maps proves (211). □

The last step is the compactness argument on Čech complexes. Use the Fréchet topology on spaces of sections and compactness of restriction to relatively compact open subsets recalled before compact finiteness and Serre duality (Lemma III.2.121). We use the compact shrinking criterion for Čech cohomology (Lemma III.2.122).

PROPOSITION III.2.160 (Finiteness with locally free coefficients). *Let X be a complex manifold and let $\Omega \subset X$ be a relatively compact domain with nonempty C^∞ strongly pseudoconvex boundary. Let $\mathcal{F}$ be a locally free sheaf of finite rank on an open neighborhood of $\overline{\Omega}$. Then $H^q(\Omega, \mathcal{F})$ is finite-dimensional and Hausdorff for $q \geq 1$, with the kernel-quotient topology of the finite Leray covering for $\mathcal{F}$ constructed below.*

PROOF. Choose finitely many pairs of concentric coordinate open balls $U_\alpha^* \subset U_\alpha \subset \Omega_N$, with each closure compact and contained in the next open set, such that the U_α^* cover $\overline{\Omega}$ and each U_α is contained either in Ω or in W. To obtain them, first cover M by sufficiently small pairs inside $W \cap \Omega_N$, and then cover the remaining compact subset of Ω by pairs inside Ω. Set

$$V := \bigcup_\alpha U_\alpha, \qquad \mathcal{U} := \{U_\alpha\}_\alpha, \qquad \mathcal{U}^* := \{U_\alpha^* \cap \Omega\}_\alpha.$$

Thus $\overline{\Omega} \subset V \subset \Omega_N$; $\mathcal{U}$ covers V, whereas $\mathcal{U}^*$ covers Ω.

Every nonempty finite intersection of the U_α is strongly 1-complete: sum the positive coordinate-ball barriers $1/(R_\alpha^2 - |z_\alpha|^2)$ on the intersection, where z_α and R_α are the coordinates and radius of U_α. The same argument applies to intersections of the smaller balls. For an intersection occurring in $\mathcal{U}^*$, if one of its balls lies in Ω, intersecting with Ω makes no further change. Otherwise all its balls lie in W; add $-1/\rho$ to their ball barriers on $\{\rho < 0\}$, as in (210). This gives a strictly plurisubharmonic exhaustion in either case. Theorem III.2.155 therefore makes both coverings acyclic for $\mathcal{F}$.

By Corollary III.2.159, the composite

$$H^q(\Omega_N, \mathcal{F}) \longrightarrow H^q(V, \mathcal{F}) \longrightarrow H^q(\Omega, \mathcal{F})$$

is surjective for $q \geq 1$. Hence the second arrow is surjective. By the naturality of the Leray isomorphisms (Theorem II.3.79), it is induced by

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^\bullet(\mathcal{U}^*, \mathcal{F}). \tag{213}$$

For a nonempty finite index tuple I, write $U_I := \bigcap_{\alpha \in I} U_\alpha$ and $U_I^* := \Omega \cap \bigcap_{\alpha \in I} U_\alpha^*$. Then $\overline{U_I^*}$ is compact and contained in U_I, so $\mathcal{F}(U_I) \longrightarrow \mathcal{F}(U_I^*)$ is compact. There are only finitely many factors in each degree. Thus (213) is a continuous morphism of Fréchet complexes, compact in every degree. The compact shrinking criterion for Čech cohomology (Lemma III.2.122), applied on V to $\mathcal{U}$ and $\mathcal{U}^*$, gives the conclusion. □

Remark III.2.161. For locally free coefficients on a complex manifold, the quotient topology obtained from smooth Dolbeault cochains agrees with that obtained from a Leray Čech covering, as used in finiteness with locally free coefficients (Proposition III.2.160). This is the natural topology on cohomology. For the definition, see (218) below; the Čech–Dolbeault comparison following that equation proves the agreement. □

The corresponding finiteness result with coherent coefficients is part of the Andreotti–Grauert theory; see [AG] and [De4, Chapter IX, Theorem (4.10) and Proposition (4.13)]. Grauert's bumping technique (Lemma III.2.158) does not depend on the coefficient sheaf, and the Mayer–Vietoris argument uses only the local vanishings (212). The coefficient sheaf is defined on a neighborhood of $\overline{\Omega}$; extension of a sheaf given only on Ω is not being assumed.

2.11 Coherent analytic sheaves and the Oka–Cartan theorem

The L^2 arguments in §2.8 were formulated for differential forms with values in holomorphic vector bundles, whose sheaves of holomorphic sections are locally free of finite rank. We now turn to arbitrary coherent analytic sheaves and examine the global consequences of coherence. Throughout, a coherent analytic sheaf on a complex space X means a coherent $\mathcal{O}_X$-module, as in Definition III.1.36. Open subsets of a complex space refer to its underlying topological space, with the restricted structure sheaves.

Theorem III.2.162 (Oka–Cartan fundamental theorems on pseudoconvex domains). *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and let $\mathcal{F}$ be a coherent analytic sheaf on Ω. Then:*

(A) *For every $x \in \Omega$, there are finitely many sections in $\Gamma(\Omega, \mathcal{F})$ whose germs generate $\mathcal{F}_x$ over $\mathcal{O}_{\Omega,x}$.*

(B) *$H^q(\Omega, \mathcal{F}) = 0$ for every $q \geq 1$.*

Proof. See [Hö, §7.2, especially Corollary 7.2.6 and Theorems 7.2.7–7.2.9; §7.4, Theorem 7.4.3], and [No1, §§4.2–4.4, Theorem 4.4.2, Corollary 4.4.21(ii), and Remark 4.4.27]. The equivalence between pseudoconvexity and holomorphic convexity is the one recalled in §2.10. Here we explain Oka's method as developed and presented by Noguchi. Noguchi starts with Cartan's merging lemma for generators near adjacent closed cubes. Generators and their coefficient representations are constructed simultaneously. Coherence allows this procedure to be repeated for the relation sheaves, giving *Oka's syzygies* of any prescribed finite length on a common neighborhood of a closed cube. The free terms have vanishing higher cohomology (Theorem III.2.155), and the induced long exact sequences identify the group to be killed with groups in successively higher degrees. A topological bound, which makes cohomology vanish in sufficiently high degrees, completes this step. Exhaustion by smaller cubes gives the coherent vanishing on a polydisc; see [No1, Lemmata 4.3.7–4.3.15].

Holomorphic convexity next gives an exhaustion by analytic polyhedra. These are relatively compact components, or finite unions of such components, of sets defined by finitely many inequalities $|h_j| < 1$ with h_j holomorphic on the ambient domain. Oka's Jôku-Ikô embeds each such polyhedron P as a closed complex submanifold of a higher-dimensional polydisc Q by adjoining the functions h_j as coordinates. Denote this embedding by ι_P. Its closed direct image of $\mathcal{F}|_P$ is coherent by the local presentation argument given below before (216); hence the polydisc result gives

$$H^q(P, \mathcal{F}) \cong H^q(Q, \iota_{P*}(\mathcal{F}|_P)) = 0 \qquad (q \geq 1).$$

Runge–Oka approximation supplies compatible primitives on the exhaustion. In degree one, their differences are sections, so approximation of their coefficients replaces the lower-degree cochain correction available in higher degrees. The limits in this proof are taken for coefficient functions; a topology on coherent sections is not assumed in advance. See [No1, Lemmata 4.4.9 and 4.4.17, and Remark 4.3.32]. In this route, A is then obtained from B by lifting generators modulo the maximal ideal and using Nakayama's lemma. □

The vanishing assertion B is called the *Oka–Cartan vanishing theorem*, sometimes *Oka's vanishing theorem*, and also *Cartan's Theorem B* or the *Oka–Cartan–Serre theorem* (See [No1, Remarks 4.3.33 and 4.4.27]). The same conclusions A and B hold on Stein manifolds, equivalently on strongly 1-complete complex manifolds (See Theorem III.2.165 below). In this form, B extends Theorem III.2.155 from locally free sheaves of finite rank to arbitrary coherent analytic sheaves.

The topology on coherent sections

With the Oka–Cartan fundamental theorems (Theorem III.2.162) now available, we can describe the topology on coherent sections directly in terms of local coefficients. On a complex manifold X, let $\mathcal{F}$ be coherent. In a sufficiently small coordinate polydisc P, choose

generators $s_1, s_2, \ldots, s_r$ and let $\mathcal{R}$ be their relation sheaf. Theorem B on P gives the exact sequence of sections

$$0 \longrightarrow \Gamma(P, \mathcal{R}) \longrightarrow \mathcal{O}_X(P)^{\oplus r} \longrightarrow \Gamma(P, \mathcal{F}) \longrightarrow 0. \tag{214}$$

The relation space is closed ([No1, Theorem 8.1.6]), so this quotient is Fréchet by Mini-Exercise I.6.42 (*iii*). For a compact exhaustion $K_1 \subset K_2 \subset \cdots$ of P, with $K_\nu \subset \operatorname{Int} K_{\nu+1}$, put

$$p_\nu(s) := \inf_{\substack{s=\sum_{j=1}^r f_j s_j \\ f_j \in \mathcal{O}_X(P)}} \sup_{x \in K_\nu} \sum_{j=1}^r |f_j(x)|. \tag{215}$$

The infimum is over all coefficient representations on P; such representations exist by (214). These are precisely the quotient seminorms of the usual Fréchet space $\mathcal{O}_X(P)^{\oplus r}$. Two generating systems can be expressed in terms of one another by holomorphic matrices. Bounds for these matrices on compact subsets show that the resulting topologies agree. For the coefficient seminorm construction and its completeness, see also [Hö, §7.2, Lemmata 7.2.3–7.2.4 and Corollary 7.2.6] and [No1, §8.1.1–2].

If $\mathcal{F}$ is finite locally free, write $\mathcal{F} = \mathcal{O}_X(E)$ for a holomorphic vector bundle E on X. This topology is compact uniform convergence in local holomorphic frames. Cauchy's estimates also control all derivatives on smaller compact sets, so it agrees with the topology induced on holomorphic sections by the Fréchet topology of C^∞ sections in §8.6. For general $\mathcal{F}$, the coefficients need not be unique; taking the infimum in (215) accounts for their relations. On an arbitrary open set, countably many such patches and the agreement conditions on overlaps give the topology.

Four fundamental coherence theorems

We have already encountered coherence repeatedly in the preceding discussion, both as a basic property of the structure sheaf and as a tool for constructing and handling analytic subspaces. It is useful at this point to pause and collect four fundamental results that provide geometric sources of coherent analytic sheaves. The first two have already appeared above, whereas the latter two will be introduced here.

Following Noguchi's terminology for Oka's first, second, and third coherence theorems, and his presentation of the classical list of four fundamental coherence theorems emphasized by Grauert and Remmert [No1, §2.5, Historical Supplements] (See also [No2, §2.2.2]), we summarize them as follows:

- Oka's first coherence theorem: coherence of the structure sheaf.
- Oka's second coherence theorem: coherence of defining ideal sheaves of analytic subsets.
- Oka's third coherence theorem: coherence of the normalization sheaf.
- Grauert's direct image theorem: coherence of higher direct images under proper holomorphic maps.

The first is Oka's first coherence theorem (Theorem III.1.41). Together with the calculus of coherent sheaves, it underlies the local presentations used above. Its passage to complex spaces was explained in Proposition III.2.36.

THEOREM III.2.163 (Oka's second coherence theorem). *For every analytic subset A of a complex space $(X, \mathcal{O}_X)$, its defining ideal sheaf $\mathcal{I}_A \subset \mathcal{O}_X$ is coherent.*

PROOF. See the Fundamental Theorem and its proof in [GrR, Chapter 4, §2, pp. 84–86]. For the presentation using powers of a discriminant, compare [No1, §§6.4–6.5, especially Theorem 6.5.1]. For an outline of the proof and its logical dependencies, see Remark III.2.164 below. □

REMARK III.2.164. In §2.4, we used Oka's second coherence theorem as a quoted result, chiefly to construct reduced structures and to obtain common local generators for the Jacobian description of singular loci. To explain why this order does not create a circular dependence, we outline the local proof of Oka's second coherence theorem (Theorem III.2.163) as follows.

The local method starts from Weierstraß preparation and division and the resulting local algebra, which give finite projections and local parametrizations of irreducible analytic set germs. The local parametrizations used here concern analytic sets in an ambient domain and do not presuppose the coherence of their full defining ideal sheaves. A discriminant separates the unramified part from an exceptional set. Using the discriminant and its powers, one compares the full defining ideal sheaf on a sufficiently small common neighborhood with an auxiliary ideal sheaf constructed from finitely many holomorphic functions. The auxiliary sheaf is coherent by Oka's first coherence theorem (Theorem III.1.41) and the basic properties of coherent sheaves, and the comparison yields Oka's second coherence theorem on an ambient open subset of $\mathbb{C}^N$.

For a general complex space, work in a quotient model $(Z, \mathcal{O}_Z)$, where $U \subset \mathbb{C}^N$ is open, $\mathcal{J}_0 \subset \mathcal{O}_U$ is coherent, and $\mathcal{O}_Z = j^{-1}(\mathcal{O}_U/\mathcal{J}_0)$ for the inclusion $j\colon Z \hookrightarrow U$ of its support. An analytic subset $A \subset Z$ is represented by an analytic subset $\widetilde{A} \subset U$ after lifting its local equations. Then $\mathcal{J}_0 \subset \mathcal{I}_{\widetilde{A}}$ and

$$\mathcal{I}_A \cong j^{-1}(\mathcal{I}_{\widetilde{A}}/\mathcal{J}_0).$$

The ambient assertion makes $\mathcal{I}_{\widetilde{A}}$ locally finitely generated, and hence $\mathcal{I}_A$ is locally finitely generated over $\mathcal{O}_Z$. Since $\mathcal{O}_Z$ is already coherent by Proposition III.2.36, its locally finitely generated subsheaf $\mathcal{I}_A$ is coherent. This passage does not first require the reduction of Z.

The local proof outlined here does not use the Oka–Cartan vanishing theorem, either in its domain form (Theorem III.2.162 (B)) or in its form for general Stein spaces (Theorem III.2.165 (B)) below. The applications in §2.4, and the global decomposition into irreducible components, are not inputs to this local proof. □

Its significance for radical ideals and reduced structures was explained in §2.4. For passage to a closed complex subspace, a similar local presentation argument gives the following fact. Let $W \subset \mathbb{C}^N$ be open, let $i\colon (A, \mathcal{O}_A) \hookrightarrow (W, \mathcal{O}_W)$ be the closed complex subspace defined by a coherent ideal $\mathcal{I}$, and let $\mathcal{F}$ be coherent on A. Then $i_*\mathcal{F}$ is coherent on W. Indeed, after

shrinking W around a point of A and restricting A and $\mathcal{F}$ accordingly, take a presentation $\mathcal{O}_A^{\oplus s} \to \mathcal{O}_A^{\oplus r} \to \mathcal{F} \to 0$ for some $r, s \in \mathbb{Z}_{\geq 0}$. After shrinking W further if necessary, lift its r by s matrix of sections of $\mathcal{O}_A = i^{-1}(\mathcal{O}_W/\mathcal{I})$ to a matrix M of sections of $\mathcal{O}_W$. Then

$$i_*\mathcal{F} \cong \mathcal{O}_W^{\oplus r}/(M\mathcal{O}_W^{\oplus s} + \mathcal{I}\mathcal{O}_W^{\oplus r}).$$

Oka's first coherence theorem (Theorem III.1.41) and the Three Lemma (Proposition III.1.39) give coherence of this quotient. Closed direct image is exact on stalks and preserves flabby sheaves. Applying the acyclicity theorem (Proposition II.3.49) to a flabby resolution, using Proposition II.3.63 (i) and equality of the corresponding section spaces, gives

$$H^q(A \cap V, \mathcal{F}) \cong H^q(V, i_*\mathcal{F}) \qquad (V \subset W \text{ open},\ q \geq 0). \tag{216}$$

In particular, the Oka–Cartan fundamental theorem B (Theorem III.2.162) applies to closed subspaces of polydiscs through their coherent direct images. This is also the sheaf-theoretic passage used in Oka's Jôku-Ikô.

We record the global form that will be used for complex spaces.

Theorem III.2.165 (Oka–Cartan fundamental theorems on Stein spaces). *Let $(X, \mathcal{O}_X)$ be a Stein complex space, equivalently a strongly 1-complete complex space, and let $\mathcal{F}$ be a coherent analytic sheaf on X. Then:*

(A) *For every $x \in X$, there are finitely many sections in $\Gamma(X, \mathcal{F})$ whose germs generate $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$.*

(B) *$H^q(X, \mathcal{F}) = 0$ for every $q \geq 1$.*

Proof. See [Gr2, §2, Sätze 3–4, pp. 17–18] for complex spaces including non-reduced ones; see also [No1, §4.5.3, Theorem 4.5.8 and the subsequent remarks]. □

Let $(X, \mathcal{O}_X)$ be a complex space and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The coefficient description of the topology also works in this setting. On a small closed model $i\colon (A, \mathcal{O}_A) \hookrightarrow (P, \mathcal{O}_P)$ for an open subset of X, where P is a polydisc, choose a presentation of $i_*\mathcal{F}$ by $\mathcal{O}_P^{\oplus r}$ with relation sheaf $\mathcal{R}$. The domain version of B gives

$$\Gamma(A, \mathcal{F}) \cong \mathcal{O}_P(P)^{\oplus r}/\Gamma(P, \mathcal{R}). \tag{217}$$

Since $\Gamma(P, \mathcal{R})$ is closed in the Fréchet space $\mathcal{O}_P(P)^{\oplus r}$, the quotient in (217) is Fréchet (Mini-Exercise I.6.42 (iii)). Its quotient topology is described by the coefficient seminorms (215) in the ambient polydisc. This topology retains the information in nilpotent sections: on the double point with ring $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, for example, the section space has its usual two-dimensional topology, rather than the topology that measures only the value at the underlying point.

Proposition III.2.166 (TVS structure of a space of coherent sections). *Let $(X, \mathcal{O}_X)$ be a complex space and let $\mathcal{F}$ be a coherent analytic sheaf on it. For every open subset $U \subset X$, the local quotient topologies defined above give a natural Fréchet topology on $\Gamma(U, \mathcal{F})$. It is independent of the presentations, embeddings, and patches. Restrictions and maps induced by morphisms of coherent sheaves are continuous. If $V \subset U$ is open with compact closure contained in U, the restriction $\Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$ is compact.*

Proof. See [No1, §8.1, Theorem 8.1.28] and [De4, Chapter IX, §4.C, Propositions (4.6)–(4.7)]; the latter uses quotient models also for non-reduced structures. □

Oka's third coherence theorem concerns *normalization*. On a reduced complex space X, a *weakly holomorphic function* on U is a holomorphic function on $U \cap X_{\mathrm{reg}}$ that is locally bounded near every point of U. Write $\widetilde{\mathcal{O}}_X$ for their sheaf. A complex space X is called *normal* if the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain for every $x \in X$.

Theorem III.2.167 (Oka's third coherence theorem). *For a reduced complex space X, the normalization sheaf $\widetilde{\mathcal{O}}_X$ is a coherent $\mathcal{O}_X$-module. There is a finite surjective holomorphic map $\nu\colon \widetilde{X} \to X$ from a normal complex space, called the* normalization*, with $\nu_*\mathcal{O}_{\widetilde{X}} \cong \widetilde{\mathcal{O}}_X$. It is an isomorphism over the normal locus of X.*

Proof. See, for example, [GrR, Chapter 8, §2.3 and §§3–4]. That construction uses a local existence theorem for one-sheeted analytic coverings, followed by gluing, to obtain normalization and the coherence of its sheaf; it precedes the general proper direct image theorem in that reference. □

Weakly holomorphic functions need not be holomorphic functions on X. The coherent normalization sheaf records these additional functions as a finite module locally.

Mini-Exercise III.2.168 (Normalization of the cusp). Let $C := \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3\}$ with its reduced structure, let $o := (0,0)$, and set $\nu\colon \mathbb{C} \to C$, $\nu(t) := (t^2, t^3)$.
(*i*) Determine the image of the pullback homomorphism $\nu^*\colon \mathcal{O}_{C,o} \to \mathcal{O}_{\mathbb{C},0}$, and show that ν is the normalization of C.
(*ii*) Give a continuous weakly holomorphic function on C which is not holomorphic at o.

Finally, let us turn to Grauert's direct image theorem, the fourth result in our list.

Theorem III.2.169 (Grauert's direct image theorem). *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be complex spaces, let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a proper holomorphic map, and let $\mathcal{F}$ be a coherent analytic sheaf on X. Then $R^q f_*\mathcal{F}$ is a coherent analytic sheaf on Y for every $q \geq 0$.*

Proof. See [Gr2, §6, Hauptsatz I, with the correction listed in the bibliography], or [GrR, Chapter 10, §4.6]. For a proof based on the properties of nuclear modules over nuclear Fréchet algebras, including relative analogues of Schwartz's theorem (Theorem I.6.56) and its cohomological consequence (Mini-Exercise I.6.57), see Kiehl–Verdier [KV]; see also Demailly [De4, Chapter IX, §5]. □

For a proper holomorphic map $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of complex spaces and a coherent analytic sheaf $\mathcal{F}$ on X, the case $q = 0$ says that holomorphic sections over $f^{-1}(U)$ form a coherent sheaf as the open set $U \subset Y$ varies. Finite maps are an important special case: their direct image preserves coherence, and can be treated by local finite-module arguments; see [GrR, Chapter 3, §1.3]. In particular, once the finite normalization map $\nu\colon \widetilde{X} \to X$ has been constructed, it is proper, and the case $q = 0$ of Grauert's direct image theorem (Theorem III.2.169), applied to the coherent structure sheaf $\mathcal{O}_{\widetilde{X}}$, gives the coherence

of $\nu_*\mathcal{O}_{\widetilde{X}} \cong \widetilde{\mathcal{O}}_X$. Thus the coherence assertion in Oka's third coherence theorem (Theorem III.2.167) is a special case of Grauert's direct image theorem (Theorem III.2.169). For the map from a compact complex space to a point, Grauert's direct image theorem (Theorem III.2.169) gives the finite-dimensionality in Cartan–Serre finiteness (Theorem III.2.174 below). The compactness argument in its proof also gives Hausdorffness.

Another application of Grauert's direct image theorem (Theorem III.2.169) is Stein factorization.

THEOREM III.2.170 (Stein factorization). *Let $(X, \mathcal{O}_X)$ and $(S, \mathcal{O}_S)$ be complex spaces, and let $f\colon (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a proper holomorphic map. There are a complex space $(Y, \mathcal{O}_Y)$ and a factorization*

$$(X, \mathcal{O}_X) \xrightarrow{\pi} (Y, \mathcal{O}_Y) \xrightarrow{\nu} (S, \mathcal{O}_S), \qquad f = \nu \circ \pi,$$

where π is proper and surjective with connected fibers, ν is finite, and the natural map $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is an isomorphism.

PROOF. See [GrR, Chapter 10, §6.1] for the complete proof. The case $q = 0$ of Grauert's direct image theorem (Theorem III.2.169) makes $\mathcal{A} := f_*\mathcal{O}_X$ a coherent $\mathcal{O}_S$-algebra. The finite analytic space associated with this algebra supplies $\nu\colon Y \to S$, with $\nu_*\mathcal{O}_Y \cong \mathcal{A}$. Locally, it is constructed from a finite presentation of the algebra, retaining its multiplication as well as its module structure. The natural evaluation of sections on X induces π. The cited construction proves its properness, connectedness of the fibers, and $\mathcal{O}_Y \cong \pi_*\mathcal{O}_X$. □

For example, let $S = \Delta$ and $X = (\mathbf{P}^1 \sqcup \mathbf{P}^1) \times \Delta$, with f the projection to Δ. This product projection is proper because its first factor $\mathbf{P}^1 \sqcup \mathbf{P}^1$ is compact. Its Stein factorization has $Y = \Delta \sqcup \Delta$: π collapses each copy of $\mathbf{P}^1$ to a point, and ν maps each disk identically onto Δ, a finite two-sheeted covering.

The case $q = 0$ of Grauert's direct image theorem (Theorem III.2.169) also gives information about images. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be complex spaces and let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a proper holomorphic map. Then

$$f(X) = \operatorname{Supp}(f_*\mathcal{O}_X).$$

Indeed, $f(X)$ is closed by properness (Mini-Exercise I.3.16 (i)); over a point of the image, the germ of the section 1 is nonzero, whereas outside the image the direct image sheaf vanishes locally. The support of a coherent sheaf is analytic, as the following exercise shows.

MINI-EXERCISE III.2.171. Let $(X, \mathcal{O}_X)$ be a complex space and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Show that $\operatorname{Supp}\mathcal{F}$ is an analytic subset of X.

Thus Grauert's direct image theorem (Theorem III.2.169) implies the proper mapping statement below, which also has a proof by simultaneous induction with the Remmert–Stein extension theorem. For this approach, see [De4, Chapter II, §8.2, Theorems (8.7)–(8.8), pp. 118–121].

- *Remmert's proper mapping theorem.* If $f\colon X \to Y$ is a proper holomorphic map of reduced complex spaces, then $f(X)$ is an analytic subset of Y.
- *Remmert–Stein extension theorem.* Let X be a reduced complex space, let $A \subset X$ be an analytic subset, and let Z be an analytic subset of $X \setminus A$. If there is an integer $p \geq 0$ such that $\dim A \leq p$ and $\dim_z Z > p$ for every $z \in Z$, then the closure $\overline{Z}$ in X is an analytic subset of X.

An application of the extension theorem relates analytic geometry to algebraic geometry. A complex space $(X, \mathcal{O}_X)$ is called *projective* if it admits a proper holomorphic embedding into some $\mathbf{P}^n$. If X is non-singular, such an embedding $i\colon X \hookrightarrow \mathbf{P}^n$ makes X Kähler: $i^*\omega_{\mathrm{FS}}$ is a Kähler form (Example III.2.128). A *projective algebraic variety* (over $\mathbb{C}$) means here a complex space biholomorphic to the common zero set in some $\mathbf{P}^n$ of finitely many homogeneous polynomials, endowed with its reduced complex-space structure. Irreducibility is not required. For a homogeneous polynomial P of degree d and $t \in \mathbb{C}$, we have $P(tZ) = t^d P(Z)$, so its zero set is well-defined in projective space. The following theorem says that, for a reduced complex space, projectivity alone already forces such an algebraic description.

Theorem III.2.172 (Chow's theorem). *Let A be an analytic subset of $\mathbf{P}^n$. There are finitely many homogeneous polynomials $P_1, P_2, \ldots, P_r$ such that*

$$A = \{[Z^0 : Z^1 : \cdots : Z^n] \in \mathbf{P}^n \mid P_j(Z^0, Z^1, \ldots, Z^n) = 0 \ (1 \leq j \leq r)\}.$$

Proof. We follow [De4, Chapter II, Theorem (8.10)]. If $A = \emptyset$, the constant polynomial 1 suffices, so assume $A \neq \emptyset$. Use the standard coordinates $Z^0, Z^1, \ldots, Z^n$ on $\mathbb{C}^{n+1}$. The corresponding projection is

$$\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbf{P}^n, \qquad Z \longmapsto [Z^0 : Z^1 : \cdots : Z^n].$$

The inverse image $Z_A := \pi^{-1}(A)$ is analytic away from 0 and invariant under multiplication by nonzero complex numbers. In a projective chart, π is a product projection with fiber $\mathbb{C}^*$, so every local dimension of Z_A is at least one. Also $0 \in \overline{Z_A}$ by scaling. Remmert–Stein, applied with the exceptional set $\{0\}$ and $p = 0$, therefore shows that $C_A := Z_A \cup \{0\}$ is analytic in $\mathbb{C}^{n+1}$.

Choose holomorphic functions $f_1, f_2, \ldots, f_s$ on a sufficiently small polydisc Δ_r^{n+1} centered at 0, with $r > 0$, such that $C_A \cap \Delta_r^{n+1} = \{f_1 = f_2 = \cdots = f_s = 0\}$. For $j \in \{1, 2, \ldots, s\}$, expand the convergent Taylor series as

$$f_j(Z) = \sum_{\mu=0}^{\infty} P_{j,\mu}(Z),$$

where $P_{j,\mu}$ is a homogeneous polynomial of degree μ. If $Z \in C_A \cap \Delta_r^{n+1}$, then $tZ \in C_A \cap \Delta_r^{n+1}$ for $|t| < 1$. Consequently $f_j(tZ) = \sum_{\mu \geq 0} t^\mu P_{j,\mu}(Z)$ vanishes identically as a function of t, and every coefficient $P_{j,\mu}(Z)$ is zero. Conversely, if all $P_{j,\mu}$ vanish at a point of Δ_r^{n+1}, then all f_j vanish there. Thus the common zero set W of the polynomials $P_{j,\mu}$ agrees with C_A in

Δ_r^{n+1}. Both sets are invariant under nonzero scalar multiplication, so scaling any point into Δ_r^{n+1} gives $W = C_A$ globally.

By Hilbert's basis theorem, the polynomial ring $\mathbb{C}[Z^0, Z^1, \dots, Z^n]$ is Noetherian. Hence the ideal generated by all the $P_{j,\mu}$ is generated by a finite subfamily of them. These generators are still homogeneous and have common zero set C_A. Their common zero set in projective space is therefore exactly A. □

Consequently, for reduced complex spaces, we use *projective* and *projective algebraic* interchangeably: a proper holomorphic embedding has a closed analytic image, and Chow's theorem makes that image algebraic. In particular, this applies to the projective manifolds obtained from Kodaira's embedding theorem (Theorem III.2.136).

Natural topology and finiteness of cohomology

Let $(X, \mathcal{O}_X)$ be a complex space and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Choose a countable open covering $\mathcal{U} = \{U_\alpha\}_\alpha$ of X whose members admit closed holomorphic embeddings into polydiscs. As in Corollary III.2.156, the nonempty finite intersections are strongly 1-complete; using the Oka–Cartan fundamental theorem B (Theorem III.2.165) instead of the locally free vanishing theorem shows that $\mathcal{U}$ is Leray for $\mathcal{F}$.

Write $\check{C}^q := \check{C}^q(\mathcal{U}, \mathcal{F})$. These cochain spaces are countable products of Fréchet section spaces, and the coboundaries are continuous. The natural topology on $H^q(X, \mathcal{F})$ is the kernel-quotient topology

$$\check{H}^q(\mathcal{U}, \mathcal{F}) = \frac{\operatorname{Ker}(\delta\colon \check{C}^q \to \check{C}^{q+1})}{\operatorname{Image}(\delta\colon \check{C}^{q-1} \to \check{C}^q)}, \tag{218}$$

transported by Leray's theorem (Theorem II.3.79); put $\check{C}^{-1} = 0$. Refinement gives continuous maps of Fréchet complexes inducing algebraic isomorphisms on cohomology. By Remark A.1.6, these are TVS isomorphisms without a Hausdorffness assumption on cohomology. A common Leray refinement proves independence of the covering.

Let X be a complex manifold, let E be a holomorphic vector bundle of finite rank on X, and put $\mathcal{F} = \mathcal{O}_X(E)$. The natural topology agrees with the Dolbeault topology of Definition III.2.115. More generally, fix $p \in \mathbb{Z}_{\geq 0}$ and choose a countable locally finite covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ by coordinate open balls on which E is trivial. Its nonempty finite intersections are strongly 1-complete by the preceding argument, so the covering is Leray for $\Omega^p_X(E)$. Put $U_{\alpha_0\alpha_1\dots\alpha_r} := U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_r}$ and consider

$$C^{r,s} := \prod_{(\alpha_0,\alpha_1,\dots,\alpha_r) \in J^{r+1}} \mathcal{A}^{p,s}(U_{\alpha_0\alpha_1\dots\alpha_r}, E), \qquad r, s \in \mathbb{Z}_{\geq 0},$$

with the zero space for an empty intersection. Each term is a countable product of Fréchet spaces. The horizontal differential is the Čech differential δ, the vertical differential is $\overline{\partial}$, and the total differential is $\delta + (-1)^r \overline{\partial}$ on $C^{r,s}$, as in Theorem A.1.1. The columns are exact in positive vertical degrees by the Oka–Cartan fundamental theorem B (Theorem III.2.165)

and the Dolbeault isomorphism (Theorem III.2.116); the rows are exact in positive horizontal degrees by the fine-sheaf argument using a partition of unity.

The bottom edge is the Čech complex of $\Omega^p_X(E)$, and the left edge is the global Dolbeault complex. These identifications also give the edge topologies: Cauchy estimates identify compact convergence of holomorphic coefficients with their induced C^∞ topology, and restriction identifies global smooth forms with the compatible families on the covering, with their subspace topology. Thus Theorem A.1.1 and Remark A.1.6 give

$$\check{H}^q(\mathcal{U}, \Omega^p_X(E)) \cong H^{p,q}(X, E), \qquad q \in \mathbb{Z}_{\geq 0},$$

as TVSs, including when these spaces are not Hausdorff. This uses the concrete Čech–Dolbeault double complex; no topology on an abstract injective resolution is needed. The two cohomologies are therefore Hausdorff simultaneously. If they are not Hausdorff, they are infinite-dimensional: otherwise Lemma I.6.55, applied to the continuous map from cochains onto the coboundaries inside the Fréchet space of cocycles, would make that image closed.

See also [Lau, Theorem 2.1 and p. 427] and [Cas, §4, Theorem 1, and p. 315] for this comparison. For compactly supported smooth forms, whose spaces are strict countable LF-spaces, and for currents, the corresponding TVS comparisons follow from the additional arguments in [Lau, §2], as used in Serre duality (Theorem III.2.132); they do not follow from the Fréchet argument alone.

Unless otherwise stated, cohomology of coherent analytic sheaves and its Dolbeault realizations will carry the natural topology. The compact shrinking criterion also has the following coherent version.

Lemma III.2.173. *Let $(X, \mathcal{O}_X)$ be a complex space and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{U} = \{U_j\}_{j\in J}$ be a finite open covering of X, and let $\mathcal{U}^* = \{U_j^*\}_{j\in J}$ be a family of open subsets such that $\overline{U_j^*}$ is compact and contained in U_j for every $j \in J$. The family $\mathcal{U}^*$ need not cover X; its Čech complex is formed on its union. Fix $q \in \mathbb{Z}_{\geq 0}$. If restriction induces a surjection*

$$\check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^q(\mathcal{U}^*, \mathcal{F}),$$

then its target is finite-dimensional and Hausdorff for the quotient topology induced by the Fréchet cochain spaces.

Proof. Repeat the proof of the compact shrinking criterion (Lemma III.2.122), using the TVS structure of coherent sections (Proposition III.2.166). The restrictions on finite intersections are compact, hence so are the maps of finite products in each cochain degree. Apply Mini-Exercise I.6.57 (*ii*) to the closed spaces of cocycles, taking degree -1 to be zero when $q = 0$. □

We now pass from compact finiteness with locally free coefficients (Lemma III.2.121) to coherent coefficients.

Theorem III.2.174 (Cartan–Serre finiteness). *Let $(X, \mathcal{O}_X)$ be a compact complex space and $\mathcal{F}$ a coherent analytic sheaf on X. Then $H^q(X, \mathcal{F})$ is finite-dimensional and Hausdorff for every $q \geq 0$.*

PROOF. Choose finite coverings $\mathcal{U} = \{U_j\}_{j\in J}$ and $\mathcal{U}^* = \{U_j^*\}_{j\in J}$ of X by strongly 1-complete open sets, with $\overline{U_j^*}$ compact and contained in U_j, using nested small polydiscs in local models. Their finite intersections are strongly 1-complete, so the Oka–Cartan fundamental theorem B (Theorem III.2.165) makes both coverings Leray for $\mathcal{F}$. Restriction induces an isomorphism on cohomology by Leray comparison and its naturality. The compact shrinking criterion with coherent coefficients (Lemma III.2.173) finishes the proof. □

The geometric bumping construction can now be reused with coherent coefficients, without changing its sets or their intersections.

COROLLARY III.2.175 (Grauert finiteness near a strongly pseudoconvex boundary). *Let X be a complex manifold, let $\Omega \subset X$ be a domain with compact closure in X and C^∞ strongly pseudoconvex boundary, and let $\mathcal{F}$ be a coherent analytic sheaf on an open neighborhood of $\overline{\Omega}$. Then $H^q(\Omega, \mathcal{F})$ is finite-dimensional and Hausdorff for every $q \geq 1$.*

PROOF. If $\partial\Omega = \emptyset$, apply Cartan–Serre finiteness (Theorem III.2.174) to Ω. Otherwise, Grauert's bumping technique (Lemma III.2.158) gives domains inside the neighborhood where $\mathcal{F}$ is defined. Their local pieces and overlaps are strongly 1-complete, hence Stein. The Oka–Cartan fundamental theorem B (Theorem III.2.165) gives (212) with these coherent coefficients. The proof of Corollary III.2.159 therefore gives surjectivity of $H^q(\Omega_N, \mathcal{F}) \to H^q(\Omega, \mathcal{F})$ for $q \geq 1$, and an isomorphism for $q \geq 2$. Use the same two coverings as in the proof of finiteness with locally free coefficients (Proposition III.2.160): the Oka–Cartan fundamental theorem B (Theorem III.2.165) makes them Leray for $\mathcal{F}$. Naturality of Leray comparison identifies the map of their Čech cohomologies with the surjective restriction above. Apply the compact shrinking criterion with coherent coefficients (Lemma III.2.173). □

We briefly recall the more general Andreotti–Grauert results. Let $(X, \mathcal{O}_X)$ be a complex space of finite dimension and let $\mathcal{F}$ be coherent. The Andreotti–Grauert theorems assert that $H^r(X, \mathcal{F})$ is finite-dimensional and Hausdorff for $r \geq q$ if X is strongly q-convex, and that it vanishes for $r \geq q$ if X is strongly q-complete. The latter statement for $q = 1$ is the Oka–Cartan fundamental theorem B (Theorem III.2.165). See [AG, §20, Théorème 14 and its Corollaire, pp. 249–250] and [De4, Chapter IX, Theorem (4.10) and Corollary (4.11)], including non-reduced structures in the latter reference. For further topics in q-convexity and q-completeness, see [OP].

2.12 Riemann surfaces revisited

We now collect what the preceding methods say about Riemann surfaces. The compact case brings together Hodge theory, degree, and curvature, whereas the non-compact case uses strong 1-completeness and the Oka–Cartan theorem.

Let X be a Riemann surface and let $p\colon \widetilde{X} \to X$ be its universal covering, as in Chapter I, §7.1. Pulling back holomorphic coordinates on evenly covered open sets gives $\widetilde{X}$ a unique complex structure for which p is locally biholomorphic. The transition maps are the same as those on X. Every deck transformation is then holomorphic, and X is the quotient of $\widetilde{X}$ by

the deck transformation group. The following theorem determines the complex structure of $\widetilde{X}$: it is one of $\mathbb{P}^1$, $\mathbb{C}$, and Δ.

THEOREM III.2.176 (Koebe's uniformization theorem). *Every simply connected Riemann surface is biholomorphic to* $\mathbb{P}^1$, $\mathbb{C}$, *or the unit disc* $\Delta := \{z \in \mathbb{C} \mid |z| < 1\}$.

PROOF. Classical potential-theoretic proofs use Green functions and suitable harmonic potentials; see, for example, [Ab2, §§4–5]. For readers familiar with the tools of this chapter, the proof in [De4, Chapter VIII, Theorem (11.12) and Lemma (11.13), pp. 401–402] also provides an accessible approach. □

Let X be a Riemann surface and let $p\colon \widetilde{X} \to X$ be its universal covering. The automorphism groups computed in Chapter I, §I.9, now describe the possible quotients of these three universal covers. A nonidentity deck transformation has no fixed point. Every nonidentity automorphism of $\mathbb{P}^1$ has a fixed point, so $\widetilde{X} \cong \mathbb{P}^1$ forces $X \cong \mathbb{P}^1$. If $\widetilde{X} \cong \mathbb{C}$, a fixed-point-free affine transformation is a translation. The deck group is consequently a discrete additive subgroup $\Lambda \subset \mathbb{C}$, of rank zero, one, or two. The corresponding quotients are

$$X \cong \mathbb{C}, \qquad X \cong \mathbb{C}^*, \qquad X \cong \mathbb{C}/\Lambda \quad \big(\Lambda = \mathbb{Z} + \mathbb{Z}\tau,\ \tau \in \mathbb{C},\ \operatorname{Im}\tau > 0\big),$$

respectively, after normalizing the lattice by a complex linear coordinate change. In rank one the identification is induced by $z \mapsto \exp(2\pi\sqrt{-1}z)$. The rank-two quotient, together with the image of 0, is an *elliptic curve*; the terminology also applies to its underlying compact Riemann surface. For the classical theory of lattices and elliptic functions, see [Ah, Chapter 7]. Every other Riemann surface has universal cover Δ and is called a *hyperbolic Riemann surface*.

Open Riemann surfaces

Let X be an *open Riemann surface*, meaning a non-compact Riemann surface. It is strongly 1-complete by Proposition III.2.154, hence Stein. The Oka–Cartan fundamental theorem B (Theorem III.2.165) therefore applies to every coherent analytic sheaf on it. In particular, $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, so the exponential sequence (193) gives

$$c_1\colon \operatorname{Pic}(X) \xrightarrow{\cong} H^2(X, \mathbb{Z}).$$

The connecting homomorphism here has the same sign as the first Chern class fixed earlier. Thus, on an open Riemann surface, holomorphic line bundles have no additional classification data beyond their first Chern classes.

Compact Riemann surfaces

Recall the genus classification of compact oriented surfaces in Chapter I, §8.4. In the smooth category, the same classification is up to orientation-preserving diffeomorphism, with genus $g \in \mathbb{Z}_{\geq 0}$. For $g \geq 1$, its smooth model Σ_g is the oriented connected sum of g tori, and Σ_0 is the sphere.

In general, the complex structure is not determined by the genus. For a compact Riemann surface X of genus g, a *marking* is an orientation-preserving diffeomorphism $\mu\colon \Sigma_g \to X$, considered up to isotopy. It records, in particular, the homotopy data of the standard generating loops of $\pi_1(\Sigma_g, *)$, for a chosen base point $* \in \Sigma_g$. The space of marked Riemann surfaces, where (X, μ) and (X', μ') are identified if a biholomorphism $F\colon X \to X'$ makes $F \circ \mu$ isotopic to μ', is *Teichmüller space* of genus g. Its analytic and geometric structures are extensively studied; see [Ab1] and [J, §4.1, Definition 4.1.2].

The degree and curvature constructions above give the following particularly concrete relation between line bundles and projectivity. Recall that a holomorphic line bundle is *positive* if it admits a C^∞ Hermitian metric whose Chern curvature satisfies $\sqrt{-1}\Theta_h > 0$.

THEOREM III.2.177. *Every compact Riemann surface X is projective. For a holomorphic line bundle L on X, positivity is equivalent to* $\deg L > 0$.

PROOF. We first exhibit one positive line bundle M of degree $r > 0$ on X, according to its uniformization.

For $X \cong \mathbb{P}^1$, take $M = L_{\mathrm{taut}}^{-1}$ with its Fubini–Study metric. Its degree is one by Remark III.2.25, and the positivity is the case $n = 1$ of Mini-Exercise III.2.26. The classification of line bundles on $\mathbb{P}^1$ is already recorded in Proposition III.2.141.

For $X \cong \mathbb{C}/\Lambda$, with $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, $\tau \in \mathbb{C}$, and $\operatorname{Im}\tau > 0$, let o be the image of 0 and let $\wp$ be the Weierstraß function of the lattice. The classical elliptic-function construction gives an embedding

$$X \longrightarrow \mathbb{P}^2, \qquad z \longmapsto [1 : \wp(z) : \wp'(z)];$$

see [Ah, Chapter 7], or [J, §5.10]. Here $1, \wp, \wp'$ are viewed as sections of $\mathcal{O}_X(3\{o\})$, so the expression also defines the map at o. The pullback M of the dual tautological bundle L_{taut}^{-1} on $\mathbb{P}^2$ has degree three, equivalently the image is a plane cubic. The restricted Fubini–Study metric has positive curvature.

In the remaining case, $\widetilde{X} \cong \Delta$. The Poincaré metric is invariant under $\mathrm{Aut}(\Delta)$ and therefore descends to X. Its tangent-line curvature was computed in (153); it is negative. Consequently the dual metric on $M = K_X$ has positive curvature. Lemma III.2.178 below computes $r = \deg K_X = 2g - 2$; positivity makes this integer strictly positive.

Let M be the positive line bundle chosen above and put $r := \deg M > 0$. Now let L be any holomorphic line bundle with $d := \deg L > 0$. The line bundle

$$F := L^r \otimes M^{-d}$$

has degree zero, so it is unitary flat by the unitary-flatness criterion from the $\partial\overline{\partial}$-lemma (Proposition III.2.131) and the degree discussion following (194). Hence $L^r \cong M^d \otimes F$ carries a Hermitian metric H of positive curvature. Taking the r-th root of its local squared norms defines a smooth Hermitian metric h on the already given bundle L, with $\Theta_h = r^{-1}\Theta_H$. Thus L is positive. Conversely, the curvature integral (196) shows that a positive line bundle has positive degree. Finally, the positive bundle M gives a projective embedding by Kodaira's embedding theorem (Theorem III.2.136). □

We next relate the compact uniformization types to the genus and compute the degree of the canonical bundle.

LEMMA III.2.178. *Let X be a compact Riemann surface of genus g, and let $p\colon \widetilde{X} \to X$ be its universal covering. Then*

$$\deg K_X = 2g - 2. \tag{219}$$

Moreover, $g = 0$ if and only if $X \cong \widetilde{X} \cong \mathbb{P}^1$; $g = 1$ if and only if X is an elliptic curve, equivalently $\widetilde{X} \cong \mathbb{C}$; and $g > 1$ if and only if X is hyperbolic, meaning that $\widetilde{X} \cong \Delta$.

PROOF. The quotient descriptions above identify the $\mathbb{P}^1$ and $\mathbb{C}$ cases with the sphere and elliptic curves, respectively. The disc case cannot have genus zero, since then X is simply connected. Nor can it have genus one: its deck group would be $\mathbb{Z}^2$, whereas a discrete abelian group of fixed-point-free disc automorphisms is cyclic. Indeed, after passing to the upper half-plane, automorphisms commuting with a non-identity such element form a group conjugate to real translations or positive dilations; a discrete subgroup of either is cyclic (Use the Möbius classification recalled in Chapter I, §I.9; see also [Ab1]). This proves the stated classification without using the degree formula.

For $g = 0$, the vector field $\partial/\partial z$ on $\mathbb{P}^1$ becomes $-w^2\partial/\partial w$ at infinity, where $w = 1/z$. Thus its zero divisor has degree two and $\deg K_X = -2$. For $g = 1$, let z be the standard coordinate on the universal cover $\mathbb{C}$. The form dz descends to a nowhere-vanishing holomorphic section of K_X, so $\deg K_X = 0$. For $g > 1$, the Hodge computation (Example III.2.127) gives $\dim H^0(X, \mathcal{O}_X(K_X)) = g$, so choose a non-zero holomorphic 1-form η. Fix a C^∞ Hermitian metric h on $T^{1,0}X$ and define its metric dual W by $\eta(v) = h(v, W)$. The identification with HX in Remark III.1.1 gives the real vector field $V = (W + \overline{W})/2$. A simple local coordinate calculation shows that, at a zero of η of order m, the index of V is $-m$. The Poincaré–Hopf theorem, recalled in Chapter I, §8.4, gives

$$2 - 2g = \sum_x \operatorname{index}_x(V) = -\sum_x \operatorname{ord}_x \eta = -\deg K_X.$$

For a proof based on the Gauß–Bonnet theorem, see [GH, Chapter 2]. □

We also recall the *Riemann–Roch theorem* for line bundles. Let X be a compact Riemann surface of genus g, and let L be a holomorphic line bundle on X. Then

$$\dim H^0(X, \mathcal{O}_X(L)) - \dim H^0(X, \mathcal{O}_X(K_X \otimes L^{-1})) = \deg L + 1 - g. \tag{220}$$

For a proof from the cohomological tools already prepared above, see Exercise III.2.

Classical function theory revisited

Let X be an open Riemann surface. The *Mittag–Leffler theorem* asserts that prescribed finite principal parts (the negative-power parts of local Laurent expansions) at a closed discrete set $A = \{p_j\}_{j\in J} \subset X$ are realized by a meromorphic function on X holomorphic on $X \setminus A$. Choose a locally finite family of pairwise disjoint coordinate neighborhoods U_j with

$U_j \cap A = \{p_j\}$, meromorphic representatives P_j of the principal parts, and C^∞ cutoffs χ_j supported compactly in U_j and equal to 1 near p_j. On $X \setminus A$ put $F = \sum_j \chi_j P_j$, extending each term by zero outside U_j. The form $v = \sum_j P_j \overline{\partial}\chi_j$ extends smoothly across A by zero and is $\overline{\partial}$-closed, since $\dim X = 1$. The vanishing recalled above and the Dolbeault isomorphism (Theorem III.2.116) give $H^{0,1}(X) \cong H^1(X, \mathcal{O}_X) = 0$. Choose a C^∞ function u with $\overline{\partial}u = v$. Then $F - u$ is holomorphic off A and has the prescribed principal parts, since u is holomorphic near every p_j.

In its classical entire-function form, *Weierstraß's theorem* realizes any closed discrete subset of $\mathbb{C}$ with prescribed positive integer multiplicities as the zero divisor of an entire function, with no additional zeros (See [Ah, Chapter 5, §2]). Likewise, *Runge's approximation theorem* says that a function holomorphic near a compact set $K \subset \mathbb{C}$ can be approximated uniformly on K by rational functions with poles outside K, and by polynomials if $\mathbb{C} \setminus K$ is connected (See [Ru1, Chapter 13]). We record the following version on a punctured compact Riemann surface.

PROPOSITION III.2.179. *Let X be a compact Riemann surface, let D be a non-zero effective divisor on X, and put $\Omega := X \setminus \operatorname{Supp} D$. Choose a C^∞ Hermitian metric h on $[D]$ with positive Chern curvature and set $\varphi := -\log |\sigma_D|_h^2$ on Ω, where σ_D is the canonical section of $[D]$. For every $c \in \mathbb{R}$ with $U_c := \{\varphi < c\} \neq \emptyset$, every holomorphic function on U_c is a limit, uniformly on compact subsets of U_c, of meromorphic functions on X with poles only in* $\operatorname{Supp} D$. *In particular, the restriction map*

$$\varinjlim_{\mu \in \mathbb{Z}_{\geq 0}} \Gamma(X, \mathcal{O}_X(\mu D)) \longrightarrow \Gamma(\Omega, \mathcal{O}_\Omega)$$

has dense image for the Fréchet topology of uniform convergence on compact subsets of Ω.

PROOF. We first explain the direct limit in the statement. The transition maps are inclusions: a meromorphic function is unchanged, while its allowed pole orders increase. Thus the direct limit consists of meromorphic functions with poles only in $\operatorname{Supp} D$ (Compare the convention following Proposition III.2.65). In the $[D]^\mu$-section description, transition and restriction are multiplication by σ_D and division by σ_D^μ, respectively.

The metric h exists by Theorem III.2.177. Put $\omega := \sqrt{-1}\Theta_h$; then $\sqrt{-1}\partial\overline{\partial}\varphi = \omega$ on Ω, and φ is an exhaustion. In particular, Ω admits a complete Kähler metric by Lemma III.2.153. Regard functions as K_X^{-1}-valued $(1,0)$-forms. Give K_X^{-1} the metric $h_{K_X^{-1}}$ induced by ω and choose $C_0 \geq 0$ with $\sqrt{-1}\Theta_{h_{K_X^{-1}}} \geq -C_0\omega$ on X. For integers $\mu > C_0$, the metric $h_{K_X^{-1},\mu} := h_{K_X^{-1}} e^{-\mu\varphi}$ has curvature at least $(\mu - C_0)\omega$.

Fix $f \in \Gamma(U_c, \mathcal{O}_{U_c})$ and a non-empty compact $K \subset U_c$. Choose real numbers a, b, d with $\max_K \varphi < a < b < d < c$ and a C^∞ function χ on $\mathbb{R}$ with $0 \leq \chi \leq 1$, equal to 1 on $(-\infty, b]$ and zero on $[d, \infty)$. Extend $F := (\chi \circ \varphi) f$ by zero to Ω. The form $v := \overline{\partial}F$ is supported in the compact band $B := \{b \leq \varphi \leq d\}$ and is $\overline{\partial}$-closed. Apply the top-degree L^2 estimate (Theorem III.2.111) with $n = q = 1$ and $(K_X^{-1}, h_{K_X^{-1},\mu})$, identifying $(0,1)$-forms

with K_X^{-1}-valued $(1,1)$-forms. It gives $\overline{\partial} u_\mu = v$ and

$$\int_\Omega |u_\mu|^2 e^{-\mu\varphi}\, dV_\omega \le \frac{C_1 e^{-\mu b}}{\mu - C_0},$$
$$\int_{\{\varphi<a\}} |u_\mu|^2\, dV_\omega \le \frac{C_1 e^{-\mu(b-a)}}{\mu - C_0} \longrightarrow 0$$

as $\mu \to \infty$, where C_1 is the squared $(h_{K_X^{-1}}, \omega)$-norm of the identified form v, integrated over B, and is independent of μ. The Cauchy–Riemann characterization (Theorem III.1.18, $(iv) \implies (i)$), applied in local coordinates to the $\overline{\partial}$-closed differences, makes u_μ holomorphic on $\{\varphi < b\}$ and $r_\mu := F - u_\mu$ holomorphic on Ω. The local submean estimate for $|u_\mu|^2$ (see (57)) now gives $u_\mu \to 0$ uniformly on K, hence $r_\mu \to f$ there.

Near a point of multiplicity d_j in D, take a coordinate z centered at that point. Here $F = 0$, and $e^{-\mu\varphi}$ differs from $|z|^{2\mu d_j}$ by a smooth positive factor. Thus $z^{\mu d_j} r_\mu$ is holomorphic and L^2 on a punctured disc, so its Laurent expansion has no negative terms (Use polar coordinates and Lemma I.4.2). Consequently $r_\mu \in \Gamma(X, \mathcal{O}_X(\mu D))$. This proves the approximation on U_c; a compact exhaustion of U_c and errors tending to zero give the asserted sequence. Every compact subset of Ω lies in some U_c, so applying the same argument to restrictions of a holomorphic function on Ω proves the final density assertion. □

Let X be a Riemann surface. A locally finite triangulation and its open-star covering (Lemma I.7.20) give a locally finite open covering whose non-empty finite intersections, including single members, are contractible. These intersections are also non-compact open Riemann surfaces, hence Stein. The covering is therefore Leray both for locally constant coefficients, by the comparison in Proposition II.3.81, and for coherent analytic sheaves, by the Oka–Cartan fundamental theorem B (Theorem III.2.165). The author does not know general existence criteria or counterexamples in higher dimensions and would welcome references concerning the following question.

Question III.2.180. *Let $(X, \mathcal{O}_X)$ be a reduced complex space of pure dimension $n \ge 2$. Under what conditions does X admit a locally finite open covering $\{U_j\}_{j\in J}$ such that every non-empty finite intersection of its members is Stein and contractible? Are there reduced complex spaces of pure dimension at least two admitting no such covering?* □

Related results appear to be discussed in [Fa, CJ], but the author has not yet examined their details.

2.13 Strongly 1-convex manifolds and exceptional analytic sets

We have seen that strongly 1-complete spaces are Stein. We now consider strongly 1-convex manifolds.

Theorem III.2.181. *Every connected strongly 1-convex complex manifold of finite dimension is holomorphically convex.*

PROOF. We use the $q = 1$ case of the global Andreotti–Grauert finiteness theorem recalled at the end of §2.11: $H^1(X, \mathcal{F})$ is finite-dimensional for every coherent analytic sheaf $\mathcal{F}$ on X; see [AG, §20, Théorème 14, p. 249] and [De4, Chapter IX, Theorem (4.10)(a)]. Fix a C^∞ exhaustion φ that is strictly plurisubharmonic outside a compact subset Γ_0. Let $\Gamma \subset X$ be a non-empty compact subset. If its holomorphic hull $\widehat{\Gamma}_X$ is not relatively compact, we can choose distinct points $x_\nu \in \widehat{\Gamma}_X$, $\nu \in \mathbb{Z}_{\geq 1}$, such that $\varphi(x_\nu) \to +\infty$. Then $S := \{x_\nu \mid \nu \in \mathbb{Z}_{\geq 1}\}$ is closed and discrete in X. The set S is analytic, since near each of its points it is a singleton and it is closed in X. Its defining ideal sheaf $\mathcal{I}_S \subset \mathcal{O}_X$ is coherent by Oka's second coherence theorem (Theorem III.2.163). The quotient sheaf is the closed direct image of the reduced structure sheaf of S, so $\Gamma(X, \mathcal{O}_X/\mathcal{I}_S) \cong \prod_{\nu \in \mathbb{Z}_{\geq 1}} \mathbb{C}$, with no boundedness condition on the sequences.

The short exact sequence

$$0 \longrightarrow \mathcal{I}_S \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I}_S \longrightarrow 0$$

therefore gives the exact sequence

$$\mathcal{O}_X(X) \xrightarrow{\mathrm{ev}_S} \prod_{\nu \in \mathbb{Z}_{\geq 1}} \mathbb{C} \xrightarrow{\delta} H^1(X, \mathcal{I}_S), \qquad \mathrm{ev}_S(f) = (f(x_\nu))_{\nu \in \mathbb{Z}_{\geq 1}}.$$

Here δ is the connecting homomorphism. Put $d := \dim_{\mathbb{C}} H^1(X, \mathcal{I}_S) \in \mathbb{Z}_{\geq 0}$. The $d+1$ classes $\delta((\nu^j)_{\nu \in \mathbb{Z}_{\geq 1}})$, $j \in \{1, \ldots, d+1\}$, are linearly dependent. Thus there are $c_1, \ldots, c_{d+1} \in \mathbb{C}$, not all zero, such that

$$P(t) := \sum_{j=1}^{d+1} c_j t^j, \qquad \delta((P(\nu))_{\nu \in \mathbb{Z}_{\geq 1}}) = 0.$$

Since P is non-zero and has zero constant term, it is non-constant. Exactness gives $f \in \mathcal{O}_X(X)$ with $f(x_\nu) = P(\nu)$ for every $\nu \in \mathbb{Z}_{\geq 1}$, whence $|f(x_\nu)| \to +\infty$. This contradicts $|f(x_\nu)| \leq \sup_\Gamma |f|$, since $x_\nu \in \widehat{\Gamma}_X$. Thus $\widehat{\Gamma}_X$ is relatively compact. It is closed in X, and hence compact, as required. □

For φ and Γ_0 as in the proof, the same conclusion holds for every sublevel $X_m := \{x \in X \mid \varphi(x) < m\}$ with $m \in \mathbb{R}$ and $m > \sup_{\Gamma_0} \varphi$. Indeed, $(m - \varphi)^{-1}$ is a smooth exhaustion of X_m and is strictly plurisubharmonic on $X_m \setminus \Gamma_0$. Thus X_m is strongly 1-convex, and the theorem applies componentwise to X_m; a compact subset meets only finitely many components. Compare [De4, Chapter IX, Theorem (2.9)(c)].

COROLLARY III.2.182. *Let X be a connected strongly 1-convex complex manifold of finite dimension. For every $x_0 \in X$, the set*

$$A_{x_0} := \{x \in X \mid f(x) = f(x_0) \text{ for every } f \in \mathcal{O}_X(X)\}$$

is a compact analytic subset of X.

PROOF. Applying the hull inequality to $f-f(x_0)$ shows that $A_{x_0} = \widehat{\{x_0\}}_X$, so the preceding theorem gives compactness. For analyticity, apply Corollary III.1.45 in local coordinates to the family $\{f - f(x_0) \mid f \in \mathcal{O}_X(X)\}$. Its common zero set is locally defined by finitely many of these functions and is exactly A_{x_0}. □

We now construct a proper map which contracts the compact positive-dimensional analytic pieces. We first work on a relatively compact neighborhood of them and then attach the unchanged exterior.

THEOREM III.2.183 (Contraction of the maximal compact analytic subset). *Let X be a connected strongly 1-convex complex manifold of complex dimension $n \in \mathbb{Z}_{\geq 1}$. There are a normal Stein space Y and a proper surjective holomorphic map $\pi\colon X \to Y$ with connected fibers such that the natural map $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is an isomorphism. There is a finite subset $E = \{p_j \mid 1 \leq j \leq r\} \subset Y$, with $r \in \mathbb{Z}_{\geq 0}$, for which*

$$A := \pi^{-1}(E), \qquad \pi\colon X \setminus A \xrightarrow{\cong} Y \setminus E.$$

Here A, endowed with its reduced structure, is a compact analytic subset without zero-dimensional irreducible components. Its connected components are the reduced fibers $A_j = (\pi^{-1}(p_j))_{\mathrm{red}}$, $1 \leq j \leq r$. Moreover, A is the union of all compact irreducible analytic subsets of X of positive dimension.

PROOF. The proof combines the holomorphic convexity just established with Stein factorization (Theorem III.2.170). For Stein factorization and normality of its intermediate space, see [GrR, Chapter 10, §6.1, pp. 213–214]. If X is compact, the constant map to a point has all the stated properties, with $A = X$: holomorphic functions are constant by Lemma III.2.91 applied to their absolute values. Assume henceforth that X is non-compact. Fix a smooth exhaustion φ that is strictly plurisubharmonic outside a compact set Γ_0. Choose $c_0 \in \mathbb{R}$ with $c_0 > \sup_{\Gamma_0} \varphi$ and $\Gamma := \{\varphi \leq c_0\} \neq \emptyset$. Every compact irreducible analytic subset of positive dimension lies in Γ. Otherwise, the maximum of φ on that subset would occur where φ is strictly plurisubharmonic, contrary to the maximum principle on a positive-dimensional analytic germ (Lemma III.2.90).

A proper map near the compact set. Theorem III.2.181 makes $\widehat{\Gamma}_X$ compact. Choose an open neighborhood $\Omega \subset X$ of $\widehat{\Gamma}_X$ with compact closure in X. For every $b \in \partial\Omega$, we have $b \notin \widehat{\Gamma}_X$. By the definition of the hull (Definition III.2.157), there is $g_b \in \mathcal{O}_X(X)$ with $\sup_\Gamma |g_b| < |g_b(b)|$. Choose a positive real number a_b strictly between these two numbers and put $f_b := g_b/a_b$. Then $\sup_\Gamma |f_b| < 1$, and $V_b := \{x \in X \mid |f_b(x)| > 1\}$ is an open neighborhood of b. The sets $V_b \cap \partial\Omega$, $b \in \partial\Omega$, cover the compact set $\partial\Omega$. Choose a finite subcover with centers $b_1, b_2, \ldots, b_N$, where $N \in \mathbb{Z}_{\geq 1}$, and set $f_j := f_{b_j}$ for $j \in \{1, 2, \ldots, N\}$. Thus every f_j satisfies $\sup_\Gamma |f_j| < 1$, while $\max_{j\in\{1,2,\ldots,N\}} |f_j| > 1$ on $\partial\Omega$. Set

$$\Delta := \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}, \qquad F := (f_1, f_2, \ldots, f_N), \qquad U := \Omega \cap F^{-1}(\Delta^N).$$

Then $\Gamma \subset U$. Continuity and the boundary inequality give $\overline{U} \cap \partial\Omega = \emptyset$, so $\overline{U}$ is compact and contained in Ω. Moreover, $\max_{j\in\{1,2,\ldots,N\}} |f_j| = 1$ on ∂U. The restriction $F\colon U \to \Delta^N$

is proper: for a compact $L \subset \Delta^N$, its inverse image is $\overline{U} \cap F^{-1}(L)$, a compact subset of U. Compare the analytic-polyhedron construction in [No1, Lemma 4.4.7].

Remmert's proper mapping theorem makes $Z := F(U)$ a closed analytic subset of Δ^N, which we give its reduced structure. Stein factorization (Theorem III.2.170) gives

$$U \xrightarrow{\pi_0} Y_0 \xrightarrow{\nu} Z, \qquad \mathcal{O}_{Y_0} \cong (\pi_0)_* \mathcal{O}_U, \tag{221}$$

where π_0 is proper and surjective with connected fibers, ν is finite, and Y_0 is normal because U is normal. The space Z is Stein, by restriction of a strictly plurisubharmonic exhaustion of the polydisc. Also Y_0 is Stein. Indeed, pullback of functions from Z confines the hull of a compact $L \subset Y_0$ to the compact set $\nu^{-1}(\widehat{\nu(L)}_Z)$. Functions from Z separate points in different fibers. For two distinct points in one finite fiber, the Oka–Cartan fundamental theorem A (Theorem III.2.165), applied to the coherent algebra $\nu_* \mathcal{O}_{Y_0}$, supplies a separating global section: its stalk is the direct sum of the local rings at the points of the fiber. If every global section had equal values at the two points, every stalk element they generate would also have equal residue values there, a contradiction. Thus Y_0 is Stein.

The non-trivial fibers. Let $A \subset U$ be the union of the positive-dimensional fibers of π_0. The reduced support of each such fiber is connected and has finitely many irreducible components (Theorem III.2.45). A zero-dimensional component would be an isolated point, contradicting connectedness. Thus every point lies on a compact positive-dimensional irreducible component, so $A \subset \Gamma$. If a fiber is zero-dimensional, its underlying set is a single point x. Properness allows us to shrink a neighborhood V of $\pi_0(x)$ so that $\pi_0^{-1}(V)$ is contained in a coordinate ball about x; use (4). All fibers over V are then finite, since a compact analytic subset of a coordinate ball cannot have a positive-dimensional component. Connectedness makes each fiber a point. Hence π_0 is a proper bijection over V, thus a homeomorphism by Mini-Exercise I.3.16 (*ii*). Together with the structure-sheaf isomorphism in (221), this makes π_0 biholomorphic over V.

It follows that A is analytic. To see this explicitly, embed a neighborhood V in Y_0 into an open subset of $\mathbb{C}^M$ by j. On $\pi_0^{-1}(V)$, the set A is precisely the locus where the Jacobian of $j \circ \pi_0$ has rank less than n. Outside A, the map π_0 is locally biholomorphic, so the rank is n. At a point of A, the fiber has positive local dimension, so rank n is impossible. The minors therefore give local holomorphic equations for A. It is closed in U and contained in $\Gamma \subset U$, hence compact and analytic also in X. Remmert's proper mapping theorem, applied to A with its reduced structure, now makes $E := \pi_0(A)$ a compact analytic subset of the Stein space Y_0, so E is finite: a strictly plurisubharmonic exhaustion excludes positive-dimensional compact components, and a compact zero-dimensional analytic set is finite. Write $E = \{p_j \mid 1 \le j \le r\}$. The sets $A_j = (\pi_0^{-1}(p_j))_{\mathrm{red}}$ are exactly the connected components of A, and $U \setminus A \cong Y_0 \setminus E$.

Every compact irreducible analytic subset $C \subset X$ of positive dimension is contained in $\Gamma \subset U$. Its image under π_0 is a connected compact analytic subset of Y_0, hence a point. Thus $C \subset A$. Conversely, every point of A lies on such a component of a fiber. This proves the asserted maximality.

Attaching the exterior and obtaining a Stein target. Glue Y_0 and $X \setminus A$ along the biholomorphism $Y_0 \setminus E \cong U \setminus A$ to obtain Y. The maps π_0 and the identity on $X \setminus A$ give

a holomorphic surjection $\pi\colon X \to Y$. Topologically, this is the quotient which collapses the finitely many pairwise disjoint compact sets A_j to points. It is Hausdorff and second countable; at a collapsed point one can use nested neighborhoods of the corresponding compact set, and away from these sets the topology is unchanged. Thus the glued charts make Y a normal complex space. The map π is closed: for a closed subset $C \subset X$, its saturation $\pi^{-1}(\pi(C))$ is C together with those finitely many A_j that meet C, and is closed. Its fibers are compact, so it is proper by Lemma I.3.15. The equality $\mathcal{O}_Y \cong \pi_*\mathcal{O}_X$ is local on Y and follows from the corresponding equality on Y_0 and the biholomorphism off E.

It remains to show that the whole space Y, not only Y_0, is Stein. Choose $c_1 > c_0$ so close to c_0 that $\{\varphi \le c_1\}$ is compact and contained in U. Take a smooth convex function χ which is zero on $(-\infty, c_1]$, has $\chi' > 0$ on (c_1, ∞), and tends to infinity. The function $\rho_X := \chi \circ \varphi$ is plurisubharmonic and is zero near A. It descends to a smooth plurisubharmonic exhaustion ρ_Y of Y: near E it is constant, and elsewhere we use the biholomorphism with $X \setminus A$. It is strictly plurisubharmonic wherever it is positive. Choose $0 < a < b$ so small that $\{\rho_Y \le b\}$ is compact and contained in Y_0, and let η be a C^∞ function on $\mathbb{R}$ with $0 \le \eta \le 1$, equal to 1 on $(-\infty, a]$ and 0 on $[b, \infty)$. Let ψ_0 be a smooth strictly plurisubharmonic exhaustion of Y_0. The function $(\eta \circ \rho_Y)\psi_0$, extended by zero outside Y_0, is smooth and compactly supported in Y. For a sufficiently large constant $C > 0$,

$$\Psi := C\rho_Y + (\eta \circ \rho_Y)\psi_0$$

is a strictly plurisubharmonic exhaustion of Y. On $\{\rho_Y \le a\}$, the second term is ψ_0. On the compact set $\{a \le \rho_Y \le b\}$, which lies in the smooth part of Y, the strictly positive Levi form of $C\rho_Y$ dominates the bounded negative part of the second term. On $\{\rho_Y \ge b\}$, $\Psi = C\rho_Y$ is strictly plurisubharmonic. The second term is bounded, so exhaustiveness is preserved. Thus Y is strongly 1-complete and hence Stein, as recalled in §2.10. This proves all the assertions. □

For a connected strongly 1-convex manifold X, the set A in Theorem III.2.183 is called the *maximal compact analytic subset of* X. Here maximality is among compact analytic subsets without zero-dimensional irreducible components. The equality $A_j = (\pi^{-1}(p_j))_{\mathrm{red}}$ describes their reduced supports, not a claim that the fibers as complex subspaces are reduced. If X is non-compact, A contains no open subset of X, so π is a proper modification. In particular, $A = \emptyset$ precisely when X is Stein.

Remark III.2.184 (Remmert reduction). Let X be a connected strongly 1-convex complex manifold and let $\pi\colon X \to Y$ be the contraction of its maximal compact analytic subset (Theorem III.2.183). All global holomorphic functions on X descend to Y, and those on Y separate distinct points. Consequently

$$\pi^{-1}(\pi(x_0)) = A_{x_0} \qquad (x_0 \in X),$$

with A_{x_0} as in Corollary III.2.182. In particular these sets are connected. More generally, every holomorphically convex complex space $(X, \mathcal{O}_X)$ has a proper holomorphic surjection onto a Stein space $(Y, \mathcal{O}_Y)$ with $\mathcal{O}_Y \cong \pi_*\mathcal{O}_X$; it is unique up to a unique isomorphism

compatible with the maps. This is the *Remmert reduction*; see [GrR, Chapter 10, §6.8, pp. 221–222]. The theorem above exhibits its strongly 1-convex manifold case: only finitely many compact connected analytic sets of positive dimension are contracted, and the rest of the manifold is unchanged. □

We next choose an exhaustion adapted to the maximal compact analytic subset.

LEMMA III.2.185. *Let X and Y be reduced complex spaces, with Y strongly 1-complete, and let $\pi\colon X \to Y$ be a proper surjective holomorphic map. Let $E = \{p_j \mid 1 \le j \le r\} \subset Y$ be a non-empty finite set and put $A := \pi^{-1}(E)$. Assume that π is biholomorphic from $X \setminus A$ onto $Y \setminus E$. Then there is a C^∞ plurisubharmonic exhaustion φ of X such that*

$$\varphi \ge 0, \qquad \varphi^{-1}(0) = A, \qquad \varphi \text{ is strictly plurisubharmonic on } X \setminus A. \tag{222}$$

PROOF. Choose a smooth strictly plurisubharmonic exhaustion $\rho \ge 1$ of Y. Take pairwise disjoint relatively compact local embedding patches U_j about p_j, with ambient coordinates $z_j = (z_j^1, z_j^2, \dots, z_j^{m_j})$ centered at p_j, and smooth cutoffs χ_j supported compactly in U_j and equal to 1 near p_j, for $1 \le j \le r$. Here z_j is the restriction of the ambient coordinate map, not a coordinate system on a possibly singular Y. On $Y \setminus E$ set

$$u := \rho + \delta \sum_{j=1}^{r} \chi_j \log |z_j|^2,$$

where each summand is extended by zero outside its patch. For sufficiently small $\delta > 0$, this function is strictly plurisubharmonic. Indeed, where $\chi_j = 1$ the logarithm is plurisubharmonic. The negative Levi terms introduced by the cutoffs are bounded on compact sets away from E, where they are dominated by the strict positivity of ρ. These estimates are made on ambient extensions in finitely many local patches.

Choose open sets $W_j \subset U_j$ containing p_j, with $\overline{W}_j$ compact and contained in $U_j \cap \{\chi_j = 1\}$. Fix a regularized maximum $\mathrm{RegMax}_{\eta,\theta}$ as in Example I.4.9. The function u is bounded below off $\bigcup_j W_j$. Thus a common constant $C > 0$ can be chosen so large that $u > -C$ there and $u > |z_j|^2 - C + 2\eta$ near ∂W_j for every j. Define

$$\psi := \begin{cases} C + \mathrm{RegMax}_{\eta,\theta}(u, |z_j|^2 - C) & \text{on } W_j \setminus \{p_j\}, \\ 0 & \text{at } p_j, \\ C + u & \text{on } Y \setminus \bigcup_j W_j. \end{cases}$$

Near ∂W_j the first expression equals $C + u$; near p_j it equals $|z_j|^2$, since the logarithmic term tends to $-\infty$. Hence ψ is smooth and strictly plurisubharmonic on Y. The regularized maximum preserves strict plurisubharmonicity by its convexity, monotonicity, and translation identity, as explained in §2.7. Moreover, its zero-mean smoothing kernel gives $\mathrm{RegMax}_{\eta,\theta}(s,t) \ge \max\{s,t\}$ by Jensen's inequality. It follows that $\psi \ge 0$ and $\psi^{-1}(0) = E$. Outside a compact set, $\psi = \rho + C$, so it remains an exhaustion. If some p_j is isolated in Y, the same definition simply sets $\psi = 0$ on that open point; the ambient function $|z_j|^2$

provides the required strictly plurisubharmonic extension there. Finally, take $\varphi := \psi \circ \pi$. Properness gives exhaustiveness, holomorphic pullback gives plurisubharmonicity, and the biholomorphism off A gives the last assertion in (222). □

Apply the lemma to the contraction of the maximal compact analytic subset (Theorem III.2.183) when $A \neq \emptyset$. The sets $X_\varepsilon := \{x \in X \mid \varphi(x) < \varepsilon\}$, $\varepsilon > 0$, form a fundamental system of neighborhoods of A. Indeed, outside any prescribed neighborhood of A, a fixed compact sublevel has a positive minimum of φ. For a positive regular value ε, the closure $\overline{X_\varepsilon}$ is a compact smooth manifold with boundary, and the boundary is strongly pseudoconvex. Such values can be chosen arbitrarily close to zero by Sard's theorem. The sublevel may be disconnected; it has only finitely many connected components, and the boundary results above apply componentwise. If X is compact, $A = X$ and $\varphi = 0$ suffice. If $A = \emptyset$, X is Stein and the Oka–Cartan fundamental theorem B (Theorem III.2.165) applies directly.

Definition III.2.186. Let X be a complex manifold and let $A \subset X$ be a non-empty compact analytic subset without zero-dimensional irreducible components. We call A an *exceptional analytic subset in the sense of Grauert* if there are a complex space $(Y, \mathcal{O}_Y)$ and a proper surjective holomorphic map $\pi\colon X \to Y$ such that $D := \pi(A)$ is finite, $\pi^{-1}(D) = A$, the restriction $X \setminus A \to Y \setminus D$ is biholomorphic, and the natural map $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is an isomorphism. Each connected component of A is contracted to one point; thus D is a singleton when A is connected (See [Gr3, §2, Definitions 2–3, pp. 339–340]). □

Grauert's criterion uses a strongly pseudoconvex neighborhood in which A is the maximal compact analytic subset (See [Gr3, §2, Satz 5, p. 340]).

Proposition III.2.187 (Grauert). *Let X be a complex manifold and let $A \subset X$ be a non-empty compact analytic subset without zero-dimensional irreducible components. Then A is exceptional in the sense of Grauert if and only if it has a fundamental system of strongly 1-convex open neighborhoods in X.*

Proof. Suppose first that A has the stated fundamental system, in the sense of Example II.3.20. Choose decreasing cofinal neighborhoods U_j, each with only finitely many connected components, all meeting A, and let E_j be their maximal compact analytic subsets. Componentwise application of the contraction of the maximal compact analytic subset (Theorem III.2.183) gives $A \subset E_{j+1} \subset E_j \subset E_1$ and $\bigcap_j E_j = A$. The descending chain of closed analytic subsets of the compact analytic space E_1 stabilizes: it has finitely many irreducible components, and a proper analytic subset of an irreducible component has smaller dimension. Thus $E_j = A$ for some j. Contract A in that U_j and attach $X \setminus A$, as in the proof of the contraction theorem; properness and the structure-sheaf isomorphism are preserved. Conversely, for a contraction π, take sufficiently small, pairwise disjoint local embedding neighborhoods of the finite set D, cut out by Euclidean balls. A strictly plurisubharmonic exhaustion on each such neighborhood pulls back to an exhaustion strictly plurisubharmonic off A. The inverse images are therefore strongly 1-convex. Properness makes them cofinal among open neighborhoods of A, proving the converse. □

The following form of Rossi's theorem says that a sufficiently high finite-order thickening of A (defined by a power of the defining ideal sheaf of A) detects every cohomology class of a fixed positive degree.

THEOREM III.2.188 (Rossi's theorem in terms of ideal powers). *Let X and $A \neq \emptyset$ be as in the contraction of the maximal compact analytic subset (Theorem III.2.183), and let φ be an exhaustion as in (222). Let $\mathcal{F}$ be a coherent analytic sheaf on a neighborhood of A, and let $\mathcal{I}_A \subset \mathcal{O}_X$ be the defining ideal sheaf of the reduced analytic subset A. For $\varepsilon > 0$, put $X_\varepsilon := \{x \in X \mid \varphi(x) < \varepsilon\}$. Choose a positive regular value ε sufficiently small that $\mathcal{F}$ is defined on a neighborhood of $\overline{X_\varepsilon}$. For each $q \in \mathbb{Z}_{>0}$ there is an $N \in \mathbb{Z}_{>0}$ such that, for every $\mu \in \mathbb{Z}_{\geq N}$, the following natural map is injective:*

$$H^q(X_\varepsilon, \mathcal{F}) \longrightarrow H^q\big(X_\varepsilon, \mathcal{F}/\mathcal{I}_A^\mu \mathcal{F}\big) \tag{223}$$

PROOF. Put $\Omega := X_\varepsilon$. If X is compact, then $A = X$ and $\mathcal{I}_A = 0$, so the assertion is immediate. We may therefore assume that X is non-compact.

We first prove, for each $q \in \mathbb{Z}_{>0}$ and every coherent $\mathcal{F}$ defined near $\overline{\Omega}$, that there is a coherent ideal sheaf $\mathcal{K} \subset \mathcal{O}_\Omega$ whose zero set $Z := \operatorname{Supp}(\mathcal{O}_\Omega/\mathcal{K})$ is compact and contained in A together with finitely many points of $\Omega \setminus A$, and such that

$$H^q(\Omega, \mathcal{K}\mathcal{F}) \longrightarrow H^q(\Omega, \mathcal{F}) \quad \text{is the zero map.} \tag{224}$$

We argue by descending induction on q, simultaneously for all such sheaves. The finite covering constructed in the proof of finiteness with locally free coefficients (Proposition III.2.160) has strongly 1-complete non-empty finite intersections. The Oka–Cartan fundamental theorem B (Theorem III.2.165) therefore makes it Leray for every coherent analytic sheaf. If the covering has s members, repeated use of Mayer–Vietoris (Proposition II.3.67) gives $H^q(\Omega, \mathcal{F}) = 0$ for $q \in \mathbb{Z}_{\geq s}$: separate the last member from the union of the preceding ones, and apply the same argument to their intersections with that member. This starts the induction, with $\mathcal{K} = \mathcal{O}_\Omega$.

For the induction step, coherent bumping (Corollary III.2.175 and its proof) gives a slightly larger open set Ω^+ with smooth strongly pseudoconvex boundary and $\overline{\Omega} \subset \Omega^+$, chosen so that $\mathcal{F}$ is defined near $\overline{\Omega^+}$ and the restriction

$$H^q(\Omega^+, \mathcal{F}) \longrightarrow H^q(\Omega, \mathcal{F})$$

is surjective. The source is finite-dimensional. Multiplication by a holomorphic function acts on cohomology, as is also visible on its Čech representatives. Define

$$I := \{f \in \mathcal{O}_X(\Omega^+) \mid f\xi = 0 \text{ for every } \xi \in H^q(\Omega^+, \mathcal{F})\}.$$

This is an ideal of finite codimension: it is the kernel of the linear map sending a function to its multiplication operator on the finite-dimensional space $H^q(\Omega^+, \mathcal{F})$.

The common zero set of I contains only finitely many classes of points not distinguished by functions in $\mathcal{O}_X(\Omega^+)$. Indeed, on representatives of any finite collection of distinct classes,

products of separating functions give arbitrary prescribed values. Evaluation on these points factors through $\mathcal{O}_X(\Omega^+)/I$ if they are common zeros of I, so their number cannot exceed its dimension. By the contraction of the maximal compact analytic subset (Theorem III.2.183), choose $\pi\colon X \to Y$ contracting the maximal compact analytic subset A. These classes in Ω^+ are its connected components and the singleton sets outside A: pullbacks of functions on the Stein space Y separate different fibers, while a holomorphic function is constant on each compact connected reduced fiber. Thus the common zero set of I in Ω is compact and is contained in A together with finitely many other points.

Apply Corollary III.1.45 in finitely many coordinate neighborhoods of $\overline{\Omega}$. We obtain $g_1, g_2, \ldots, g_t \in I$, with $t \in \mathbb{Z}_{>0}$, whose generated ideal sheaf has the same zero set on Ω. If necessary a zero function may be included to make the list non-empty. Put $\mathcal{G} := (g_1, g_2, \ldots, g_t)$ and consider

$$G\colon \mathcal{F}^{\oplus t} \longrightarrow \mathcal{F}, \qquad G(s_1, s_2, \ldots, s_t) = \sum_{j=1}^{t} g_j s_j, \qquad \mathcal{R} := \operatorname{Ker} G.$$

The functions g_j and the coherent sheaf $\mathcal{R}$ are defined near $\overline{\Omega}$. By the surjectivity just obtained, each g_j also acts as zero on $H^q(\Omega, \mathcal{F})$. Consequently the map induced by G on degree-q cohomology is zero.

Apply the induction hypothesis in degree $q+1$ to $\mathcal{R}$. It gives a coherent ideal $\mathcal{J}$ on Ω, with compact zero set of the indicated type, such that

$$H^{q+1}(\Omega, \mathcal{J}\mathcal{R}) \longrightarrow H^{q+1}(\Omega, \mathcal{R})$$

is zero. Artin–Rees (Proposition I.5.14) at each stalk of $\mathcal{R} \subset \mathcal{F}^{\oplus t}$ gives, for some $a \in \mathbb{Z}_{>0}$,

$$\mathcal{R} \cap \mathcal{J}^a \mathcal{F}^{\oplus t} \subset \mathcal{J}\mathcal{R}. \tag{225}$$

The exponent can be chosen on all of Ω: coherence extends each stalkwise inclusion to a neighborhood, compactness of $\operatorname{Supp}(\mathcal{O}_\Omega/\mathcal{J})$ gives a finite subcover, and outside that set $\mathcal{J} = \mathcal{O}_\Omega$. Increasing the exponent preserves the inclusion, so the maximum of the finitely many exponents works.

Set $\mathcal{K} := \mathcal{G}\mathcal{J}^a$. The two short exact sequences obtained from G and its restriction to $\mathcal{J}^a\mathcal{F}^{\oplus t}$ induce the following commutative diagram, whose rows are exact.

$$\begin{array}{ccccc} H^q(\Omega, \mathcal{J}^a\mathcal{F}^{\oplus t}) & \longrightarrow & H^q(\Omega, \mathcal{K}\mathcal{F}) & \longrightarrow & H^{q+1}(\Omega, \mathcal{R} \cap \mathcal{J}^a\mathcal{F}^{\oplus t}) \\ \downarrow & & \downarrow & & \downarrow 0 \\ H^q(\Omega, \mathcal{F}^{\oplus t}) & \longrightarrow & H^q(\Omega, \mathcal{G}\mathcal{F}) & \longrightarrow & H^{q+1}(\Omega, \mathcal{R}). \end{array}$$

The right vertical map is zero by (225), since it factors through $H^{q+1}(\Omega, \mathcal{J}\mathcal{R})$. Exactness therefore shows that the image of any class in $H^q(\Omega, \mathcal{K}\mathcal{F})$ in $H^q(\Omega, \mathcal{G}\mathcal{F})$ comes from $H^q(\Omega, \mathcal{F}^{\oplus t})$. Its further image in $H^q(\Omega, \mathcal{F})$ is zero, because that composite is induced by G. This proves (224). The zero set of $\mathcal{K}$ is the union of the zero sets of $\mathcal{G}$ and $\mathcal{J}$, so it has the required form. The induction is complete.

The long exact sequence associated with $0 \to \mathcal{K}\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{K}\mathcal{F} \to 0$ shows that

$$H^q(\Omega, \mathcal{F}) \longrightarrow H^q(\Omega, \mathcal{F}/\mathcal{K}\mathcal{F})$$

is injective. Write $S := Z \setminus A$, a finite set disjoint from A, and put $\mathcal{J}_0 := \mathcal{I}_{A\cup S}$. Rückert's Nullstellensatz (Theorem III.2.37) gives $\mathcal{J}_0 \subset \sqrt{\mathcal{K}}$. Finite generation at each stalk and compactness of $A \cup S$ therefore give $N \in \mathbb{Z}_{>0}$ such that $\mathcal{J}_0^N \subset \mathcal{K}$ on Ω. For clarity, if local generators have respective powers in $\mathcal{K}$, every monomial of sufficiently high total degree contains one of those powers. This gives an inclusion locally; coherence and a finite covering make its exponent uniform. Outside $A \cup S$ the ideal $\mathcal{K}$ is the unit ideal. It follows that the natural map

$$H^q(\Omega, \mathcal{F}) \longrightarrow H^q(\Omega, \mathcal{F}/\mathcal{J}_0^N\mathcal{F})$$

is injective, since its composite with the quotient map to $H^q(\Omega, \mathcal{F}/\mathcal{K}\mathcal{F})$ is injective.

Since $A \cap S = \emptyset$, we have $\mathcal{J}_0 = \mathcal{I}_A\mathcal{I}_S$. The natural map gives a sheaf isomorphism

$$\mathcal{F}/\mathcal{J}_0^N\mathcal{F} \cong \mathcal{F}/\mathcal{I}_A^N\mathcal{F} \oplus \mathcal{F}/\mathcal{I}_S^N\mathcal{F}.$$

This can also be checked stalkwise: at every point at least one of $\mathcal{I}_A$ and $\mathcal{I}_S$ is the unit ideal. The second summand is supported on finitely many points, so its positive-degree cohomology vanishes. Indeed, a sheaf supported on a finite closed discrete set is flabby, and Proposition II.3.63 applies. We obtain (223) for $\mu = N$. For every $\mu \in \mathbb{Z}_{\geq N}$, compose the map in (223) with the map induced by $\mathcal{F}/\mathcal{I}_A^\mu\mathcal{F} \to \mathcal{F}/\mathcal{I}_A^N\mathcal{F}$. The composite is the already injective map for N, so the first map is injective as well.

For the original proof, see [Ros, Theorem 3, pp. 17–20]; see also [Koi2, Proposition 3.1 (1) and its proof, pp. 1963–1964] for the passage to ideal powers with structure-sheaf coefficients in degree one. □

2.14 The Levi problem, Hartogs extension, and Runge-type approximation

Here, let us recall three classical problems in several complex variables. The expression "the three main problems" is used with somewhat different meanings in the literature. Following Noguchi's account [No2], we use it here for the following problems, organized by Behnke–Thullen and pursued by Oka:

- *The Cousin problems.* The additive and multiplicative Cousin problems, corresponding to the Mittag–Leffler theorem and the Weierstraß product theorem in one variable.
- *The Levi problem.* Also called the pseudoconvexity problem or the Hartogs inverse problem: to characterize domains of holomorphy by pseudoconvexity.
- *The Runge approximation problem.* Approximation of holomorphic functions of several variables, corresponding to Runge's theorem in one variable.

These problems ask how phenomena governed by classical theorems in one variable should be understood in several variables. We briefly follow their subsequent development, with the later chapters of these notes in view.

On the Cousin problems

The Cousin problems concern the globalization of the additive and multiplicative data already considered in these notes. In the language of Čech cohomology, the additive problem leads more generally to asking when a 1-cocycle with coefficients in a coherent analytic sheaf is a coboundary, and hence to the structure and vanishing of sheaf cohomology. We have already encountered several basic vanishing theorems of this kind.

Multiplicative Cousin data are directly related to holomorphic line bundles and divisors. The Oka principle appearing in their solution became a starting point of *Oka theory*, including the theory of *Oka manifolds*. The existence and classification of holomorphic line bundles, and more generally of holomorphic vector bundles, on a fixed complex manifold are also basic questions in complex geometry. Divisors and the exponential sequence have already provided descriptions of some of these data. They are closely related to the main subjects of the later chapters, and will reappear from Chapter IV onward.

On the Levi problem

Much of the background of the Levi problem was discussed in §2.10. Starting with the Hartogs phenomenon and the Kugelsatz, we described domains of holomorphy in $\mathbb{C}^n$ through pseudoconvexity, holomorphic convexity, and plurisubharmonic exhaustions. Oka's characterization for domains in $\mathbb{C}^n$, and then for unramified Riemann domains over $\mathbb{C}^n$, is the affirmative solution of the classical Levi problem in this setting.

A natural next question is what function-theoretic or geometric properties pseudoconvex domains possess in other ambient complex manifolds. In complex projective space, affirmative results were obtained by Fujita, Kiselman, and Takeuchi; see Takeuchi's paper [Ta] and the historical discussion in [Ad]. For general ambient manifolds, the same conclusion no longer holds. Grauert's examples [Gr4] illustrate this distinction. We give a concrete torus example, using the following elementary observation.

LEMMA III.2.189. *Let X be a connected complex manifold and let $M \subset X$ be a nonempty compact C^∞ Levi-flat real hypersurface. Suppose that every leaf of its Levi foliation is dense in M. Then $\mathcal{O}_X(X) = \mathbb{C}$.*

PROOF. For $f \in \mathcal{O}_X(X)$, compactness gives a point $p \in M$ such that $|f(p)| = \sup_M |f|$. Let $\iota\colon L \to X$ be the natural immersion of the Levi leaf through p. By Mini-Exercise III.2.10, L has a complex manifold structure for which ι is holomorphic. The maximum modulus principle makes $f \circ \iota$ constant. Density of L and continuity show that $f|_M = c$ for some $c \in \mathbb{C}$.

The analytic subset $\{f = c\} \subset X$ contains M. If it were proper, its local complex dimension would be at most $n-1$, where $n = \dim X$, and its real dimension at most $2n-2$. It cannot contain the smooth real hypersurface M, of dimension $2n-1$. Thus $f \equiv c$. □

EXAMPLE III.2.190 (Grauert). Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and use coordinates (z, w) on $\mathbb{C}^2$. Put

$$\Gamma_\theta := \mathbb{Z}(1,0) + \mathbb{Z}(\sqrt{-1},0) + \mathbb{Z}(\sqrt{-1}\theta,1) + \mathbb{Z}(0,\sqrt{-1}), \qquad T_\theta := \mathbb{C}^2/\Gamma_\theta,$$

and let $\pi\colon \mathbb{C}^2 \to T_\theta$ be the quotient map. The four generators are linearly independent over $\mathbb{R}$, so T_θ is a compact complex torus. Set $M_c := \pi(\{\operatorname{Re} z = c\})$ for $c \in \mathbb{R}$ and $\Omega := T_\theta \setminus M_0 = \pi(\{0 < \operatorname{Re} z < 1\})$. Each M_c is a compact C^ω Levi-flat real hypersurface, since its local defining functions are pluriharmonic. The function

$$\rho(z,w) := -\log \sin^2(\pi \operatorname{Re} z)$$

descends to a C^∞ plurisubharmonic exhaustion of Ω. Indeed, its second derivative in $x = \operatorname{Re} z$ is $2\pi^2/\sin^2(\pi x) > 0$, and it tends to $+\infty$ as $x \to 0$ or $x \to 1$.

For $0 < c < 1$, write $z = c + \sqrt{-1}y$ and $w = u + \sqrt{-1}v$. The hypersurface M_c is the real torus with identifications

$$(y,u,v) \sim (y+1,u,v) \sim (y+\theta,u+1,v) \sim (y,u,v+1).$$

Its Levi leaves are locally $z =$ constant. Returning once in the u-direction shifts the transverse y-coordinate by θ modulo $\mathbb{Z}$. Irrationality of θ therefore makes every leaf dense in M_c. Applying Lemma III.2.189 gives $\mathcal{O}_\Omega(\Omega) = \mathbb{C}$. In particular, Ω is not holomorphically convex and is not Stein.

Here M_0 is the topological boundary of Ω, but Ω lies on both local sides of M_0. To obtain a domain with a smooth boundary in the usual one-sided sense, take instead $\Omega_{a,b} := \pi(\{a < \operatorname{Re} z < b\})$, where $0 < a < b < 1$. Its boundary is $M_a \sqcup M_b$, its exhaustion is $-\log \sin^2(\pi(\operatorname{Re} z - a)/(b-a))$, and the same argument shows that $\mathcal{O}_{\Omega_{a,b}}(\Omega_{a,b}) = \mathbb{C}$. □

Rather different behavior is also possible. There are compact complex surfaces separated by a compact Levi-flat hypersurface into two strongly 1-convex domains; see the disc-bundle constructions of Diederich–Ohsawa [DO2]. Nemirovski constructed examples in which both domains are Stein [Nem]; see also [Oh2]. In contrast, for a compact Kähler manifold X of dimension $n \geq 3$ and a compact C^ω Levi-flat hypersurface $M \subset X$, Ohsawa proved that $X \setminus M$ admits no C^2 plurisubharmonic exhaustion whose Levi form has at least three positive eigenvalues outside a compact subset [Oh3, Theorem 0.1]. In particular, it cannot be strongly 1-convex: a strongly 1-convex exhaustion can be made plurisubharmonic everywhere by a convex modification on a compact set, as explained after Definition III.2.151. Thus the two-sided strongly 1-convex phenomenon just described for surfaces does not occur in this compact Kähler, C^ω setting when $n \geq 3$. For background on Levi-flat hypersurfaces and their complements, see [Ad]; for further counterexamples in several complex variables, see [FS].

The existence of a plurisubharmonic exhaustion does not imply that of a plurisubharmonic boundary defining function ψ, with the domain locally given by $\{\psi < 0\}$ and $d\psi \neq 0$ on the boundary. The following result of Nemirovski makes this distinction particularly clear.

THEOREM III.2.191 (Nemirovski). *Let X be a complex manifold of pure dimension 2, and let $U \subset X$ be a Stein domain with nonempty compact C^∞ Levi-flat boundary. There is no C^∞ plurisubharmonic function ψ on a neighborhood V of ∂U such that $U \cap V = \{\psi < 0\}$, $\partial U = \{\psi = 0\}$, and $d\psi \neq 0$ on ∂U.*

PROOF. We combine Proposition 2 and its following corollary in [Nem]. Suppose that ψ exists. Shrinking to a tubular neighborhood of the compact boundary, choose $\varepsilon > 0$ so that the levels $M_t := \{\psi = t\}$ in this neighborhood are compact smooth hypersurfaces for $-\varepsilon < t \leq 0$. Orient them as boundaries of the sublevel side, and put

$$F(t) := \int_{M_t} d^c\psi \wedge dd^c\psi.$$

Choose an auxiliary C^∞ Hermitian metric on X, and let ν be the real unit normal vector field along M_t with respect to the associated Riemannian metric, pointing towards the side $\{\psi > t\}$. Then $J\nu$ is tangent to M_t, and $d^c\psi$ vanishes on HM_t, while $\langle d^c\psi, J\nu\rangle = \langle d\psi, \nu\rangle/(4\pi) > 0$ (Mini-Exercise III.1.16 (i)). The direction $J\nu$, followed by the complex orientation of HM_t, gives the boundary orientation. Since the restriction of $dd^c\psi$ to HM_t represents the tangential Levi form (Mini-Exercise III.1.16 (iii)), plurisubharmonicity implies that $d^c\psi \wedge dd^c\psi$ is nonnegative on M_t with this orientation. Thus $F(t) \geq 0$, and Stokes' theorem gives

$$F(t) - F(s) = \int_{\{s<\psi<t\}} dd^c\psi \wedge dd^c\psi \geq 0 \qquad (-\varepsilon < s < t \leq 0),$$

where the slab $\{s < \psi < t\}$ has compact closure in this neighborhood. Levi-flatness gives $F(0) = 0$, so $F(t) = 0$ for every such t. The integrand on M_t is nonnegative and is a positive multiple of its tangential Levi form. Hence each M_t is Levi-flat.

Fix $t < 0$. A smooth strictly plurisubharmonic function on the Stein manifold U attains a maximum on the compact hypersurface M_t. Its restriction to the Levi leaf through a maximum point is strictly subharmonic and has a local maximum, contrary to the maximum principle. This proves the assertion. □

On a general complex manifold, pseudoconvexity in the local boundary-Levi sense need not give Steinness or even holomorphic convexity. One can nevertheless ask how much global function theory remains. The following theorem of Ohsawa gives an extension result in precisely this direction [Oh4, Theorem 0.1]. His original boundary regularity is C^2. We state the C^∞ case so that the construction of the complete metric does not require a separate smoothing argument.

THEOREM III.2.192 (Ohsawa's Hartogs-type extension theorem). *Let X be a Kähler manifold of pure complex dimension $n \geq 2$, and let $\Omega \subset X$ be a domain with compact closure in X and non-empty C^∞ boundary. Suppose that the Levi form of $\partial\Omega$, with the sign determined by Ω, is everywhere semi-positive and has a positive eigenvalue at some boundary point. For every compact subset $K \subset \Omega$ such that $\Omega \setminus K$ has no connected component relatively compact in Ω, every holomorphic function on $\Omega \setminus K$ extends uniquely to Ω. Moreover, $\partial\Omega$ is connected.*

We prepare a complete metric and a top-degree estimate before proving the theorem. Fix a Kähler form ω on X, and write $\delta(x) := d_\omega(x, \partial\Omega)$ for $x \in \Omega$. For sufficiently small $b > 0$, δ is smooth on $V_b := \{\delta < b\} \subset \Omega$. Choose $a, \varepsilon > 0$ with $4\varepsilon < a$, $a + 2\varepsilon < b$, and $a < \min\{1/2, e^{-2}\}$. By Example I.4.9, the function $\rho := \mathrm{RegMax}_{\varepsilon,\theta}(-\delta, -a)$ on V_b extends

smoothly as $-a$ to the rest of Ω. It equals $-\delta$ near the boundary and satisfies $-1/2 < \rho < 0$ and $-\rho < e^{-2}$. Put $\Phi := 1/\log(-\rho)$. Direct differentiation gives

$$\sqrt{-1}\partial\overline{\partial}\Phi = -\frac{\sqrt{-1}\partial\overline{\partial}\rho}{\rho(\log(-\rho))^2} + \frac{1}{\rho^2}\left(\frac{1}{(\log(-\rho))^2} + \frac{2}{(\log(-\rho))^3}\right)\sqrt{-1}\partial\rho\wedge\overline{\partial}\rho. \tag{226}$$

The coefficient of the last term is positive.

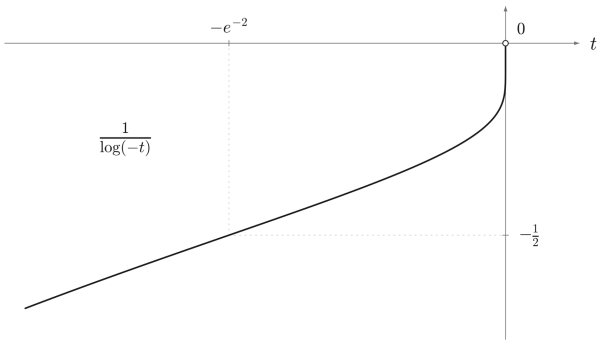


Figure III.6: The graph of $t \mapsto 1/\log(-t)$ on the relevant negative range.

Sufficiently near $\partial\Omega$ inside Ω, where $\rho = -\delta$, write (226) as

$$\sqrt{-1}\partial\overline{\partial}\Phi = \alpha\sqrt{-1}\partial\overline{\partial}\rho + \beta\sqrt{-1}\partial\rho\wedge\overline{\partial}\rho,$$
$$\alpha := \frac{1}{\delta(\log\delta)^2}, \qquad \beta := \frac{1-2/|\log\delta|}{\delta^2(\log\delta)^2} > 0.$$

In the following estimates, write $\mathcal{L}_\rho$ and $\mathcal{L}_\Phi$ for the Hermitian forms corresponding to $\sqrt{-1}\partial\overline{\partial}\rho$ and $\sqrt{-1}\partial\overline{\partial}\Phi$ via (95). At each point, decompose the holomorphic tangent space into $\operatorname{Ker}\partial\rho$ and its ω-orthogonal complement. The first space is the complex tangent space to the level hypersurface of ρ; the second is its complex normal line. Boundary Levi semi-positivity, smoothness, and compactness show that the negative tangential part of $\mathcal{L}_\rho$ is $O(\delta)$, while its mixed and normal coefficients are uniformly bounded. Moreover, $|\partial\rho|_\omega$ is bounded away from zero. Thus, for a vector $v = v_T + v_N$ with tangential and normal parts,

$$\mathcal{L}_\Phi(v,v) \geq -C\alpha\delta|v_T|_\omega^2 - 2C\alpha|v_T|_\omega|v_N|_\omega + (c\beta - C\alpha)|v_N|_\omega^2$$

with constants $c, C > 0$ independent of the point near the boundary. Since $\frac{\alpha}{\beta} = O(\delta)$, the negative normal term uses at most one quarter of $c\beta|v_N|_\omega^2$ sufficiently near the boundary. For a constant $C' > 0$, the mixed term satisfies

$$2C\alpha|v_T|_\omega|v_N|_\omega \leq \frac{c\beta}{4}|v_N|_\omega^2 + C'\frac{\alpha^2}{\beta}|v_T|_\omega^2.$$

The remaining tangential loss is bounded by a constant times $\alpha\delta + \frac{\alpha^2}{\beta} = O(|\log\delta|^{-2})$, while half of the positive normal term remains. Consequently, for suitable constants $M, c_0 > 0$,

$$\sqrt{-1}\partial\overline{\partial}\Phi \geq -M\omega + \frac{c_0\sqrt{-1}\partial\delta\wedge\overline{\partial}\delta}{\delta^2|\log\delta|^2}$$

near the boundary. Since the remaining part of Ω is compact, increasing M gives $\sqrt{-1}\partial\overline{\partial}\Phi \geq -M\omega$ on all of Ω. Choose $A > M$ sufficiently large that $\frac{1}{2} - \frac{(n-1)M}{A-M} > 0$. Then

$$\omega_A := A\omega + \sqrt{-1}\partial\overline{\partial}\Phi \geq c_1\omega + \frac{c_2\sqrt{-1}\partial\delta\wedge\overline{\partial}\delta}{\delta^2|\log\delta|^2} \tag{227}$$

sufficiently near $\partial\Omega$ inside Ω, with $c_1, c_2 > 0$, and ω_A is a Kähler form on all of Ω. The divergence of $\int_0 d\delta/(\delta|\log\delta|)$ proves completeness. To check the trace estimate, let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the ω-eigenvalues of $\sqrt{-1}\partial\overline{\partial}\Phi$. The preceding lower bound gives $b_j \geq -M$ for all j. Evaluation on a unit complex normal vector shows that the largest eigenvalue b_n tends to $+\infty$ as $\delta \searrow 0$. The eigenvalues relative to ω_A are $\mu_j = \frac{b_j}{A+b_j}$, so sufficiently near $\partial\Omega$ inside Ω we have

$$\mathrm{tr}_{\omega_A}(\sqrt{-1}\partial\overline{\partial}\Phi) = \sum_{j=1}^{n}\mu_j \geq \frac{1}{2} - \frac{(n-1)M}{A-M} > 0.$$

Mini-Exercise III.2.193. Keep this configuration, and let $\mathbb{I}_\Omega \to \Omega$ be a holomorphically trivial line bundle with a fixed global holomorphic frame denoted by 1. Let h_0 be the C^∞ Hermitian metric on $\mathbb{I}_\Omega$ defined by $|1|^2_{h_0} = 1$.

(*i*) Use the Kähler case of the proof of Proposition III.2.154 and the regularized maximum to construct a bounded real smooth function φ on Ω and a constant $c > 0$ such that $\mathrm{tr}_{\omega_A}(\sqrt{-1}\partial\overline{\partial}\varphi) \geq c$ everywhere.

(*ii*) Show that a constant $C > 0$ exists such that, for every $g \in L^{n,n}_{(2)}(\Omega, \mathbb{I}_\Omega; h_0, \omega_A)$, there is $u \in L^{n,n-1}_{(2)}(\Omega, \mathbb{I}_\Omega; h_0, \omega_A)$ satisfying $\overline{\partial}u = g$ and $\|u\|_{h_0,\omega_A} \leq C\|g\|_{h_0,\omega_A}$.

Proof of Theorem III.2.192. We give a proof following [Oh4, proof of Theorem 1.3] and [Oh5, §5.2.2]. The proof of the Kugelsatz after Example III.2.147 suggests the same goal here: $H^{0,1}_c(\Omega) = 0$. We obtain it from L^2 solvability for $(0,1)$-forms on (Ω, ω_A).

We retain the holomorphically trivial line bundle $\mathbb{I}_\Omega \to \Omega$, its flat metric h_0, and the global holomorphic frame 1 fixed in Mini-Exercise III.2.193, with $|1|^2_{h_0} = 1$. We identify scalar forms with $\mathbb{I}_\Omega$-valued forms using this frame.

Use maximal distributional realizations of $\overline{\partial}$, and put $T := \overline{\partial}\colon L^{0,0}_{(2)}(\Omega, \mathbb{I}_\Omega; h_0, \omega_A) \xrightarrow{\mathrm{op}} L^{0,1}_{(2)}(\Omega, \mathbb{I}_\Omega; h_0, \omega_A)$. By completeness, the Hilbert-space adjoint $T^\dagger$ agrees with the maximal realization of the formal adjoint, as explained before Theorem III.2.107. Conjugation and the Hodge-star identity of Mini-Exercise III.2.105 transfer the top-degree estimate of Mini-Exercise III.2.193 (*ii*) to the assertion that, for every $f \in L^{0,0}_{(2)}(\Omega, \mathbb{I}_\Omega; h_0, \omega_A)$, there is $w \in \mathrm{Dom}\, T^\dagger$ with $T^\dagger w = f$ and $\|w\|_{h_0,\omega_A;\Omega} \leq C\|f\|_{h_0,\omega_A;\Omega}$.

We first show that $\operatorname{Ran} T$ is closed. By Theorem I.6.19, with its second operator taken to be zero, it suffices to prove $\|v\|_{h_0,\omega_A;\Omega} \leq C\|T^\dagger v\|_{h_0,\omega_A;\Omega}$ for $v \in \operatorname{Dom} T^\dagger \cap (\operatorname{Ker} T^\dagger)^\perp$. Use the constant C from the preceding assertion, applied to $f := T^\dagger v$, and project the resulting solution w onto $(\operatorname{Ker} T^\dagger)^\perp$. This preserves the equation and does not increase the norm. The projected solution is v, since their difference belongs to both $\operatorname{Ker} T^\dagger$ and its orthogonal complement. This gives the required estimate and $L^{0,1}_{(2)}(\Omega, \mathbb{I}_\Omega; h_0, \omega_A) = \operatorname{Ran} T \oplus \operatorname{Ker} T^\dagger$, hence closedness.

We next show that every $\overline{\partial}$-closed element of $L^{0,1}_{(2)}(\Omega, \mathbb{I}_\Omega; h_0, \omega_A)$ has an L^2 primitive. Here we use the maximal distributional realization

$$\overline{\partial}\colon L^{0,1}_{(2)}(\Omega, \mathbb{I}_\Omega; h_0, \omega_A) \xrightarrow{\text{op}} L^{0,2}_{(2)}(\Omega, \mathbb{I}_\Omega; h_0, \omega_A).$$

Since $\operatorname{Image} T$ is closed and contained in the kernel of this operator, von Neumann's orthogonal decomposition theorem (Theorem I.6.18) shows that it suffices to prove that every $\overline{\partial}$-closed element in $\operatorname{Ker} T^\dagger$ is zero. Note that this is the same strategy of representing the obstruction by harmonic forms as in Proposition III.2.120 and the alternative proof of Kodaira vanishing following it.

The flat metric h_0 gives no positive curvature term in Nakano's inequality (178), but we may replace it by a uniformly equivalent C^∞ Hermitian metric on $\mathbb{I}_\Omega$, preserving the L^2 spaces as sets and the maximal $\overline{\partial}$ graphs, so as to make the sum of the smallest $n-1$ eigenvalues of the negative of its Chern curvature form relative to ω_A non-negative everywhere and positive on a nonempty open set (See (183)). We shall use boundary non-flatness to choose a bounded change of the bundle metric and then apply Nakano's inequality.

Fix a boundary point p with a positive Levi eigenvalue. Use the coefficients α, β and the Hermitian forms $\mathcal{L}_\rho, \mathcal{L}_\Phi$ from the preceding estimates. On a smaller boundary patch and its inner collar, choose an ω-unit complex tangential vector field e_T with $\mathcal{L}_\rho(e_T, e_T) \geq c > 0$, and an ω-unit complex normal vector field e_N. For $v = se_T + te_N$, the same absorption estimate gives

$$\mathcal{L}_\Phi(v, v) \geq \left(c\alpha - C\frac{\alpha^2}{\beta}\right)|s|^2 + c'\beta|t|^2$$

with constants $C, c' > 0$. Since $\alpha \to +\infty$, $\frac{\beta}{\alpha} \to +\infty$, and $\frac{\alpha^2}{\beta} = O(|\log \delta|^{-2})$, this is at least $c''\alpha(|s|^2 + |t|^2)$ near the boundary, for a constant $c'' > 0$. Thus $\mathcal{L}_\Phi$ is uniformly large on the complex two-plane spanned by e_T, e_N. At each point this plane intersects the span of eigenvectors for b_j, $1 \leq j \leq n-1$, nontrivially. On a unit vector in the intersection, $\mathcal{L}_\Phi$ is at most b_{n-1} and at least $c''\alpha$. Hence $b_{n-1} \geq c''\alpha \to +\infty$, and $b_n \geq b_{n-1} \to +\infty$ as well. The ω_A-eigenvalues are $\mu_j = \frac{b_j}{A+b_j}$, so $\mu_{n-1}, \mu_n \to 1$. By the choice of A, at points of Ω in this patch sufficiently close to $\partial\Omega$, we have

$$\sum_{j=1}^{n-1} \mu_j \geq \frac{1}{2} - \frac{(n-2)M}{A-M} > 0.$$

Choose a smooth cutoff $0 \leq \chi \leq 1$ supported in the ambient patch in X and equal to 1 near p. For sufficiently small $\epsilon > 0$, put $s := \Phi + \epsilon^2\chi$ on Ω. The non-empty set $V := \{s > 0\}$

lies in the locus where the preceding lower bound for the eigenvalue sum holds, and the bounded Hessian of χ, multiplied by ϵ^2, is absorbed by the positive lower bound for the sum of the smallest $n-1$ eigenvalues. Take a smooth convex function τ which is zero on $(-\infty, 0]$ and satisfies $\tau', \tau'' > 0$ on $(0, \infty)$. For $\Psi := \tau(s)$, the identity

$$\sqrt{-1}\partial\overline{\partial}\Psi = \tau'(s)\sqrt{-1}\partial\overline{\partial}s + \tau''(s)\sqrt{-1}\partial s \wedge \overline{\partial}s$$

shows that every sum of $n-1$ eigenvalues is positive on V and that the Hessian vanishes outside V. Moreover, $0 \leq \Psi \leq \tau(\epsilon^2)$, so Ψ is bounded.

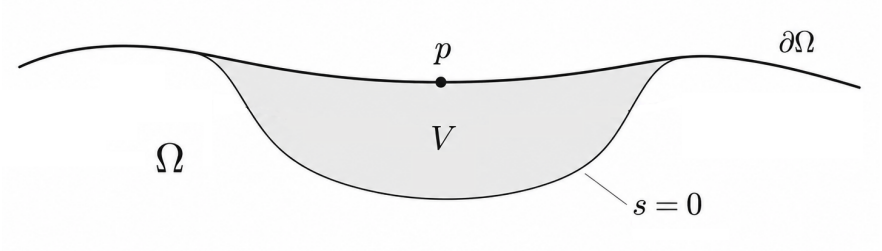


Figure III.7: $V = \{s > 0\} \subset \Omega$ near $p \in \partial\Omega$.

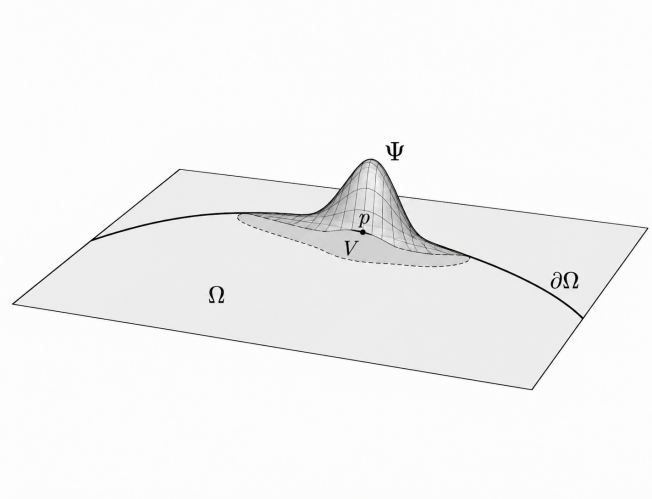


Figure III.8: $\Psi = \tau(s)$ near $p \in \partial\Omega$, with $\{\Psi > 0\} = V$.

On $\mathbb{I}_\Omega$ put $|1|^2_{h_\Psi} = e^\Psi$. Its curvature is $\sqrt{-1}\Theta_{h_\Psi} = -\sqrt{-1}\partial\overline{\partial}\Psi$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\sqrt{-1}\partial\overline{\partial}\Psi$ relative to ω_A, formula (183), in bidegree $(0,1)$, gives the eigenvalues

$$-\lambda_j - \sum_{k=1}^{n}(-\lambda_k) = \sum_{k\neq j}\lambda_k \quad \text{of } A^{0,1}_{h_\Psi,\omega_A}.$$

Consequently $A^{0,1}_{h_\Psi,\omega_A} \geq 0$ on Ω and $A^{0,1}_{h_\Psi,\omega_A} > 0$ on V. The h_Ψ and h_0 norms are equivalent, so the L^2 spaces as sets, the maximal $\overline{\partial}$ graphs, and closedness of the range are unchanged. The adjoint and orthogonal decomposition below are taken with respect to h_Ψ and ω_A.

Let $\alpha \in L^{0,1}_{(2)}(\Omega, \mathbb{I}_\Omega; h_\Psi, \omega_A)$ be $\overline{\partial}$-closed. Orthogonal decomposition with respect to this closed range, using von Neumann's orthogonal decomposition theorem (Theorem I.6.18), gives $\alpha = \overline{\partial} v + u$, where $\overline{\partial} u = 0$ and $\overline{\partial}^\dagger_{h_\Psi, \omega_A} u = 0$. The complete-metric Nakano inequality gives

$$0 \geq \int_\Omega \langle A^{0,1}_{h_\Psi, \omega_A} u, u \rangle_{h_\Psi, \omega_A} \, dV_{\omega_A} \geq 0.$$

Thus $u = 0$ almost everywhere on V. The symbol calculation in the proof of the Hodge decomposition for the Dolbeault complex (Proposition III.2.120) gives $\sigma_{\square_{h_\Psi, \omega_A}}(x, \xi) = -\frac{1}{2}|\xi|^2_{g_{\omega_A}} \mathbf{I}$. Since $\square_{h_\Psi, \omega_A} u = 0$ in the sense of distributions, local elliptic regularity (Remark I.8.40) makes u smooth. Aronszajn's unique continuation (Theorem I.8.65) then gives $u \equiv 0$. Hence every such α has an L^2 primitive.

It remains to recover compact support. The metric lower bound (227) implies

$$dV_{\omega_A} \geq \frac{c \, dV_\omega}{\delta^2 |\log \delta|^2} \geq \frac{c' \, dV_\omega}{\delta} \tag{228}$$

near the boundary. A holomorphic function f on $\{x \in \Omega \mid 0 < \delta(x) < \varepsilon_0\}$ for some $\varepsilon_0 > 0$, with $\int |f|^2/\delta \, dV_\omega < \infty$ must vanish sufficiently near $\partial\Omega$ inside Ω. See Ohsawa [Oh5, Lemma 5.1] for a proof using one-variable function theory. Here we give a direct distributional argument. Choose a connected holomorphic coordinate neighborhood U of a boundary point, with coordinates $(z^1, z^2, \dots, z^n)$, small enough that f is defined and δ is smooth on $U \cap \Omega$. Let $\widetilde{f}$ be the extension of $f|_{U \cap \Omega}$ by zero to U. The integrability assumption gives $\widetilde{f} \in L^2_{\mathrm{loc}}(U)$. It suffices to prove that $\overline{\partial} \widetilde{f} = 0$ in the sense of distributions. Indeed, by Theorem III.1.18, $(iv) \implies (i)$, $\widetilde{f}$ would then be holomorphic on U. Since it vanishes on the nonempty open set $U \setminus \overline{\Omega}$, the identity principle would give $\widetilde{f} = 0$ on U.

Let $d\lambda$ denote the Lebesgue measure in these coordinates, and fix $\varphi \in C^\infty_c(U)$. Choose $\vartheta \in C^\infty(\mathbb{R})$ with $0 \leq \vartheta \leq 1$, equal to zero on $(-\infty, 1]$ and to one on $[2, \infty)$, and put $\vartheta_\varepsilon := \vartheta(\delta/\varepsilon)$ on $U \cap \Omega$. The product $\vartheta_\varepsilon \varphi$ has compact support in $U \cap \Omega$. For each $j \in \{1, 2, \dots, n\}$, integration by parts with this test function, using $\overline{\partial} f = 0$, and dominated convergence give

$$\left\langle \frac{\partial \widetilde{f}}{\partial \overline{z^j}}, \varphi \right\rangle = -\int_U \widetilde{f} \, \frac{\partial \varphi}{\partial \overline{z^j}} \, d\lambda = -\lim_{\varepsilon \searrow 0} \int_{U \cap \Omega} \vartheta_\varepsilon f \, \frac{\partial \varphi}{\partial \overline{z^j}} \, d\lambda = \lim_{\varepsilon \searrow 0} \int_{U \cap \Omega} f \varphi \, \frac{\partial \vartheta_\varepsilon}{\partial \overline{z^j}} \, d\lambda.$$

The derivative of ϑ_ε is supported where $\varepsilon < \delta < 2\varepsilon$ and is bounded by C/ε on the support of φ. Comparing $d\lambda$ with dV_ω on this compact set and applying the Cauchy–Schwarz inequality, we obtain

$$\left| \int_{U \cap \Omega} f \varphi \, \frac{\partial \vartheta_\varepsilon}{\partial \overline{z^j}} \, d\lambda \right| \leq \frac{C_\varphi}{\varepsilon} \left(\int_{\{\varepsilon < \delta < 2\varepsilon\}} \frac{|f|^2}{\delta} \, dV_\omega \right)^{1/2} \left(\int_{\{\varepsilon < \delta < 2\varepsilon\}} \delta \, dV_\omega \right)^{1/2} \to 0$$

as $\varepsilon \searrow 0$, where C_φ is independent of ε. The last factor is $O(\varepsilon)$, while the first tends to zero by the integrability of $|f|^2/\delta$. Since φ and j were arbitrary, this proves $\overline{\partial} \widetilde{f} = 0$ in the sense of distributions. Thus $f = 0$ on $U \cap \Omega$. Finitely many such charts cover the compact boundary.

Now take $\alpha \in \mathcal{A}_c^{0,1}(\Omega)$ with $\overline{\partial}\alpha = 0$, and solve $\overline{\partial} v = \alpha$ in L^2 as above. Outside $\operatorname{Supp}\alpha$, v is holomorphic. Boundedness of Ψ and (228) imply that it vanishes sufficiently near $\partial\Omega$ inside Ω. Its support is therefore compact in Ω. Finally v is smooth by Theorem I.8.37 in local coordinates, since $\overline{\partial} v = \alpha$ is smooth. This proves $H_c^{0,1}(\Omega) = 0$.

The exact sequence for cohomology at infinity (Proposition II.3.72) and the Dolbeault isomorphism (Theorem III.2.116) now show that every $f \in \mathcal{O}_\Omega(\Omega \setminus K)$ agrees outside a compact subset with a holomorphic function on Ω. Every connected component of $\Omega \setminus K$ is not relatively compact and therefore meets the set where these functions agree. The identity principle gives agreement on all of $\Omega \setminus K$ and uniqueness of the extension.

If the boundary were disconnected, then, for sufficiently small $\varepsilon > 0$, the inward collar $\{x \in \Omega \mid 0 < \delta(x) < \varepsilon\}$ would have at least two connected components: the tubular neighborhood theorem (Theorem I.8.4) identifies it diffeomorphically with $\partial\Omega \times (0, \varepsilon)$. Its complement in Ω is compact, and each collar component is not relatively compact in Ω. A function taking different constants on these components would extend holomorphically to the connected manifold Ω, contradicting the identity principle. Thus $\partial\Omega$ is connected. □

On the Runge approximation problem

The finite-pole approximation in Proposition III.2.179 is a one-dimensional model. In several variables, the approximation problem developed with Stein theory, extending to holomorphic sections and $\overline{\partial}$-cohomology classes. Ohsawa's finiteness, isomorphism, and approximation theorems on weakly 1-complete manifolds connect these questions with the L^2 method [Oh1, Oh6]. We record an approximation result on the complement of the support of an effective divisor satisfying the positivity condition below.

THEOREM III.2.194 (Ohsawa's Runge-type approximation theorem in the Kähler case). *Let X be a compact connected Kähler manifold of complex dimension 2, and let $D \neq 0$ be an effective divisor on X. Put $Y := \operatorname{Supp} D$ and $M := X \setminus Y$. Suppose that $[D]$ admits a C^∞ Hermitian metric whose Chern curvature is positive definite at every point of Y, as a form on $T_X|_Y$. Let σ_D be the canonical section. Then the natural map*

$$\varinjlim_{\mu \geq 0} \Gamma(X, \mathcal{O}_X(K_X \otimes [D]^\mu)) \longrightarrow \Gamma(M, \mathcal{O}_M(K_M)), \qquad \sigma \longmapsto \left.\frac{\sigma}{\sigma_D^\mu}\right|_M,$$

has dense image for uniform convergence on compact subsets of M. The transition maps in the direct limit are multiplication by σ_D. Equivalently, holomorphic 2-forms on M can be approximated on compact subsets by meromorphic 2-forms on X with finite-order poles only along D.

To prove Theorem III.2.194, we shall cut off the given holomorphic form away from D and correct its $\overline{\partial}$-error. The correction must be small on the prescribed compact set, while its growth near D must allow only a finite-order pole. We first establish the following weighted L^2 estimate, which requires strict positivity only on the support of the error.

LEMMA III.2.195. *Let $(M, \widehat{\omega})$ be a complete Kähler manifold of pure complex dimension 2, and let ψ be a real-valued C^∞ plurisubharmonic function on M. Let $\mathbb{I}_M$ be a holomorphically trivial line bundle on M, and let 1 denote a fixed global nowhere-vanishing holomorphic frame. Give $\mathbb{I}_M$ the C^∞ Hermitian metric $|1|^2_{h_\psi} = e^{-\psi}$. Suppose that $v \in \mathcal{A}^{2,1}_c(M)$ satisfies $\overline{\partial} v = 0$ and that, for a constant $c > 0$, $\sqrt{-1}\partial\overline{\partial}\psi \geq c\widehat{\omega}$ on $\operatorname{Supp} v$. Then there is an L^2 $(2,0)$-current a such that*

$$\overline{\partial} a = v, \qquad \|a\|^2_{h_\psi;M} \leq c^{-1}\|v\|^2_{h_\psi,\widehat{\omega};M}. \tag{229}$$

Here we identify scalar forms with $\mathbb{I}_M$-valued forms by its constant section 1, and omit the base metric from the norm of a $(2,0)$-form as in (182).

PROOF. Put $H_j = L^{2,j}_{(2)}(M, \mathbb{I}_M; h_\psi, \widehat{\omega})$ for $j = 0, 1, 2$, and use the maximal distributional realizations $T = \overline{\partial}\colon H_0 \xrightarrow{\text{op}} H_1$ and $S = \overline{\partial}\colon H_1 \xrightarrow{\text{op}} H_2$. They are densely defined and closed, with $\operatorname{Image} T \subset \operatorname{Ker} S$. In bidegree $(2,1)$, the eigenvalues of $A^{2,1}_{h_\psi,\widehat{\omega}}$ are those of $\sqrt{-1}\partial\overline{\partial}\psi$ relative to $\widehat{\omega}$, by (183). Thus this operator is semi-positive on M and bounded below by c on $\operatorname{Supp} v$. The complete-metric passage to the joint domain described before Theorem III.2.107 allows us to use Nakano's inequality (178) for $u \in \operatorname{Dom} T^\dagger \cap \operatorname{Ker} S$. It gives

$$c\int_{\operatorname{Supp} v} |u|^2_{h_\psi,\widehat{\omega}}\, dV_{\widehat{\omega}} \leq \int_M \langle A^{2,1}_{h_\psi,\widehat{\omega}} u, u\rangle_{h_\psi,\widehat{\omega}}\, dV_{\widehat{\omega}} \leq \|T^\dagger u\|^2_{h_\psi;M}.$$

Cauchy–Schwarz on $\operatorname{Supp} v$ therefore yields

$$|\langle\!\langle u, v\rangle\!\rangle_{h_\psi,\widehat{\omega};M}|^2 \leq c^{-1}\|v\|^2_{h_\psi,\widehat{\omega};M}\|T^\dagger u\|^2_{h_\psi;M}. \tag{230}$$

To apply the range criterion, take an arbitrary $u \in \operatorname{Dom} T^\dagger$ and decompose $u = u_0 + u_1$ with $u_0 \in \operatorname{Ker} S$ and $u_1 \perp \operatorname{Ker} S$. Since u_1 annihilates $\operatorname{Image} T$, it belongs to $\operatorname{Dom} T^\dagger$ and $T^\dagger u_1 = 0$. Hence $u_0 \in \operatorname{Dom} T^\dagger$, and $Sv = 0$ shows that (230) holds for this arbitrary u as well. The Hörmander range criterion (10) now gives a with the stated bound (Equivalently, extend the bounded functional $T^\dagger u \mapsto \langle\!\langle u, v\rangle\!\rangle_{h_\psi,\widehat{\omega};M}$ by Hahn–Banach and use Riesz representation and $T^{\dagger\dagger} = T$). □

Proof of Theorem III.2.194. We give the argument in the present Kähler setting, following Ohsawa's method of using a family of weights [Oh1, Theorem 4.6, p. 867] (See also [Oh6, §2, pp. 276–278]). Fix a Kähler form ω on X, and a metric h as in the statement. On M, put $\varphi_0 = -\log|\sigma_D|^2_h$, where σ_D is the canonical section of $[D]$. This is a smooth exhaustion, and $\sqrt{-1}\partial\overline{\partial}\varphi_0 = \sqrt{-1}\Theta_h([D])$ is positive near Y. Choose a compact subset outside which this positivity holds. Taking the regularized maximum with a constant larger than $\sup\varphi_0$ on that compact subset gives a smooth plurisubharmonic exhaustion $\varphi \geq 1$, equal to φ_0 near Y and strictly plurisubharmonic outside a compact subset. Here the maximum agrees with the constant wherever φ_0 need not be plurisubharmonic.

By the construction in Lemma III.2.153, fix the complete Kähler form $\widehat{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}(\varphi^2)$ on M. Fix $\eta \in \Gamma(M, \mathcal{O}_M(K_M))$ and a compact subset $K \subset M$. Choose $a < b$ sufficiently large that $K \subset \{\varphi < a\}$ and φ is strictly plurisubharmonic on $\{\varphi \geq a\}$. Choose a smooth convex non-decreasing function λ on $\mathbb{R}$ such that $\lambda = 0$ on $(-\infty, a]$, $\lambda' > 0$ on (a, ∞), and

$\lambda' = 1$ for large arguments (For example, take λ' to be a smooth non-decreasing function which is zero on $(-\infty, a]$, positive on (a, ∞), and one on $[a+1, \infty)$, and integrate from a). Take a C^∞ function χ on $\mathbb{R}$ with $0 \le \chi \le 1$, equal to 1 on $(-\infty, b]$ and to 0 on $[b+1, \infty)$, and set $v = \overline{\partial}((\chi \circ \varphi)\eta)$. Then v is $\overline{\partial}$-closed and its support lies in the compact set $B = \{b \le \varphi \le b+1\}$.

The form $\sqrt{-1}\partial\overline{\partial}(\lambda \circ \varphi)$ is semi-positive on M and positive on B. Consequently it is bounded below by $\delta\widehat{\omega}$ on B for some $\delta > 0$. Fix a holomorphically trivial line bundle $\mathbb{I}_M$ on M with a global holomorphic frame 1. For each integer $\mu \ge 1$, put $|1|^2_{h_\mu} = e^{-\mu\lambda\circ\varphi}$ on $\mathbb{I}_M$. Lemma III.2.195, applied with $\psi = \mu\lambda \circ \varphi$ and $c = \mu\delta$, gives, for every positive integer μ, a $(2,0)$-current η_μ satisfying

$$\overline{\partial}\eta_\mu = v, \qquad \|\eta_\mu\|^2_{h_\mu;M} \le \frac{1}{\mu\delta}\|v\|^2_{h_\mu,\widehat{\omega};M}. \tag{231}$$

Since $\lambda(\varphi) \ge \lambda(b) > 0$ on B,

$$\|\eta_\mu\|^2_{h_\mu;M} \le \frac{e^{-\mu\lambda(b)}}{\mu\delta}\int_B |v|^2_{\widehat{\omega}}\, dV_{\widehat{\omega}} \longrightarrow 0$$

as $\mu \to \infty$, where δ and the last integral are independent of μ.

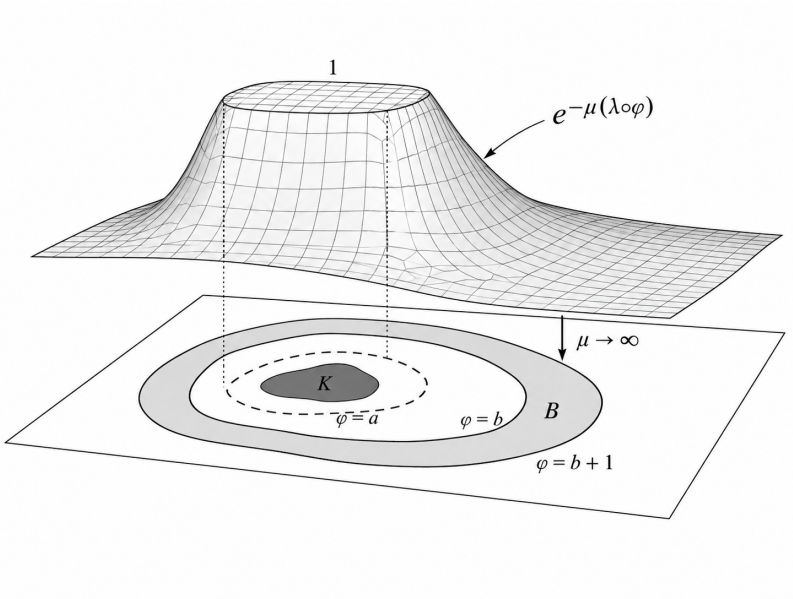


Figure III.9: Schematic picture of the weight $e^{-\mu(\lambda\circ\varphi)}$, which equals 1 near K and tends uniformly to 0 on the shaded band B as $\mu \to \infty$.

Now $\xi_\mu = (\chi \circ \varphi)\eta - \eta_\mu$ is $\overline{\partial}$-closed as a current, hence is a holomorphic 2-form by local regularity (Theorem III.1.18, $(iv) \implies (i)$). On a neighborhood of K, the functions $\lambda\circ\varphi$ and $\chi \circ \varphi$ equal 0 and 1, respectively. Thus $\eta - \xi_\mu = \eta_\mu$ is holomorphic there and its unweighted local L^2 norm tends to zero. Applying the local L^2 estimate for holomorphic functions to its coefficients in finitely many relatively compact coordinate charts gives $\sup_K |\eta - \xi_\mu|_\omega \to 0$.

Finally, we verify that $\sigma_D^\mu \xi_\mu$ extends holomorphically across Y. Write $\lambda(t) = t + c_\lambda$ for large t. Near Y, we have $\varphi = -\log|\sigma_D|_h^2$ and $\chi \circ \varphi = 0$, so that

$$e^{-\mu\lambda\circ\varphi} = e^{-\mu c_\lambda}|\sigma_D|_h^{2\mu}.$$

The metric-independence of the $(2,0)$-density (182) and (231) show that $\sigma_D^\mu \xi_\mu$ is locally square integrable across Y. At a smooth point of the reduced set Y, the Laurent expansions of its coefficients in a transverse coordinate have no negative powers by square integrability (Use polar coordinates and Lemma I.4.2). The remaining set has complex codimension at least two, so Hartogs extension applies (See the discussion after Example III.2.147). The extensions agree on overlaps and give $\sigma_\mu \in \Gamma(X, \mathcal{O}_X(K_X \otimes [D]^\mu))$ with $\xi_\mu = \sigma_\mu/\sigma_D^\mu$ on M. This proves the required approximation. □

Remark III.2.196. Under the hypotheses above, the exhaustion φ is plurisubharmonic and strictly plurisubharmonic outside a compact subset. The Lefschetz isomorphism in [De2, Theorem 3 and (8), pp. 284 and 294], with $n = 2$, $q = 1$, and $(r,s) = (1,0)$, gives $H_c^{1,0}(M) \cong H^{2,1}(M)$. The left side is zero, since a compactly supported holomorphic 1-form on a connected non-compact complex manifold vanishes. Thus $H^{2,1}(M) = 0$. The weighted estimate above additionally controls the growth of the primitive; this growth control is what allows the approximating forms to have finite-order poles along D. □

2.15 Weakly 1-complete manifolds

The preceding results use strict positivity in several different places: in an exhaustion, in the Levi form of a boundary, or in the curvature of a bundle. It gives estimates for the $\overline{\partial}$-equation and, through these estimates, vanishing, extension, and approximation. We now look at examples for which we have a plurisubharmonic exhaustion but do not assume any further strict positivity. For domains with smooth boundary, Levi-flat boundaries are a natural case to keep in mind. This is a motivation, not a characterization of weak 1-completeness: the examples of Diederich–Ohsawa and Nemirovski in §2.14 show that a Levi-flat boundary need not prevent strong 1-convexity or Steinness.

Compact spaces and a suspension over an elliptic curve

A compact complex space is weakly 1-complete, with a constant exhaustion. On a connected reduced compact complex space, holomorphic functions and plurisubharmonic functions are constant (Lemma III.2.91). Its coherent cohomology is finite-dimensional and Hausdorff by the Cartan–Serre finiteness theorem (Theorem III.2.174). Thus few holomorphic functions do not by themselves imply poorly behaved cohomology. The following non-compact examples exhibit both possibilities.

Fix $a \in \mathbb{C}$ with $|a| > 1$, and put $C := \mathbb{C}^*/\!\sim_a$, where $\sim_a$ is the equivalence relation on $\mathbb{C}^*$ defined by $z \sim_a z'$ if and only if $z' = a^m z$ for some $m \in \mathbb{Z}$. This quotient is an elliptic curve. We also write $C = \mathbb{C}^*/\langle a\rangle$, where $\langle a\rangle := \{a^m \mid m \in \mathbb{Z}\}$ is the subgroup generated by

a, acting on $\mathbb{C}^*$ by multiplication. For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, set $\lambda := e^{2\pi\sqrt{-1}\theta}$; we keep a fixed and vary θ, and hence λ. The transformation

$$T_\theta(z, w) := (az, \lambda w) \tag{232}$$

generates a free properly discontinuous action on $\mathbb{C}^* \times \mathbb{C}$. Its quotient is the total space of a unitary flat line bundle $F_\theta \to C$. Put $Y_\theta := (\text{Zero Section})_{F_\theta} \subset F_\theta$ and

$$M_\theta := F_\theta \setminus Y_\theta = (\mathbb{C}^* \times \mathbb{C}^*)/\langle T_\theta \rangle, \qquad \pi \colon M_\theta \longrightarrow C.$$

The other generator, coming from a loop around 0 in the first $\mathbb{C}^*$, has trivial monodromy. We deliberately keep just this one-parameter family (Compare [KoT, §2 and §4.1]).

Since $|\lambda| = 1$, the functions $|w|$ and $t := \log |w|^2$ descend to M_θ. The function $\varphi := t^2$ is a C^∞ plurisubharmonic exhaustion: the base is compact and t^2 tends to $+\infty$ at both ends of each fiber. Thus M_θ is weakly 1-complete. Moreover,

$$\sqrt{-1}\partial\overline{\partial}\varphi = 2\sqrt{-1}\frac{dw}{w} \wedge \frac{d\overline{w}}{\overline{w}}.$$

The invariant form $\sqrt{-1}(dz/z) \wedge (d\overline{z}/\overline{z}) + \sqrt{-1}(dw/w) \wedge (d\overline{w}/\overline{w})$ is a complete Kähler form: on the logarithmic covering $\mathbb{C}^2$ it is a constant positive form, and the deck transformations are translations. The displayed Hessian has a positive eigenvalue everywhere, so φ also makes M_θ strongly 2-complete.

Each hypersurface $\{|w| = r\}$ is compact and Levi-flat. Its Levi leaves are induced by $w =$ constant upstairs, and every leaf is dense: on a transversal its returns are the rotations $w \mapsto \lambda^n w$, whose orbit is dense in the circle of radius r. Thus Lemma III.2.189 gives $\mathcal{O}_{M_\theta}(M_\theta) = \mathbb{C}$. In particular M_θ is not holomorphically convex, and hence not strongly 1-convex by Theorem III.2.181. It is a *toroidal group*: a connected complex Lie group with no nonconstant holomorphic functions. Its group structure is inherited from the multiplicative group $\mathbb{C}^* \times \mathbb{C}^*$.

The following observation also gives a coefficientwise explanation.

Lemma III.2.197. *Let $(X, \mathcal{O}_X)$ be a connected compact reduced complex space, and let F be a unitary flat holomorphic line bundle on X. Then $\dim H^0(X, \mathcal{O}_X(F))$ is either 0 or 1, and it is 1 if and only if F is holomorphically trivial.*

Proof. Choose a flat metric h on F. For a non-zero holomorphic section s, the function $|s|_h^2$ is globally plurisubharmonic: in a unitary frame it is the squared absolute value of a holomorphic function. By Lemma III.2.91, it is a positive constant. Thus s never vanishes and trivializes F. Conversely, when F is trivial, its sections are constant multiples of a fixed nowhere vanishing section, since $\mathcal{O}_X(X) = \mathbb{C}$. □

To apply this observation, use the unitary flatness of $F_\theta \to C$ to choose a coordinate open covering $\{U_j\}_{j \in J}$ of C with unitary flat trivializations. For each $j \in J$, let z_j be a holomorphic coordinate on U_j, and let w_j be the fiber coordinate on $F_\theta|_{U_j}$ induced by the chosen trivialization. On overlaps, $w_j = t_{jk} w_k$ for $j, k \in J$, where t_{jk} is locally constant on

$U_j \cap U_k$ and $|t_{jk}| = 1$. A holomorphic function on M_θ has local Laurent expansions $f_j(z_j, w_j) = \sum_{n\in\mathbb{Z}} c_{j,n}(z_j) w_j^n$. Uniqueness of Laurent coefficients on an overlap gives $c_{k,n} = t_{jk}^n c_{j,n}$, so the coefficients of degree n define a holomorphic section of F_θ^{-n}. For $n \neq 0$, this bundle has non-trivial unitary monodromy λ^{-n} and is not holomorphically trivial, by the uniqueness of unitary flat structures (Lemma III.2.130). The preceding lemma makes these coefficients zero. The degree-zero coefficient is constant on C, proving the constancy again. The same argument works on the disc and punctured-disc subbundles used below. This coefficientwise description will be developed for unitary flat line bundles over compact Riemann surfaces, including its application to first cohomology, in Proposition III.2.205 and the subsequent discussion.

A difference equation and shrinking neighborhoods

Keep $a \in \mathbb{C}$ with $|a| > 1$ fixed, let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and put $\lambda = e^{2\pi\sqrt{-1}\theta}$ and M_θ as above. Write $\delta_n := |1 - \lambda^n|$ for $n \geq 1$. We shall use the condition

$$\text{there exist } C_0, b > 0 \text{ such that } \delta_n \geq C_0 e^{-bn} \quad \text{for every } n \geq 1. \tag{233}$$

It is equivalent to the same exponential lower bound for $d(n\theta, \mathbb{Z})$, since $4d(n\theta, \mathbb{Z}) \leq \delta_n \leq 2\pi d(n\theta, \mathbb{Z})$ (See also the discussion following (247) in §3.1). In particular, the Diophantine condition of Exercise I.4 implies (233); failure of the Diophantine condition alone does not imply failure of this weaker condition.

Theorem III.2.198 (Kazama, for the above two-dimensional family). *Fix $a \in \mathbb{C}$ with $|a| > 1$. For every $\theta \in \mathbb{R} \setminus \mathbb{Q}$, set $\lambda = e^{2\pi\sqrt{-1}\theta}$ and $M_\theta = (\mathbb{C}^* \times \mathbb{C}^*)/\langle T_\theta \rangle$, where $T_\theta(z, w) = (az, \lambda w)$. Then $H^q(M_\theta, \mathcal{O}_{M_\theta}) = 0$ for $q \geq 2$. If (233) holds, pullback induces a TVS isomorphism*

$$\pi^* \colon H^1(C, \mathcal{O}_C) \longrightarrow H^1(M_\theta, \mathcal{O}_{M_\theta}) \cong \mathbb{C}.$$

Otherwise $H^1(M_\theta, \mathcal{O}_{M_\theta})$ is infinite-dimensional and non-Hausdorff for its natural quotient topology.

This is the special case of [Ka, Theorems 4.3–4.4] discussed in [KoT, §2, Proposition 2.3]. We give a calculation adapted to this suspension, and then pass from shrinking disc-bundle neighborhoods to their complement by Mayer–Vietoris, following the geometric organization of [Koi12, §§5–6].

For a positive real number r, put $\Delta_r := \{|w| < r\}$ and $\Delta_r^* := \Delta_r \setminus \{0\}$. For $A = \Delta_r, \Delta_r^*$, or $\mathbb{C}^*$, let $V_{\theta,A} := (\mathbb{C}^* \times A)/\langle T_\theta \rangle$. Define the continuous linear map

$$D_\theta \colon \mathcal{O}_A(A) \longrightarrow \mathcal{O}_A(A), \qquad (D_\theta b)(w) := b(\lambda w) - b(w).$$

We give these function spaces the topology of uniform convergence on compact subsets.

Lemma III.2.199. *In the above setting, for every $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and each of the above choices of A, there are natural identifications*

$$\begin{aligned} H^0(V_{\theta,A}, \mathcal{O}_{V_{\theta,A}}) &\cong \operatorname{Ker} D_\theta, \\ H^1(V_{\theta,A}, \mathcal{O}_{V_{\theta,A}}) &\cong \mathcal{O}_A(A)/\operatorname{Image} D_\theta, \end{aligned} \tag{234}$$

and $H^q(V_{\theta,A}, \mathcal{O}_{V_{\theta,A}}) = 0$ for $q \geq 2$. The identifications are TVS isomorphisms for the natural cohomology topologies and the indicated subspace and quotient topologies. They commute with restriction of the fiber domain A.

PROOF. Put $V := V_{\theta,A}$, and let $\mathcal{F}$ be the foliation induced by $w =$ constant upstairs. Its sheaf $\mathcal{C}_{\mathcal{F}}$ of holomorphic functions locally constant along the leaves fits into the exact sequence of sheaves of complex vector spaces

$$0 \longrightarrow \mathcal{C}_{\mathcal{F}} \longrightarrow \mathcal{O}_V \xrightarrow{P} \mathcal{O}_V(K_V \otimes [Y_\theta]) \longrightarrow 0, \qquad P(f) = df \wedge \frac{dw}{w}$$

when $A = \Delta_r$. Off the zero section, replace $K_V \otimes [Y_\theta]$ by K_V (Compare [Koi12, §2 and Proposition 3.1]). To identify the last sheaf explicitly, use the additive leaf coordinate ζ with $z = e^{2\pi\sqrt{-1}\zeta}$. The form $d\zeta \wedge dw/w$, where $d\zeta = (2\pi\sqrt{-1})^{-1} dz/z$, is a global frame of this line bundle. In this frame $P = \partial/\partial\zeta = 2\pi\sqrt{-1} z\partial/\partial z$, and the sequence becomes

$$0 \longrightarrow \mathcal{C}_{\mathcal{F}} \longrightarrow \mathcal{O}_V \xrightarrow{P} \mathcal{O}_V \longrightarrow 0.$$

Local integration in ζ, with w as a parameter, proves exactness. We first isolate the obstruction to integration in the base variable on an annulus, and then apply this calculation to the two-annulus covering of the elliptic curve.

First consider a concentric annulus $B = \{r_- < |z| < r_+\}$, where $0 < r_- < r_+$, and put $W := B \times A$. Fix $0 < \eta < \pi/2$ and cover B by the two sectors

$$U_0 = \{re^{\sqrt{-1}t} \mid r_- < r < r_+,\ -\eta < t < \pi + \eta\},$$
$$U_1 = \{re^{\sqrt{-1}t} \mid r_- < r < r_+,\ \pi - \eta < t < 2\pi + \eta\}.$$

Denote the overlap components near the negative and positive real axes by U_π and U_+, in this order. Choose logarithms ζ_i on U_i such that $z = e^{2\pi\sqrt{-1}\zeta_i}$ and $\zeta_1 - \zeta_0 = 0$ on U_π, while $\zeta_1 - \zeta_0 = 1$ on U_+. On $U_0 \times A$, $U_1 \times A$, and each connected component of their intersection, integration in the simply connected ζ-domain, with w as a holomorphic parameter, makes P surjective on sections. Since these sets are Stein, the Oka–Cartan fundamental theorem B (Theorem III.2.165) and the long exact sequence show that this covering is Leray for all three sheaves.

A section of $\mathcal{C}_{\mathcal{F}}$ on each of these products depends only on w. Thus the two-term complex has both terms equal to $\mathcal{O}_A(A) \oplus \mathcal{O}_A(A)$, with components as follows:

Degree	Pieces	Components
0	$U_0 \times A$, $U_1 \times A$	$b_0(w)$, $b_1(w)$
1	$U_\pi \times A$, $U_+ \times A$	$c_\pi(w)$, $c_+(w)$

Its differential is $(b_0, b_1) \mapsto (b_1 - b_0, b_1 - b_0)$. Thus $H^0(W, \mathcal{C}_{\mathcal{F}}) \cong \mathcal{O}_A(A)$, $H^1(W, \mathcal{C}_{\mathcal{F}}) \cong \mathcal{O}_A(A)$ by $[(c_\pi, c_+)] \mapsto c_+ - c_\pi$, and the higher cohomology vanishes. To compute the connecting map, let $g \in \mathcal{O}_W(W)$ and choose local primitives f_i with $Pf_i = g$ on $U_i \times A$. Their differences belong to $\mathcal{C}_{\mathcal{F}}$ and hence depend only on w. The connecting class is represented by $c_\pi = f_1 - f_0$

on $U_\pi \times A$ and $c_+ = f_1 - f_0$ on $U_+ \times A$ (Remark II.3.80). For $r_- < \rho < r_+$, integration along the upper and lower semicircles gives

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_\rho} g(z,w)\frac{dz}{z} &= f_0(-\rho,w) - f_0(\rho,w)\\ &\quad + f_1(\rho,w) - f_1(-\rho,w)\\ &= c_+(w) - c_\pi(w).\end{aligned}$$

Here $\gamma_\rho\colon [0,2\pi] \to B$ is the positively oriented loop $\gamma_\rho(t) = \rho e^{\sqrt{-1}t}$, which traverses the circle centered at 0 of radius ρ once. The integral defines the period average $m_B(g)(w)$ and the connecting map

$$m_B\colon \mathcal{O}_{B\times A}(B\times A) \longrightarrow \mathcal{O}_A(A).$$

For $b \in \mathcal{O}_A(A)$, put $\iota(b)(z,w) := b(w)$. The primitives $b(w)\zeta_i$ of $\iota(b)$ give the cocycle $(0,b)$, so $m_B(\iota(b)) = b$. The long exact sequence therefore gives

$$0 \longrightarrow \mathcal{O}_A(A) \xrightarrow{\iota} \mathcal{O}_W(W) \xrightarrow{P} \mathcal{O}_W(W) \xrightarrow{m_B} \mathcal{O}_A(A) \longrightarrow 0,$$

In particular, $\operatorname{Image} P = \operatorname{Ker} m_B$ and $m_B\iota = \mathrm{id}$. Cauchy's theorem makes m_B independent of ρ. It is continuous for compact convergence and commutes with restriction to smaller annuli and smaller fiber domains. Put $R\colon \mathcal{O}_A(A) \to \mathcal{O}_A(A)$, $(Rb)(w) := b(\lambda w)$. For $g \in \mathcal{O}_{aB\times A}(aB\times A)$, where $aB := \{az \mid z \in B\}$, change of variable in the contour integral gives $m_B(g\circ T_\theta) = R\big(m_{aB}(g)\big)$.

Now fix $0 < \varepsilon < 1/4$. The images in C of the annuli

$$B_0 = \{|a|^{-\varepsilon} < |z| < |a|^{1/2+\varepsilon}\}, \qquad B_1 = \{|a|^{1/2-\varepsilon} < |z| < |a|^{1+\varepsilon}\}$$

form a two-element open covering, with each annulus mapping injectively. Its overlap components are represented by $B_m = \{|a|^{1/2-\varepsilon} < |z| < |a|^{1/2+\varepsilon}\}$ and $B_s = \{|a|^{-\varepsilon} < |z| < |a|^{\varepsilon}\}$. The subscripts m and s stand for the middle and seam overlaps, respectively. The induced covering of V has pieces represented by $B_i \times A$, with the following coordinate identifications and Čech components:

Overlap representative	Coordinates in the two charts	Čech component
$B_m = B_0 \cap B_1$	(z,w) in both charts	$f_1(z,w) - f_0(z,w)$
$B_s = B_0 \cap a^{-1}B_1$	(z,w) and $T_\theta(z,w) = (az,\lambda w)$	$f_1(az,\lambda w) - f_0(z,w)$

These pieces and overlaps are Stein. The Oka–Cartan fundamental theorem B (Theorem III.2.165) and the Mayer–Vietoris sequence (Proposition II.3.67) therefore compute $H^0(V,\mathcal{O}_V)$ and $H^1(V,\mathcal{O}_V)$ by the complex

$$\begin{aligned}E^0 &= \mathcal{O}_{B_0\times A}(B_0\times A) \oplus \mathcal{O}_{B_1\times A}(B_1\times A),\\ E^1 &= \mathcal{O}_{B_m\times A}(B_m\times A) \oplus \mathcal{O}_{B_s\times A}(B_s\times A),\\ \delta(f_0,f_1) &= (f_1 - f_0,\ f_1\circ T_\theta - f_0),\end{aligned}$$

Here $\delta\colon E^0 \to E^1$ is the differential; all other terms are zero. The same argument gives $H^q(V, \mathcal{O}_V) = 0$ for $q \geq 2$. We use the natural cohomology topology of (218).

For $j = 0, 1$, apply P componentwise to obtain $P\colon E^j \to E^j$. Write $F^j := \operatorname{Ker} P$ and $Z^j := \operatorname{Image} P$. Then $F^0 = F^1 = \mathcal{O}_A(A) \oplus \mathcal{O}_A(A)$, with differential $\delta_F(b_0, b_1) = (b_1 - b_0, Rb_1 - b_0)$. The componentwise averages give a continuous cochain map $p\colon E^\bullet \to F^\bullet$, with $p^0(f_0, f_1) = \big(m_{B_0}(f_0), m_{B_1}(f_1)\big)$ and $p^1(g_m, g_s) = \big(m_{B_m}(g_m), m_{B_s}(g_s)\big)$. Inclusion gives $i\colon F^\bullet \to E^\bullet$ with $pi = \mathrm{id}$. The calculation on $B \times A$ shows that $Z^j = \operatorname{Ker} p^j$ and $E^j = i(F^j) \oplus Z^j$ as TVSs. Since $P\delta = \delta P$, there is a short exact sequence of complexes

$$0 \longrightarrow F^\bullet \xrightarrow{i} E^\bullet \xrightarrow{P} Z^\bullet \longrightarrow 0.$$

Its long exact sequence is

$$\begin{aligned} 0 \longrightarrow H^0(F^\bullet) &\longrightarrow H^0(E^\bullet) \longrightarrow H^0(Z^\bullet) \\ &\longrightarrow H^1(F^\bullet) \longrightarrow H^1(E^\bullet) \longrightarrow H^1(Z^\bullet) \longrightarrow 0 \end{aligned}$$

(See Chapter I, §5.2). We compute its terms and then check their topologies.

(*i*) Since $D_\theta = R - \mathrm{id}$, the map $b \mapsto (b, b)$ identifies $\operatorname{Ker} D_\theta$ with $H^0(F^\bullet)$. For degree one, use $[(c_m, c_s)] \mapsto [c_s - c_m]$ to identify $H^1(F^\bullet)$ with the quotient $\mathcal{O}_A(A)/\operatorname{Image} D_\theta$. Its inverse is $[b] \mapsto [(0, b)]$.

(*ii*) A cocycle in Z^0 defines a global holomorphic function on V. By the coefficient argument following Lemma III.2.197, this function is constant. Its averages vanish, so it is zero. Thus $H^0(Z^\bullet) = 0$.

(*iii*) Take $(g_m, g_s) \in Z^1$. Its Laurent expansions in z have zero constant terms. Write $g_\nu = g_\nu^+ + g_\nu^-$ for $\nu = m, s$, where the positive part extends across 0 and vanishes there, and the negative part extends across ∞ and vanishes there. Define

$$\begin{aligned} f_1^+(z, w) &= \sum_{j \geq 1} (g_s^+ - g_m^+)(a^{-j}z, \lambda^{-j}w), & f_0^+ &= f_1^+ - g_m^+, \\ f_0^-(z, w) &= -\sum_{j \geq 0} g_s^-(a^j z, \lambda^j w) + \sum_{j \geq 1} g_m^-(a^j z, \lambda^j w), & f_1^- &= f_0^- + g_m^-. \end{aligned}$$

The positive parts f_1^+ and f_0^+ are defined for $|z| < |a|^{1+\varepsilon}$ and $|z| < |a|^{1/2+\varepsilon}$, respectively; the negative parts f_0^- and f_1^- are defined for $|z| > |a|^{-\varepsilon}$ and $|z| > |a|^{1/2-\varepsilon}$, respectively. The vanishing at 0 or ∞ makes the terms $O(|a|^{-j})$ uniformly on compact subsets. All rotations of a compact subset of A remain in a compact subset of A, so the series converge normally. Cauchy estimates also show continuity of this construction. For $f_i = f_i^+ + f_i^-$, telescoping gives $f_1 - f_0 = g_m$ and $f_1 \circ T_\theta - f_0 = g_s$. Each f_i has zero average. Hence $\delta(Z^0) = Z^1$, and $H^1(Z^\bullet) = 0$.

(*iv*) The long exact sequence now makes the maps induced by i isomorphisms in degrees zero and one. On the original degree-one cochains, the comparison and its inverse are

$$[(g_m, g_s)] \longmapsto [m_{B_s}(g_s) - m_{B_m}(g_m)], \qquad [b] \longmapsto [(0, b)].$$

Indeed, a coboundary maps to $D_\theta\big(m_{B_1}(f_1)\big)$, and the difference between a cochain and ip of that cochain is a coboundary in Z^1 by (iii). Both maps are induced by continuous maps before passage to the quotients. They are therefore inverse TVS isomorphisms, even when the quotients are not Hausdorff. The degree-zero assertion follows in the same way from the continuous inclusion and average.

All these maps commute with restriction of A. Cochains independent of w give the corresponding complex on the base C, so the same comparison also identifies pullback from C. □

In particular, only the fiber coefficients can produce small denominators. On a sufficiently fine trivializing cover of C, a cocycle has expansions $g_{jk}(z_j, w_j) = \sum_n g_{jk,n}(z_j) w_j^n$; the degree-$n$ coefficients form a cocycle with values in F_θ^{-n}. The lemma computes those coefficient equations at once in the present suspension.

For $r > 0$, let $V_{\theta,r} := V_{\theta,\Delta_r} \subset F_\theta$ and $V_{\theta,r}^* := V_{\theta,r} \setminus Y_\theta$. Recall the directed set $\mathcal{N}_{F_\theta}(Y_\theta)$ of open neighborhoods of Y_θ, ordered by reverse inclusion (Chapter II, Example II.3.20). Since Y_θ is compact, the $V_{\theta,r}$ form a cofinal family as $r \searrow 0$.

Proposition III.2.200. *Under (233), restriction gives isomorphisms of vector spaces*

$$\varinjlim_{V \in \mathcal{N}_{F_\theta}(Y_\theta)} H^1(V, \mathcal{O}_V) \longrightarrow \varinjlim_{V \in \mathcal{N}_{F_\theta}(Y_\theta)} H^1(V \setminus Y_\theta, \mathcal{O}_{V \setminus Y_\theta}) \cong \mathbb{C}.$$

Both copies of $\mathbb{C}$ are represented by the constant Laurent coefficient in (234).

Proof. For $g(w) = \sum_{n \geq 1} g_n w^n$ holomorphic on Δ_r, the unique formal solution with zero constant term is $b_n = g_n/(\lambda^n - 1)$. If $R < r$, Cauchy's estimate gives

$$|g_n| \leq M_R R^{-n}, \qquad |b_n| \leq C_0^{-1} M_R (e^b/R)^n.$$

It therefore defines a holomorphic solution on every disc of radius less than Re^{-b}. Every nonconstant part vanishes after restriction to a smaller disc; the constant coefficient cannot be in the image of D_θ.

On Δ_r^* split a Laurent series into its nonnegative and negative parts. The negative part is an entire function of $1/w$ with zero constant term. Since $|1 - \lambda^{-n}| = \delta_n$, division by the negative-degree multipliers has at most exponential growth and preserves this entire convergence. The positive part is handled by the same shrinking as above. The lemma now gives the assertions. □

Remark III.2.201. The proposition does not assert that $H^1(V_{\theta,r}, \mathcal{O}_{V_{\theta,r}})$ is one-dimensional for a fixed r. A solution can lose radius of convergence. The distinction between Cases I and II in [Koi12, §2.3 and Proposition 5.1] concerns precisely what may happen before shrinking; both are covered by (233). Passing to a neighborhood germ absorbs a finite loss of radius, but it still requires a positive radius. It does not replace convergent functions by arbitrary formal series. □

Proof of Theorem III.2.198. Compactify the fibers to obtain the ruled surface

$$S_\theta := (\mathbb{C}^* \times \mathbb{P}^1)/\langle T_\theta \rangle.$$

Here T_θ acts by multiplication by λ in the affine fiber coordinate. Write C_0, C_∞ for the images of the zero and infinity sections, respectively; then $M_\theta = S_\theta \setminus (C_0 \sqcup C_\infty)$. Use the same two-annulus covering of C as in Lemma III.2.199. Each piece and overlap upstairs is a product $B_i \times \mathbb{P}^1$. Although these products are not Stein, their two affine fiber charts are Stein, and Cauchy splitting in the fiber coordinate shows that their positive-degree $\mathcal{O}$-cohomology vanishes (Exercise I.6, with the annulus coordinate as a holomorphic parameter). Their holomorphic functions are constant along the compact fibers. Hence their Čech complex is that of the base, giving, by pullback,

$$H^1(S_\theta, \mathcal{O}_{S_\theta}) \cong H^1(C, \mathcal{O}_C) \cong \mathbb{C}, \qquad H^q(S_\theta, \mathcal{O}_{S_\theta}) = 0 \quad (q \geq 2).$$

Assume (233). For $0 < r < 1$ take disjoint neighborhoods $V_{0,r}$ and $V_{\infty,r}$ of the two sections, given by $|w| < r$ and $|1/w| < r$, respectively, and put $V^*_{\nu,r} := V_{\nu,r} \setminus C_\nu$ for $\nu = 0, \infty$. The Mayer–Vietoris sequence (Proposition II.3.67) for $S_\theta = M_\theta \cup (V_{0,r} \sqcup V_{\infty,r})$ gives

$$\begin{aligned} 0 \longrightarrow H^1(S_\theta, \mathcal{O}_{S_\theta}) &\longrightarrow H^1(M_\theta, \mathcal{O}_{M_\theta}) \oplus H^1(V_{0,r}, \mathcal{O}_{V_{0,r}}) \oplus H^1(V_{\infty,r}, \mathcal{O}_{V_{\infty,r}}) \\ &\longrightarrow H^1(V^*_{0,r}, \mathcal{O}_{V^*_{0,r}}) \oplus H^1(V^*_{\infty,r}, \mathcal{O}_{V^*_{\infty,r}}) \longrightarrow 0. \end{aligned}$$

Indeed, the degree-zero groups on these pieces are constants by Lemma III.2.199, and their difference map $\mathbb{C} \oplus \mathbb{C}^2 \to \mathbb{C}^2$ is surjective; also $H^2(S_\theta, \mathcal{O}_{S_\theta}) = 0$. The neighborhoods $V_{\nu,r}$ are cofinal as $r \searrow 0$. Taking their direct limit preserves exactness by Proposition II.3.43. With

$$G_\nu := \varinjlim_{V \in \mathcal{N}_{S_\theta}(C_\nu)} H^1(V, \mathcal{O}_V), \qquad G^*_\nu := \varinjlim_{V \in \mathcal{N}_{S_\theta}(C_\nu)} H^1(V \setminus C_\nu, \mathcal{O}_{V \setminus C_\nu}),$$

we obtain

$$0 \to H^1(S_\theta, \mathcal{O}_{S_\theta}) \to H^1(M_\theta, \mathcal{O}_{M_\theta}) \oplus G_0 \oplus G_\infty \to G^*_0 \oplus G^*_\infty \to 0. \tag{235}$$

The proposition makes $G_\nu \to G^*_\nu$ an isomorphism for each ν (at infinity replace λ by λ^{-1}). Exactness therefore makes restriction $H^1(S_\theta, \mathcal{O}_{S_\theta}) \to H^1(M_\theta, \mathcal{O}_{M_\theta})$ bijective. It follows that π^* is a linear bijection.

To check its topology, use the lemma with $A = \mathbb{C}^*$. For a Laurent series convergent on all of $\mathbb{C}^*$, both its positive part and its negative part, written in $1/w$, are entire. A finite exponential loss preserves both convergences. Hence Image D_θ is exactly the closed hyperplane of functions with zero constant coefficient. The quotient is therefore $\mathbb{C}$ with its usual topology. This step, rather than the algebraic exactness of the direct limit alone, proves Hausdorffness.

Suppose instead that (233) fails. Choose strictly increasing integers $n_j \geq 2$ with $\delta_{n_j}^{1/n_j} \leq e^{-2j}$. Then

$$g(w) := \sum_{j \geq 1} \sqrt{\delta_{n_j}}\, w^{n_j}$$

is entire. But any Laurent solution of $D_\theta b = g$ must have coefficient $\sqrt{\delta_{n_j}}/(\lambda^{n_j}-1)$ in degree n_j, whose absolute value has n_j-th root at least e^j. Its positive part has no positive radius of convergence, so no such solution exists even on a punctured neighborhood of 0. Every finite partial sum of g is in Image D_θ, and these partial sums converge on compact subsets. Thus the quotient in the lemma is non-Hausdorff. If it were finite-dimensional, Serre's closed-image criterion (Lemma I.6.55), applied to the continuous linear map D_θ between the Fréchet spaces $\mathcal{O}_{\mathbb{C}^*}(\mathbb{C}^*)$, would make its image closed. This contradiction proves infinite-dimensionality. The vanishing in degrees at least two follows from the lemma. □

Remark III.2.202. An irrational number θ is called *asymptotically positive* if (233) holds, and *asymptotically zero* otherwise. In the family considered here, M_θ is a *toroidal theta group* in the former case and a *toroidal wild group* in the latter ([AK, §2.2]). For these terms and further generalizations, see [Koi12]. □

A global $\partial\overline{\partial}$-potential problem

The compact Kähler $\partial\overline{\partial}$-lemma has several equivalent exactness formulations. Here we ask just whether every d-exact smooth $(1,1)$-form is $\partial\overline{\partial}$ of a global smooth function, the question studied by Kazama and Takayama [KaT1, KaT2].

Proposition III.2.203. *Every d-exact smooth $(1,1)$-form on M_θ is $\partial\overline{\partial}$ of a smooth function if and only if (233) holds. In that case, a holomorphic line bundle L on M_θ is unitary flat if and only if $c_1^{\mathbb{R}}(L) = 0$, where $c_1^{\mathbb{R}}(L)$ denotes the image of the integral first Chern class of Example II.3.86 under $\mathbb{Z} \to \mathbb{R}$, and we identify de Rham cohomology with $H^2(M_\theta, \mathbb{R})$ by the de Rham isomorphism.*

Proof. This is the $(1,1)$ case of [KaT1, Theorem 3.3]. In our example it follows directly from the preceding calculation. Write $\eta := dz/z$, a nowhere vanishing holomorphic form on C, and also write η for its pullback. Under (233), the pullback isomorphism in Kazama's theorem (Theorem III.2.198) and the Hodge calculation for the elliptic curve (Example III.2.127) give $H^{0,1}(M_\theta) = \mathbb{C}\{\overline{\eta}\}$. If $\alpha = d\gamma$ has type $(1,1)$, then $\overline{\partial}\gamma^{0,1} = 0$ and $\partial\gamma^{1,0} = 0$. By this cohomology computation and its complex conjugate, there are constants c, c' and smooth functions u, v such that

$$\gamma^{0,1} = c\overline{\eta} + \overline{\partial}u, \qquad \gamma^{1,0} = c'\eta + \partial v.$$

Since $d\eta = 0$, we obtain $\alpha = \partial\overline{\partial}(u - v)$.

Conversely, suppose that every d-exact smooth $(1,1)$-form has such a potential. For any $\overline{\partial}$-closed $(0,1)$-form β, apply this assertion to $d\beta = \partial\beta$. If $\partial\beta = \partial\overline{\partial}u$, then $\beta - \overline{\partial}u$ is d-closed. Thus the map taking the $(0,1)$ component induces a surjection $H^1_{\mathrm{DR}}(M_\theta, \mathbb{C}) \to H^{0,1}(M_\theta)$. Since the flat line bundle F_θ on the elliptic curve is topologically trivial, M_θ is homotopy equivalent to the real three-dimensional torus $(\mathbb{R}/\mathbb{Z})^3$. Its first de Rham cohomology with complex coefficients is therefore three-dimensional. The non-Hausdorff case of the theorem, in which $H^{0,1}$ is infinite-dimensional, is impossible. The assertion about line bundles is now the unitary-flatness criterion from the $\partial\overline{\partial}$-lemma (Proposition III.2.131). □

Ueda's estimate for unitary flat line bundles

The preceding suspension separates the equations for the fiber coefficients from the question of convergence. Ueda's estimate gives an analogous separation for a unitary flat line bundle, without choosing one generator of its monodromy in advance.

Let Y be a compact connected complex manifold. Write

$$\mathcal{P}(Y) := \mathrm{Image}\big(H^1(Y, \underline{\mathrm{U}(1)}_Y) \longrightarrow H^1(Y, \mathcal{O}_Y^*) = \mathrm{Pic}(Y)\big),$$

where $\underline{\mathrm{U}(1)}_Y$ is the sheaf of locally constant $\mathrm{U}(1)$-valued functions. By Lemma III.2.130, the map in this display is injective. We identify its source with its image. Let $\mathbb{I}_Y$ be a holomorphically trivial line bundle on Y. We also write $\mathbb{I}_Y$ for its class, which is the identity of $\mathcal{P}(Y)$. The character description identifies $\mathcal{P}(Y)$ with $\mathrm{Hom}(H_1(Y,\mathbb{Z}), \mathrm{U}(1))$ (Use the constant-sheaf comparison of Proposition II.3.81 and the universal coefficient theorem in degree one). Since $H_1(Y,\mathbb{Z})$ is finitely generated, this is a compact real torus times a finite group. This description specifies its topology; no topology on the whole Picard group is needed here. In general $\mathcal{P}(Y)$ is not $\mathrm{Pic}^0(Y)$.

Fix a translation-invariant distance d as follows. On the identity component use a Euclidean quotient distance on $H^1(Y,\mathbb{R})/\mathrm{Image}\big(H^1(Y,\mathbb{Z}) \longrightarrow H^1(Y,\mathbb{R})\big)$, where the map is induced by $\mathbb{Z} \hookrightarrow \mathbb{R}$. Transport this distance to the other components, and give points in distinct components a fixed distance larger than the diameter of a component. This is a choice, and it satisfies $d(\mathbb{I}_Y, F^{-1}) = d(\mathbb{I}_Y, F)$ for every $F \in \mathcal{P}(Y)$. For a sufficiently fine fixed finite connected open covering $\mathcal{U}_0 = \{U_j\}_{j\in J}$ trivializing unitary flat bundles, Ueda's distance can also be written

$$d_{\mathcal{U}_0}(\mathbb{I}_Y, F) = \inf_{(a_j)\in \mathrm{U}(1)^J} \max_{j,k} \sup_{U_j\cap U_k} |t_{jk}a_k - a_j|, \tag{236}$$

where $F \in \mathcal{P}(Y)$ is represented by locally constant unitary transition functions t_{jk}. Empty intersections contribute zero. Changing unitary frames leaves this value unchanged. The distances d and $d_{\mathcal{U}_0}$ are bi-Lipschitz equivalent. On the identity component this is [KUeh3, Appendix A.3, Proposition A.3]; the other finitely many components are compact and separated from the identity, so the comparison constants can be enlarged to include them. In particular, replacing d by this distance, or by another such Euclidean distance, only changes fixed multiplicative constants.

For a finite open covering $\mathcal{U} = \{U_j\}_{j\in J}$ and $F \in \mathcal{P}(Y)$ admitting unitary flat trivializations on its members, fix a compatible unitary flat structure on F, and let h be its flat metric. Choose local frames of this structure on the members of $\mathcal{U}$. Put $U_{jk} := U_j \cap U_k$. For cochains

$$f = \{(U_j, f_j)\} \in \check{C}^0(\mathcal{U}, \mathcal{O}_Y(F)),$$
$$g = \{(U_{jk}, g_{jk})\} \in \check{C}^1(\mathcal{U}, \mathcal{O}_Y(F)),$$

the components $f_j \in \mathcal{O}_Y(F)(U_j)$ and $g_{jk} \in \mathcal{O}_Y(F)(U_{jk})$ are sections. Put

$$\|f\|_{L^\infty(\mathcal{U})} := \max_j \sup_{x\in U_j} |f_j(x)|_h, \qquad \|g\|_{L^\infty(\mathcal{U})} := \max_{j,k} \sup_{x\in U_{jk}} |g_{jk}(x)|_h.$$

Empty intersections contribute zero, and these quantities can be infinite. Denote the Čech coboundary map by

$$\check{\delta}_F \colon \check{C}^0(\mathcal{U}, \mathcal{O}_Y(F)) \longrightarrow \check{C}^1(\mathcal{U}, \mathcal{O}_Y(F)).$$

Thus $(\check{\delta}_F f)_{jk} = f_k|_{U_{jk}} - f_j|_{U_{jk}}$.

THEOREM III.2.204 (Ueda's estimate). *Let Y be a compact connected complex manifold, let $\mathbb{I}_Y$ be a holomorphically trivial line bundle on Y, and fix the distance d on $\mathcal{P}(Y)$ as above. Let $\mathcal{U} = \{U_j\}_{j\in J}$ be a fixed finite open covering of Y. There exists $K > 0$, depending on Y, $\mathcal{U}$, and d, with the following property. For every $F \in \mathcal{P}(Y)$ admitting unitary flat trivializations on $\mathcal{U}$ and every coboundary*

$$g = \{(U_{jk}, g_{jk})\} \in \check{B}^1(\mathcal{U}, \mathcal{O}_Y(F)), \qquad \|g\|_{L^\infty(\mathcal{U})} < \infty,$$

any $f = \{(U_j, f_j)\} \in \check{C}^0(\mathcal{U}, \mathcal{O}_Y(F))$ with $\check{\delta}_F f = g$ satisfies

$$d(\mathbb{I}_Y, F)\, \|f\|_{L^\infty(\mathcal{U})} \le K\, \|g\|_{L^\infty(\mathcal{U})}. \tag{237}$$

PROOF. See Ueda [U1, Lemma 4], and [KUeh3, Lemma 8.1] for the statement in this form. Another proof by Ueda, included with his permission in [KUeh3, §8.3], makes the constant more explicit. The distance comparison above converts that estimate to our choice of d (See also [KUeh3, Appendix A.3, Proposition A.3] and [Koi12, §2.3]). For the L^2 version of the $\overline{\partial}$-equation, see [HaK, Theorem 1.1 and §2.3]: in the Kähler setting, the uniform L^2 estimate gives the Čech estimate. Conversely, [HKM, Theorem 1.3] uses Ueda's Čech estimate and the Čech–Dolbeault comparison to obtain the L^2 estimate on a general compact complex manifold with a Hermitian metric. In both formulations the prescribed cohomology class is assumed to be zero.

Choose a finite open covering $\{U_k^*\}_{k\in J}$ of Y such that $\overline{U_k^*}$ is a compact subset of U_k for every $k \in J$. The identities $f_j = f_k - g_{jk}$ on overlaps and the bounds of f_k on U_k^* show that $\|f\|_{L^\infty(\mathcal{U})}$ is finite. □

The estimate controls a primitive of a cocycle already known to be a coboundary; it does not assert the vanishing of $H^1(Y, \mathcal{O}_Y(F))$. We will divide by the distance only when F is non-trivial; in that case its primitive is unique by Lemma III.2.197.

Fiberwise expansions and cohomology

We now fix a compact Riemann surface Y and a unitary flat holomorphic line bundle $F \to Y$. In this setting $\mathcal{P}(Y) = \mathrm{Pic}^0(Y)$: a unitary flat bundle has degree zero, and the converse follows from Proposition III.2.138 and the degree calculation for curves. Thus (194) gives its compact complex torus structure. Use a Euclidean distance d on this torus as above.

Fix a finite covering $\mathcal{U} = \{U_j\}_{j\in J}$ by coordinate open sets such that every non-empty finite intersection is contractible. One can choose it independently of F by taking a sufficiently fine triangulation and its open stars (Lemma I.7.20 and the discussion preceding Question III.2.180). These open sets and their intersections are Stein. Fix a compatible

unitary flat structure on F, and let h be its flat metric. For each $j \in J$, choose a holomorphic coordinate z_j on U_j and a local frame e_j of this structure on U_j. Write w_j for the corresponding fiber coordinate. On overlaps our conventions give

$$e_k = t_{jk} e_j, \qquad w_j = t_{jk} w_k, \qquad |t_{jk}| = 1, \tag{238}$$

where t_{jk} is locally constant. The frames are chosen separately for the bundle under consideration, not as a family over the Picard torus.

Let $p\colon X_F \to Y$ be the projection of the total space of F. Using the fixed metric h, put

$$\begin{aligned} X_F^* &:= X_F \setminus (\text{Zero Section})_F, \\ V_{F,r} &:= \{v \in X_F \mid |v|_h < r\}, \qquad V_{F,r}^* := V_{F,r} \setminus (\text{Zero Section})_F \quad (r > 0). \end{aligned}$$

For $\Delta_r := \{w \in \mathbb{C} \mid |w| < r\}$ and $\Delta_r^* := \Delta_r \setminus \{0\}$, we use $V_{F,A}$ to denote X_F, X_F^*, $V_{F,r}$, or $V_{F,r}^*$ when A is $\mathbb{C}$, $\mathbb{C}^*$, Δ_r, or Δ_r^*, respectively. Set $V_{F,A,j} := p^{-1}(U_j) \cap V_{F,A}$ and $V_{F,A,jk} := V_{F,A,j} \cap V_{F,A,k}$. Each member and each non-empty finite intersection of this covering is a product of a Stein open subset of Y with A, hence is Stein. Cartan's theorem B (Theorem III.2.165) and Leray comparison (Theorem II.3.79) compute $H^q(V_{F,A}, \mathcal{O}_{V_{F,A}})$ by this covering. We give it the natural cohomology topology (218).

These four spaces are weakly 1-complete. For example, the fiber functions $|w|^2$, $(\log|w|^2)^2$, $-\log(r^2-|w|^2)$, and $-\log(r^2-|w|^2)-\log|w|^2$, respectively, glue to smooth plurisubharmonic exhaustions, since the base is compact.

Holomorphic functions and their coefficients. A holomorphic function f on $V_{F,A}$ has local expansions

$$f_j(z_j, w_j) = \sum_{\nu} s_{j,\nu}(z_j) w_j^{\nu},$$

with $\nu \geq 0$ for $A = \mathbb{C}, \Delta_r$, and $\nu \in \mathbb{Z}$ for $A = \mathbb{C}^*, \Delta_r^*$. Cauchy's integral formula makes the coefficients holomorphic in z_j and the expansions locally uniformly convergent. Write $z_k = \varphi_{kj}(z_j)$ for the base coordinate change. Comparing coefficients in $f_j = f_k$ gives

$$s_{k,\nu}(\varphi_{kj}(z_j)) = t_{jk}^{\nu} s_{j,\nu}(z_j).$$

Hence $\{(U_j, s_{j,\nu} e_j^{-\nu})\}$ defines a global section of $\mathcal{O}_Y(F^{-\nu})$. In particular, the coefficient bundle is $F^{-\nu}$, with the same sign as in the preceding suspension. In subsequent overlap formulas, functions in another base coordinate are understood to have been pulled back by φ_{kj}.

Proposition III.2.205. *Let Y, F, and $V_{F,A}$ be as above. If F is non-torsion in* $\mathrm{Pic}(Y)$, *then $H^0(V_{F,A}, \mathcal{O}_{V_{F,A}}) = \mathbb{C}$ for each of the four choices of A. If F has finite order $m \geq 1$, choose a unitary trivialization of F^m. It defines a holomorphic function u on X_F, locally $u = w_j^m$ in suitable unitary frames, and composition with u gives a TVS isomorphism*

$$\mathcal{O}_{A^{(m)}}(A^{(m)}) \longrightarrow H^0(V_{F,A}, \mathcal{O}_{V_{F,A}}), \qquad A^{(m)} := \{w^m \mid w \in A\}.$$

Here $A^{(m)}$ is, respectively, $\mathbb{C}$, $\mathbb{C}^*$, Δ_{r^m}, or $\Delta_{r^m}^*$, and the function spaces have the compact-open topologies.

PROOF. If F is non-torsion, then $F^{-\nu}$ is non-trivial for $\nu \neq 0$. Lemma III.2.197 annihilates these coefficients; the degree-zero coefficient is a holomorphic function on the compact Riemann surface Y, hence is constant.

If F has order m, adjust the unitary frames so that $t_{jk}^m = 1$, using the chosen trivialization of F^m. Then $u = w_j^m$ glues, and the same lemma leaves precisely the coefficients of degrees divisible by m. In these frames those coefficients are constants. The Taylor or Laurent expansion is therefore the expansion of one holomorphic function of u on $A^{(m)}$. Cauchy's formula, or restriction to a fixed fiber, shows continuity of the inverse map; composition with u is continuous. The function u depends on the chosen trivialization of F^m. □

For the non-torsion suspension M_θ above, this recovers the constancy first obtained from the dense Levi leaves. No density assertion is required for the coefficient calculation on a general compact Riemann surface.

The coefficient equations for a 1-cocycle. Keep the compact Riemann surface Y, bundle F, and covering fixed. Represent a class in $H^1(V_{F,A}, \mathcal{O}_{V_{F,A}})$ by $[\{(V_{F,A,jk}, f_{jk})\}]$, and expand f_{jk} in the j-th fiber coordinate as $\sum_\nu s_{jk,\nu}(z_j) w_j^\nu$. On a triple intersection the scalar cocycle equation, expressed in the j-th coordinates, is

$$f_{k\ell}(\varphi_{kj}(z_j), t_{jk}^{-1} w_j) - f_{j\ell}(z_j, w_j) + f_{jk}(z_j, w_j) = 0.$$

Consequently

$$t_{jk}^{-\nu} s_{k\ell,\nu} - s_{j\ell,\nu} + s_{jk,\nu} = 0. \tag{239}$$

Thus $s_\nu := \{(U_{jk}, s_{jk,\nu} e_j^{-\nu})\}$ is an $\mathcal{O}_Y(F^{-\nu})$-valued 1-cocycle. Its class is the first obstruction to solving for a primitive. If $[s_\nu] = 0$, Leray comparison gives a 0-cochain $b_\nu = \{(U_j, b_{j,\nu} e_j^{-\nu})\}$ satisfying

$$t_{jk}^{-\nu} b_{k,\nu} - b_{j,\nu} = s_{jk,\nu}. \tag{240}$$

The second question is whether the functions $\sum_\nu b_{j,\nu} w_j^\nu$ converge. Changing the original representative adds a coboundary to each s_ν, so the coefficient classes are well-defined on cohomology. Coefficient extraction also commutes with restriction of A.

When Y is an elliptic curve and F is non-torsion, the non-zero-degree coefficient classes always vanish. Indeed, compact Serre duality (Lemma III.2.121), Lemma III.2.178, and Lemma III.2.197 give, for $\nu \neq 0$,

$$H^1(Y, \mathcal{O}_Y(F^{-\nu})) \cong H^0(Y, \mathcal{O}_Y(K_Y \otimes F^\nu))^\vee = 0. \tag{241}$$

This explains the coefficientwise solvability in the elliptic suspension. On a general compact Riemann surface we instead keep the hypothesis $[s_\nu] = 0$; Ueda's estimate does not supply it.

Approach to the trivial bundle and convergence. Let F now be a non-torsion unitary flat line bundle on the compact Riemann surface Y, and let $\mathbb{I}_Y$ be a holomorphically trivial line bundle on Y. Set

$$d_\nu := d(\mathbb{I}_Y, F^\nu) = d(\mathbb{I}_Y, F^{-\nu}) > 0, \qquad \alpha(F) := \liminf_{\nu \to \infty} d_\nu^{1/\nu}. \tag{242}$$

Since the distance is bounded on the torus, $0 \leq \alpha(F) \leq 1$. We call F *asymptotically positive* when $\alpha(F) > 0$, and *asymptotically zero* when $\alpha(F) = 0$. The former condition is equivalent to the existence of $c, b > 0$ with $d_\nu \geq ce^{-b\nu}$ for every $\nu \geq 1$. These conditions are unchanged by a bi-Lipschitz change of distance, and agree with the terminology used above for the preceding one-parameter family: there the positive condition is (233). A polynomial lower bound implies asymptotic positivity, but is stronger. Asymptotic zero does not mean that d_ν itself tends to zero.

LEMMA III.2.206. *Let Y be a compact Riemann surface, let F be a non-torsion unitary flat line bundle, and let $[\{(V_{F,A,jk}, f_{jk})\}]$ be a class as above whose coefficient classes $[s_\nu]$ are all zero. If $\alpha(F) = 1$, this class is zero on $V_{F,A}$ for all four choices of A. If $\alpha(F) > 0$, it is zero on X_F and X_F^* when defined there. When defined on $V_{F,r}$ or $V_{F,r}^*$, it becomes zero on $V_{F,s}$ or $V_{F,s}^*$, respectively, for every $0 < s < \alpha(F)r$.*

PROOF. Choose the coefficient primitives in (240). For $\nu \neq 0$ they are unique, since the difference of two of them is a global section of $F^{-\nu}$, and Lemma III.2.197 applies. Choose the degree-zero primitive separately; no division by a zero distance is involved in that degree.

First consider the positive degrees on a disc of radius r. Use a smaller finite base covering $\mathcal{U}' = \{U_j'\}_{j\in J}$ of Y such that $\overline{U_j'}$ is a compact subset of U_j for every $j \in J$. Apply Ueda's estimate (Theorem III.2.204) to $\mathcal{U}'$. For $0 < R < r$, the original cocycle is bounded on the finitely many compact sets $\overline{U_j' \cap U_k'} \times \{|w| = R\}$ in the original trivializations. Cauchy's formula and Ueda's estimate therefore give constants M_R and K, independent of $\nu \geq 1$, with

$$\|s_\nu\|_{L^\infty(\mathcal{U}')} \leq M_R R^{-\nu}, \qquad \|b_\nu\|_{L^\infty(\mathcal{U}')} \leq K M_R d_\nu^{-1} R^{-\nu}. \tag{243}$$

The coefficient primitive restricted from the larger covering is bounded when its coboundary is bounded, as checked in the proof of that theorem. Thus the second inequality is applicable. For every a with $0 < a < \alpha(F)$ there is $c_a > 0$ such that $d_\nu \geq c_a a^\nu$ for all $\nu \geq 1$. The positive-degree primitive consequently converges normally for $|w| < aR$. Given $s < \alpha(F)r$, choose $a < \alpha(F)$ and $R < r$ with $s < aR$. If $\alpha(F) = 1$, this proves convergence on the whole open disc, by letting $s \nearrow r$. For the entire positive-degree series one may take R arbitrarily large, so any positive $\alpha(F)$ preserves entire convergence.

For a punctured domain, treat the negative-degree part separately. Writing $\nu = -\ell$ with $\ell > 0$, its series is an entire power series in $1/w$ with zero constant term. Cauchy's estimate on any circle $|w| = R > 0$ contained in A gives $\|s_{-\ell}\|_{L^\infty(\mathcal{U}')} \leq M_R R^\ell$. The same bound for its primitive includes the factor d_ℓ^{-1}. Since R may be arbitrarily small, a finite exponential loss preserves convergence of this negative-degree series for every $w \neq 0$. The positive-degree part is handled as above. These normally convergent series are holomorphic primitives on the smaller base covering and satisfy the original cocycle equation. That covering still covers the same fiber domain; Leray comparison proves the assertions. □

Cocycles with divergent primitives. The loss in (243) is not just an artifact of an upper bound. We construct cocycles realizing it, following the lower-bound argument of [Koi12, §2.3, Proposition 2.15]. Keep a non-torsion F on a compact Riemann surface Y, and the

fixed finite contractible covering and its unitary frames. For each $\nu \geq 1$, formula (236) allows constants $a_{j,\nu} \in \mathrm{U}(1)$ such that, on putting

$$a_\nu := \{(U_j, a_{j,\nu} e_j^{-\nu})\}, \qquad \varepsilon_\nu := \|\check{\delta}_{F^{-\nu}} a_\nu\|_{L^\infty(\mathcal{U})},$$

one has

$$d_{\mathcal{U}}(\mathbb{I}_Y, F^{-\nu}) \leq \varepsilon_\nu < 2 d_{\mathcal{U}}(\mathbb{I}_Y, F^{-\nu}).$$

In particular $\varepsilon_\nu > 0$ and $c_1 d_\nu \leq \varepsilon_\nu \leq c_2 d_\nu$ for fixed $c_1, c_2 > 0$. There is no need for the infimum to be attained. We are choosing 0-cochains, not changing the fiber coordinates with ν. Define the coboundary

$$s_\nu := \varepsilon_\nu^{-1} \check{\delta}_{F^{-\nu}} a_\nu = \{(U_{jk}, s_{jk,\nu} e_j^{-\nu})\}. \tag{244}$$

It has norm one. Its unique holomorphic primitive is a_ν/ε_ν, again by Lemma III.2.197.

Proposition III.2.207. *Let Y be a compact Riemann surface and let F be a non-torsion asymptotically zero unitary flat line bundle. There is a holomorphic 1-cocycle on the covering $\{V_{F,\mathbb{C},j}\}$ of X_F which is a limit of coboundaries, but whose restriction is not a coboundary on any $V_{F,r}$ or $V_{F,r}^*$ with $r > 0$, nor on X_F^*. In particular, the naturally topologized first cohomology of each of these fixed domains is non-Hausdorff and infinite-dimensional.*

Proof. Use (244), and take strictly increasing positive integers ν_k such that $\varepsilon_{\nu_k}^{1/\nu_k} \to 0$. Set

$$f_{j\ell}(z_j, w_j) := \sum_{k \geq 1} \sqrt{\varepsilon_{\nu_k}}\, s_{j\ell,\nu_k}(z_j) w_j^{\nu_k}. \tag{245}$$

Since $|s_{j\ell,\nu_k}| \leq 1$, the root test gives normal convergence on the whole total space in each chart. Equation (239) gives the scalar cocycle condition.

Suppose that its restriction to one of the stated domains were zero in cohomology. Since the chosen covering is Leray, there would be a holomorphic 0-cochain primitive on it. Comparing its Taylor or Laurent coefficients with (240), uniqueness forces its coefficient of degree ν_k on U_j to be $a_{j,\nu_k}/\sqrt{\varepsilon_{\nu_k}}$. The absolute values of these coefficients have ν_k-th roots $\varepsilon_{\nu_k}^{-1/(2\nu_k)} \to \infty$. Their positive-degree series has radius zero, contradicting holomorphy even on a punctured disc. Allowing negative Laurent coefficients does not change this contradiction.

Every finite truncation of (245) is the coboundary of the finite 0-cochain with local expressions $\sum_{k \leq N} a_{j,\nu_k} w_j^{\nu_k}/\sqrt{\varepsilon_{\nu_k}}$. The truncations converge in the compact-open cochain topology to f. Thus $[f] \neq 0$ belongs to the closure of zero in the cohomology quotient, for each fixed domain. Its cochain spaces and closed cocycle space are Fréchet. If their first cohomology were finite-dimensional, Lemma I.6.55, applied to the differential from 0-cochains to 1-cocycles, would make its image closed. This contradicts the constructed class. □

Remark III.2.208. Let Y and F be as in the coefficient calculation, but assume $0 < \alpha(F) < 1$. Fix $r > 0$ and choose ν_k with $\varepsilon_{\nu_k}^{1/\nu_k} \to \alpha(F)$. Replacing the coefficient $\sqrt{\varepsilon_{\nu_k}} s_{j\ell,\nu_k}$ in (245) by $r^{-\nu_k} s_{j\ell,\nu_k}$ gives a cocycle on $V_{F,r}$. Its forced primitive has coefficients $r^{-\nu_k} a_{j,\nu_k}/\varepsilon_{\nu_k}$,

and hence has radius exactly $\alpha(F)r$ on every base chart. The cocycle is therefore non-zero on $V_{F,r}$ and $V^*_{F,r}$, but is a limit of its finite coboundary truncations. Their first cohomologies are non-Hausdorff and infinite-dimensional. In contrast with Proposition III.2.207, this particular class becomes zero on every disc subbundle of radius less than $\alpha(F)r$ (And likewise on the punctured subbundles). Thus fixed-radius cohomology and cohomology after shrinking can have different separation properties. □

Torsion bundles over an elliptic curve. Finite monodromy reduces this case to a product after a finite covering. We record the calculation over an elliptic curve, where the deck translations act trivially on the base cohomology.

PROPOSITION III.2.209. *Let Y be an elliptic curve, let F be a unitary flat line bundle of order $m \geq 1$, and choose the unitary trivialization of F^m defining u as in Proposition III.2.205. For $A = \mathbb{C}, \mathbb{C}^*, \Delta_r$, or Δ_r^*, pullback from Y and multiplication by functions of u give a TVS isomorphism*

$$H^1(Y, \mathcal{O}_Y) \otimes_{\mathbb{C}} \mathcal{O}_{A^{(m)}}(A^{(m)}) \;\cong\; H^1(V_{F,A}, \mathcal{O}_{V_{F,A}}). \tag{246}$$

The left side is a finite-dimensional tensor factor times the compact-open function space, and the right side has its natural cohomology topology. These isomorphisms commute with restriction of A.

PROOF. The unitary character of F has image of order m, by the uniqueness of the unitary flat structure (Lemma III.2.130). Its kernel determines a connected cyclic unramified covering $\pi\colon \widetilde{Y} \to Y$ trivializing F. The curve $\widetilde{Y}$ is again elliptic and its deck transformations are translations: in a presentation of Y as a quotient of $\mathbb{C}$, replace the period lattice by a sublattice of finite index. Thus $V_{F,A}$ is the quotient of $\widetilde{Y} \times A$ by a finite group G acting by these translations and rotations of order m on A. For the product, the fixed harmonic projection and Green operator in the compact-base Dolbeault Hodge decomposition (Proposition III.2.120) give

$$H^1(\widetilde{Y} \times A, \mathcal{O}_{\widetilde{Y}\times A}) \cong H^1(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}) \otimes_{\mathbb{C}} \mathcal{O}_A(A).$$

These continuous operators preserve holomorphic dependence on the fiber variable. The resulting TVS isomorphism is induced by pullback and multiplication and commutes with restriction of A.

Averaging over G on cochains identifies the downstairs cohomology, with its quotient topology, with the invariant part upstairs. Translations act trivially on $H^1(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}})$, as seen from translation-invariant harmonic representatives, and pullback by π identifies this space with $H^1(Y, \mathcal{O}_Y)$. The invariant functions on A are precisely the functions of w^m, namely $\mathcal{O}_{A^{(m)}}(A^{(m)})$. This proves (246) and its compatibility with restriction. □

For an elliptic curve Y, the preceding proposition gives Hausdorff infinite-dimensional first cohomology in the torsion case. For non-torsion F, degree-zero coefficient extraction and pullback, together with (241) and Lemma III.2.206, give $H^1(V_{F,r}, \mathcal{O}_{V_{F,r}}) \cong H^1(Y, \mathcal{O}_Y)$ as TVSs when $\alpha(F) = 1$. If $\alpha(F) < 1$, this fixed-radius cohomology is non-Hausdorff and

infinite-dimensional by Proposition III.2.207 and Remark III.2.208; the same alternatives hold on punctured disc subbundles. When $0 < \alpha(F) < 1$, the nonconstant coefficient classes vanish after shrinking, whereas for $\alpha(F) = 0$ the class (245) remains non-zero on every smaller neighborhood. We return to these restriction-compatible calculations for neighborhood germs in Chapter IV, Example IV.1.44. For related cohomology calculations in the fibration case, see [Koi12, §2].

3 Local dynamics in one complex variable

What happens to a point when the same holomorphic map is applied repeatedly? This question belongs to the classical theory of iteration, which already included the work of Schröder and Kœnigs in the nineteenth century. The iteration of rational functions was the subject of the 1918 Grand Prix des sciences mathématiques of the French Académie des sciences. Against this background, the work of Fatou and Julia, with Montel's theory of normal families as an important tool, laid the foundations of the theory now expressed in terms of Fatou and Julia sets (See [Au] for the historical development).

In the language of discrete dynamical systems, iteration means studying the successive positions of a point under a self-map. For a complex manifold X, holomorphic self-maps form the monoid $\mathrm{End}(X)$, and its invertible elements form the group $\mathrm{Aut}(X)$ of holomorphic automorphisms. Their dynamics, and its relation to the complex analytic geometry of X, are also studied in several variables and higher dimensions (For the local viewpoint, see [Re]). Beginning with rotations of the circle as a real one-dimensional model, we follow a path into the iteration theory of holomorphic maps in one complex variable, concentrating on the local behavior near a fixed point. For a systematic introduction to one-variable complex dynamics, we refer to Carleson–Gamelin [CG] and Milnor [Mi3].

Let X be a set and $f\colon X \to X$ a map. Its *iterates* are defined by $f^{\circ 0} := \mathrm{id}_X$ and $f^{\circ(n+1)} := f \circ f^{\circ n}$ for $n \in \mathbb{Z}_{\geq 0}$. For $x \in X$, the points $x, f(x), f^{\circ 2}(x), \ldots$ form its *forward orbit*, also called its *orbit* when the direction is clear. A point is *periodic* if $f^{\circ q}(x) = x$ for some positive integer q; the least such q is its *period*, and its finite orbit is a *cycle*. A point of period one is a *fixed point*. If f is bijective, put $f^{\circ(-n)} := (f^{-1})^{\circ n}$ for $n \in \mathbb{Z}_{\geq 0}$. These negative iterates give the *backward orbit*; allowing all $n \in \mathbb{Z}$ gives the *two-sided orbit*.

In this section, we shall also often consider a map $f\colon U \to X$ defined only on a subset $U \subset X$. When considering its iterates, care is needed because $f(U)$ need not be contained in U. For $n \geq 1$, the value $f^{\circ n}(x)$ is defined when $x, f(x), \ldots, f^{\circ(n-1)}(x)$ all belong to U. The last value may lie outside U, in which case the next iterate is not defined. We use the same orbit terminology whenever the required iterates exist. Forward iteration does not require injectivity; if f is injective, negative iteration is treated in the same way using $f^{-1}\colon f(U) \to U$. For example, a power series centered at 0 with radius of convergence 1 defines a holomorphic map $f\colon \Delta \to \mathbb{C}$, where $\Delta = \{w \in \mathbb{C} \mid |w| < 1\}$. Its second iterate at $w \in \Delta$ is defined only when $f(w) \in \Delta$; further iterates require the intermediate points to remain in Δ.

When X is equipped with a probability measure μ, a measurable self-map $f\colon X \to X$ is

called *non-singular* if $\mu(f^{-1}(E)) = 0$ holds if and only if $\mu(E) = 0$, for every measurable set $E \subset X$. Such a map is called *ergodic* with respect to μ if every measurable set E satisfying $f^{-1}(E) = E$ up to a set of measure zero has $\mu(E) = 0$ or 1. Informally, iteration stirs the space in the sense that it cannot be separated into two invariant measurable parts of positive measure.

3.1 Circle rotations and rational approximation

Let us first view the number $\lambda = e^{2\pi\sqrt{-1}\theta}$ as a moving point of the unit circle $\mathrm{U}(1) = \partial\Delta$. Recall the exponential covering (Example I.7.3) and the notion of a lift (Theorem I.7.4). For $\theta \in \mathbb{R}$, let $R_\theta \colon \mathrm{U}(1) \to \mathrm{U}(1)$ be the rotation given by

$$p \colon \mathbb{R} \longrightarrow \mathrm{U}(1), \qquad p(x) := e^{2\pi\sqrt{-1}x}, \qquad R_\theta(\zeta) := \lambda\zeta.$$

Since $p(x+\theta) = R_\theta(p(x))$, the rotation R_θ lifts to $x \mapsto x + \theta$. The covering identifies $\mathbb{R}/\mathbb{Z}$ with $\mathrm{U}(1)$: the real number θ is a representative of the class $\theta + \mathbb{Z}$ determining the rotation. A rotation is a translation in the group $\mathrm{U}(1)$, not a group automorphism unless it is the identity.

We use two distance functions:

$$d_{\mathrm{U}(1)}(\zeta,\eta) := |\zeta - \eta| \quad (\zeta,\eta \in \mathrm{U}(1)), \qquad d_{\mathbb{R}/\mathbb{Z}}(x+\mathbb{Z}, y+\mathbb{Z}) := \min_{k\in\mathbb{Z}} |x - y - k|.$$

The first is the Euclidean chord distance restricted to the circle; the second measures shortest length on a circle of circumference one. Changing the real representatives x, y only changes the integer in the minimum, so the latter is well-defined and takes values in $[0, 1/2]$. Under the exponential identification they satisfy

$$\begin{aligned} 4d_{\mathbb{R}/\mathbb{Z}}(x+\mathbb{Z}, y+\mathbb{Z}) &\le d_{\mathrm{U}(1)}(p(x), p(y)) \\ &\le 2\pi d_{\mathbb{R}/\mathbb{Z}}(x+\mathbb{Z}, y+\mathbb{Z}). \end{aligned} \tag{247}$$

Indeed, if $s = d_{\mathbb{R}/\mathbb{Z}}(x+\mathbb{Z}, y+\mathbb{Z})$, then $d_{\mathrm{U}(1)}(p(x), p(y)) = 2\sin(\pi s)$; use $(2/\pi)t \le \sin t \le t$ for $0 \le t \le \pi/2$. The notation $d(t,\mathbb{Z}) := \min_{k\in\mathbb{Z}} |t-k|$, used in Chapter II in the discussion after Remark II.1.2, is related by $d_{\mathbb{R}/\mathbb{Z}}(t+\mathbb{Z}, 0+\mathbb{Z}) = d(t,\mathbb{Z})$.

The element λ is *torsion* in $\mathrm{U}(1)$ precisely when θ is rational. If $\theta = p/q$ in lowest terms with $q > 0$, every orbit of R_θ has period q (For example, $\theta = 2/5$ gives a cycle of five points). For irrational θ there are no periodic points and every forward orbit is dense. For density, the pigeonhole principle applied to $0, \theta, \ldots, N\theta$ modulo $\mathbb{Z}$ gives non-zero integer multiples arbitrarily close to 0. Their successive multiples form arbitrarily fine nets in the circle, so the subgroup generated by θ is dense. The closure of the forward orbit contains the negative multiples as well: choose a sequence $\{n_j\}_{j\in\mathbb{Z}_{\ge 1}}$ of positive integers tending to infinity with $n_j\theta \to 0$ modulo $\mathbb{Z}$, and use $(n_j - k)\theta \to -k\theta$ for each fixed positive integer k.

An irrational rotation never returns exactly to its starting point. After how many iterates does it return close to that point, and how close? For every $\zeta \in \mathrm{U}(1)$ and positive integer q,

$$d_{\mathrm{U}(1)}(R_\theta^{\circ q}(\zeta), \zeta) = |\lambda^q - 1|.$$

Thus the quantity $\delta_q = |1 - \lambda^q|$ appearing in Chapter III, §2.15, is a return distance. By (247), this is controlled by how close $q\theta$ is to an integer p, or equivalently by how well p/q approximates θ.

Continued fractions and return times

Consider $\theta = \sqrt{2} - 1$. We have

$$\theta = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

Indeed, $1/\theta = 2 + \theta$, so repeated substitution of $\theta = 1/(2+\theta)$ produces the nested expression above. The infinite expression is understood as the limit of its finite truncations. This is an example of a *continued fraction expansion*, which we write as

$$\theta = [0; 2, 2, 2, \ldots].$$

Stopping after successively more occurrences of 2 gives

$$\frac{1}{2}, \qquad \frac{1}{2 + 1/2} = \frac{2}{5}, \qquad \frac{1}{2 + 1/(2 + 1/2)} = \frac{5}{12}, \qquad \frac{12}{29}, \ldots.$$

These fractions describe increasingly close returns. For example, $5\theta - 2 \approx 0.0711$, whereas $12\theta - 5 \approx -0.0294$: after five and twelve iterates the point is close to its starting point, on opposite sides. The sign records the side of return, while the absolute value measures the short displacement on $\mathbb{R}/\mathbb{Z}$.

For general $\theta \in (0, 1) \setminus \mathbb{Q}$, repeat the same algorithm. Define $\{\theta_n\}_{n \in \mathbb{Z}_{\geq 0}}$ and the positive integers $\{a_n\}_{n \in \mathbb{Z}_{\geq 1}}$ by $\theta_0 = \theta$, $a_{n+1} = \lfloor 1/\theta_n \rfloor$, and $\theta_{n+1} = 1/\theta_n - a_{n+1}$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Each remainder lies in $(0, 1)$. The integers a_n are the *partial quotients*. Truncating $[0; a_1, a_2, \ldots]$ gives its *convergents* $\{p_n/q_n\}_{n \in \mathbb{Z}_{\geq 1}}$, determined by the following recurrence, initialized with auxiliary values at indices -1 and 0:

$$\begin{gathered} p_{-1} = 1, \quad p_0 = 0, \quad q_{-1} = 0, \quad q_0 = 1, \\ p_{n+1} = a_{n+1} p_n + p_{n-1}, \qquad q_{n+1} = a_{n+1} q_n + q_{n-1} \quad (n \geq 0). \end{gathered}$$

Here is a precise summary of the resulting approximation.

LEMMA III.3.1. *Let $\theta \in (0, 1) \setminus \mathbb{Q}$, and let p_n and q_n be the continued-fraction numerators and denominators defined by the preceding recurrence. Then p_n and q_n are relatively prime, $q_n \to \infty$, and $p_n/q_n \to \theta$. Define $\{e_n\}_{n \in \mathbb{Z}_{\geq 0}}$ by $e_n := q_n\theta - p_n$. For every $n \geq 1$, consecutive denominators $q = q_n$ and $Q = q_{n+1}$ satisfy*

$$q < Q, \qquad e_n e_{n+1} < 0, \qquad |e_{n+1}| < |e_n| < 1/Q,$$

and $|j\theta - k| \geq |e_n|$ whenever $0 < j < Q$ and $k \in \mathbb{Z}$.

PROOF. The determinant identity $p_{n+1}q_n - p_nq_{n+1} = (-1)^n$ follows by induction. With $A_{n+1} := 1/\theta_n > 1$, substitution gives

$$\theta = \frac{p_nA_{n+1} + p_{n-1}}{q_nA_{n+1} + q_{n-1}}, \qquad |e_n| = \frac{1}{q_nA_{n+1} + q_{n-1}}.$$

Since $A_{n+1} > 1$, the relation $e_{n-1} = -A_{n+1}e_n$ shows that the errors alternate in sign and strictly decrease in absolute value. Also $Q|e_n| + q|e_{n+1}| = 1$, so $|e_n| < 1/Q$. The denominator recurrence implies $q_n \to \infty$ and $q_{n+1} > q_n$ for $n \geq 1$, and the error bound gives $p_n/q_n \to \theta$.

For the last assertion, the determinant identity gives unique $a, b \in \mathbb{Z}$ such that $k = ap_n + bp_{n+1}$ and $j = aq_n + bq_{n+1}$. If $0 < j < Q$, then $b \geq 1$ forces $a \leq -1$, and $b \leq -1$ forces $a \geq 1$. In both cases ae_n and be_{n+1} have the same sign and $|ae_n + be_{n+1}| \geq |e_n|$. For $b = 0$, the positive integer $j = aq$ gives the same bound. □

For $n \geq 1$ we have $|e_n| < 1/q_{n+1} \leq 1/2$, so p_n is the nearest integer to $q_n\theta$ and $|e_n| = d_{\mathbb{R}/\mathbb{Z}}(q_n\theta + \mathbb{Z}, 0 + \mathbb{Z})$. Thus q_n are increasingly good return times for the rotation. The initial denominators can coincide: $q_0 = q_1 = 1$ when $a_1 = 1$.

3.2 Circle diffeomorphisms and rotation numbers

A rotation advances every point by the same angle; for a general orientation-preserving circle homeomorphism the displacement depends on the point. By the lifting theorem (Theorem I.7.4), an orientation-preserving homeomorphism g of $\mathrm{U}(1)$ has a lift $G\colon \mathbb{R} \to \mathbb{R}$ with $p \circ G = g \circ p$. It is increasing and satisfies $G(x + 1) = G(x) + 1$: orientation preservation gives an increasing lift of degree one. Its mean displacement is the following invariant.

LEMMA III.3.2. *Let $g\colon \mathrm{U}(1) \to \mathrm{U}(1)$ be an orientation-preserving homeomorphism, and let $G\colon \mathbb{R} \to \mathbb{R}$ be an increasing lift, so that $p \circ G = g \circ p$ and $G(x + 1) = G(x) + 1$, where $p(x) := e^{2\pi\sqrt{-1}x}$. The limit*

$$\rho(G) := \lim_{n\to\infty} \frac{G^{\circ n}(x) - x}{n}$$

exists, is independent of x, is called the translation number *of the lift, and satisfies $|G^{\circ n}(x) - x - n\rho(G)| \leq 1$ for $n \geq 1$. The class $\rho(g) := \rho(G) + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$, called the* rotation number*, is independent of the lift and invariant under orientation-preserving topological conjugacy. If a sequence $\{g_j\}_{j\in\mathbb{Z}_{\geq 1}}$ of such homeomorphisms converges uniformly to g, then $d_{\mathbb{R}/\mathbb{Z}}(\rho(g_j), \rho(g)) \to 0$. Moreover, $\rho(R_\theta) = \theta + \mathbb{Z}$, $\rho(g^{\circ q}) = q\rho(g)$ for every positive integer q, and g has a periodic point if and only if $\rho(g) \in \mathbb{Q}/\mathbb{Z}$.*

PROOF. For the rotation-number properties, see [He1, Chapter II, §2, pp. 20–22, and §5, pp. 24–25]. We recall how the mean displacement controls them. Put $M_n := \sup_x(G^{\circ n}(x) - x)$ and $m_n := \inf_x(G^{\circ n}(x) - x)$. Monotonicity and periodicity give $M_n - m_n \leq 1$, $M_{n+k} \leq M_n + M_k$, and $m_{n+k} \geq m_n + m_k$. Writing $n = ak + r$ for fixed k shows that M_n/n and m_n/n have the same limit, with $m_n \leq n\rho(G) \leq M_n$. This proves the uniform error bound. Replacing G by $G + p$, $p \in \mathbb{Z}$, changes $\rho(G)$ by p. For convergent lifts, approximate $\rho(G)$ by one fixed finite iterate with error $1/n$ and then pass to the limit in that iterate. This gives continuity

for the quotient distance. A lift H of a conjugacy has $H(x) - x$ bounded; comparison of iterates and division by n gives conjugacy invariance. The formulas for rotations and iterates follow directly from the limit.

A periodic point gives $G^{\circ q}(x) = x + p$ and hence $\rho(G) = p/q$. Conversely, if $\rho(G) = p/q$ and $G^{\circ q}(x) - x - p$ has no zero, it is everywhere at least $\delta > 0$ or everywhere at most $-\delta$. In the first case, iteration gives

$$\frac{G^{\circ qm}(x) - x - mp}{m} = \frac{1}{m} \sum_{j=0}^{m-1} \bigl(G^{\circ q(j+1)}(x) - G^{\circ qj}(x) - p \bigr) \geq \delta \qquad (m \geq 1),$$

since G commutes with integer translations. Passing to the limit as $m \to \infty$ gives $q\rho(G) - p \geq \delta > 0$, contrary to $\rho(G) = p/q$. The negative case is analogous. □

For a circle homeomorphism g as above with $\rho(g) = \theta + \mathbb{Z}$, we compare g with the rotation R_θ having the same rotation number. We ask whether a change of coordinate on the circle transforms g into R_θ: if $g \circ \eta = \eta \circ R_\theta$, then $g^{\circ n}(\eta(\zeta)) = \eta(R_\theta^{\circ n}(\zeta))$. The regularity required of η is part of this question.

Lemma III.3.3 (Denjoy). *Let $g\colon \mathrm{U}(1) \to \mathrm{U}(1)$ be an orientation-preserving C^2 diffeomorphism with $\rho(g) = \theta + \mathbb{Z}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$. There is an orientation-preserving homeomorphism $\eta\colon \mathrm{U}(1) \to \mathrm{U}(1)$ such that $g \circ \eta = \eta \circ R_\theta$. Moreover, $g^{\circ q_n} \to \mathrm{id}$ uniformly for $d_{\mathrm{U}(1)}$, where $\{q_n\}_{n \in \mathbb{Z}_{\geq 1}}$ are the continued-fraction denominators of the representative of $\theta + \mathbb{Z}$ in $(0, 1)$.*

Proof. For a complete proof, see [He1, Chapter VI, §5, Theorem 5.5, p. 76, with §4]; its class P includes C^2 circle diffeomorphisms. The main idea is to begin with a *semiconjugacy*, a continuous monotone surjection H satisfying $H \circ g = R_\theta \circ H$, which need not be injective. If H collapses an interval, its distinct iterates are disjoint. At continued-fraction return times, the distortion estimate compares the stretching at different points and prevents these intervals from shrinking as disjoint intervals must. Thus no interval is collapsed, and $\eta = H^{-1}$ gives the conjugacy. The asserted uniform convergence follows from uniform continuity of η and the return estimates for the rotation (Lemma III.3.1 and (247)). □

For measurable dynamics we use the normalized Lebesgue measure m: it is length on $\mathbb{R}/\mathbb{Z}$, or arc length divided by 2π on $\mathrm{U}(1)$. A topological conjugacy need not preserve null sets, so Denjoy's conclusion alone does not imply the following assertion about this measure.

Theorem III.3.4 (Non-singular ergodicity). *Let $g\colon \mathrm{U}(1) \to \mathrm{U}(1)$ be an orientation-preserving C^2 diffeomorphism with irrational rotation number, and let m be normalized Lebesgue measure on $\mathrm{U}(1)$. Then g is non-singular and ergodic with respect to m.*

Proof. A complete proof is given in [He1, Chapter VII, Theorem 1.4 and §§1.6–1.15, pp. 86–92], using Hurewicz's ergodic theorem and estimates for averages of derivatives. □

Linearization of circle diffeomorphisms

Write $R_\theta(\zeta) = e^{2\pi\sqrt{-1}\theta}\zeta$ on U(1); in additive circle coordinates it is $x \mapsto x + \theta$. We record a case of analytic linearization that will also be used for external circle maps later.

THEOREM III.3.5 (Herman's circle linearization theorem, a special case). *Let $g\colon \mathrm{U}(1) \to \mathrm{U}(1)$ be an orientation-preserving C^ω diffeomorphism, where C^ω means real analytic. If $\rho(g) = \theta \bmod \mathbb{Z}$ for a real quadratic irrational θ, then there is an orientation-preserving C^ω diffeomorphism $\eta\colon \mathrm{U}(1) \to \mathrm{U}(1)$ such that $g \circ \eta = \eta \circ R_\theta$.*

PROOF. See [He1, Chapter IX, Theorem 5.1, p. 127]. Quadratic irrationals satisfy Herman's condition A ([He1, Chapter V, §10, pp. 67–68]). Keep g and θ as in the theorem.

For the local analytic problem, assume in addition that g is sufficiently close to R_θ in an analytic norm on a complex neighborhood of the circle. Newton iteration then constructs a real analytic conjugacy: one divides Fourier coefficients by $e^{2\pi\sqrt{-1}k\theta} - 1$, $k \neq 0$, and uses quadratic error reduction to compensate for the loss of strip width ([He1, Appendix, §§4–5]). For these angles, a fixed point approach instead uses the logarithm of the derivative ([He2, Appendix VIII, §§6–8]; see also [He3]). The auxiliary translation parameter vanishes because the rotation number is prescribed. The Schwarzian variant for a fixed point is described in Remark III.3.13. These constructions concern the whole circle, but require a closeness assumption absent from the present theorem.

The global smooth theorem removes this restriction. Continued-fraction return times and estimates for derivatives of iterates give an orientation-preserving C^∞ diffeomorphism $h\colon \mathrm{U}(1) \to \mathrm{U}(1)$ with $g \circ h = h \circ R_\theta$, without assuming that g is close to a rotation. At this stage the conjugacy is exact, but its analyticity has not yet been obtained.

To combine the results, approximate h by orientation-preserving real analytic circle diffeomorphisms h_j. Then $g_j := h_j^{-1} \circ g \circ h_j$ has the same rotation number and converges to R_θ in C^∞. Here the local theorem applies to analytic diffeomorphisms sufficiently close to R_θ in a suitable finite C^r topology ([He1, Chapter XI, Theorem 5.1 and Proposition 6.2; Appendix, Theorem 2.2]; see also [Y1, p. 335]). For a sufficiently good approximant, let u_j be an orientation-preserving real analytic circle diffeomorphism satisfying $g_j \circ u_j = u_j \circ R_\theta$. Then $\eta := h_j \circ u_j$ is the required real analytic conjugacy of g. □

REMARK III.3.6. Let $g\colon \mathrm{U}(1) \to \mathrm{U}(1)$ be an orientation-preserving circle diffeomorphism, and write $\rho(g) = \theta + \mathbb{Z}$. For g of class C^2 and θ irrational, Denjoy's theorem (Lemma III.3.3) supplies a topological conjugacy to R_θ. The regularity of g and that of this conjugacy are different questions. If θ is Diophantine, Yoccoz's theorem gives a C^∞ conjugacy when g is C^∞, and a real analytic conjugacy when g is real analytic (See [Y1, p. 335, Corollaire]. The Diophantine condition is the one defined after Theorem I.1.4). These are global statements: g is not required to be close to a rotation. □

Retain the circle diffeomorphisms g and η in Theorem III.3.5. These maps and their inverses extend holomorphically to open neighborhoods of U(1) by complexifying their convergent Taylor expansions in real analytic coordinates on the circle. We use the same symbols

for these extensions. Compactness and the local inverse function theorem make them univalent after shrinking their domains. Put $\lambda := e^{2\pi\sqrt{-1}\theta}$ and define $R_\lambda(w) := \lambda w$ on $\mathbb{C}$, so that $R_\theta = R_\lambda|_{\mathrm{U}(1)}$. The identity theorem extends the equation to $g \circ \eta = \eta \circ R_\lambda$ wherever these compositions are defined near the circle. Thus, for $s > 1$ sufficiently close to 1,

$$\eta(\{|\zeta| = s\}) \tag{248}$$

is a g-invariant real analytic Jordan curve on the exterior side of $\mathrm{U}(1)$.

3.3 Fixed points and normal forms

Iteration and local normal forms

Let f be a non-constant holomorphic function defined near $0 \in \mathbb{C}$ with $f(0) = 0$. We identify two such functions if they agree on a possibly smaller neighborhood of 0, and thus regard f as a germ. Its iterates and orbits are understood wherever the compositions are defined in a chosen representative. Although every finite iterate is defined as a germ at 0, there need not be a single neighborhood on which all forward orbits remain.

As in the circle problem, a change of coordinates may make the orbits easier to follow. Here we use local biholomorphic coordinates near the fixed point and seek a simpler normal form, not necessarily a linear map. If h is a local biholomorphism fixing 0, put

$$g := h^{-1} \circ f \circ h, \qquad f^{\circ n}(h(w)) = h\big(g^{\circ n}(w)\big). \tag{249}$$

The second equality follows by induction wherever the compositions are defined. Thus knowledge of the orbits of g gives knowledge of those of f by applying h. We call f and g *conjugate*. A simpler representative obtained in this way is called a *normal form*; the aim is to make the motion of points visible in the expression for the map, rather than to change that motion.

Definition III.3.7 (Multiplier and types of fixed points). Let f be the germ at $0 \in \mathbb{C}$ of a non-constant holomorphic function with $f(0) = 0$. The number $\lambda := f'(0)$ is the *multiplier* at the fixed point 0. The fixed point is *attracting* if $|\lambda| < 1$, *repelling* if $|\lambda| > 1$, and *neutral* if $|\lambda| = 1$. An attracting fixed point with $\lambda = 0$ is called *superattracting*. A neutral fixed point is *rationally neutral* if λ is a root of unity, and *irrationally neutral* otherwise. □

These types do not depend on the coordinate: for the conjugacy in (249), the chain rule gives $g'(0) = (h^{-1})'(0) f'(0) h'(0) = f'(0)$. The names attracting and repelling reflect the local behavior when $|\lambda| \neq 1$. If $|\lambda| < 1$, then $|f(w)| \leq c|w|$ near 0 for some $c < 1$, so sufficiently nearby points tend to 0 under forward iteration. If $|\lambda| > 1$, then f has a local inverse with multiplier λ^{-1}, and this inverse is attracting. When $|\lambda| = 1$, finer information is needed to determine the dynamics.

Superattracting fixed points and Böttcher coordinates

First let f be the germ at 0 of a non-constant holomorphic function with $f(0) = f'(0) = 0$. Its zero at 0 has a finite order $d \geq 2$, and $f(w) = aw^d + O(w^{d+1})$ with $a \neq 0$. Here and

below $O(w^m)$ denotes the germ at 0 of a holomorphic function divisible by w^m. The simplest normal form with this order is $w \mapsto w^d$, whose n-th iterate is $w \mapsto w^{d^n}$.

THEOREM III.3.8 (Böttcher coordinates). *Let f be the germ at $0 \in \mathbb{C}$ of a holomorphic function with expansion $f(w) = aw^d + O(w^{d+1})$, where $a \in \mathbb{C}^*$ and d is an integer with $d \geq 2$. For each choice of $c \in \mathbb{C}^*$ with $c^{d-1} = a$, there is a unique germ Φ at 0 of a holomorphic function such that*

$$\Phi(0) = 0, \qquad \Phi'(0) = c, \qquad \Phi(f(w)) = \Phi(w)^d. \tag{250}$$

PROOF. We give the logarithmic construction (Compare [Gam, pp. 4–5]). Write $f(w) = aw^d(1 + \eta(w))$, where $\eta(0) = 0$. Choose a sufficiently small disk D centered at 0 such that $f(D) \subset D$ and there is a bounded holomorphic function $u = \log(1 + \eta)$ on D with $u(0) = 0$. The series

$$S(w) := \sum_{j=0}^{\infty} \frac{u(f^{\circ j}(w))}{d^{j+1}}$$

converges uniformly on compact subsets of D: its terms are bounded by a constant times d^{-j-1}. It defines a holomorphic function with $S(0) = 0$ and $d \cdot S = u + S \circ f$. Therefore $\Phi(w) := cw \exp S(w)$ has derivative c at 0, and

$$\Phi(f(w)) = caw^d \exp\big(u(w) + S(f(w))\big) = c^d w^d \exp(d \cdot S(w)) = \Phi(w)^d.$$

The holomorphic inverse function theorem makes Φ a coordinate near 0.

For uniqueness, use Φ as a coordinate. A second solution Φ_1 gives $H := \Phi_1 \circ \Phi^{-1}$ with $H(0) = 0$, $H'(0) = 1$, and $H(w^d) = H(w)^d$. If $H(w) = w + bw^m + O(w^{m+1})$ with $b \neq 0$ and $m \geq 2$, comparison in degree $d + m - 1 < dm$ gives $d \cdot b = 0$, a contradiction. □

For f and Φ in Theorem III.3.8, we call Φ a *Böttcher coordinate*.

Kœnigs and Siegel linearization theorems

Let f now be the germ at 0 of a holomorphic function with $f(0) = 0$ and multiplier $\lambda := f'(0) \in \mathbb{C}^*$, so that f is locally biholomorphic. The simplest normal form compatible with the multiplier is the linear map $R_\lambda \colon \mathbb{C} \to \mathbb{C}$, $R_\lambda(w) := \lambda w$. When $\lambda = e^{2\pi\sqrt{-1}\theta}$ with $\theta \in \mathbb{R}$, its restriction to $\mathrm{U}(1)$ is the rotation R_θ. We say that f is *linearizable* if there is a local biholomorphism h, fixing 0, such that

$$h^{-1} \circ f \circ h = R_\lambda, \qquad \text{equivalently} \qquad f(h(w)) = h(\lambda w). \tag{251}$$

We call h a *linearizing function*. Here linearization, unless another regularity is specified, means linearization by a holomorphic local coordinate; formal or topological conjugacies will be named explicitly. The inverse coordinate $\Phi = h^{-1}$ satisfies *Schröder's equation*

$$\Phi(f(w)) = \lambda\Phi(w).$$

We shall solve this equation in the equivalent form (251) for h. There is no loss in requiring $h'(0) = 1$: if a solution h_0 has $h_0'(0) = b \neq 0$, then $h(w) := h_0(w/b)$ is again a solution and has derivative 1. This uses the fact that scalar multiplications commute with R_λ.

MINI-EXERCISE III.3.9. Let $f(w) := 3w - 4w^3$. Show that f is linearizable at 0 by finding explicitly a holomorphic function h defined on a neighborhood of 0 such that $h(0) = 0$, $h'(0) = 1$, and $f(h(w)) = h(3w)$ for all w sufficiently close to 0.

For the germ f at 0 of a holomorphic function fixing 0 with multiplier $\lambda = f'(0) \in \mathbb{C}^*$, write its Taylor expansion and seek a normalized formal power series $h \in \mathbb{C}[[w]]$ as

$$f(w) = \lambda w + \sum_{j=2}^{\infty} a_j w^j, \qquad h(w) = w + \sum_{m=2}^{\infty} b_m w^m,$$

and set $b_1 := 1$. The coefficients of w^2 and w^3 in $f(h(w)) = h(\lambda w)$ give, when the denominators are non-zero,

$$b_2 = \frac{a_2}{\lambda^2 - \lambda}, \qquad b_3 = \frac{2a_2 b_2 + a_3}{\lambda^3 - \lambda}.$$

More generally, the coefficient of w^m gives

$$(\lambda^m - \lambda) b_m = \sum_{j=2}^{m} a_j \sum_{\substack{k_1 + \cdots + k_j = m \\ k_1, k_2, \ldots, k_j \geq 1}} b_{k_1} b_{k_2} \cdots b_{k_j}. \tag{252}$$

Every k_i on the right is at most $m - 1$, so this determines the next coefficient from the preceding ones whenever $\lambda^m - \lambda \neq 0$.

Two different difficulties can occur. If a denominator is zero, the right-hand side must vanish in order to proceed; if it does vanish, that equation leaves a free coefficient. If all denominators are non-zero but some are very small, the formally determined coefficients may grow too fast for convergence. We first assume

$$\lambda^n \neq 1 \qquad (n \in \mathbb{Z}_{>0}), \tag{253}$$

keeping $\lambda \in \mathbb{C}^*$. Recursion (252) then gives a unique formal series $h \in \mathbb{C}[[w]]$ with $h(0) = 0$ and $h'(0) = 1$ satisfying the conjugacy equation. Such a series has a formal inverse under composition. Its convergence is a separate question.

THEOREM III.3.10 (Kœnigs and Siegel linearization theorems). *Let f be the germ at 0 of a holomorphic function fixing 0, with multiplier $\lambda := f'(0) \in \mathbb{C}^*$. Suppose that one of the following conditions holds:*

(*i*) $|\lambda| \neq 1$ (Kœnigs);

(*ii*) $\lambda = e^{2\pi\sqrt{-1}\theta}$ *for a Diophantine real number* θ (Siegel).

Then there is a unique linearizing germ h with $h(0) = 0$ and $h'(0) = 1$.

PROOF. For the classical results, see [Kœ, Sie] and [CG, Chapter II]. We derive both cases from the Siegel–majorant theorem (Theorem II.1.1). Both hypotheses imply (253), so

the formal series h has already been determined. By convergence of the series for f, choose $R, M > 0$ such that $|a_j| \le RM^{j-2}$ for all $j \ge 2$.

We first choose a sequence $\{e_k\}_{k\in\mathbb{Z}_{\ge 1}}$ of positive numbers with $e_{m-1} \le |\lambda^m - \lambda|$ for $m \ge 2$, satisfying the two hypotheses of the Siegel–majorant theorem (Theorem II.1.1). In case (i), put $c := |\lambda|\,\big||\lambda| - 1\big|$. For $m \ge 2$, the reverse triangle inequality gives

$$|\lambda^m - \lambda| \ge |\lambda|\,\big||\lambda|^{m-1} - 1\big| \ge c > 0.$$

Take $e_k := c$ for all k. Then $e_{p-q} \le e_p + e_q$ for $p > q > 0$, and $e_k \ge (2k)^{-\alpha}$ for a sufficiently large $\alpha > 0$.

In case (ii), put $e_k := |1 - \lambda^k| = d_{\mathrm{U}(1)}(\lambda^k, 1)$. For $p > q > 0$ we have $e_{p-q} = |\lambda^p - \lambda^q| \le e_p + e_q$. Since θ is Diophantine, choose $C > 0$ and $\tau > 1$ such that $\min_{m\in\mathbb{Z}} |k\theta - m| \ge Ck^{-\tau}$ for all $k \ge 1$. The distance comparison (247) then gives

$$e_k \ge 4d_{\mathbb{R}/\mathbb{Z}}(k\theta + \mathbb{Z}, 0 + \mathbb{Z}) \ge 4Ck^{-\tau}.$$

Choosing $\alpha \ge \tau$ with $2^{-\alpha} \le 4C$ gives $e_k \ge (2k)^{-\alpha}$. Also $|\lambda^m - \lambda| = e_{m-1}$ in this case.

Now define $A(u) = u + \sum_{m\ge 2} A_m u^m$, with $A_1 := 1$, by

$$\sum_{m=2}^{\infty} e_{m-1} A_m u^m = R\frac{A(u)^2}{1 - MA(u)}. \tag{254}$$

The coefficient recursion in Chapter II gives $A_m > 0$. We prove $|b_m| \le A_m$ by induction. The assertion holds for $m = 1$. If it holds for smaller indices, then (252) yields

$$|b_m| \le \frac{1}{e_{m-1}} \sum_{j=2}^{m} RM^{j-2} \sum_{\substack{k_1+\cdots+k_j=m \\ k_1,k_2,\ldots,k_j\ge 1}} A_{k_1} A_{k_2} \cdots A_{k_j} = A_m,$$

where the last equality is the coefficient of u^m in (254). The Siegel–majorant theorem (Theorem II.1.1) gives A a positive radius of convergence, and hence gives the same conclusion for h. Its formal equation is therefore an identity between germs at 0 of holomorphic functions. Since $h'(0) = 1$, the inverse function theorem proves that it is a linearizing function. Uniqueness follows from the coefficient recursion. □

Thus an attracting fixed point which is not superattracting, or a repelling fixed point, has a linear normal form. Let f be an irrationally neutral germ at 0 of a holomorphic function which is linearizable, and let $h\colon \Delta_R \to D$ be a normalized linearizing function, where $\Delta_R := \{w \in \mathbb{C} \mid |w| < R\}$. Thus $h(0) = 0$, $h'(0) = 1$, and h conjugates f on D to $w \mapsto \lambda w$, where $\lambda := f'(0)$. For a map given on a fixed ambient domain, a maximal such rotation domain is called a *Siegel disc* at 0 (See [Mi3]). The images $h(\Delta_r)$, $0 < r < R$, are called *Siegel subdiscs*; their orbits lie on $h(\{|w| = s\})$, $0 < s < r$. We specify the ambient domain when maximality is used. The Diophantine condition guarantees these discs, but is not necessary for an individual germ to be linearizable.

REMARK III.3.11. Let $g\colon \mathrm{U}(1) \to \mathrm{U}(1)$ be an orientation-preserving circle diffeomorphism, and write $\rho(g) = \theta + \mathbb{Z}$. Analyticity of g alone does not even ensure absolute continuity of a topological conjugacy, as the examples recalled in [Y1, Introduction, p. 334] show. Nor does a C^∞ conjugacy of an analytic circle diffeomorphism have to be analytic. Such examples can be obtained from the quadratic Siegel discs with smooth boundary constructed in [ABC, Main Theorem and the first proof of Corollary 1] (The introduction of that paper describes Pérez-Marco's earlier construction).

Here is the connection. Choose such a maximal Siegel disc whose closure lies in a domain on which the polynomial is univalent, as in the cited proof. Conjugation by the exterior conformal map gives a circle map which extends holomorphically across the circle by Schwarz reflection (See [P, §II.1]). The boundary regularity theorem for conformal maps of smooth Jordan domains makes both the interior and exterior boundary parametrizations C^∞ diffeomorphisms; their composition is a C^∞ conjugacy of this circle map to the rotation (See [Pom, Chapter 3]). If an analytic conjugacy existed, it would extend to a complex neighborhood of the circle. The images of nearby exterior circles would then give invariant Jordan curves strictly surrounding the Siegel disc, still inside the domain of univalence. The polynomial would be an automorphism of their interiors. A Riemann map and the Schwarz lemma at its interior fixed point would linearize it on a larger disc, contradicting maximality. Thus the smooth conjugacy in this example cannot be analytic. Irrationality alone is therefore insufficient for analytic linearization. This does not say that every map with a given non-Diophantine rotation number fails to be linearizable; the rotation itself is an immediate counterexample to that assertion. For a rational rotation number there is a periodic point, by Lemma III.3.2, but there need not be a conjugacy of the whole circle to a rational rotation.

For comparison with linearization at a fixed point, an irrational number θ satisfies the *Brjuno condition* if the denominators q_n of its continued-fraction convergents satisfy

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty. \tag{255}$$

The continued-fraction notation is that of §3.1. In the analytic circle problem, the Brjuno condition occurs in local theorems for maps sufficiently close to the rotation in an analytic topology. The optimal arithmetic condition for global analytic linearization is the more restrictive *Herman condition*, not just (255); see [Y3] for the distinct local and global statements and the definition of that condition (See also the introduction of [ABC]). The quadratic irrational case in Theorem III.3.5 will suffice for the approximation argument below. □

REMARK III.3.12 (The Brjuno–Yoccoz criterion). Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda := e^{2\pi\sqrt{-1}\theta}$. If θ satisfies (255), every germ at 0 of a holomorphic function $f(w) = \lambda w + O(w^2)$ is linearizable. Moreover, the quadratic polynomial

$$P_\theta(w) := \lambda w + w^2$$

is linearizable at 0 if and only if θ satisfies that condition. The sufficiency is Brjuno's theorem; the necessity for this quadratic family is due to Yoccoz. In particular, the linearizability of

P_θ implies the linearizability of every holomorphic germ with the same multiplier. For the statements and their relation, see [Mar, Theorem 3.1 and §5]. The polynomial written there as $\lambda(z - z^2/2)$ is conjugate to P_θ by $z = -2w/\lambda$.

For proofs, the majorant argument for sufficiency is given in [Mar, §5.1]; the discussion in §5.2 of that reference states Yoccoz's sharper theorem, including the obstruction in the non-Brjuno case, but does not reproduce its full proof. The latter is in [Y2, Part I, §2.6 and Part II, §1.8]. Chéritat gave a different proof of necessity for the quadratic family using *parabolic explosion*, the appearance of periodic cycles when a rationally neutral parameter is perturbed. A linearization domain cannot contain a non-zero periodic point; estimates for the accumulated effect of these cycles give upper bounds for the radius of its normalized parameter disc which tend to zero when the Brjuno series diverges (See [Ch, Theorem 1.2 and §2] and [BC, Theorem 1 and §2]). □

Remark III.3.13. For the germ f and multiplier λ in Theorem III.3.10, the proof uses Chapter II's version of Siegel's original majorant argument: one must control the products of factors $|1 - \lambda^n|^{-1}$ arising in the coefficient recursion, not merely show that each denominator is non-zero. This is the *small denominator problem* ([Sie]). Kœnigs's case admits simpler proofs ([CG, Chapter II, Theorem 2.1]). Other approaches to Siegel's theorem include the following.

(*i*) The *Kolmogorov–Arnol'd–Moser (KAM) method* uses Newton-type corrections: solve a linearized conjugacy equation, reduce the remaining error quadratically, and control the loss of domain through the iteration ([CG, Chapter II, Theorem 6.4]; [Mar, §6.3]).

(*ii*) Yoccoz's geometric approach first treats a quadratic polynomial and transfers linearizability to other germs with the same multiplier ([Mar, §3]; see also Remark III.3.12).

(*iii*) For stronger arithmetic bounds, Herman's differential-invariant method uses fixed point arguments ([He2, Appendix VIII, §§6–8, 12]; [He3]); a contraction formulation is described below.

For another short proof of a particular case, see [Ll].

Here is the Schwarzian version of the last idea ([Mar, §6.2]). For locally biholomorphic holomorphic maps f and g, the *Schwarzian derivative* and its chain rule, on open sets where the composition is defined, are

$$S(f) := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2, \qquad S(f \circ g) = (S(f) \circ g)(g')^2 + S(g).$$

Returning to the germ f in the theorem, a locally biholomorphic holomorphic solution h of $f \circ h = h \circ R_\lambda$, where $R_\lambda(w) = \lambda w$, satisfies

$$D_\lambda S(h) = (S(f) \circ h)(h')^2, \qquad D_\lambda \varphi := \lambda^2 \varphi \circ R_\lambda - \varphi.$$

The coefficient of w^n is divided by $\lambda^{n+2} - 1$ when D_λ is inverted. After removing the quadratic term of f, normalize $h(0) = 0$, $h'(0) = 1$, $h''(0) = 0$; these three conditions remove the Möbius

ambiguity in inverting S. The conjugacy problem then becomes the fixed point problem

$$h = \mathcal{T}(h) := S^{-1}\big(D_\lambda^{-1}[(S(f)\circ h)(h')^2]\big)\,.$$

For example, under $|1-\lambda^n| \geq cn^{-2}$ for $n \geq 1$, $c > 0$, suitable *Hardy–Sobolev spaces*, defined by square summability of weighted Taylor coefficients, make the derivative count balance: h' costs one derivative, D_λ^{-1} at most two, and S^{-1} restores three. Rescaling f near 0 makes $\mathcal{T}$ a contraction on a closed ball of normalized functions, so the Banach fixed point theorem (Theorem I.3.8) applies. The function-space estimates are given in the cited references. This direct contraction argument has the stated arithmetic restriction; it is not the proof for every Diophantine exponent used above. □

Small cycles and non-linearizable germs

For the germ f at 0 of a holomorphic function fixing 0 with irrationally neutral multiplier $\lambda = f'(0)$, the unique normalized formal linearizing series need not converge, as Cremer's examples of rational maps show ([Cr, pp. 154–158]). Here we follow Ueda's ingenious refinement of Cremer's argument, which yields a particularly transparent construction of small cycles for polynomials ([U1, §5, no. 4, p. 606]).

Proposition III.3.14 (Small cycles under a strong Cremer condition). *Let $d \geq 2$ be an integer and let $f\colon \mathbb{C} \to \mathbb{C}$ be the monic polynomial*

$$f(w) = \lambda w + \sum_{j=2}^{d-1} a_j w^j + w^d,$$

where $a_j \in \mathbb{C}$, $|\lambda| = 1$, and $\lambda^\ell \neq 1$ for every $\ell \geq 1$ (The intermediate sum is empty when $d = 2$). Suppose that there is $A > 1$ such that

$$\liminf_{\ell\to\infty} A^\ell |1-\lambda^\ell|^{1/(d^\ell-1)} = 0. \tag{256}$$

Then f has a sequence $\{C_\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ of cycles with periods m_ν, such that $m_\nu \to \infty$ and $\max_{w\in C_\nu}|w| \to 0$. Consequently f is not linearizable at 0.

Proof. For each $\ell \in \mathbb{Z}_{>0}$, the polynomial $f^{\circ\ell}(w) - w$ is monic of degree d^ℓ, with linear coefficient $\lambda^\ell - 1 \neq 0$. Thus 0 is a simple root, and the product of the remaining roots, counted with multiplicity, has absolute value $|1-\lambda^\ell|$. At least one of these roots, denoted by c_ℓ, satisfies

$$0 < |c_\ell| \leq |1-\lambda^\ell|^{1/(d^\ell-1)}, \qquad f^{\circ\ell}(c_\ell) = c_\ell. \tag{257}$$

It remains to control its whole orbit, not only this one point.

Fix $A > 1$ satisfying (256), and choose $r > 0$ sufficiently small that

$$K := \sup_{0<|w|<r}\left|\frac{f(w)}{w}\right| < A.$$

This is possible because $f(w)/w \to \lambda$ and $|\lambda| = 1$; also $K \geq 1$. By (256) and (257), choose a strictly increasing sequence $\{\ell_\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ of positive integers and roots $c_\nu := c_{\ell_\nu}$ with $0 < |c_\nu| < rA^{-\ell_\nu}$. Induction, using $|f(w)| \leq K|w|$ while $|w| < r$, gives

$$|f^{\circ j}(c_\nu)| \leq K^j|c_\nu| < (K/A)^{\ell_\nu} r < r \qquad (0 \leq j \leq \ell_\nu). \tag{258}$$

Indeed, each preceding point remains in the disk, so that the next application of the estimate is legitimate. No point of this orbit is 0, since that would imply $c_\nu = 0$. Its least period m_ν divides ℓ_ν, and (258) shows that the whole cycle C_ν tends to 0. For each fixed integer M, the union of the non-zero roots of $f^{\circ m}(w) - w$, $1 \leq m \leq M$, is finite and avoids 0. Hence $m_\nu > M$ for all sufficiently large ν, which proves $m_\nu \to \infty$.

If a linearization existed, 0 would lie in a Siegel subdisc U. For large ν, the whole non-trivial cycle C_ν would lie in U. Its image would be a non-zero periodic orbit of that rotation, contrary to $\lambda^m \neq 1$ for every $m > 0$. □

The cycles in this proposition are called *small cycles*. For the polynomials with small cycles under a strong Cremer condition (Proposition III.3.14), the normalized formal linearization exists, but has radius of convergence zero. In particular, there is no Siegel subdisc at 0, and hence no Siegel disc centered there. This does not exclude Siegel discs elsewhere.

Proposition III.3.15. *For every fixed integer $d \geq 2$ and real number $A > 1$, there are uncountably many classes $\theta + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$, represented by irrational real numbers θ, such that $\lambda = e^{2\pi\sqrt{-1}\theta}$ satisfies (256) with this A.*

Proof. Use the Baire category theorem recalled in Chapter I, §3.1 on the circle $\mathbb{R}/\mathbb{Z}$. For positive integers N, j, let

$$U_{N,j} := \bigcup_{\ell\geq N} \left\{ \theta + \mathbb{Z} \;\middle|\; \theta \in \mathbb{R},\ |1 - e^{2\pi\sqrt{-1}\ell\theta}| < \left(j^{-1}A^{-\ell}\right)^{d^\ell - 1} \right\}.$$

Each $U_{N,j}$ is open. It is dense because the rational points k/ℓ, with arbitrarily large denominators $\ell \geq N$, are dense and belong to it. The intersection of all $U_{N,j}$ is therefore a dense G_δ set (i.e., a countable intersection of open sets). Removing the countable set $\mathbb{Q}/\mathbb{Z}$ still leaves a comeagre set. This set is uncountable: otherwise it and its complement would both be meagre, contradicting the Baire theorem. For each remaining θ, the defining inequalities for every N, j say precisely that the liminf in (256) is zero. □

This strengthens, for the present purpose, the non-countability in Mini-Exercise I.1.7. A Diophantine θ cannot satisfy (256): the lower bound $|1 - \lambda^\ell| \geq 4C\ell^{-\tau}$ makes its $(d^\ell - 1)$-st root tend to 1, so that the additional factor A^ℓ tends to infinity. Exercise I.4 therefore contrasts a full-measure set of Diophantine multipliers with the comeagre, measure-zero set furnished here. This is not a dichotomy for all germs with a given multiplier: $w \mapsto \lambda w$ is itself linear for every λ. For instance, the proposition applies to the specific family $f(w) = \lambda w + w^2$ when the condition is imposed with $d = 2$.

Rationally neutral fixed points and Fatou coordinates

We return to the case postponed in (253). Let f be the germ at 0 of a holomorphic function fixing 0, and suppose its multiplier $\lambda := f'(0)$ is a root of unity of order $q \geq 1$. This is a condition on the multiplier; it does not imply that $f^{\circ q} = \mathrm{id}$. In fact, f is linearizable if and only if $f^{\circ q} = \mathrm{id}$ as germs. Necessity follows by conjugating $R_\lambda^{\circ q} = \mathrm{id}$. Conversely, if $f^{\circ q} = \mathrm{id}$, define

$$\Phi(w) := \frac{1}{q} \sum_{j=0}^{q-1} \lambda^{-j} f^{\circ j}(w).$$

Every summand has derivative 1 at 0, so $\Phi'(0) = 1$. Shifting the index in the sum, and using $\lambda^q = 1$ and $f^{\circ q} = \mathrm{id}$, gives $\Phi \circ f = \lambda \Phi$. Thus Φ is a linearizing coordinate by the inverse function theorem.

Example III.3.16. Let f be the germ at 0 of $-w/(1+w) = -w + w^2 + O(w^3)$. Then $f'(0) = -1$ and

$$f(f(w)) = \frac{-(-w/(1+w))}{1 - w/(1+w)} = w.$$

Thus f has order 2 and is linearizable. The function $h(w) := w/(1 - w/2)$ satisfies $h(0) = 0$, $h'(0) = 1$, and $f \circ h = h \circ R_{-1}$ near 0. □

Returning to a general germ f with multiplier of order q, suppose that $f^{\circ q} \neq \mathrm{id}$. Then the iterate $F := f^{\circ q}$ is *tangent to the identity* and has the form

$$F(w) = w + aw^{k+1} + O(w^{k+2}), \qquad a \neq 0, \quad k \geq 1. \tag{259}$$

We call this the *parabolic* case, distinguishing it from the finite-order case just considered. A linear normal form at the fixed point is impossible, but another normal form becomes available on suitable domains approaching that point.

Consider the example

$$f(w) = \frac{w}{1-w}, \qquad \zeta := \frac{1}{w}, \qquad \frac{1}{f(w)} = \zeta - 1. \tag{260}$$

In the ζ-coordinate the map is translation by -1, and $f^{\circ n}(w) = w/(1 - nw)$. For $R > 0$, the half-plane $\{\mathrm{Re}\,\zeta < -R\}$ is forward invariant, and its points move towards infinity under iteration. Its inverse image under $w \mapsto 1/w$ is a domain with 0 on its boundary, and every orbit in it tends to 0. It is an example of an attracting *petal*.

We now return to a general germ $F = f^{\circ q}$ as in (259). The same change of viewpoint applies to it. On a sector where $\zeta(w) := -1/(kaw^k)$ is injective, direct expansion gives

$$\zeta(F(w)) = \zeta(w) + 1 + O(w).$$

The *Leau–Fatou theory* replaces this approximate translation by an exact one on petals: there are k attracting petals, one for each direction in which aw^k is negative real, on which $F^{\circ n}(w) \to 0$; on each such petal there is an injective holomorphic function Φ satisfying

$$\Phi(F(w)) = \Phi(w) + 1. \tag{261}$$

The corresponding statement for F^{-1} gives the repelling petals. For proofs of these assertions, see [CG, Chapter II, §5] and [Gam, pp. 3, 7–9]. Such a function Φ is called a *Fatou coordinate*, and (261) is *Abel's equation*. In (260), changing ζ to $-\zeta$ gives the $+1$ convention.

3.4 Siegel compacta and hedgehogs

An irrationally neutral fixed point need not admit a linearizing coordinate, as the preceding subsection shows. Nevertheless, a compact invariant set can replace the closures of Siegel subdiscs. We explain Pérez-Marco's construction of such sets and some of their dynamical properties ([P]). The maps will be germs at 0 of holomorphic functions, not necessarily polynomials.

Write $\Delta_r := \{w \in \mathbb{C} \mid |w| < r\}$ and $\mathrm{U}(1) := \partial\Delta$, where $\Delta = \Delta_1$. Fix the germ f at 0 of a holomorphic function with $f(0) = 0$ and $f'(0) \neq 0$. Throughout this subsection, f^{-1} denotes the inverse germ fixing 0, with a specified *univalent* (that is, holomorphic and injective) representative when a domain is under consideration. The representatives are taken from one local biholomorphism and its inverse. This notation does not denote all inverse branches of a polynomial. For a non-empty compact set $K \subset \mathbb{C}$, its diameter is $\operatorname{diam} K := \sup\{|z - w| \mid z, w \in K\}$, and $d(z, K) := \inf\{|z - w| \mid w \in K\}$ for $z \in \mathbb{C}$.

Definition III.3.17. Let f be the germ at 0 of a holomorphic function with $f(0) = 0$ and $f'(0) \neq 0$, and let f^{-1} be its inverse germ fixing 0. A bounded domain $U \subset \mathbb{C}$ containing 0, whose boundary is a Jordan curve, is *admissible* for f if f and f^{-1} have holomorphic univalent representatives on open neighborhoods of $\overline{U}$. A compact set $K \subset \mathbb{C}$ is *full* if $\mathbb{C} \setminus K$ is connected. For a compact set L, write $\mathrm{Fill}(L)$ for the union of L and all bounded connected components of $\mathbb{C} \setminus L$. A *Siegel compactum* for f is a connected full compact set $K \neq \{0\}$ containing 0, on a neighborhood of which f and f^{-1} are holomorphic and univalent, such that $f(K) = K = f^{-1}(K)$. □

Sufficiently small disks centered at 0 are admissible. The neighborhoods on which the two representatives are defined need not be equal. We first state the existence result.

Theorem III.3.18 (Pérez-Marco). *Let f be the germ at $0 \in \mathbb{C}$ of a holomorphic function with $f(w) = e^{2\pi\sqrt{-1}\theta}w + O(w^2)$, where $\theta \in \mathbb{R}$, and let U be admissible for f in the sense of Definition III.3.17. There is a Siegel compactum K such that*

$$0 \in K \subset \overline{U}, \qquad K \cap \partial U \neq \emptyset, \qquad f(K) = K = f^{-1}(K). \tag{262}$$

The existence statement is [P, Theorem 1]. For two-sided orbits in the compactum we use the chosen inverse of f for negative iterates.

When θ is irrational and f is not linearizable at 0, we call such a Siegel compactum a *hedgehog*. We concentrate on this case of the terminology (There is a broader usage for compacta extending beyond a linearization domain; see [P, §V.1]). Let f, θ, and U be as in Theorem III.3.18. The existence proof will approximate θ by real quadratic irrationals and perturb f accordingly. Siegel's theorem gives an invariant disc for each approximating map. Starting with its Siegel subdiscs, we continue the linearization inside U, take the

filled closure of the resulting invariant domain, and then pass to a Hausdorff convergent subsequence. Filling the limit will give the desired compactum.

Two points require care. Invariance must be preserved under the limit and the filling operation, and the compacta must not shrink to $\{0\}$. To prevent collapse, we will use Herman's circle linearization theorem, applied to the external circle map, to show that the closures of the continued invariant domains meet ∂U. Since they contain 0, this gives the positive lower bound given by the Euclidean distance $d(0, \partial U)$ for the diameters of the compacta used in the limit. We therefore make the following four preparations.

Preparation 1. *Hausdorff limits and filling holes.* We check how invariance and boundary contact pass to Hausdorff limits, and how filling bounded complementary components preserves invariance (Lemma III.3.19).

Preparation 2. *The external circle map.* Exterior uniformization and uniform boundary estimates turn the dynamics near an invariant compactum into a circle diffeomorphism (Lemmata III.3.20 and III.3.21).

Preparation 3. *Estimates and continuity of the external construction.* We give coefficient estimates and prove convergence of the external maps on a common annulus, so that their rotation numbers can be followed through the limit (Lemmata III.3.22 and III.3.23).

Preparation 4. *Boundary contact for quadratic irrational angles.* For these angles, Herman's theorem prevents the invariant domain obtained from Siegel's theorem from stopping strictly inside U (Proposition III.3.24).

After proving the existence theorem, we study the position of 0 and periodic points for irrational θ. We then define exterior harmonic measure and state Pérez-Marco's boundary-orbit theorem.

Preparation 1: Hausdorff limits and filling holes

We use the Hausdorff distance d_H induced by the Euclidean distance $d(z, w) := |z - w|$, as in Mini-Exercise I.3.7, including the compactness of $\mathcal{H}_X$ when X is a compact metric space and the preservation of connectedness under Hausdorff limits. Here are the additional observations needed for invariance.

LEMMA III.3.19. *Let $U \subset \mathbb{C}$ be a bounded domain containing 0 whose boundary is a Jordan curve. Let L and L_j, $j \in \mathbb{Z}_{\geq 1}$, be non-empty compact subsets of $\overline{U}$, with $0 \in L_j$, $L_j \cap \partial U \neq \emptyset$, and $L_j \to L$ in d_H. Then $0 \in L$ and $L \cap \partial U \neq \emptyset$. Let $Q \subset \mathbb{C}$ be a compact set containing L and all L_j. If continuous maps $F_j, F \colon Q \to \mathbb{C}$ satisfy $F_j \to F$ uniformly on Q, then $F_j(L_j) \to F(L)$ in d_H. In particular, if $F_j(L_j) = L_j$ for every j, then $F(L) = L$.*

Independently, let f be the germ at 0 of a holomorphic function with $f(0) = 0$ and $f'(0) \neq 0$, and let U be admissible for f. Let L be a non-empty connected compact subset of $\overline{U}$ with $f(L) = L = f^{-1}(L)$ for the chosen representatives. Then $K := \mathrm{Fill}(L)$ is connected and full, is contained in $\overline{U}$, and satisfies $f(K) = K = f^{-1}(K)$.

PROOF. If $\{x_j\}_{j\in\mathbb{Z}_{\geq 1}}$ satisfies $x_j \in L_j$ and $x_j \to x$, then $d(x, L) \leq d(x, x_j) + d_H(L_j, L) \to 0$, so $x \in L$. Conversely, every $x \in L$ is a limit of points $x_j \in L_j$. Choose a sequence $\{x_j\}_{j\in\mathbb{Z}_{\geq 1}}$ with $x_j \in L_j \cap \partial U$ and take a convergent subsequence to obtain boundary contact. Fix $\varepsilon > 0$. Since F is continuous on the compact set Q, there is $\omega > 0$ such that $|F(x) - F(y)| < \varepsilon$ whenever $x, y \in Q$ and $|x - y| < \omega$. For all sufficiently large j, we have $d_H(L_j, L) < \omega$, and hence

$$d_H(F_j(L_j), F(L)) \leq \sup_{x\in Q} |F_j(x) - F(x)| + \varepsilon.$$

Taking the upper limit as $j \to \infty$ and then letting $\varepsilon \searrow 0$ proves the assertion about images.

Since the boundary of U is a Jordan curve, its exterior is connected and unbounded. Thus all bounded complementary components of L lie in $\overline{U}$. Their boundaries lie in L, and filling them produces a connected compact set with connected complement. For any such component D, univalence near $\overline{U}$ gives $f(D) \cap L = \emptyset$ and $\partial f(D) = f(\partial D) \subset L$. Consequently $f(D)$ is open and closed in the component of $\mathbb{C} \setminus L$ containing it. It equals that component, which is bounded because $f(\overline{D})$ is compact. Apply the same argument to f^{-1} to obtain both inclusions needed for invariance of the filled set. □

Fullness itself is not preserved by Hausdorff convergence. Closed arcs obtained by removing successively shorter open arcs from U(1) are full, whereas their limit U(1) is not. This is why the filling operation is performed after taking the limit.

Preparation 2: The external circle map

LEMMA III.3.20. *Let $K \subset \mathbb{C}$ be a connected full compact set containing at least two points, and let*

$$h\colon \mathbb{P}^1 \setminus \overline{\Delta} \longrightarrow \mathbb{P}^1 \setminus K, \qquad h(\zeta) = c\zeta + b_0 + \sum_{m\geq 1} b_m \zeta^{-m}, \quad c > 0, \tag{263}$$

be its normalized exterior uniformization, with $h(\infty) = \infty$. Suppose that f and its chosen inverse f^{-1} have holomorphic univalent representatives on open neighborhoods of K and satisfy $f(K) = K = f^{-1}(K)$. Then the composition

$$g := h^{-1} \circ f \circ h,$$

initially defined on a sufficiently thin annulus outside $\mathrm{U}(1) = \partial\Delta$, extends holomorphically across $\mathrm{U}(1)$. Its restriction $g|_{\mathrm{U}(1)}\colon \mathrm{U}(1) \to \mathrm{U}(1)$ is an orientation-preserving C^ω diffeomorphism. The external construction for f^{-1} gives its inverse.

The assertion concerns the extension of the *composition* $g = h^{-1} \circ f \circ h$, not an extension of h itself. In particular, no local connectedness of K and no continuous extension of h to all of U(1) are assumed. We call the restriction $g|_{\mathrm{U}(1)}$ the *external circle map* of (f, K) (See [P, §II.1]). When the same symbol g denotes the holomorphic extension in the lemma, we write $\rho(g)$ for the rotation number of $g|_{\mathrm{U}(1)}$. The same convention applies to the external maps g_j below.

We first make explicit the uniform boundary estimates needed for the reflection argument.

LEMMA III.3.21. *Let $K \subset \mathbb{C}$ be a connected full compact set containing at least two points, and let h be the normalized exterior map in (263). Let f and f^{-1} be chosen holomorphic univalent representatives on neighborhoods of K, with $f(K) = K = f^{-1}(K)$, as in Lemma III.3.20. Write $d(w, K) := \inf_{x \in K} |w - x|$. Then the following assertions hold.*

(*i*) *The exterior map approaches K uniformly:*

$$\lim_{r \searrow 1} \sup_{1<|\zeta|<r} d(h(\zeta), K) = 0.$$

(*ii*) *Conversely, its inverse approaches* $\mathrm{U}(1)$ *uniformly from the exterior of K:*

$$\lim_{\varepsilon \searrow 0} \sup_{\substack{w \notin K \\ 0<d(w,K)<\varepsilon}} \big(|h^{-1}(w)| - 1\big) = 0.$$

(*iii*) *The composition $g = h^{-1} \circ f \circ h$ is well-defined on $1 < |\zeta| < r_0$ for some $r_0 > 1$, takes values in $\{|\zeta| > 1\}$, and satisfies*

$$\lim_{r \searrow 1} \sup_{1<|\zeta|<r} \big(|g(\zeta)| - 1\big) = 0.$$

PROOF. We first recall why h exists and fix a useful family of neighborhoods of K. Since K is connected and full, its complement in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is simply connected. Indeed, for a Jordan curve in that complement, the connected set K lies in one component of the complement of that curve in $\mathbb{P}^1$, and the other component is disjoint from K; decomposing a polygonal loop into simple loops gives the same conclusion for arbitrary loops. Choose a point of K and send it to infinity by a Möbius transformation. Since K contains another point, the transformed complement is a proper simply connected plane domain. The Riemann mapping theorem recalled in Chapter I, §I.9, therefore gives a biholomorphism $\widetilde{h}\colon \Delta \to \mathbb{P}^1 \setminus K$ with $\widetilde{h}(0) = \infty$. Near 0, the reciprocal $1/\widetilde{h}$ has a simple zero. After a rotation of the disc, write

$$\frac{1}{\widetilde{h}(z)} = az + O(z^2), \qquad a > 0.$$

Taking the reciprocal gives a Laurent expansion with principal part c/z, where $c = 1/a > 0$; its remaining part is holomorphic on Δ. Putting $\zeta = 1/z$ gives the normalized exterior map $h(\zeta) := \widetilde{h}(1/\zeta)$ in (263). The normalization makes it unique, since an automorphism of Δ fixing 0 is a rotation.

For $R > 1$, let V_R be the bounded side of the analytic Jordan curve $h(\{|\zeta| = R\})$. Explicitly,

$$V_R = K \cup h(\{1 < |\zeta| < R\}).$$

In particular, V_R is a bounded open neighborhood of K, its boundary is disjoint from K, and $1 < |h^{-1}(w)| < R$ for $w \in V_R \setminus K$.

(*i*) Suppose that the asserted uniform convergence fails. There are $\varepsilon_0 > 0$ and points ζ_j with $1 < |\zeta_j| < 1 + 1/j$ such that $d(h(\zeta_j), K) \geq \varepsilon_0$. Fix $R > 1$. For large j, $h(\zeta_j) \in V_R$; since

V_R is bounded, we may pass to a subsequence with $h(\zeta_j) \to w \in \mathbb{C}$. The distance inequality gives $w \notin K$. Continuity of h^{-1} on the exterior of K then gives

$$\zeta_j = h^{-1}(h(\zeta_j)) \longrightarrow h^{-1}(w), \qquad |h^{-1}(w)| > 1,$$

contradicting $|\zeta_j| \to 1$. This proves (i), with no restriction on the arguments of ζ_j.

(ii) Fix $R > 1$. Since V_R is open and contains the compact set K, there is $\delta_R > 0$ such that $\{w \in \mathbb{C} \mid d(w, K) < \delta_R\} \subset V_R$. For every exterior point $w \notin K$ with $d(w, K) < \delta_R$, we therefore have $0 < |h^{-1}(w)| - 1 < R - 1$. Taking the supremum and then letting $R \searrow 1$ proves (ii).

(iii) Choose $\eta > 0$ so small that the closed 2η-neighborhood of K lies in a domain on which f is holomorphic and univalent, and put

$$C := \max\left\{1, \sup_{d(w,K)\le 2\eta} |f'(w)|\right\} < \infty.$$

If $d(w, K) < \eta$, choose $x \in K$ with $|w - x| = d(w, K)$. The line segment from x to w is contained in the 2η-neighborhood of K, so $f(x) \in K$ and

$$d(f(w), K) \le |f(w) - f(x)| \le C\, d(w, K).$$

Since f is injective on the chosen neighborhood and $f(K) = K$, it sends exterior points in that neighborhood to exterior points: indeed, $f(w) = f(x)$ with $x \in K$ would imply $w = x$. By (i), $h(\{1 < |\zeta| < r_0\})$ lies in the η-neighborhood of K for a suitable $r_0 > 1$. Thus g is defined on this outer annulus and $|g| > 1$ there.

For $1 < r < r_0$ put

$$a(r) := \sup_{1<|\zeta|<r} d(h(\zeta), K), \qquad b(\varepsilon) := \sup_{\substack{w\notin K \\ 0<d(w,K)<\varepsilon}} \big(|h^{-1}(w)| - 1\big).$$

Statements (i) and (ii) say that $a(r) \to 0$ as $r \searrow 1$ and $b(\varepsilon) \to 0$ as $\varepsilon \searrow 0$. For sufficiently small $r - 1$, the preceding Lipschitz bound implies

$$0 \le \sup_{1<|\zeta|<r} \big(|g(\zeta)| - 1\big) \le b\big(2C\, a(r)\big) \longrightarrow 0.$$

The factor 2 accommodates the strict inequality in the definition of b. This proves (iii). □

Proof of Lemma III.3.20. By Lemma III.3.21 (iii), $g = h^{-1} \circ f \circ h$ is holomorphic and non-vanishing on a sufficiently thin outer annulus, takes values in $\{|\zeta| > 1\}$, and has $|g(\zeta)| \to 1$ uniformly as $|\zeta| \searrow 1$. Thus $u(\zeta) := \log|g(\zeta)| > 0$ is harmonic on that annulus and extends continuously to $\mathrm{U}(1)$ with boundary value zero.

Fix $\zeta_0 \in \mathrm{U}(1)$. In a sufficiently small logarithmic coordinate rectangle $\zeta = \zeta_0 e^{s+\sqrt{-1}t}$, the outer side is $s > 0$ and the circle is $s = 0$. The odd reflection

$$\widetilde{u}(s,t) = \begin{cases} u(\zeta_0 e^{s+\sqrt{-1}t}), & s \ge 0, \\ -u(\zeta_0 e^{-s+\sqrt{-1}t}), & s < 0 \end{cases}$$

is harmonic across $s = 0$ by the harmonic reflection principle for continuous zero boundary data. On the outer half-rectangle $g(\zeta_0 e^{s+\sqrt{-1}t})$ has a holomorphic logarithm H, with $\operatorname{Re} H = u$. The harmonic function $\widetilde{u}$ has a harmonic conjugate on the full rectangle. Its associated holomorphic function differs from H by a purely imaginary constant on the outer half; adjust that constant and exponentiate. This extends g holomorphically across the chosen arc of U(1). The local extensions agree on overlaps by the identity theorem, so they glue to an extension on a full annulus around U(1). Moreover $|g| = 1$ on U(1). On a smaller symmetric annulus, the extension satisfies

$$g(\zeta) = \frac{1}{\overline{g(1/\bar{\zeta})}}.$$

Apply the same construction to f^{-1} and denote the resulting extended external map by g_-. On a sufficiently thin outer annulus the two maps are inverse to one another; after shrinking the neighborhoods if necessary, the identity theorem extends

$$g_- \circ g = \mathrm{id}, \qquad g \circ g_- = \mathrm{id}$$

across U(1). Thus $g|_{\mathrm{U}(1)}$ is a real analytic diffeomorphism with real analytic inverse and nowhere vanishing derivative. Finally, the restriction preserves orientation. Indeed, locally write $g(e^{s+\sqrt{-1}t}) = e^{U(s,t)+\sqrt{-1}V(s,t)}$. The Cauchy–Riemann equations give $\partial_t V = \partial_s U$. Since $U(0,t) = 0$ and $U(s,t) > 0$ for $s > 0$, we have $\partial_s U(0,t) \geq 0$; it is strictly positive because the boundary restriction is a diffeomorphism. Thus the angular coordinate is increasing. □

Preparation 3: Estimates and continuity of the external construction

The purpose of this preparation is twofold. First, we show that a normalized exterior map cannot degenerate as long as the diameter of the corresponding compact set stays bounded away from zero. Second, when the compact sets K_j and holomorphic maps f_j vary, we prove convergence of the associated external circle maps g_j and their rotation numbers. The second step requires convergence of g_j on a common annulus containing U(1), not merely on the exterior of the circle.

We first recall the following three classical facts from one-variable complex analysis.

- *Grönwall's area theorem.* Let $H(\zeta) = \zeta + B_0 + \sum_{m=1}^{\infty} B_m \zeta^{-m}$ be holomorphic and univalent on $\{|\zeta| > 1\}$. Then
$$\sum_{m=1}^{\infty} m|B_m|^2 \leq 1$$
(See [CG, Chapter I, Theorem 1.1]).

- *Koebe's quarter theorem and the second coefficient estimate.* If $F\colon \Delta \to \mathbb{C}$ is univalent and $F(z) = z + a_2 z^2 + \cdots$, then $|a_2| \leq 2$ and $\Delta_{1/4} \subset F(\Delta)$ (See [CG, Chapter I, Theorems 1.2 and 1.3]).

- *Hurwitz's theorem.* Let u_j be holomorphic functions on a domain D which converge locally uniformly to u. Then u is holomorphic and $u_j' \to u'$ locally uniformly. If,

moreover, every u_j is nowhere zero, then either $u \equiv 0$ or u is nowhere zero. In particular, a locally uniform limit of univalent holomorphic functions on D is either univalent or constant (See [Ah, Chapter 5, §1, Theorems 1 and 2]).

We now apply these facts to the normalized exterior maps in (263).

LEMMA III.3.22. *Let $M > 0$, and let $K \subset \overline{\Delta}_M$ be a connected full compact set containing 0 and at least one other point. Let $h\colon \mathbb{P}^1 \setminus \overline{\Delta} \to \mathbb{P}^1 \setminus K$ be its normalized exterior map, with Laurent coefficients $c > 0$, b_0, and b_m as in (263). Then*

$$\frac{\operatorname{diam} K}{4} \le c \le M, \qquad |b_0| \le 2c, \qquad \sum_{m=1}^{\infty} m|b_m|^2 \le c^2. \tag{264}$$

PROOF. Applying Grönwall's area theorem to h/c gives $\sum_{m=1}^{\infty} m|b_m|^2 \le c^2$. For each $a \in K$, the function

$$F_a(z) := \frac{c}{h(1/z) - a} = z - \frac{b_0 - a}{c} z^2 + \cdots$$

is holomorphic and univalent on Δ and satisfies $F_a(0) = 0$ and $F_a'(0) = 1$. Taking $a = 0$ and using the second coefficient estimate gives $|b_0| \le 2c$. For distinct $a, b \in K$, the value $c/(b-a)$ is omitted by F_a. Koebe's quarter theorem therefore gives $|c/(b-a)| \ge 1/4$, or $|b - a| \le 4c$. Taking the supremum over $a, b \in K$ yields $\operatorname{diam} K \le 4c$. Finally, $z \mapsto 1/h^{-1}(M/z)$ extends holomorphically across $z = 0$, maps Δ into Δ, fixes 0, and has derivative c/M at 0. Schwarz's lemma implies $c \le M$. □

LEMMA III.3.23 (Continuity on a common annulus). *Fix $M > 0$ and $d > 0$. Let $\{K_j\}_{j \in \mathbb{Z}_{\ge 1}}$ be a sequence of connected full compact subsets of $\overline{\Delta}_M$ containing 0, with $\operatorname{diam} K_j \ge d$. Suppose $K_j \to L$ in the Hausdorff metric and put $K := \operatorname{Fill}(L)$. Let f_j and f be holomorphic maps with chosen holomorphic inverses f_j^{-1} and f^{-1}. Suppose these maps and inverses are univalent on a common open neighborhood of K and all sufficiently large K_j, with $f_j \to f$ and $f_j^{-1} \to f^{-1}$ locally uniformly there. Assume K_j is invariant under f_j and f_j^{-1}, and K is invariant under f and f^{-1}. Let h_j and h be the normalized exterior maps of K_j and K in (263), and let g_j and g be the holomorphic extensions across $\mathrm{U}(1)$ of $h_j^{-1} \circ f_j \circ h_j$ and $h^{-1} \circ f \circ h$, respectively, given by Lemma III.3.20. Then the following assertions hold.*

(*i*) *The normalized exterior maps satisfy $h_j \to h$ locally uniformly on $\{|\zeta| > 1\}$, and $h_j^{-1} \to h^{-1}$ locally uniformly on $\mathbb{P}^1 \setminus K$, in the appropriate spherical charts.*

(*ii*) *There is an open annulus A containing $\mathrm{U}(1)$ such that, for all sufficiently large j, both g_j and g are holomorphic on A, and $g_j \to g$ locally uniformly there. In particular, $d_{\mathbb{R}/\mathbb{Z}}(\rho(g_j), \rho(g)) \to 0$.*

PROOF. We first prove (*i*). Write $h_j(\zeta) = c_j\zeta + b_{0,j} + \sum_{m=1}^{\infty} b_{m,j}\zeta^{-m}$. By (264) and the Cauchy–Schwarz inequality, for $|\zeta| > 1$ we have

$$\begin{aligned}\left|\sum_{m=1}^{\infty} b_{m,j}\zeta^{-m}\right| &\le \left(\sum_{m=1}^{\infty} m|b_{m,j}|^2\right)^{1/2} \left(\sum_{m=1}^{\infty} \frac{|\zeta|^{-2m}}{m}\right)^{1/2} \\ &\le M\sqrt{-\log(1 - |\zeta|^{-2})}.\end{aligned}$$

Since $d/4 \le c_j \le M$ and $|b_{0,j}| \le 2M$, Montel's theorem gives a locally uniformly convergent subsequence on $\{|\zeta| > 1\}$, whose limit we denote by h_*. Passing to a further subsequence if necessary, $c_j \to c_* \in [d/4, M]$. The leading Laurent coefficient of h_* is c_*, so h_* is non-constant; Hurwitz's theorem implies that it is univalent.

For $a \in L$, choose $a_j \in K_j$ with $a_j \to a$. The functions $h_j - a_j$ have no zeros on $\{|\zeta| > 1\}$. By Hurwitz's theorem and the non-constancy of h_*, the function $h_* - a$ has no zeros there. Thus the image of h_* avoids L and lies in the unbounded component $\mathbb{P}^1 \setminus K$ of $\mathbb{P}^1 \setminus L$.

Conversely, let W be a compact subset of $\mathbb{P}^1 \setminus K$. For all sufficiently large j, a fixed open neighborhood of W is disjoint from K_j. On this neighborhood, $u_j(w) := 1/h_j^{-1}(w)$ is holomorphic and satisfies $|u_j| \le 1$. A diagonal normal-family argument on $\mathbb{P}^1 \setminus K$ gives a subsequential limit u. Near infinity, $u_j(w) = c_j/w + O(w^{-2})$, so $u(\infty) = 0$ and its derivative at infinity, in the coordinate $1/w$, is $c_* > 0$. In particular, u is not constant. Since u_j has no zero at any finite exterior point, Hurwitz's theorem gives $u(w) \neq 0$ there. The maximum modulus principle also gives $|u(w)| < 1$ at all finite exterior points. Passing to the limit in $h_j(1/u_j(w)) = w$ now gives $h_*(1/u(w)) = w$. Thus h_* maps onto the exterior of K, with inverse $w \mapsto 1/u(w)$. Its normalization at infinity implies $h_* = h$. Since every convergent subsequence has the same limit, the full sequence satisfies $h_j \to h$ locally uniformly, and the same argument gives convergence of the inverse maps on compact subsets of $\mathbb{P}^1 \setminus K$. This proves (i).

We next prove (ii), namely convergence on a common annulus that crosses U(1). For $s > 1$, write

$$V_s := K \cup h(\{1 < |\zeta| < s\}), \qquad V_{j,s} := K_j \cup h_j(\{1 < |\zeta| < s\}).$$

These are the bounded sides of the respective analytic Jordan curves $h(\{|\zeta| = s\})$ and $h_j(\{|\zeta| = s\})$. Let $\mathcal{W}$ be a common open neighborhood as in the statement. Choose $1 < R_0 < R$ sufficiently close to 1 that $\overline{V}_R$ is a compact subset of $\mathcal{W}$. By invariance of K and local uniform convergence of the maps and their inverses, there is an open neighborhood N of K, with compact closure in $\mathcal{W}$, such that for all sufficiently large j,

$$f_j(N) \cup f_j^{-1}(N) \subset V_{R_0}.$$

Choose $1 < r < R_0$ so close to 1 that $\overline{V}_r \subset N$. Local uniform convergence $h_j \to h$ implies smooth convergence on the two fixed circles $|\zeta| = r$ and $|\zeta| = R$. Stability of winding numbers off these Jordan curves then gives, for all sufficiently large j,

$$V_{j,r} \subset N, \qquad \overline{V}_{R_0} \subset V_{j,R}.$$

Since $f_j(K_j) = K_j = f_j^{-1}(K_j)$ and both maps are univalent on $\mathcal{W}$, they send nearby exterior points to exterior points. Consequently,

$$1 < |g_j(\zeta)| < R \qquad (1 < |\zeta| < r).$$

The same estimate holds for the external map of f_j^{-1}, and also for the limiting pair (f, K).

The reflection argument of Lemma III.3.20 extends all these external maps to the fixed annulus $\{1/r < |\zeta| < r\}$. On its outer half the maps g_j have modulus between 1 and R; on its inner half, the reflection formula gives modulus between $1/R$ and 1. Thus the g_j are uniformly bounded on this common annulus, and Montel's theorem applies. On the outer half, the already established convergence of h_j, h_j^{-1}, and f_j gives

$$g_j = h_j^{-1} \circ f_j \circ h_j \longrightarrow h^{-1} \circ f \circ h = g$$

locally uniformly. The identity theorem determines every subsequential limit on the whole annulus, and hence $g_j \to g$ there locally uniformly. In particular, the circle maps converge uniformly, so continuity of the rotation number (Lemma III.3.2) gives $d_{\mathbb{R}/\mathbb{Z}}(\rho(g_j), \rho(g)) \to 0$.

The two radii $R_0 < R$ leave room for the moving exterior curves. Convergence on compact subsets strictly outside U(1) alone would not give a common annulus across the circle (Compare [P, Lemma II.1.1]). This completes the proof of (ii). □

Preparation 4: Boundary contact for quadratic irrational angles

Let F be the germ at 0 of a holomorphic function with $F(w) = e^{2\pi\sqrt{-1}\theta}w + O(w^2)$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$. First suppose that a representative of F has an invariant disc Ω parametrized by a univalent map $\phi\colon \Delta_R \to \Omega$ with $\phi(0) = 0$, $\phi'(0) = 1$, and $F(\phi(w)) = \phi(e^{2\pi\sqrt{-1}\theta}w)$. For $0 < r < R$ the closed Siegel subdisc $K_r = \phi(\overline{\Delta}_r)$ has analytic Jordan boundary. Its exterior map h_r extends across U(1) by reflection in that analytic boundary. The two coordinates on the same boundary give an orientation-preserving C^ω diffeomorphism $\eta_r\colon \mathrm{U}(1) \to \mathrm{U}(1)$,

$$\eta_r(\zeta) := h_r^{-1}(\phi(r\zeta)), \qquad |\zeta| = 1.$$

Its external map g_r satisfies $g_r \circ \eta_r = \eta_r \circ R_\theta$, so $\rho(g_r) = \theta \bmod \mathbb{Z}$. If $\overline{\Omega}$ is compact and lies in the domain of F and its inverse, $K_r \to \overline{\Omega}$ in the Hausdorff metric as $r \nearrow R$. Continuity on a common annulus (Lemma III.3.23), applied to F fixed, therefore gives

$$\rho\big(g_{\mathrm{Fill}(\overline{\Omega})}\big) = \theta \bmod \mathbb{Z}. \tag{265}$$

This compares internal and external circle coordinates before taking a limit; it is not obtained by differentiating the external map at the fixed point.

Proposition III.3.24. *Let F be the germ at 0 of a holomorphic function with $F(w) = e^{2\pi\sqrt{-1}\theta}w + O(w^2)$, where θ is a real quadratic irrational, and let U be admissible for F. There is a Siegel disc Ω in U with $\overline{\Omega} \cap \partial U \neq \emptyset$. Its filled closure is a Siegel compactum as in (262).*

Proof. Kœnigs–Siegel linearization (Theorem III.3.10) gives the unique normalized linearizing germ ϕ. Let R be the supremum of radii r for which this germ extends univalently to Δ_r, satisfies the conjugacy equation, and has image in U. These extensions agree on overlaps, and define $\phi\colon \Delta_R \to \Omega \subset U$. If $U \subset \Delta_M$, Schwarz's lemma applied to $\phi(r\zeta)/M$ gives $r \le M$, so $0 < R < \infty$.

Suppose $\overline{\Omega} \cap \partial U = \emptyset$. Then $K := \mathrm{Fill}(\overline{\Omega})$ is a compact subset of U and is invariant by Lemma III.3.19, and its external map g has rotation number θ by (265). Apply Herman's

circle linearization theorem (Theorem III.3.5). The invariant curves in (248), transported by h, give an F-invariant analytic Jordan curve whose bounded side V satisfies

$$K \subset V \subset U,$$

with $\overline{V}$ compact and contained in U. Indeed, the images of the exterior annuli approach K uniformly as their outer radii tend to 1. Univalence of F near $\overline{V}$ and invariance of its boundary give $F(V) = V$.

A Riemann map to V, normalized at 0, conjugates $F|_V$ to a disk automorphism fixing 0. Schwarz's lemma makes this a rotation, whose multiplier is $e^{2\pi\sqrt{-1}\theta}$. Rescale the disk to write this map as $\psi\colon \Delta_S \to V$ with $\psi'(0) = 1$. The inverse-coordinate change $\psi^{-1} \circ \phi$ equals the identity near 0 by uniqueness of the normalized linearizing germ. It therefore equals the identity throughout Δ_R. Since $\overline{\Omega}$ is compact and contained in V, this implies $S > R$, contrary to the definition of R. Thus boundary contact holds. Finally, $F(\Omega) = \Omega$ implies $F(\overline{\Omega}) = \overline{\Omega}$, and similarly for the inverse. Filling preserves invariance. □

The two linearization results have different roles in this proof. The local Siegel theorem starts an invariant disc near 0; the global circle theorem prevents its maximal continuation in U from stopping strictly inside U. Quadratic irrational rotation numbers give a dense family for which both steps apply. They are sufficient for this argument, not a claimed necessary condition for boundary contact.

Remark III.3.25 (Smooth boundaries and the size of Siegel discs). For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $P_\theta(w) = e^{2\pi\sqrt{-1}\theta}w + w^2$. When θ is Brjuno, denote its maximal Siegel disc in $\mathbb{C}$ by Ω_θ, and write its normalized linearization as $\phi_\theta\colon \Delta_{r_\theta} \to \Omega_\theta$, with $\phi_\theta(0) = 0$ and $\phi'_\theta(0) = 1$. The number r_θ is the *conformal radius* of this pointed disc. Avila–Buff–Chéritat prove the following approximation statement. Given such a θ, a number $0 < r < r_\theta$, and $\varepsilon > 0$, one can choose a Brjuno number θ' with $|\theta' - \theta| < \varepsilon$ and $r_{\theta'} = r$ such that the boundary of $\Omega_{\theta'}$ is a C^∞ Jordan curve. Its parametrization $t \mapsto \phi_{\theta'}(re^{2\pi\sqrt{-1}t})$ can be made arbitrarily close to $t \mapsto \phi_\theta(re^{2\pi\sqrt{-1}t})$ in the C^∞ topology on maps $\mathbb{R}/\mathbb{Z} \to \mathbb{C}$ (See [ABC, Main Theorem]).

Here a smaller conformal radius is not, by itself, a statement about Euclidean diameter. To obtain a disc in a small Euclidean neighborhood of 0, choose a ball centered at 0 whose closure is contained in that neighborhood. Then choose r so that the reference curve $\phi_\theta(r\mathrm{U}(1))$ lies in this ball and make the boundary approximation close enough to remain there. The maximum principle puts the resulting Siegel disc in the same ball. In particular, a local linearization alone need not continue as an invariant disc meeting the boundary of an arbitrary admissible domain. This quadratic family varies its multiplier while keeping the coefficient of w^2 equal to 1. It is different from the family used in the next proof, where multiplication of a fixed function by a constant also changes its nonlinear coefficients. □

Proof of the existence theorem

Proof of Theorem III.3.18. Let f, θ, and the admissible domain U be as in Theorem III.3.18. Use representatives of f and its inverse near $\overline{U}$. Choose a sequence $\{\theta_j\}_{j\in\mathbb{Z}_{\geq 1}}$ of real quadratic

irrationals with $\theta_j \to \theta$ and put

$$f_j(w) := e^{2\pi\sqrt{-1}(\theta_j-\theta)} f(w), \qquad f_j^{-1}(w) = f^{-1}\bigl(e^{-2\pi\sqrt{-1}(\theta_j-\theta)} w\bigr).$$

Because f^{-1} is defined on an open neighborhood of the compact set $\overline{U}$, these maps and inverses are defined and univalent on a common neighborhood of $\overline{U}$ for all large j. Both sequences converge locally uniformly there. Proposition III.3.24 gives $K_j = \mathrm{Fill}(\overline{\Omega}_j) \subset \overline{U}$ with $0 \in K_j$ and $K_j \cap \partial U \neq \emptyset$. Mini-Exercise I.3.7 gives a Hausdorff convergent subsequence, again denoted by $\{K_j\}_{j\in\mathbb{Z}_{\geq 1}}$, with limit L connected. Lemma III.3.19 preserves boundary contact and gives $f(L) = L = f^{-1}(L)$. The set $K = \mathrm{Fill}(L)$ has all the required properties. □

The existence theorem is now proved, also for rational θ. For the dynamical properties below, retain the germ f, the approximants f_j and K_j, and the compactum K just constructed, and assume that θ is irrational. For each approximant, (265) gives $\rho(g_{K_j}) = \theta_j \bmod \mathbb{Z}$. Since $\operatorname{diam} K_j \geq d(0, \partial U)$, continuity on a common annulus (Lemma III.3.23) and rotation-number continuity give

$$\rho(g_K) = \theta \bmod \mathbb{Z}. \tag{266}$$

Here the map and the compactum both vary, whereas in (265) only the radius of a linearized disk varied. We use (266) for compacta obtained by this construction.

The position of the fixed point and periodic orbits

Proposition III.3.26. *Let f be the germ at 0 of a holomorphic function with $f(w) = e^{2\pi\sqrt{-1}\theta} w + O(w^2)$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let K be any Siegel compactum of f. Then f is linearizable at 0 if and only if $0 \in \operatorname{Int} K$. In particular, $0 \in \partial K$ in the non-linearizable case.*

Proof. Suppose $\Delta_r \subset K$. All non-negative iterates are holomorphic on Δ_r and take values in the same bounded set K. With $\lambda = f'(0)$, define the sequence $\{H_N\}_{N\in\mathbb{Z}_{\geq 1}}$ by

$$H_N(w) := \frac{1}{N} \sum_{j=0}^{N-1} \lambda^{-j} f^{\circ j}(w).$$

These functions are uniformly bounded, $H_N(0) = 0$, and $H_N'(0) = 1$. A subsequence converges to a holomorphic H with $H'(0) = 1$. Where w and $f(w)$ belong to Δ_r,

$$H_N(f(w)) - \lambda H_N(w) = \frac{\lambda}{N}\bigl(\lambda^{-N} f^{\circ N}(w) - w\bigr) \longrightarrow 0.$$

Thus $H \circ f = \lambda H$, and H is a local coordinate. This implication in fact needs only $|\lambda| = 1$.

Conversely, take a small closed linearized disk about 0. Since K is connected and contains a point other than 0, its boundary meets K when the disk is sufficiently small. The orbit of an intersection point is dense in this invariant circle, because it is conjugate to an irrational rotation. Hence the whole circle lies in K, and fullness puts its bounded side in K as well. Thus 0 is an interior point. Here we use the density of irrational rotations proved in §3.1. □

PROPOSITION III.3.27. *Let f be the germ at 0 of a holomorphic function with $f(w) = e^{2\pi\sqrt{-1}\theta}w + O(w^2)$, where $\theta \in \mathbb{R}\setminus\mathbb{Q}$. Let K be a Siegel compactum obtained by the construction in the proof of Theorem III.3.18. Then 0 is the only periodic point of f in K.*

PROOF. Put $\lambda := f'(0)$. Let h be the normalized exterior uniformization of K in (263), write $c > 0$ for its leading Laurent coefficient, and let g be the corresponding external circle map and its holomorphic extension from Lemma III.3.20. Fix $q \in \mathbb{Z}_{\geq 1}$. We shall show that $f^{\circ q}(w) - w$ has exactly one zero, counted with multiplicity, in a suitable neighborhood of K.

For $s > 1$, let V_s be the bounded domain enclosed by the analytic Jordan curve $\Gamma_s := h(\{\zeta \mid |\zeta| = s\})$, positively oriented by $t \mapsto h(se^{\sqrt{-1}t})$, $0 \leq t \leq 2\pi$. Compactness and invariance give a neighborhood of K on which $f^{\circ q}$ is defined and univalent. By Lemma III.3.21 (i), we may take s sufficiently close to 1 that $f^{\circ q}$ is holomorphic on an open neighborhood of $\overline{V}_s$. Recall the winding number about 0 from Chapter I, §7.1, and its argument-integral interpretation in §I.9. By the argument principle recalled in the latter section, it suffices to prove

$$\operatorname{wind}\big(t \mapsto f^{\circ q}(h(se^{\sqrt{-1}t})) - h(se^{\sqrt{-1}t})\big) = 1.$$

Here and below the closed curves are parametrized by $0 \leq t \leq 2\pi$.

Put $v := g^{\circ q}$ on a sufficiently small neighborhood of $\mathrm{U}(1)$. By (266) and the iteration rule for rotation numbers (Lemma III.3.2), $\rho(v) = q\theta + \mathbb{Z}$ is irrational, so v has no fixed point on $\mathrm{U}(1)$. Compactness therefore gives $v(\zeta) - \zeta \neq 0$ on a sufficiently thin annulus containing $\mathrm{U}(1)$. After decreasing $s - 1$ further if necessary, all the required iterates and conjugacy relations are defined on $|\zeta| = s$, and $|v(\zeta)| > 1$ there. On this circle we have

$$f^{\circ q}(h(\zeta)) - h(\zeta) = (v(\zeta) - \zeta)\frac{h(v(\zeta)) - h(\zeta)}{v(\zeta) - \zeta}.$$

We first compute the winding number of $v(\zeta) - \zeta$. On $\mathrm{U}(1)$, write $v(e^{\sqrt{-1}t}) = e^{\sqrt{-1}\Phi(t)}$, where Φ is a continuous lift with $\Phi(t + 2\pi) = \Phi(t) + 2\pi$. Since v has no fixed point on the circle, $\Phi(t) - t$ never belongs to $2\pi\mathbb{Z}$. After subtracting a fixed integer multiple of 2π, we obtain a continuous 2π-periodic function δ with $0 < \delta(t) < 2\pi$ and

$$v(e^{\sqrt{-1}t}) - e^{\sqrt{-1}t} = e^{\sqrt{-1}t}\big(e^{\sqrt{-1}\delta(t)} - 1\big).$$

The factor $e^{\sqrt{-1}\delta(t)} - 1$ lies in the open left half-plane, since its real part is $\cos\delta(t) - 1 < 0$. Its winding number is zero, whereas that of $e^{\sqrt{-1}t}$ is one. Thus the product has winding number one. The same holds on $|\zeta| = s$ by homotopy invariance, since $v(\zeta) - \zeta$ is non-zero throughout the intervening annulus.

For the second factor, define the continuous map

$$H\colon [1,\infty) \times \{\zeta \in \mathbb{C} \mid |\zeta| = s\} \longrightarrow \mathbb{C}^*,$$
$$(\tau, \zeta) \longmapsto \frac{h(\tau v(\zeta)) - h(\tau\zeta)}{\tau(v(\zeta) - \zeta)}.$$

Both $\tau v(\zeta)$ and $\tau\zeta$ lie in the exterior of $\overline{\Delta}$ and are distinct, so univalence of h shows that the displayed value is non-zero. Write $H_\tau(\zeta) := H(\tau, \zeta)$ for $\tau \geq 1$. The Laurent expansion (263) gives

$$H_\tau(\zeta) = c + \sum_{m=1}^{\infty} b_m \tau^{-m-1} \frac{v(\zeta)^{-m} - \zeta^{-m}}{v(\zeta) - \zeta}.$$

For this fixed s, the points ζ and $v(\zeta)$ range over compact subsets of $\{|\xi| > 1\}$, and $v(\zeta) - \zeta$ is bounded away from zero. It follows that $H_\tau \to c$ uniformly on $|\zeta| = s$ as $\tau \to \infty$. Choose $T > 1$ so large that H_T takes values in the disk $\{z \in \mathbb{C} \mid |z - c| < c/2\}$. Its winding number is zero. Restricting H to $[1, T] \times \{\zeta \in \mathbb{C} \mid |\zeta| = s\}$ and using homotopy invariance gives the same conclusion for H_1.

Winding numbers add under multiplication, so the factorization above gives the required value $1 + 0 = 1$. The argument principle now yields

$$\frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_s} \frac{(f^{\circ q})'(w) - 1}{f^{\circ q}(w) - w}\, dw = 1.$$

Thus $f^{\circ q}(w) - w$ has exactly one zero in V_s, counted with multiplicity. The zero at 0 is simple because $(f^{\circ q})'(0) - 1 = \lambda^q - 1 \neq 0$. Since $K \subset V_s$ and $q \geq 1$ was arbitrary, 0 is the only periodic point of f in K. □

The approximating Siegel discs in U, $\{\Omega_j\}_{j \in \mathbb{Z}_{\geq 1}}$, where $\Omega_j = \phi_j(\Delta_{R_j})$ and $\phi_j'(0) = 1$, also distinguish two notions of size. If f is non-linearizable, then $R_j \to 0$. Otherwise, on a subsequence $R_j \geq r > 0$, the bounded maps $\phi_j|_{\Delta_r}$ have a holomorphic limit with derivative 1. Passing to the limit in the conjugacy equation would linearize f. Thus the conformal radii tend to zero while the diameters of K_j stay bounded below. Neither this observation nor $0 \in \partial K$ alone asserts that K has empty interior.

Boundary values and harmonic measure

Let $K \subset \mathbb{C}$ again be a connected full compact set containing at least two points, and let $h\colon \mathbb{P}^1 \setminus \overline{\Delta} \to \mathbb{P}^1 \setminus K$ be its normalized exterior map in (263). A topological conjugacy on the external circle cannot simply be transported to all of ∂K, since h need not extend continuously there. Instead we use boundary values almost everywhere.

Here almost everywhere refers to the normalized arc-length measure m on $\mathrm{U}(1)$ fixed above. An approach $\zeta \to \zeta_0 \in \mathrm{U}(1)$ from $|\zeta| > 1$ is called *non-tangential* if $|\zeta - \zeta_0| \leq C(|\zeta| - 1)$ for some fixed $C > 1$.

Lemma III.3.28. *Let $K \subset \mathbb{C}$ be a connected full compact set containing at least two points, and let $h\colon \mathbb{P}^1 \setminus \overline{\Delta} \to \mathbb{P}^1 \setminus K$ be its normalized exterior map in (263). For almost every $t \in [0, 2\pi)$, the limit*

$$h^*(e^{\sqrt{-1}t}) := \lim_{r \searrow 1} h(re^{\sqrt{-1}t})$$

exists and belongs to ∂K. At almost every such point the same limit is obtained on non-tangential approach from the exterior.

PROOF. Let $c > 0$ be the leading Laurent coefficient of h, and put $B(z) := h(1/z) - c/z$ for $0 < |z| < 1$. The Laurent expansion extends B holomorphically across 0. This extension is bounded on Δ, since $h(\{1 < |\zeta| < 2\})$ is bounded.

By Fatou's theorem for bounded holomorphic functions, B has finite non-tangential boundary limits at almost every point of U(1) (See [Ru1, Theorem 17.11(b)]). Since $h(\zeta) = c\zeta + B(1/\zeta)$ and inversion preserves non-tangential approaches to the circle, the same conclusion holds for h from the exterior. Its radial limits exist and agree with these limits almost everywhere. The uniform approach to K in Lemma III.3.21 (i) puts these limits in K. Since $h(\zeta) \notin K$ for $|\zeta| > 1$, they belong to ∂K. □

DEFINITION III.3.29. Let $K \subset \mathbb{C}$ be a connected full compact set containing at least two points, and let h be its normalized exterior map in (263). Let h^* be the radial boundary map furnished by Lemma III.3.28. The *exterior harmonic measure of K with pole at infinity* is the probability measure μ on ∂K given, for Borel sets $A \subset \partial K$, by

$$\mu(A) := \frac{1}{2\pi} m_{\mathbb{R}}(\{t \in [0, 2\pi) \mid h^*(e^{\sqrt{-1}t}) \in A\}),$$

where $m_{\mathbb{R}}$ denotes Lebesgue measure on $\mathbb{R}$. Equivalently, $\mu(A) = m((h^*)^{-1}(A))$ for the normalized circle measure m fixed above. Values of h^* on its exceptional null set may be chosen arbitrarily in ∂K; they do not change μ. □

Every non-empty relatively open subset of ∂K has positive μ measure; equivalently, the support of μ is ∂K (See [P, p. 274]).

Boundary dynamics

THEOREM III.3.30 (Pérez-Marco). *Let f be the germ at 0 of a holomorphic function with $f(w) = e^{2\pi\sqrt{-1}\theta} w + O(w^2)$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let K be a Siegel compactum obtained by the construction in the proof of Theorem III.3.18, and let μ be its exterior harmonic measure with pole at infinity. The set of points in ∂K whose two-sided orbit is dense in ∂K is a dense G_δ set and has μ measure one.*

PROOF. Apply the argument in [P, Theorem IV.2.3 and its proof, pp. 274–275], using (266). The G_δ description is the countable-intersection construction used in that proof. □

Let f and K be as in the theorem, and suppose that f is not linearizable. Proposition III.3.26 gives $0 \in \partial K$. A point with dense two-sided orbit is non-zero, since the orbit of 0 is $\{0\}$. Thus there are $w_0 \in \partial K \setminus \{0\}$ and integers n_ν with $|n_\nu| \to \infty$ such that $f^{\circ n_\nu}(w_0) \to 0$: no finite iterate of w_0 equals 0, so finitely many iterates cannot provide arbitrarily close approaches to 0.

Exercises for Chapter III

Exercise III.1. Let $f\colon X \to Y$ be a proper surjective holomorphic submersion between complex manifolds. Explain which step in the proof of Ehresmann's fibration theorem (Theorem I.8.7) cannot in general be carried out holomorphically.

Exercise III.2. Let X be a compact Riemann surface of genus g, and let L be a holomorphic line bundle on X. For a coherent sheaf $\mathcal{F}$ on X, put $\chi(\mathcal{F}) := \dim H^0(X,\mathcal{F}) - \dim H^1(X,\mathcal{F})$. The following steps give a proof of (220).
(*i*) Let $\mathbb{I}_X$ be a holomorphically trivial line bundle on X. Verify (220) for $L = \mathbb{I}_X$, and show that $\chi(\mathcal{O}_X) = 1 - g$.
(*ii*) Show, without using Riemann–Roch, that L has a nonzero meromorphic section and hence is isomorphic to $[D]$ for a divisor D on X. Write $D = A - B$ with A and B effective (Use principal parts at one point and finite-dimensionality of $H^1(X,\mathcal{O}_X(L))$).
(*iii*) For an arbitrary divisor D and a point $p \in X$, identify the quotient $\mathcal{O}_X(D+p)/\mathcal{O}_X(D)$ and prove $\chi(\mathcal{O}_X(D+p)) = \chi(\mathcal{O}_X(D)) + 1$.
(*iv*) Prove (220).

Exercise III.3. (*i*) Let X be a complex manifold of pure dimension 2 and let $Z \subset X$ be a closed non-singular complex curve. Prove the isomorphism called the *adjunction formula*, $K_Z \cong K_X|_Z \otimes N_{Z/X}$ by using adapted local coordinates.
(*ii*) Let $P(Z_0,Z_1,Z_2)$ be a nonzero homogeneous polynomial of degree $d \geq 1$ whose zero divisor $Z \subset \mathbb{P}^2$ is reduced and non-singular. Show that $K_{\mathbb{P}^2} \cong L_{\mathrm{taut}}^3$ and $[Z] \cong L_{\mathrm{taut}}^{-d}$. Compute $\deg(L_{\mathrm{taut}}^{-1}|_Z)$, interpreting degree as a sum over connected components until connectedness has been proved.
(*iii*) Prove that this Z is connected and that its genus is $g = (d-1)(d-2)/2$. Check the values for $d = 1, 2, 3$.

Chapter IV

Complex analytic geometry and dynamics on analytic neighborhood spaces

We introduce in these notes the notion of an *analytic neighborhood space* and study its basic properties. We use this framework, together with finite and formal jets, to organize and interpret several known results in complex analytic geometry on neighborhoods of analytic subsets and in one-dimensional complex dynamics. This viewpoint allows us to investigate the relation between global problems on complex manifolds and problems concerning neighborhoods of their analytic subsets, which need not be points. It may therefore be regarded as a semi-local approach to complex geometry and dynamics.

A common question will be how local morphism data can be brought into a specified form by compatible changes. The diagrams introduced below distinguish the compatibility of charts from that of iterates of one map.

1 Analytic neighborhood spaces

We first introduce a category which records the germ of a complex analytic space, together with a coherent ideal sheaf, along the analytic subset defined by that ideal.

1.1 The category (ANM)

Definition IV.1.1. An *analytic neighborhood model* is a pair $(X, \mathcal{I})$ satisfying the following conditions:

- X is a reduced complex analytic space;
- $\mathcal{I} \subset \mathcal{O}_X$ is a coherent ideal sheaf.

□

Definition IV.1.2. Let $(X,\mathcal{I})$ be an analytic neighborhood model. For its support $S := \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$, denote the inclusion by $j\colon S \hookrightarrow X$. The *reduced center* of $(X,\mathcal{I})$ is the reduction of the closed complex subspace defined by $\mathcal{I}$. It is denoted by $Z(X,\mathcal{I})$:

$$Z(X,\mathcal{I}) := \big(S,\ j^{-1}(\mathcal{O}_X/\mathcal{I})\big)_{\mathrm{red}}.$$

□

The coherence of $\sqrt{\mathcal{I}}$ follows from Rückert's Nullstellensatz (Theorem III.2.37) and Oka's second coherence theorem (Theorem III.2.163). The reduced closed subspace is obtained as in Proposition III.2.36 and the explanation following Definition III.2.38.

Definition IV.1.3. Let $(X,\mathcal{I})$ and $(Y,\mathcal{J})$ be analytic neighborhood models. A *representative of a morphism* $(X,\mathcal{I}) \to (Y,\mathcal{J})$ is a holomorphic map $f\colon U \to Y$, where $U \subset X$ is an open neighborhood of $Z(X,\mathcal{I})$, such that

$$f^{-1}\mathcal{J}\cdot\mathcal{O}_U \subset \mathcal{I}|_U,$$

where $f^{-1}\mathcal{J}\cdot\mathcal{O}_U$ denotes the ideal sheaf of $\mathcal{O}_U$ generated by the image of $f^{-1}\mathcal{J}$ under the adjoint structure homomorphism $f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_U$.

Two representatives $f_\nu\colon U_\nu \to Y$ $(\nu = 1,2)$ are called *equivalent* if there exists an open neighborhood $U' \subset U_1 \cap U_2$ of $Z(X,\mathcal{I})$ such that $f_1|_{U'} = f_2|_{U'}$. A *morphism of analytic neighborhood models* is an equivalence class of such representatives. □

Lemma IV.1.4. *Let $f\colon U \to Y$ represent a morphism $(X,\mathcal{I}) \to (Y,\mathcal{J})$ of analytic neighborhood models. Then*

$$f\big(Z(X,\mathcal{I})\big) \subset Z(Y,\mathcal{J}).$$

Consequently, f induces a holomorphic morphism $f_{\mathrm{center}}\colon Z(X,\mathcal{I}) \to Z(Y,\mathcal{J})$ between the reduced centers.

Proof. Let $x \in Z(X,\mathcal{I})$. If $f(x) \notin Z(Y,\mathcal{J})$, then $\mathcal{J}_{f(x)} = \mathcal{O}_{Y,f(x)}$. The condition $f^{-1}\mathcal{J}\cdot\mathcal{O}_U \subset \mathcal{I}|_U$ would then imply $\mathcal{I}_x = \mathcal{O}_{X,x}$, contrary to $x \in Z(X,\mathcal{I})$. Thus $f(x) \in Z(Y,\mathcal{J})$.

Moreover,

$$f^{-1}\mathcal{J}\cdot\mathcal{O}_U \subset \mathcal{I}|_U$$

implies

$$f^{-1}\sqrt{\mathcal{J}}\cdot\mathcal{O}_U \subset \sqrt{\mathcal{I}|_U}.$$

Hence the restriction of f to the underlying space of the reduced center induces a morphism of the corresponding reduced closed analytic subspaces. □

Definition IV.1.5. We denote by (ANM) the category whose objects are analytic neighborhood models and whose morphisms are those of Definition IV.1.3. The identity of $(X,\mathcal{I})$ is represented by $\mathrm{id}_U\colon U \to X$ on any open neighborhood U of its center. If $f\colon U \to Y$ and $g\colon V \to T$ represent successive morphisms, their composition is represented by

$$g \circ f\colon U \cap f^{-1}(V) \to T.$$

The domain is a neighborhood of the source center by Lemma IV.1.4. □

Proposition IV.1.6. *The data in Definition IV.1.5 define a well-defined category* (ANM).

Proof. All representatives of the identity determine the same germ, and the identity laws follow from the identity maps on neighborhoods.

Let

$$(X,\mathcal{I}) \xrightarrow{f} (Y,\mathcal{J}) \xrightarrow{g} (T,\mathcal{K})$$

be morphisms represented by $f\colon U \to Y$ and $g\colon V \to T$, where U is a neighborhood of $Z(X,\mathcal{I})$ and V is a neighborhood of $Z(Y,\mathcal{J})$. By Lemma IV.1.4, one has

$$f\big(Z(X,\mathcal{I})\big) \subset Z(Y,\mathcal{J}) \subset V.$$

Therefore

$$U' := U \cap f^{-1}(V)$$

is an open neighborhood of $Z(X,\mathcal{I})$, and $g \circ f\colon U' \to T$ is defined. Furthermore,

$$(g\circ f)^{-1}\mathcal{K}\cdot\mathcal{O}_{U'} = f^{-1}(g^{-1}\mathcal{K}\cdot\mathcal{O}_V)\cdot\mathcal{O}_{U'} \subset f^{-1}\mathcal{J}\cdot\mathcal{O}_{U'} \subset \mathcal{I}|_{U'}.$$

Thus $g \circ f$ represents a morphism in (ANM).

The equivalence class of this composition does not depend on the chosen representatives. Associativity follows from the associativity of the composition of holomorphic maps after restricting to a sufficiently small common neighborhood of the relevant reduced center. □

Lemma IV.1.7. *A morphism $(X,\mathcal{I}) \to (Y,\mathcal{J})$ of analytic neighborhood models is an isomorphism in* (ANM) *if and only if it can be represented by a biholomorphic map $f\colon U \to V$ between open neighborhoods*

$$Z(X,\mathcal{I}) \subset U \subset X, \qquad Z(Y,\mathcal{J}) \subset V \subset Y,$$

such that

$$f^{-1}\mathcal{J}\cdot\mathcal{O}_U = \mathcal{I}|_U.$$

Proof. Such a biholomorphic map clearly determines an isomorphism, since its inverse satisfies the corresponding ideal-sheaf condition.

Conversely, suppose that a morphism represented by $f\colon U \to Y$ has an inverse in (ANM), represented by $g\colon V \to X$. After shrinking U and V around the reduced centers, the equalities

$$g\circ f = \mathrm{id}_U \qquad \text{and} \qquad f\circ g = \mathrm{id}_V$$

hold as actual holomorphic maps. Thus f is a biholomorphism between suitable neighborhoods of the reduced centers. The ideal inclusions for f and g then give both inclusions between $f^{-1}\mathcal{J}\cdot\mathcal{O}_U$ and $\mathcal{I}|_U$, and hence give the required equality. □

Remark IV.1.8. Let $(X,\mathcal{I})$ be an analytic neighborhood model. For every open neighborhood U of $Z(X,\mathcal{I})$, the inclusion $(U,\mathcal{I}|_U) \to (X,\mathcal{I})$ is an isomorphism in (ANM). Its inverse is represented by the identity on U. □

Definition IV.1.9. An *analytic neighborhood space* is an isomorphism class $\mathcal{X} = [(X, \mathcal{I})]$ of objects of (ANM). A representative $(X, \mathcal{I})$ is called a *model* of $\mathcal{X}$. □

We shall regard $\mathcal{X}$ as the geometric object under study. A model supplies an ambient space in which to represent functions, sheaves, and maps; it may be shrunk without changing $\mathcal{X}$. This keeps the neighborhood of the whole center, rather than only its pointwise or formal data.

Proposition IV.1.10. *Let $\mathcal{X}$ be an analytic neighborhood space and choose a model $(X, \mathcal{I})$. The isomorphism class of $Z(X, \mathcal{I})$ depends only on $\mathcal{X}$.*

Proof. An isomorphism of models and its inverse induce inverse maps of their reduced centers, by Lemma IV.1.4. □

Definition IV.1.11. Let $\mathcal{X}$ be an analytic neighborhood space. Its *reduced center* $Z(\mathcal{X})$ is the isomorphism class of $Z(X, \mathcal{I})$ for a model of $\mathcal{X}$. We also use $Z(\mathcal{X})$ for a chosen representative. □

Remark IV.1.12. Morphisms, sheaves, and automorphisms of $\mathcal{X}$ are interpreted after choosing a model in (ANM). A specified isomorphism of models transports these data; independence of the model is understood up to this isomorphism or equivalence. □

Example IV.1.13. Let A be an analytic subset of a complex space X, let $x \in A$, and give A its canonical reduced structure. The ideal sheaf $\mathcal{I}_{\{x\}} \subset \mathcal{O}_A$ of the point x is coherent by Oka's second coherence theorem (Theorem III.2.163). Thus $\mathcal{X} := [(A, \mathcal{I}_{\{x\}})]$ is an analytic neighborhood space with reduced center $\{x\}$. Replacing A by an open neighborhood of x gives an isomorphic model, so $\mathcal{X}$ depends only on the germ $\underline{A}_x$ introduced in Chapter III, §2.4. We shall also use $\underline{A}_x$ for this analytic neighborhood space.

For another reduced complex space B with $y \in B$, morphisms $(A, \mathcal{I}_{\{x\}}) \to (B, \mathcal{I}_{\{y\}})$ are precisely germs of holomorphic maps f near x with $f(x) = y$: the ideal condition says exactly that functions vanishing at y pull back to functions vanishing at x. By Lemma IV.1.7, these models are isomorphic exactly when such a map is biholomorphic after shrinking around the distinguished points. Thus the isomorphism class records the pointed reduced analytic germ up to biholomorphism, not its chosen embedding in X. □

The reduced center and a comparison question

The reduced center can itself be regarded as an analytic neighborhood model. This also expresses the meaning of prescribing a map of centers. Write (RedCpxSp) for the full subcategory of (CpxSp) consisting of reduced complex spaces.

Proposition IV.1.14. *The functor $\mathscr{Z}\colon (\mathsf{ANM}) \xrightarrow{\text{functor}} (\mathsf{RedCpxSp})$ which takes a model to its reduced center has a fully faithful left adjoint $\mathscr{E}$, given by $\mathscr{E}(A) = (A, 0)$. More explicitly, for a reduced complex space A and a model $(X, \mathcal{I})$ there is a natural bijection*

$$\mathrm{Hom}_{(\mathsf{ANM})}\big((A,0),(X,\mathcal{I})\big) \cong \mathrm{Hom}_{(\mathsf{RedCpxSp})}\big(A, Z(X,\mathcal{I})\big).$$

PROOF. Functoriality of $\mathscr{Z}$ follows from Lemma IV.1.4. The center of $(A,0)$ is all of A, so a morphism from this model is represented by a holomorphic map $f\colon A \to X$ with $f^{-1}\mathcal{I}\cdot\mathcal{O}_A = 0$. If a local section g belongs to $\sqrt{\mathcal{I}}$, a positive power of f^*g is zero. Reducedness of A gives $f^*g = 0$. Thus f factors uniquely through the reduced closed subspace $Z(X,\mathcal{I})$. Conversely, such a factorization annihilates $\mathcal{I}$. The constructions commute with composition in both variables. Taking $\mathcal{I} = 0$ also proves full faithfulness (Use Definition II.3.16). □

For example, a model with a specified morphism from $(A,0)$ has a specified map $A \to Z(X,\mathcal{I})$. Requiring this map to be an isomorphism gives neighborhoods with a fixed identification of their reduced center with A. The ideal remains part of the data. If D is an effective divisor in a complex manifold X with reduced support Z, the inclusions of ideals give morphisms

$$(Z,0) \longrightarrow (X,\mathcal{I}_Z) \longrightarrow (X,\mathcal{O}_X(-D)) \longrightarrow (X,0).$$

The specified ideal is retained on each model.

We shall also compare collections of morphisms. Recall from Example II.3.8 that a diagram indexed by a small category $\mathfrak{I}$ is a functor. Fix models U_i for $i \in \mathrm{ob}(\mathfrak{I})$ and two diagrams $F, F_{\mathrm{std}}\colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{ANM})$ with $F(i) = F_{\mathrm{std}}(i) = U_i$. These equalities mean that local identifications of the objects have already been chosen. For each i, specify a subgroup $G_i \subset \mathrm{Aut}_{(\mathsf{ANM})}(U_i)$ of changes allowed in the comparison.

DEFINITION IV.1.15. For the diagrams and groups just fixed, an *admissible comparison* from F_{std} to F is a family $H_i \in G_i$ such that, for every morphism $a\colon i \to j$ in $\mathfrak{I}$,

$$F(a)\circ H_i = H_j \circ F_{\mathrm{std}}(a). \tag{267}$$

Thus it is a natural isomorphism $H\colon F_{\mathrm{std}} \xrightarrow{\text{nat}} F$ with the prescribed admissibility of its components. □

The question is whether such a comparison exists, and how its existence depends on the permitted changes and the selected normal form. A condition on the center, a specified normal frame, or an additional map or foliation may restrict the changes. All equations here are equations of morphisms in (ANM), hence of germs along the whole source center.

EXAMPLE IV.1.16. Three shapes of diagram illustrate the compatibility required in (267).

(*i*) Let $(X,\mathcal{I})$ be an analytic neighborhood model, let $f \in \mathrm{Aut}_{(\mathsf{ANM})}((X,\mathcal{I}))$, and let $\mathfrak{I}$ be the category generated by one arrow $a\colon 0 \to 1$. Define $F, F_{\mathrm{std}}\colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{ANM})$ by sending both objects to $(X,\mathcal{I})$ and putting $F(a) = f$ and $F_{\mathrm{std}}(a) = \mathrm{id}_{(X,\mathcal{I})}$. With unrestricted changes at the two endpoints, $H_0 = \mathrm{id}_{(X,\mathcal{I})}$ and $H_1 = f$ give $F(a)\circ H_0 = f = H_1 \circ F_{\mathrm{std}}(a)$. Thus the invertible arrow is equivalent to the identity when its source and target are indexed separately.

(*ii*) Let G be a group, let BG be its one-object category, and fix an analytic neighborhood model $(X,\mathcal{I})$ with actions $\alpha, \alpha_{\mathrm{std}}\colon G \to \mathrm{Aut}_{(\mathsf{ANM})}((X,\mathcal{I}))$. The corresponding functors

send the unique object to $(X,\mathcal{I})$ and $g \in G$ to $F(g) = \alpha(g)$ and $F_{\mathrm{std}}(g) = \alpha_{\mathrm{std}}(g)$, respectively. A comparison has only one component H, and naturality requires $\alpha(g) \circ H = H \circ \alpha_{\mathrm{std}}(g)$ for every $g \in G$. For $G = \mathbb{Z}$, put $f := \alpha(1)$ and $f_{\mathrm{std}} := \alpha_{\mathrm{std}}(1)$. The equation $f \circ H = H \circ f_{\mathrm{std}}$ then implies the corresponding equation for every positive and negative iterate. All arrows use the same H, not independent left–right changes.

(*iii*) Let $(X,\mathcal{I})$ be an analytic neighborhood model, put $Z := Z(X,\mathcal{I})$, and fix an open covering $\{U_j\}_{j\in J}$ of Z. Choose open subsets $V_j \subset X$ with $V_j \cap Z = U_j$ and biholomorphic charts $\phi_j \colon V_j \to W_j$ onto reduced complex spaces. Transport $\mathcal{I}|_{V_j}$ by ϕ_j to the ideal $\mathcal{J}_j$ on W_j, and put $Q_j := (W_j, \mathcal{J}_j)$. Write $U_{jk} := U_j \cap U_k$, $U_{jk\ell} := U_j \cap U_k \cap U_\ell$, and $V_{jk} := V_j \cap V_k$, with the analogous notation for further intersections. The transition

$$F_{jk} := \phi_j \circ \phi_k^{-1} \colon \phi_k(V_{jk}) \longrightarrow \phi_j(V_{jk})$$

is considered as a germ along $\phi_k(U_{jk})$. It is a morphism between the restricted chart models, not in general a morphism $Q_k \to Q_j$ along their whole centers.

For each j, choose one automorphism H_j of Q_j fixing its reduced center pointwise. After taking representatives and shrinking the charts where needed, set $\phi'_j := H_j^{-1} \circ \phi_j$. The new transitions satisfy

$$F'_{jk} = H_j^{-1} \circ F_{jk} \circ H_k, \qquad F'_{jk} \circ F'_{k\ell} = F'_{j\ell}$$

on the corresponding overlaps and triple overlaps. The same H_j is used on every overlap meeting U_j. For prescribed standard transitions $F_{\mathrm{std},jk}$ on these restricted models, requiring $F'_{jk} = F_{\mathrm{std},jk}$ is therefore the equation $F_{jk} \circ H_k = H_j \circ F_{\mathrm{std},jk}$.

To express this as a diagram, include the chart models restricted to every non-empty finite intersection as objects, and include both their inclusion morphisms and the chart transitions as arrows. The diagrams F and F_{std} have these same objects and inclusion morphisms, and use the given and the prescribed standard transitions, respectively. Naturality for the inclusions says that the components on intersections are restrictions of H_j; naturality for the transitions is exactly the preceding equation. □

Remark IV.1.17. Let $\mathfrak{I}$ be finitely presented in the sense of Definition II.3.24, and let F, F_{std} and $H_i \in G_i$ be as in Definition IV.1.15. It suffices to check (267) on the chosen generating morphisms. Indeed, if it holds for $a \colon i \to j$ and $b \colon j \to k$, then

$$F(b \circ a) \circ H_i = F(b) \circ H_j \circ F_{\mathrm{std}}(a) = H_k \circ F_{\mathrm{std}}(b \circ a).$$

Induction gives the identity for every path. For this check, finite generation suffices, since the two functors are already given. When constructing a functor from the images of the vertices and generating edges, one must also check the stated path relations.

The arrow category and $B\mathbb{Z}$ were described in Example II.3.25. For a finite open covering of a center, there are only finitely many labelled non-empty intersections of distinct members; their inclusions, the chart comparisons, and the restriction and triple-overlap identities give a

finitely presented indexing category for the covering diagram. Compactness permits a finite covering to be chosen for the local comparisons. Also, if Z is a connected compact analytic subset of a complex manifold, Theorem III.2.55 gives a finite triangulation. Fix a basepoint $* \in Z$. A maximal tree in the 1-skeleton leaves finitely many generators of $\pi_1(Z, *)$, and its 2-simplices give finitely many relations (See [Ha1, Propositions 1.26 and 1A.2]). Thus the one-object indexing category for this monodromy is also finitely presented.

Finitely many generating germs and the paths in the stated relations can be represented on suitable neighborhoods, and the finitely many equalities hold after shrinking around their source centers. The representatives may have different domains. □

REMARK IV.1.18. There are two choices in this question: which coordinate changes are allowed, and which part of a transition must be prescribed. A comparison of neighborhoods matches all components with those of a fixed diagram F_{std}. In the Ueda problem below, only the transverse defining functions are required to have specified linear transitions. The tangential components of the new transitions are not specified in advance and may still depend on transverse variables. Viewed as a problem about full transitions, the target is a class of diagrams with the required transverse components, rather than one fixed full diagram. Alternatively, use the defining maps as the components of a comparison with one fixed linear transverse diagram. That comparison need not be invertible; the source and standard objects can differ, as explained in § 4.1. These two coordinate problems are called *full* and *vertical linearization*, respectively, in [GS1, §§2.3–2.5]. □

1.2 Sheaves and cohomology on an analytic neighborhood space

DEFINITION IV.1.19. Let $\mathcal{X}$ be an analytic neighborhood space, choose a model $(X, \mathcal{I})$, and put $Z := Z(X, \mathcal{I})$. Recall that $\mathcal{N}_X(Z)$ is the set of open neighborhoods of Z, ordered by reverse inclusion (Example II.3.20). A *sheaf of abelian groups on* $\mathcal{X}$ is represented by a sheaf of abelian groups $\mathcal{F}_V$ on some $V \in \mathcal{N}_X(Z)$, with restriction to a smaller such neighborhood giving a canonically isomorphic germ. For two representatives on V and V', morphisms are represented by morphisms of sheaves of abelian groups on a common smaller neighborhood of the whole center. Their set is

$$\operatorname{Hom}_{\mathcal{X}}(\mathcal{F}, \mathcal{G}) := \varinjlim_{W \in \mathcal{N}_{V \cap V'}(Z)} \operatorname{Hom}_W(\mathcal{F}_V|_W, \mathcal{G}_{V'}|_W).$$

Thus two morphisms agree when they agree on one smaller neighborhood of the whole center. Two sheaves represent isomorphic germs when they are isomorphic on such a neighborhood.

The germ represented by $\mathcal{O}_X$ is denoted by $\mathcal{O}_{\mathcal{X}}$. An *analytic sheaf* on $\mathcal{X}$ is represented by an $\mathcal{O}_V$-module, with $\mathcal{O}_W$-linear morphisms in the formula above. It is *coherent*, or *finite locally free*, if it has a representative with that property. We write $(\mathsf{Ab}(\mathcal{X}))$ for the category of abelian sheaves, $(\mathsf{Mod}(\mathcal{O}_{\mathcal{X}}))$ for analytic sheaves, and $\operatorname{Coh}(\mathcal{X})$ for its full subcategory of coherent analytic sheaves. □

For a representative $\mathcal{F}_V$ on V, let $j_Z \colon Z \hookrightarrow V$ be the inclusion; then $(j_Z^{-1}\mathcal{F}_V)_z \cong (\mathcal{F}_V)_z$ for $z \in Z$, so the inverse image retains the ambient germs. Finite generators and their relations

for a coherent representative are available on the same ambient neighborhoods along Z; in particular, $\mathcal{O}_{\mathcal{X}}$ is coherent because $\mathcal{O}_V$ is coherent (Proposition III.2.36).

Composition is defined after restricting to a common neighborhood. Finitely many representatives and identities can always be handled there. In particular, an isomorphism of germs has representatives which are inverse on a sufficiently small common neighborhood. Changing the model transports these categories by the biholomorphism of neighborhoods in Lemma IV.1.7.

Definition IV.1.20. Let $\mathcal{X}$ be an analytic neighborhood space and let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{X}$. Choose a model $(X,\mathcal{I})$, put $Z := Z(X,\mathcal{I})$, and represent $\mathcal{F}$ by $\mathcal{F}_V$ on $V \in \mathcal{N}_X(Z)$. For $q \in \mathbb{Z}_{\geq 0}$, set

$$H^q(\mathcal{X},\mathcal{F}) := \varinjlim_{W \in \mathcal{N}_V(Z)} H^q(W, \mathcal{F}_V|_W), \qquad \Gamma(\mathcal{X},\mathcal{F}) := H^0(\mathcal{X},\mathcal{F}). \tag{268}$$

Restriction to a smaller representative is cofinal, so it does not change these groups. The same definition applies to analytic sheaves, using their usual sheaf cohomology. □

Proposition IV.1.21. *Let $\mathcal{X}$ be an analytic neighborhood space. The categories $(\mathsf{Ab}(\mathcal{X}))$ and $(\mathsf{Mod}(\mathcal{O}_{\mathcal{X}}))$ are abelian and have enough injectives. In either category, the functors in (268) are the right derived functors of $\Gamma(\mathcal{X},-)$. Moreover, $\mathrm{Coh}(\mathcal{X})$ is abelian; kernels, images, cokernels, extensions, tensor products, and sheaf Hom of coherent analytic sheaves remain coherent.*

Proof. Restriction to an open subset is exact and preserves the stalks at its points. Kernels and cokernels can therefore be computed on representatives and restricted to a common neighborhood, giving the abelian assertions by Proposition II.3.58. A zero germ is represented by zero on a sufficiently small neighborhood, so an exact finite sequence can be represented on one common neighborhood. Applying the coherent-sheaf results of Chapter III there gives the stated coherence properties.

To obtain enough injectives, embed a representative $\mathcal{F}_V$ into an injective sheaf $\mathcal{J}$ on V. Restriction to an open subset preserves injectives because it has the exact left adjoint given by extension by zero (The stalks of this extension are the original stalks on the open subset and zero outside; see [St, Tag 009Z]). Now represent a monomorphism and a morphism from its source to the germ of $\mathcal{J}$ on one common neighborhood. After shrinking, the former is a monomorphism there, and injectivity of the restriction of $\mathcal{J}$ extends the latter across it. Thus the germ of $\mathcal{J}$ is injective, as required.

An injective resolution on V consequently gives one in the germ category, and its restrictions remain injective resolutions on every smaller open neighborhood. Each section functor is left exact, and filtered colimits of groups are exact; hence $\Gamma(\mathcal{X},-)$ is left exact. Taking cohomology of the complex of germ sections gives

$$R^q\Gamma(\mathcal{X},\mathcal{F}) \cong \varinjlim_{W \in \mathcal{N}_V(Z)} H^q\big(\Gamma(W, \mathcal{J}^\bullet|_W)\big) = H^q(\mathcal{X},\mathcal{F}),$$

by Corollary II.3.44. □

Let $\mathcal{X}$ be an analytic neighborhood space, choose a model $(X,\mathcal{I})$, and put $Z := Z(X,\mathcal{I})$. For a representative $\mathcal{F}_V$ of a sheaf on $\mathcal{X}$, let $j_Z\colon Z \hookrightarrow V$ be the inclusion. The sheaf $j_Z^{-1}\mathcal{F}_V$ retains the ambient stalks at points of Z. For a compact center its cohomology gives the following description of the neighborhood cohomology.

PROPOSITION IV.1.22. *Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center and let $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$. Choose a model $(X,\mathcal{I})$ on which $\mathcal{F}$ is represented, put $Z := Z(X,\mathcal{I})$, and let $j_Z\colon Z \hookrightarrow X$ be the inclusion. Then $\mathcal{F}$ satisfies the cofinal finite Leray refinement condition along Z, and restriction induces isomorphisms*

$$\Theta^q_{Z,\mathcal{F}}\colon H^q(\mathcal{X},\mathcal{F}) \longrightarrow H^q(Z, j_Z^{-1}\mathcal{F}), \qquad q \geq 0.$$

PROOF. Given $V \in \mathcal{N}_X(Z)$ and a finite open covering $\{A_\ell\}_{\ell\in L}$ of V, choose a Stein neighborhood $U_z \subset A_{\ell(z)} \cap V$ of each $z \in Z$. Compactness gives finitely many $U_{z_1},\dots,U_{z_m}$ covering Z. They form a finite covering of the neighborhood $W := \bigcup_{j=1}^m U_{z_j}$ subordinate to the given covering. Their nonempty finite intersections are Stein, by the discussion preceding (218). Cartan's Theorem B (Theorem III.2.165) therefore makes this a Leray covering for $\mathcal{F}|_W$. This proves the required cofinal refinement condition. The isomorphism now follows from Theorem II.3.112 and Definition IV.1.20. □

REMARK IV.1.23. In particular, $H^q(\mathcal{X},\mathcal{O}_\mathcal{X}) \cong H^q(Z, j_Z^{-1}\mathcal{O}_X)$, by taking $\mathcal{F} = \mathcal{O}_X$. A class is represented on an actual neighborhood of Z, and two representatives give the same class precisely when they agree after restriction to a smaller common neighborhood. For arbitrary abelian coefficients the same comparison follows from Remark II.3.113; the proposition gives the coherent case by finite Leray coverings. □

NOTATION IV.1.24. Let $\mathcal{X}$ be an analytic neighborhood space, choose a model $(X,\mathcal{I})$, and put $Z := Z(X,\mathcal{I})$. We write $\mathcal{X}^*$ as shorthand for the punctured neighborhoods $W \setminus Z$ considered as the open neighborhood W of Z shrinks (Here the difference uses the underlying set of Z). Let $\mathcal{F} \in (\mathsf{Ab}(\mathcal{X}))$ be represented by $\mathcal{F}_V$ on $V \in \mathcal{N}_X(Z)$. For $q \in \mathbb{Z}_{\geq 0}$, we define

$$\begin{aligned} H^q(\mathcal{X}^*,\mathcal{F}) &:= \varinjlim_{W\in\mathcal{N}_V(Z)} H^q(W\setminus Z, \mathcal{F}_V|_{W\setminus Z}), \\ \Gamma(\mathcal{X}^*,\mathcal{F}) &:= H^0(\mathcal{X}^*,\mathcal{F}). \end{aligned} \tag{269}$$

The transition maps are induced by restriction to smaller neighborhoods. In the second argument we suppress restriction to each punctured neighborhood. For analytic sheaves we use their underlying sheaves of abelian groups. Shrinking a representative gives a cofinal system, so it does not change these groups; a specified isomorphism of models identifies the corresponding complements and groups. The usual restrictions induce

$$r^q_{\mathcal{X},\mathcal{F}}\colon H^q(\mathcal{X},\mathcal{F}) \to H^q(\mathcal{X}^*,\mathcal{F}),$$

which for $q = 0$ is restriction of sections. □

Remark IV.1.25. A morphism of models need not map the complement of its center into the complement of the target center. For example, $f(w) = 0$ defines an endomorphism of $(\mathbb{C}, \mathcal{I}_{\{0\}})$, where $\mathcal{I}_{\{0\}}$ is the defining ideal sheaf of the origin, but sends every point of a punctured neighborhood to the center. □

Example IV.1.26. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and take the unitary-flat line bundle $F_\theta \to C$ over the elliptic curve C constructed in Chapter III, §2.15. Let X be its total space and Z the image of its zero section (the subset denoted by Y_θ there), and put $\mathcal{X}_\theta := [(X, \mathcal{I}_Z)]$. Set $\lambda = e^{2\pi\sqrt{-1}\theta}$ and assume the exponential lower bound (233): $|1 - \lambda^n| \geq C_0 e^{-bn}$ for all $n \in \mathbb{Z}_{>0}$, with some $C_0, b > 0$. Then restriction gives isomorphisms of vector spaces

$$H^q(\mathcal{X}_\theta, \mathcal{O}_{\mathcal{X}_\theta}) \to H^q(\mathcal{X}_\theta^*, \mathcal{O}_{\mathcal{X}_\theta}) \qquad (q \in \mathbb{Z}_{\geq 0}).$$

Both sides are $\mathbb{C}$ for $q = 0, 1$ and zero for $q \geq 2$; thus the kernel and cokernel are zero in every degree. For $q = 1$ this is Proposition III.2.200. For $q = 0$ and $q \geq 2$ it follows from Lemma III.2.199 and its compatibility with restriction: the difference operator has kernel consisting of constants, and the higher groups vanish on every disc or punctured-disc subbundle. These subbundles are cofinal because Z is compact. □

1.3 The natural topology on cohomology with coherent analytic coefficients

We now equip the cohomology of an analytic neighborhood space with compact center and coherent analytic coefficients with a natural topology, using the inductive limit in Definition IV.1.20. We first collect the functional-analytic facts needed to compare this topology with the quotient topology obtained from a complex.

Inductive limits of Fréchet complexes

For each $\lambda \in \mathbb{Z}_{\geq 0}$, let $C_\lambda^\bullet = (C_\lambda^q, \delta_\lambda^q)_{q \geq 0}$ be a cochain complex of Fréchet spaces and continuous linear maps. Thus $\delta_\lambda^q \colon C_\lambda^q \to C_\lambda^{q+1}$ and $\delta_\lambda^{q+1} \circ \delta_\lambda^q = 0$. Put $C_\lambda^{-1} = 0$ and $\delta_\lambda^{-1} = 0$. Suppose that continuous linear maps $r_\lambda^q \colon C_\lambda^q \to C_{\lambda+1}^q$ are given and satisfy

$$\delta_{\lambda+1}^q \circ r_\lambda^q = r_\lambda^{q+1} \circ \delta_\lambda^q \qquad (\lambda, q \in \mathbb{Z}_{\geq 0}).$$

They define an inductive sequence of complexes. For $\lambda \leq \mu$, write $r_{\lambda,\mu}^q$ for the corresponding composite, taking $r_{\lambda,\lambda}^q = \mathrm{id}_{C_\lambda^q}$. The maps δ_λ^q change the cochain degree, whereas the maps r_λ^q change the stage of the inductive sequence.

For every $\lambda, q \in \mathbb{Z}_{\geq 0}$, put

$$Z_\lambda^q := \operatorname{Ker} \delta_\lambda^q, \qquad B_\lambda^q := \operatorname{Image} \delta_\lambda^{q-1}, \qquad H_\lambda^q := Z_\lambda^q / B_\lambda^q.$$

Give Z_λ^q and B_λ^q the subspace topologies induced from C_λ^q, and give H_λ^q the quotient topology. Then Z_λ^q is Fréchet, by Mini-Exercise I.6.42 (*iii*). The quotient H_λ^q is an LCTVS. The

maps r^q_λ preserve cocycles and coboundaries and induce continuous linear maps between the corresponding quotient spaces.

We use the *locally convex final topology* recalled in Chapter I, § 6.4 for all inductive limits below. On the algebraic inductive limit of LCTVSs, this topology is defined by all seminorms whose pullbacks to every stage are continuous. A linear map from this limit to an LCTVS is continuous if and only if its composite with every canonical map from a stage is continuous.

For the inductive sequence of complexes just fixed, denote the canonical maps by

$$u^q_\lambda \colon C^q_\lambda \longrightarrow C^q_\infty := \varinjlim_\lambda C^q_\lambda, \qquad v^q_\lambda \colon Z^q_\lambda \longrightarrow Z^q_\infty := \varinjlim_\lambda Z^q_\lambda,$$
$$h^q_\lambda \colon H^q_\lambda \longrightarrow H^q_\infty := \varinjlim_\lambda H^q_\lambda.$$

The induced differential is $\delta^q_\infty \colon C^q_\infty \to C^{q+1}_\infty$. There is also a continuous map $\beta^q_\infty \colon C^{q-1}_\infty \to Z^q_\infty$, characterized by $\beta^q_\infty u^{q-1}_\lambda = v^q_\lambda \delta^{q-1}_\lambda$. For $q = 0$ its source is zero. Put $B^q_\infty := \operatorname{Image} \beta^q_\infty$.

Proposition IV.1.27. *Let $C^\bullet_\lambda$ and $r^\bullet_\lambda$ be an inductive sequence of Fréchet complexes and continuous cochain maps as above, and fix $q \in \mathbb{Z}_{\geq 0}$. With the locally convex final topologies just specified, the map*

$$Z^q_\infty / B^q_\infty \longrightarrow H^q_\infty, \qquad v^q_\lambda(z) + B^q_\infty \longmapsto h^q_\lambda(z + B^q_\lambda) \tag{270}$$

is a TVS isomorphism.

Proof. Filtered inductive limits of vector spaces are exact (Proposition II.3.43). Applying this fact to the kernels, images, and quotients at each stage shows that (270) is a well-defined linear isomorphism. For example, if the class of a cocycle $z \in Z^q_\lambda$ is zero in H^q_∞, then for some $\mu \geq \lambda$ there is $c \in C^{q-1}_\mu$ with $r^q_{\lambda,\mu} z = \delta^{q-1}_\mu c$. Hence $v^q_\lambda(z) \in B^q_\infty$.

We check the topologies as well. The compatible continuous maps $Z^q_\lambda \to H^q_\lambda \to H^q_\infty$ give a continuous map $Z^q_\infty \to H^q_\infty$. It vanishes on B^q_∞, so it factors continuously through the quotient on the left of (270). Conversely, the continuous map $Z^q_\lambda \to Z^q_\infty / B^q_\infty$ vanishes on B^q_λ. The quotient topology therefore gives a continuous map $H^q_\lambda \to Z^q_\infty / B^q_\infty$. These maps are compatible with the transition maps, and the locally convex final topology gives a continuous map from H^q_∞. It is the inverse of (270). □

For this sequence of complexes, exactness of filtered inductive limits also identifies Z^q_∞ algebraically with $\operatorname{Ker} \delta^q_\infty$, by the map induced by $Z^q_\lambda \subset C^q_\lambda$. The resulting algebraic cohomology comparison is Corollary II.3.44. The map on cocycles is continuous, but its inverse need not yet be continuous when the kernel has the subspace topology induced from C^q_∞. Thus Proposition IV.1.27 concerns the topology on the inductive limit of the cocycle spaces. The comparison with the subspace topology on a kernel will be made below under additional assumptions.

Proposition IV.1.28. *Let $C^\bullet_\lambda$ and $r^\bullet_\lambda$ be an inductive sequence of Fréchet complexes and continuous cochain maps as in Proposition IV.1.27, and fix $q \geq 0$. Then H^q_∞, with its locally convex final topology, is the cokernel of a continuous linear map between strict countable LF-spaces, where the cokernel is given its quotient topology.*

PROOF. For this fixed degree, set

$$S^q := \bigoplus_{\lambda\geq 0} Z^q_\lambda, \qquad T^q := \bigoplus_{\lambda\geq 0} (Z^q_\lambda \oplus C^{q-1}_\lambda).$$

Give both direct sums the locally convex final topology of their finite partial sums. Each finite partial sum is Fréchet, and its inclusion in the next one is a TVS isomorphism onto a closed direct summand. Consequently S^q and T^q are strict countable LF-spaces in the sense of Chapter I, § 6.4.

Let $e_\lambda\colon Z^q_\lambda \to S^q$ be the inclusion of the λ-th summand. Define $L^q\colon T^q \to S^q$ by prescribing its restriction to that summand of T^q:

$$(z, c) \longmapsto e_\lambda(z) - e_{\lambda+1}(r^q_\lambda z) + e_\lambda(\delta^{q-1}_\lambda c). \tag{271}$$

Its restrictions to finite partial sums are continuous, so L^q is continuous. The map

$$Q^q\colon S^q \longrightarrow H^q_\infty, \qquad e_\lambda(z) \longmapsto h^q_\lambda(z + B^q_\lambda)$$

is also continuous and is surjective. Its kernel is Image L^q: the first two terms of (271) impose the relations of the inductive limit, and the last term kills the coboundaries. More explicitly, a finite sum in S^q can be transported, modulo the first relations, to a single later stage. If its class is zero, then at a further stage it is a coboundary, which is a relation of the last kind.

The quotient topology on $S^q/\operatorname{Image} L^q$ is the same as the locally convex final topology on H^q_∞: a linear map from this quotient to an LCTVS is continuous exactly when its composites with Z^q_λ are continuous, and these factor continuously through H^q_λ precisely because they vanish on B^q_λ. Hence $S^q/\operatorname{Image} L^q \cong H^q_\infty$ as TVSs. □

COROLLARY IV.1.29. *Let $C^\bullet_\lambda$ be an inductive sequence of Fréchet complexes with continuous differentials and continuous cochain transition maps $r^\bullet_\lambda$, as in Proposition IV.1.28. Fix $q \in \mathbb{Z}_{\geq 0}$ and give $H^q_\infty = \varinjlim_\lambda H^q(C^\bullet_\lambda)$ its locally convex final topology. If $d := \dim_{\mathbb{C}} H^q_\infty < \infty$, then H^q_∞ is Hausdorff and is TVS isomorphic to $\mathbb{C}^d$ with its usual topology. Consequently, if H^q_∞ is non-Hausdorff, it is infinite-dimensional.*

PROOF. Use the continuous linear map $L^q\colon T^q \to S^q$ constructed in the proof of Proposition IV.1.28. Both spaces are strict countable LF-spaces, and its cokernel, with the quotient topology, is H^q_∞. Lemma I.6.55 therefore makes Image L^q closed in S^q and H^q_∞ Hausdorff. The identification with $\mathbb{C}^d$ is then Theorem I.6.4 (*i*). □

The same cokernel presentations also allow us to use the open mapping theorem before passing to a quotient. We record the precise form needed for such comparisons.

PROPOSITION IV.1.30. *For $i = 1, 2$, let E_i and F_i be strict countable LF-spaces, let $L_i\colon E_i \to F_i$ be a continuous linear map, and give $Q_i := F_i/\operatorname{Image} L_i$ its quotient topology. Suppose that a continuous linear map $A\colon F_1 \to F_2$ satisfies $A(\operatorname{Image} L_1) \subset \operatorname{Image} L_2$. If the induced map $\overline{A}\colon Q_1 \to Q_2$ is surjective, then it is continuous and open. In particular, if it is an isomorphism of vector spaces, it is a TVS isomorphism.*

PROOF. Let $p_i\colon F_i \to Q_i$ be the quotient maps. The relation $p_2 \circ A = \overline{A} \circ p_1$ proves continuity of $\overline{A}$. Surjectivity of $\overline{A}$ says that $F_2 = A(F_1) + \operatorname{Image} L_2$. Hence the continuous linear map

$$\Theta\colon F_1 \oplus E_2 \longrightarrow F_2, \qquad \Theta(x,y) := A(x) + L_2(y)$$

is surjective. A finite direct sum of strict countable LF-spaces is again such a space: take the direct sums of the corresponding Fréchet stages. Thus the open mapping theorem (Theorem I.6.46) makes Θ open. For an open subset $W \subset Q_1$, we have

$$p_2^{-1}(\overline{A}(W)) = \Theta\big(p_1^{-1}(W) \times E_2\big).$$

The right-hand side is open in F_2, and the quotient topology makes $\overline{A}(W)$ open in Q_2. □

We next use compactness of the transition maps for a different purpose. The following form of Komatsu's results is enough for the comparison of kernels that occurs here.

THEOREM IV.1.31 (Komatsu). *Let $(E_\lambda, a_\lambda)_{\lambda\ge 0}$ be an inductive sequence of Fréchet spaces whose transition maps $a_\lambda\colon E_\lambda \to E_{\lambda+1}$ are injective and compact. Put $E = \varinjlim_\lambda E_\lambda$ with its locally convex final topology, and denote the canonical maps by $a_{\lambda,\infty}\colon E_\lambda \to E$. Then E is Hausdorff and complete, and this topology is also the final topology of the underlying topological spaces. Thus a subset $O \subset E$ is open if and only if $a_{\lambda,\infty}^{-1}(O)$ is open for every λ.*

If $Y \subset E$ is a closed vector subspace and $Y_\lambda := a_{\lambda,\infty}^{-1}(Y)$ has the subspace topology in E_λ, then the canonical map $\varinjlim_\lambda Y_\lambda \to Y$ is a TVS isomorphism, where Y has the subspace topology in E.

PROOF. See [Kom, Lemma 3 and Theorems 6′ and 7′, pp. 372 and 375–376]. The word "injective" in the definition of a sequence in that paper includes injectivity of its transition maps (p. 368). A compact linear map sends some neighborhood of zero to a relatively compact set, as in [Kom, p. 367]. □

COROLLARY IV.1.32. *Let $C_\lambda^\bullet$ be an inductive sequence of Fréchet complexes with continuous differentials and continuous cochain transition maps $r_\lambda^\bullet$. Assume, in addition, that every r_λ^p is injective and compact. For $q \ge 0$, use the notation C_∞^p, δ_∞^p, Z_∞^q, and H_∞^q introduced above. Then*

$$\varinjlim_\lambda Z_\lambda^q \longrightarrow \operatorname{Ker}(\delta_\infty^q\colon C_\infty^q \to C_\infty^{q+1}) \tag{272}$$

is a TVS isomorphism for the subspace topology on the right. Consequently,

$$H_\infty^q \cong \frac{\operatorname{Ker}\delta_\infty^q}{\operatorname{Image}\delta_\infty^{q-1}} \tag{273}$$

is a TVS isomorphism for the kernel-quotient topology on the right. The topology on H_∞^q is also the final topology of the underlying spaces H_λ^q.

Proof. Fix q. Theorem IV.1.31 makes both C^q_∞ and C^{q+1}_∞ Hausdorff. The map δ^q_∞ is continuous by the final-topology universal property, so its kernel is closed. All canonical maps from the stages into these algebraic inductive limits are injective. Therefore

$$(u^q_\lambda)^{-1}(\operatorname{Ker}\delta^q_\infty) = \{c \in C^q_\lambda \mid u^{q+1}_\lambda(\delta^q_\lambda c) = 0\} = Z^q_\lambda.$$

Apply the closed-subspace assertion of Theorem IV.1.31 to obtain (272). Proposition IV.1.27 then gives (273).

The restrictions $r^q_\lambda|_{Z^q_\lambda}$ are also injective and compact. Indeed, the cocycle spaces are closed, and the image of a relatively compact set is still relatively compact when viewed in the closed target subspace. Thus Z^q_∞ has the final topology of the spaces Z^q_λ. For a subset $O \subset H^q_\infty$, its inverse image in Z^q_∞ is open exactly when all its inverse images in Z^q_λ are open. Since $Z^q_\lambda \to H^q_\lambda$ is a quotient map, this is equivalent to $(h^q_\lambda)^{-1}(O)$ being open for every λ. □

Under the assumptions of Corollary IV.1.32, H^q_∞ is Hausdorff precisely when $\operatorname{Image}\delta^{q-1}_\infty$ is closed in $\operatorname{Ker}\delta^q_\infty$.

Admissible pairs of stages

We return to the inductive sequence of Fréchet complexes $C^\bullet_\lambda$ of Proposition IV.1.27. Fix $q \in \mathbb{Z}_{\geq 0}$, and retain Z^q_λ, B^q_λ, H^q_λ, and H^q_∞ with their previously specified topologies. For $\lambda \leq \mu$, write

$$h^q_{\lambda,\mu}\colon H^q_\lambda \longrightarrow H^q_\mu, \qquad z + B^q_\lambda \longmapsto r^q_{\lambda,\mu} z + B^q_\mu.$$

These maps are continuous, and the canonical maps to the inductive limit satisfy $h^q_\lambda = h^q_\mu \circ h^q_{\lambda,\mu}$.

Definition IV.1.33. For the inductive sequence and the degree q just fixed, a pair of integers (λ, μ) with $\lambda < \mu$ is called a *q-admissible pair* if

$$\operatorname{Ker} h^q_{\lambda,\mu} = \operatorname{Ker} h^q_\lambda.$$

It is called a *fully q-admissible pair* if, in addition, h^q_λ is surjective. Thus for a fully q-admissible pair, h^q_μ restricts to a linear bijection from $\operatorname{Image} h^q_{\lambda,\mu}$ onto H^q_∞. □

For a q-admissible pair, all classes at stage λ which vanish in the inductive limit already vanish at the same stage μ.

Proposition IV.1.34. *Let $C^\bullet_\lambda$ be an inductive sequence of Fréchet complexes with continuous cochain transition maps as above, and fix $q \in \mathbb{Z}_{\geq 0}$. If H^q_∞ with its locally convex final topology is Hausdorff, then, for every $\lambda \in \mathbb{Z}_{\geq 0}$, there exists $\mu > \lambda$ such that (λ, μ) is q-admissible. Every larger choice of μ is then also q-admissible with the same λ.*

Proof. Fix λ, and let

$$K^q_\lambda := \operatorname{Ker}\bigl(Z^q_\lambda \longrightarrow H^q_\infty\bigr),$$

where the map sends a cocycle to its class in the inductive limit. It is a closed subspace of the Fréchet space Z^q_λ, since this map is continuous and its target is Hausdorff. For $\mu \geq \lambda$ define

$$G^q_{\lambda,\mu} := \{(z,c) \in K^q_\lambda \oplus C^{q-1}_\mu \mid r^q_{\lambda,\mu} z = \delta^{q-1}_\mu c\}, \qquad \pi^q_{\lambda,\mu}(z,c) := z. \tag{274}$$

Here $C^{-1}_\mu = 0$ when $q = 0$. The space $G^q_{\lambda,\mu}$ is closed in the displayed Fréchet direct sum, and hence is Fréchet (Mini-Exercise I.6.42 (*iii*)). The projection $\pi^q_{\lambda,\mu} \colon G^q_{\lambda,\mu} \to K^q_\lambda$ is continuous. The algebraic description of an inductive limit gives

$$K^q_\lambda = \bigcup_{\mu \geq \lambda} \operatorname{Image} \pi^q_{\lambda,\mu} :$$

a class represented by $z \in Z^q_\lambda$ vanishes in the limit precisely when its restriction is a boundary at some later stage. By Theorem I.6.47, each image in this countable union is either meagre in K^q_λ or equals K^q_λ. The Baire category theorem therefore makes at least one of the projections surjective. This gives $\operatorname{Ker} h^q_\lambda \subset \operatorname{Ker} h^q_{\lambda,\mu}$; the opposite inclusion always holds. Composing with later transition maps preserves the equality, so we may also require $\mu > \lambda$. □

Corollary IV.1.35. *Let the sequence and degree q be as in Proposition IV.1.34, assume that H^q_∞ is Hausdorff, and let (λ,μ) be q-admissible. Use K^q_λ, $G^q_{\lambda,\mu}$, and $\pi^q_{\lambda,\mu}$ from (274). Then $\pi^q_{\lambda,\mu}$ is an open surjection. In particular, for every neighborhood W of 0 in C^{q-1}_μ there exists a neighborhood U of 0 in K^q_λ such that, for every $z \in U$, one can choose $c \in W$ with $r^q_{\lambda,\mu} z = \delta^{q-1}_\mu c$.*

Proof. Admissibility makes the projection surjective. Apply the open mapping theorem for Fréchet spaces (Theorem I.6.46). The image of $G^q_{\lambda,\mu} \cap (K^q_\lambda \times W)$ is then a neighborhood of 0 with the required property. □

Proposition IV.1.36. *Let $C^\bullet_\lambda$ be an inductive sequence of Fréchet complexes with continuous cochain transition maps, and fix $q \in \mathbb{Z}_{\geq 0}$. Assume that $\dim_{\mathbb{C}} H^q_\infty < \infty$. Then, for every $N \in \mathbb{Z}_{\geq 0}$, there exist $N < \lambda < \mu$ forming a fully q-admissible pair. For every such pair the induced maps give TVS isomorphisms*

$$H^q_\lambda / \operatorname{Ker} h^q_{\lambda,\mu} \;\cong\; \operatorname{Image} h^q_{\lambda,\mu} \xrightarrow[\cong]{h^q_\mu} H^q_\infty, \tag{275}$$

where the quotient has its quotient topology and the image has the subspace topology from H^q_μ.

Proof. Corollary IV.1.29 shows that H^q_∞ is Hausdorff. A finite basis of this vector space has representatives at a common stage, so h^q_λ is surjective for every sufficiently large λ (This also holds when $H^q_\infty = 0$). Choose such a $\lambda > N$, and apply Proposition IV.1.34 to obtain μ.

For any fully q-admissible pair, the last map in (275) is a continuous linear bijection. Its domain is Hausdorff, because it admits this continuous injection into the Hausdorff space H^q_∞, and it is finite-dimensional. Thus it is a TVS isomorphism by Theorem I.6.4. Moreover, the first quotient in (275) is naturally $Z^q_\lambda / K^q_\lambda$ with its quotient topology. The kernel K^q_λ is closed, so this is a finite-dimensional Hausdorff Fréchet quotient. Its continuous linear bijection to H^q_∞ is also a TVS isomorphism, which proves the assertion. □

COROLLARY IV.1.37. *Let $C_\lambda^\bullet$ be an inductive sequence of Fréchet complexes with continuous cochain transition maps, and fix $q \in \mathbb{Z}_{\geq 0}$. Assume that H_∞^q is Hausdorff and that each restriction $r_\lambda^q|_{Z_\lambda^q} \colon Z_\lambda^q \to Z_{\lambda+1}^q$ is compact. Then some h_λ^q is surjective if and only if H_∞^q is finite-dimensional. Consequently, if H_∞^q is infinite-dimensional, there are q-admissible pairs beginning at every stage, but there is no fully q-admissible pair.*

PROOF. The finite-dimensional case follows from Proposition IV.1.36. Conversely, suppose h_λ^q is surjective. Then $h_{\lambda+1}^q$ is also surjective. For $\nu = \lambda, \lambda+1$, put $K_\nu^q = \operatorname{Ker}(Z_\nu^q \to H_\infty^q)$ and $\widetilde{H}_\nu^q = Z_\nu^q / K_\nu^q$ with the quotient topology. These are Fréchet spaces, since K_ν^q is closed. The induced map $\widetilde{r}_\lambda^q \colon \widetilde{H}_\lambda^q \to \widetilde{H}_{\lambda+1}^q$ is a continuous linear bijection: both spaces map bijectively to the same algebraic limit, and the transition commutes with these maps.

This induced map is compact. Indeed, choose a neighborhood of zero in Z_λ^q whose image has compact closure in $Z_{\lambda+1}^q$, and pass to the two open quotient maps. Its image in $\widetilde{H}_\lambda^q$ is a neighborhood of zero, and its further image has compact closure in $\widetilde{H}_{\lambda+1}^q$. The open mapping theorem (Theorem I.6.46) makes $\widetilde{r}_\lambda^q$ a TVS isomorphism. Its target therefore has a compact neighborhood of zero, so it is finite-dimensional by Theorem I.6.4 (iii). The same holds algebraically for H_∞^q. The last assertion now follows from Proposition IV.1.34. □

Analytic neighborhood spaces with compact center

Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center $Z = Z(\mathcal{X})$, and let $\mathcal{F}$ be a coherent analytic sheaf on $\mathcal{X}$. We now give $H^q(\mathcal{X}, \mathcal{F})$ its natural topology. Choose a model $(X, \mathcal{I})$ on which $\mathcal{F}$ is represented by a coherent $\mathcal{O}_X$-module, shrinking the model if necessary. We use Z for its chosen reduced center in X, and use $\mathcal{F}$ also for this representative. Restrictions of $\mathcal{F}$ will be suppressed when the domain is specified. If Z is empty, the cohomology groups are zero, so we assume $Z \neq \emptyset$ while constructing the neighborhoods.

LEMMA IV.1.38. *Let $(X, \mathcal{I})$ be an analytic neighborhood model with nonempty compact reduced center Z. There are a finite index set J and Stein open subsets $V_j^{(\lambda)} \subset X$ for $j \in J$ and $\lambda \in \mathbb{Z}_{\geq 0}$ such that, on putting $V_j := V_j^{(0)}$ and*

$$\mathfrak{V}^{(\lambda)} := \{V_j^{(\lambda)}\}_{j \in J}, \qquad V^{(\lambda)} := \bigcup_{j \in J} V_j^{(\lambda)}, \tag{276}$$

we have $Z \subset V^{(\lambda)}$, and the closure of $V_j^{(\lambda+1)}$ is a compact subset of $V_j^{(\lambda)}$ for all j, λ. Moreover, the sequence $\{V^{(\lambda)}\}_{\lambda \geq 0}$ is cofinal in $\mathcal{N}_X(Z)$, ordered by reverse inclusion. Every nonempty finite intersection of members of $\mathfrak{V}^{(\lambda)}$ is Stein.

PROOF. Use local closed embeddings into polydiscs. After shrinking a local model, we may choose an open subset $\Omega_j \subset X$, a closed holomorphic embedding $\iota_j \colon \Omega_j \to P_j$ into a polydisc $P_j \subset \mathbb{C}^{N_j}$, and finitely many holomorphic functions $f_{j,1}, f_{j,2}, \ldots, f_{j,m_j}$ on Ω_j whose common zero set is $Z \cap \Omega_j$. Such functions are obtained from local generators of $\mathcal{I}$. If the ideal is zero on a patch, one zero function may be used.

Choose smaller concentric polydiscs $P_j^* \subset P_j$ with compact closure in P_j so that finitely many sets $\iota_j^{-1}(P_j^*)$ cover Z; denote their finite index set by J. For $1 \le k \le N_j$, write $a_{j,k}$ for the radii of P_j^*, and choose $b_{j,k} > a_{j,k}$ strictly smaller than the corresponding radii of P_j. Let $P_j^{(\lambda)}$ have radii $a_{j,k} + 2^{-\lambda}(b_{j,k} - a_{j,k})$, and put

$$V_j^{(\lambda)} := \{x \in \Omega_j \mid \iota_j(x) \in P_j^{(\lambda)}, \quad |f_{j,\ell}(x)| < 2^{-\lambda} \ (1 \le \ell \le m_j)\}.$$

These sets contain $Z \cap \iota_j^{-1}(P_j^*)$, and hence their unions cover Z. The strict inequalities between consecutive radii and thresholds show that the closure of $V_j^{(\lambda+1)}$ is a compact subset of $V_j^{(\lambda)}$. To see that each set is Stein, use the graph of $(f_{j,1}, \dots, f_{j,m_j})$ over the closed analytic subspace $\iota_j(\Omega_j) \cap P_j^{(\lambda)}$. The relevant graph is a closed analytic subspace of the product of $P_j^{(\lambda)}$ with m_j discs of radius $2^{-\lambda}$. It is therefore Stein, by the Stein-space facts used before (218).

Let $V \subset X$ be an open neighborhood of Z. If no $V^{(\lambda)}$ were contained in V, choose $x_\lambda \in V^{(\lambda)} \setminus V$. Passing to a subsequence, use finiteness of J to place all x_λ in the same patch, and then use its compact closure to obtain a limit x. The inequalities $|f_{j,\ell}(x_\lambda)| < 2^{-\lambda}$ imply that $f_{j,\ell}(x) = 0$ for every ℓ, so $x \in Z$. But $x \in X \setminus V$, a contradiction. This proves cofinality. Finite intersections are Stein by the same closed-diagonal argument as before (218). □

For the analytic neighborhood space $\mathcal{X}$ and sheaf $\mathcal{F}$ fixed above, choose one family from Lemma IV.1.38. Cofinality and Definition IV.1.20 give

$$H^q(\mathcal{X}, \mathcal{F}) \cong \varinjlim_{\lambda \ge 0} H^q(V^{(\lambda)}, \mathcal{F}) \qquad (q \ge 0). \tag{277}$$

Each space on the right has the natural topology of (218), and restriction induces continuous linear maps. We first define the topology using these cohomology spaces themselves.

Definition IV.1.39. Let $\mathcal{X}$ have compact reduced center, let $\mathcal{F}$ be a coherent analytic sheaf on $\mathcal{X}$, and fix $q \ge 0$. Choose a representative and a cofinal shrinking family as in (276). The *natural topology* on $H^q(\mathcal{X}, \mathcal{F})$ is the locally convex final topology, under (277), of the natural topologies on $H^q(V^{(\lambda)}, \mathcal{F})$. In particular, it is the strongest locally convex TVS topology for which all canonical maps from these spaces are continuous. □

This specializes the construction discussed in Chapter II after Example II.3.117 to coherent analytic coefficients with their natural topologies. The independence of choices is part of the following assertion.

Proposition IV.1.40. *Let $\mathcal{X}$ have compact reduced center and let $\mathcal{F}$ be coherent. For every $q \ge 0$, the topology in Definition IV.1.39 is independent of the chosen cofinal shrinking family and of the representative of $\mathcal{F}$. A specified isomorphism of models and of the represented sheaves induces a TVS isomorphism. Morphisms of coherent sheaves on the fixed analytic neighborhood space induce continuous maps on cohomology.*

PROOF. The natural topology in (218) is independent of the chosen Leray covering, and restrictions and sheaf morphisms induce continuous cohomology maps. For instance, on a smaller open set choose a countable Stein refinement of the restricted covering; restriction and refinement are continuous maps of the Fréchet cochain complexes.

For two cofinal sequences of neighborhoods, every term of either sequence contains a term of the other. The corresponding continuous restriction maps show, by the locally convex final universal property, that the canonical identification of the two limits is continuous in both directions. Replacing a representative by its restriction amounts to passing to a cofinal subsystem of neighborhoods. An isomorphism of representatives is realized on a common neighborhood of the whole center and acts continuously there. The same reasoning applies to a morphism of coherent sheaves. For a specified model isomorphism, transport the neighborhoods through its biholomorphic representative and apply this argument (Compare Remark IV.1.12). □

We now describe the topology of Definition IV.1.39 by a complex. Keep the model, representative, and shrinking family fixed. For $\lambda, q \geq 0$, put

$$C^q_\lambda := \check{C}^q(\mathfrak{V}^{(\lambda)}, \mathcal{F}) = \prod_{(j_0,j_1,\dots,j_q)\in J^{q+1}} \mathcal{F}(V^{(\lambda)}_{j_0} \cap V^{(\lambda)}_{j_1} \cap \cdots \cap V^{(\lambda)}_{j_q}),$$

with the zero space for an empty intersection, and let δ^q_λ be its Čech differential. We use all ordered tuples, as in Chapter II, § 3.3. Put $C^{-1}_\lambda = \check{C}^{-1}(\mathfrak{V}^{(\lambda)}, \mathcal{F}) = 0$ and $\delta^{-1}_\lambda = 0$. Thus

$$Z^q_\lambda = \check{Z}^q(\mathfrak{V}^{(\lambda)}, \mathcal{F}) = \operatorname{Ker} \delta^q_\lambda, \qquad B^q_\lambda = \check{B}^q(\mathfrak{V}^{(\lambda)}, \mathcal{F}) = \operatorname{Image} \delta^{q-1}_\lambda.$$

The space C^q_λ is Fréchet by Proposition III.2.166 and finiteness of J. The differential is continuous, so Z^q_λ is Fréchet as well. Cartan's theorem B (Theorem III.2.165) makes $\mathfrak{V}^{(\lambda)}$ Leray for $\mathcal{F}$. Leray comparison (Theorem II.3.79) therefore identifies Z^q_λ/B^q_λ with $H^q(V^{(\lambda)}, \mathcal{F})$, including their natural topologies.

The restrictions define continuous cochain maps $r^q_\lambda \colon C^q_\lambda \to C^q_{\lambda+1}$. Each map is compact. Indeed, on every finite intersection, the next open set has compact closure in the preceding one, so Proposition III.2.166 applies. A finite product of compact linear maps is compact. The same reasoning applies to the closed cocycle spaces Z^q_λ (Compare Lemma III.2.173).

PROPOSITION IV.1.41. *Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center, and let $\mathcal{F}$ be a coherent analytic sheaf on it. Choose a model and a cofinal finite Stein shrinking family as above. For every $q \geq 0$, there is a TVS isomorphism*

$$H^q(\mathcal{X}, \mathcal{F}) \cong \frac{\varinjlim_\lambda \check{Z}^q(\mathfrak{V}^{(\lambda)}, \mathcal{F})}{\operatorname{Image}\left(\varinjlim_\lambda \check{C}^{q-1}(\mathfrak{V}^{(\lambda)}, \mathcal{F}) \longrightarrow \varinjlim_\lambda \check{Z}^q(\mathfrak{V}^{(\lambda)}, \mathcal{F})\right)}. \tag{278}$$

The inductive limits in this display have their locally convex final topologies, and the outer quotient has its quotient topology. Moreover, $H^q(\mathcal{X}, \mathcal{F})$ is the cokernel of a continuous linear map between strict countable LF-spaces, with its quotient topology.

PROOF. For the chosen representative and covering sequence, use the Fréchet complexes $C^\bullet_\lambda$ just defined. Their cohomology topologies are the natural topologies on $H^q(V^{(\lambda)},\mathcal{F})$. Proposition IV.1.27 gives (278), and Proposition IV.1.28 gives the asserted cokernel presentation. □

COROLLARY IV.1.42. *Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center, let $\mathcal{F}$ be a coherent analytic sheaf on $\mathcal{X}$, and fix $q \in \mathbb{Z}_{\geq 0}$. Give $H^q(\mathcal{X},\mathcal{F})$ its natural topology. If $d := \dim_{\mathbb{C}} H^q(\mathcal{X},\mathcal{F}) < \infty$, then this cohomology is Hausdorff and is TVS isomorphic to $\mathbb{C}^d$. In particular, non-Hausdorff cohomology is necessarily infinite-dimensional.*

PROOF. Choose a representative and a cofinal finite Stein shrinking family as in Proposition IV.1.41. The associated Fréchet Čech complexes realize $H^q(\mathcal{X},\mathcal{F})$ as H^q_∞ with its locally convex final topology. Apply Corollary IV.1.29. Equivalently, apply Lemma I.6.55 to the strict countable LF-space cokernel presentation in Proposition IV.1.41. □

For $\mathcal{X}$, $\mathcal{F}$, and q as in Corollary IV.1.42, exactly one of the following possibilities occurs:

- $H^q(\mathcal{X},\mathcal{F})$ is finite-dimensional;
- $H^q(\mathcal{X},\mathcal{F})$ is infinite-dimensional and Hausdorff;
- $H^q(\mathcal{X},\mathcal{F})$ is non-Hausdorff.

The first case is automatically Hausdorff, and the third is infinite-dimensional.

PROPOSITION IV.1.43. *Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center, let $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, and fix $q \in \mathbb{Z}_{\geq 0}$. Choose a model $(X,\mathcal{I})$, a coherent representative also denoted by $\mathcal{F}$, and a fixed finite Stein shrinking family $\mathfrak{V}^{(\lambda)} = \{V_j^{(\lambda)}\}_{j\in J}$ as in Lemma IV.1.38. Put $V^{(\lambda)} = \bigcup_{j\in J} V_j^{(\lambda)}$, $H^q_\lambda = H^q(V^{(\lambda)},\mathcal{F})$, and $H^q_\infty = H^q(\mathcal{X},\mathcal{F})$, with their natural topologies. Write $h^q_{\lambda,\mu}$ and h^q_λ for restriction to a smaller neighborhood and to the neighborhood germ, respectively. Then the following holds.*

(i) If H^q_∞ is Hausdorff, every λ admits $\mu > \lambda$ with $\operatorname{Ker} h^q_{\lambda,\mu} = \operatorname{Ker} h^q_\lambda$. Thus all classes on $V^{(\lambda)}$ which vanish after some shrinking vanish on the same $V^{(\mu)}$.

(ii) If H^q_∞ is finite-dimensional, fully q-admissible pairs can be chosen arbitrarily late in the sequence, and (275) gives its TVS presentation as a two-stage image.

(iii) If H^q_∞ is infinite-dimensional and Hausdorff, no h^q_λ is surjective, although the pairs in (i) exist.

For the Čech cocycles of this shrinking family, an admissible pair in (i) also gives the primitive control of Corollary IV.1.35.

PROOF. By Proposition IV.1.41, these cohomology spaces and maps are realized by the inductive sequence of Fréchet Čech complexes of the fixed finite covers. Apply Proposition IV.1.34 and Corollary IV.1.35 for (i) and the last assertion, and Proposition IV.1.36 for (ii).

The cochain restrictions are compact: there are only finitely many components in each degree, and each is a relatively compact restriction of sections of a coherent sheaf (Proposition III.2.166). Their restrictions to the closed subspaces of cocycles are compact as well. Corollary IV.1.37 therefore proves (iii). □

EXAMPLE IV.1.44. Let Y be an elliptic curve, let $F \to Y$ be a unitary flat holomorphic line bundle, and let X_F be its total space, with projection $p\colon X_F \to Y$. Put $Z := (\text{Zero Section})_F$, let $\mathcal{I}_Z$ be its reduced defining ideal, and consider the analytic neighborhood space $\mathcal{X}_F = [(X_F, \mathcal{I}_Z)]$. We use the disc neighborhoods $V_{F,r}$ and the distance on $\mathcal{P}(Y)$ fixed in Chapter III, §2.15. The sets $V_{F,r}$ form a cofinal system as $r \searrow 0$. Definition IV.1.39 and Proposition IV.1.40 therefore give

$$H^1(\mathcal{X}_F, \mathcal{O}_{\mathcal{X}_F}) = \varinjlim_{r \searrow 0} H^1(V_{F,r}, \mathcal{O}_{V_{F,r}})$$

with the locally convex final topology. This limit realizes all three possibilities discussed above.

If F has order $m \geq 1$, choose a unitary trivialization of F^m and the resulting fiber function $u = w_j^m$. The restriction-compatible TVS isomorphisms of Proposition III.2.209 induce

$$H^1(\mathcal{X}_F, \mathcal{O}_{\mathcal{X}_F}) \cong H^1(Y, \mathcal{O}_Y) \otimes_{\mathbb{C}} \mathbb{C}\{u\}. \tag{279}$$

Here $\mathbb{C}\{u\}$ has the locally convex final topology of the spaces $\mathcal{O}_{\Delta_\rho}(\Delta_\rho)$ as $\rho \searrow 0$. Coefficient evaluation at each order is continuous on every such space, and hence on this limit. These evaluations separate convergent power series, so this locally convex topology is Hausdorff. The space is infinite-dimensional. Formula (279) follows as a TVS isomorphism from the universal property of the locally convex final topology.

If F is non-torsion and asymptotically positive, pullback and the degree-zero coefficient give inverse continuous maps

$$H^1(Y, \mathcal{O}_Y) \cong H^1(\mathcal{X}_F, \mathcal{O}_{\mathcal{X}_F}) \cong \mathbb{C}. \tag{280}$$

Indeed, after subtracting the pullback of the degree-zero class, Lemma III.2.206 kills the remaining class after shrinking. Continuity of the coefficient map follows on every stage from Cauchy's integral formula; continuity of pullback follows by factoring it through any one stage. In particular this is the finite-dimensional Hausdorff case, even when $0 < \alpha(F) < 1$ and each fixed disc neighborhood has non-Hausdorff cohomology.

If F is non-torsion and asymptotically zero, the cocycle of Proposition III.2.207 defines a non-zero class in this limit. On each fixed neighborhood it is a limit of coboundaries. The map from that stage to the limit is continuous, so this non-zero class belongs to the closure of zero also in $H^1(\mathcal{X}_F, \mathcal{O}_{\mathcal{X}_F})$. Thus the limit is non-Hausdorff and is infinite-dimensional by Corollary IV.1.42.

For the admissible-pair conclusions of Proposition IV.1.43, use the finite Stein shrinking family of Lemma IV.1.38, which shrinks the base charts as well as the fibers. Proposition IV.1.40 identifies the resulting topology with the one used in this calculation. □

For the Čech complexes of the finite Stein shrinking family in Lemma IV.1.38, Komatsu's injectivity hypothesis gives the following comparison.

Corollary IV.1.45. *Let $\mathcal{X}$ have compact reduced center, let $\mathcal{F}$ be coherent, and fix a model and a shrinking family as in Lemma IV.1.38. Assume that all the cochain restriction maps $r_\lambda^p\colon C_\lambda^p \to C_{\lambda+1}^p$ are injective. For $q \geq 0$, the natural topology on $H^q(\mathcal{X},\mathcal{F})$ then agrees with the kernel-quotient topology on*

$$H^q(C_\infty^\bullet) = \frac{\operatorname{Ker}\delta_\infty^q}{\operatorname{Image}\delta_\infty^{q-1}}, \qquad C_\infty^p := \varinjlim_\lambda C_\lambda^p,$$

where the kernel has its subspace topology in C_∞^q. It is also the final topology of the underlying topological spaces $H^q(V^{(\lambda)},\mathcal{F})$.

Proof. Compactness of all the cochain restriction maps was checked above. The additional injectivity assumption therefore permits the application of Corollary IV.1.32 to this very sequence of Čech complexes. Its cohomology limit is $H^q(\mathcal{X},\mathcal{F})$ with the topology in Definition IV.1.39. □

On a complex manifold with a locally free coefficient sheaf, the identity principle gives the injectivity in Corollary IV.1.45 whenever every connected component of an intersection at stage λ meets the corresponding intersection at stage $\lambda+1$.

Remark IV.1.46. Let $\mathcal{X}$ now be an analytic neighborhood space whose center need not be compact, and let $\mathcal{F}$ be coherent. Choose a model $(X,\mathcal{I})$ and a coherent representative, again denoted by $\mathcal{F}$, on X; put $Z = Z(X,\mathcal{I})$. One can still give

$$\varinjlim_{V\in\mathcal{N}_X(Z)} H^q(V,\mathcal{F}) = H^q(\mathcal{X},\mathcal{F})$$

the locally convex final topology of the natural cohomology topologies. Here $\mathcal{N}_X(Z)$ is ordered by reverse inclusion as in Example II.3.20. This makes the cohomology an LCTVS, and the same cofinality argument proves independence of its representative.

Compactness was not required for the existence of this topology. In the preceding construction it provided a finite Stein covering and a countable cofinal shrinking sequence. Relative compactness of the smaller intersections then gave compact restrictions, and finiteness of the covering made the cochain transition maps compact. Countable products of Fréchet spaces are still Fréchet (Mini-Exercise I.6.42 (*iv*)), but an infinite product of compact linear maps need not be compact. Moreover, a non-compact center need not have a countable cofinal system of open neighborhoods. Thus the locally convex final topology can still be defined, while the sequential presentations and the compact-transition comparisons require separate consideration. □

Problem IV.1.47. *For an analytic neighborhood space $\mathcal{X}$ with non-compact center and a coherent analytic sheaf $\mathcal{F}$, to what extent do the preceding topological descriptions of $H^q(\mathcal{X},\mathcal{F})$ remain valid? What geometric conditions on the center and its neighborhoods give analogous comparisons, and what kinds of failure can occur?* □

2 Finite jets and formal completions

We now pass from an analytic neighborhood space $\mathcal{X}$ to its finite jets $\mathcal{X}^{(\mu)}$ and its formal jet $\mathcal{X}^{(\infty)}$. A model gives concrete representatives of these spaces and specifies which order of vanishing the index μ records.

2.1 Finite jets

Definition IV.2.1. Let $\mathcal{X}$ be an analytic neighborhood space, choose a model $(X,\mathcal{I})$, put $Z := Z(X,\mathcal{I})$, and let $j_Z\colon Z \hookrightarrow X$ be the inclusion. For $\mu \in \mathbb{Z}_{\geq 0}$ define

$$\mathscr{J}_\mu(X,\mathcal{I}) := \big(Z, j_Z^{-1}(\mathcal{O}_X/\mathcal{I}^{\mu+1})\big).$$

The *μ-th jet* of $\mathcal{X}$, also called its *μ-th infinitesimal neighborhood*, is its isomorphism class $\mathcal{X}^{(\mu)}$ in (CpxSp). The functoriality below shows that it depends only on $\mathcal{X}$. We also write $\mathscr{J}_\mu(\mathcal{X})$ for this class. After choosing a model we use $\mathcal{X}^{(\mu)}$ for that representative as well, whose structure sheaf is $\mathcal{O}_{\mathcal{X}^{(\mu)}} = j_Z^{-1}(\mathcal{O}_X/\mathcal{I}^{\mu+1})$. □

Remark IV.2.2. If $\mathcal{I}$ is radical, then $\mathcal{X}^{(0)} = Z(\mathcal{X})$ with its reduced structure. Otherwise even $\mathcal{X}^{(0)}$ may retain nilpotent information. □

Proposition IV.2.3. *For every integer $\mu \geq 0$, the assignment*

$$(X,\mathcal{I}) \longmapsto \mathscr{J}_\mu(X,\mathcal{I})$$

extends to a functor

$$\mathscr{J}_\mu\colon (\mathsf{ANM}) \xrightarrow{\text{functor}} (\mathsf{CpxSp}).$$

Proof. Represent a morphism of models by $f\colon U \to Y$, where U is a neighborhood of $Z(X,\mathcal{I})$. Its ideal condition gives

$$f^{-1}\mathcal{J}^{\mu+1}\cdot\mathcal{O}_U \subset \mathcal{I}^{\mu+1}|_U.$$

Together with the map of reduced centers, the induced homomorphism of quotient sheaves defines $\mathscr{J}_\mu(f)\colon \mathscr{J}_\mu(X,\mathcal{I}) \to \mathscr{J}_\mu(Y,\mathcal{J})$. It depends only on the germ of f, since shrinking U does not change these quotient sheaves on the center. Identities and compositions are preserved by the corresponding homomorphisms of quotients. Thus a morphism f between chosen models of $\mathcal{X}$ and $\mathcal{Y}$ induces a morphism $f^{(\mu)}\colon \mathcal{X}^{(\mu)} \to \mathcal{Y}^{(\mu)}$; this is $\mathscr{J}_\mu(f)$ in the chosen representatives. □

Remark IV.2.4. The quotient homomorphisms $\mathcal{O}_X/\mathcal{I}^{\mu+2} \to \mathcal{O}_X/\mathcal{I}^{\mu+1}$ give closed embeddings

$$\mathcal{X}^{(\mu)} \longrightarrow \mathcal{X}^{(\mu+1)}.$$

Thus the finite jets form an increasing system of infinitesimal neighborhoods of the reduced center, while their structure sheaves form an inverse system. □

Notation IV.2.5. Let $\mathcal{X}$ be an analytic neighborhood space, choose a model $(X, \mathcal{I})$, and let $j_Z \colon Z(\mathcal{X}) \hookrightarrow X$ be the inclusion of its reduced center. For $\mu \in \mathbb{Z}_{\geq 0}$, put

$$\mathcal{S}^{\mu}_{\mathcal{X}} := j_Z^{-1}(\mathcal{I}^{\mu}/\mathcal{I}^{\mu+1}), \qquad \mathcal{N}^{*}_{\mathcal{X}} := \mathcal{S}^{1}_{\mathcal{X}}.$$

These are coherent $\mathcal{O}_{\mathcal{X}^{(0)}}$-modules; $\mathcal{S}^{0}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}^{(0)}}$. We call $\mathcal{N}^{*}_{\mathcal{X}}$ the *conormal sheaf*. The notation $\mathcal{S}^{\mu}_{\mathcal{X}}$ denotes the degree-μ piece of the associated graded algebra of the specified ideal, not a symmetric-power operation. Shrinking the model leaves these sheaves unchanged, and a specified isomorphism of models identifies them. □

Multiplication gives a natural surjection

$$S^{\mu}_{\mathcal{O}_{\mathcal{X}^{(0)}}}(\mathcal{N}^{*}_{\mathcal{X}}) \longrightarrow \mathcal{S}^{\mu}_{\mathcal{X}}.$$

Here $S^{\mu}_{\mathcal{R}}\mathcal{E}$ denotes the μ-th symmetric tensor power of an $\mathcal{R}$-module $\mathcal{E}$. Indeed, the products of μ local generators of $\mathcal{I}$ generate $\mathcal{I}^{\mu}/\mathcal{I}^{\mu+1}$, and changing a factor modulo $\mathcal{I}^2$ changes the product by an element of $\mathcal{I}^{\mu+1}$. The surjection need not be an isomorphism. For instance, on the nodal curve with local ring $\mathbb{C}\{u,v\}/(uv)$ and ideal $\mathcal{I} = (u,v)$, the symmetric square of $\mathcal{I}/\mathcal{I}^2$ has dimension three, whereas $\mathcal{I}^2/\mathcal{I}^3$ has basis u^2, v^2 and dimension two. For a non-radical ideal the coefficient ring must also be retained: if $\mathcal{I} = (w^2)$, then $w[w^2] = [w^3] \neq 0$ in $\mathcal{I}/\mathcal{I}^2 = (w^2)/(w^4)$, so this module action does not factor through the structure sheaf of the reduced point.

Example IV.2.6. Let Z be a closed complex submanifold of a complex manifold X, and consider the analytic neighborhood space $\mathcal{X} := [(X, \mathcal{I}_Z)]$. The differential of a defining function identifies $\mathcal{N}^{*}_{\mathcal{X}}$ with $\mathcal{O}_Z(N^{*}_{Z/X})$. For each $\mu \in \mathbb{Z}_{\geq 0}$, the monomials w^{α} with $|\alpha| = \mu$ in adapted coordinates (z, w) give a local basis of $\mathcal{S}^{\mu}_{\mathcal{X}}$. Taking the degree-$\mu$ part of a change of defining functions therefore gives

$$\mathcal{S}^{\mu}_{\mathcal{X}} \cong S^{\mu}_{\mathcal{O}_Z}(\mathcal{N}^{*}_{\mathcal{X}}) \cong \mathcal{O}_Z(S^{\mu}N^{*}_{Z/X}). \tag{281}$$

Here $S^{\mu}N^{*}_{Z/X}$ denotes the μ-th symmetric tensor power of the conormal bundle, formed fiberwise. In particular these are finite locally free sheaves. This is the geometric meaning of the notation in the submanifold case. □

2.2 Formal complex spaces and formal jets

The structure sheaves of the finite jets form an inverse system. We first describe the category in which their limit will be considered, using the locally ringed spaces of Chapter II and the complex spaces of Chapter III.

For an inverse system $\{\mathcal{F}_\mu\}_{\mu\in\mathbb{Z}_{\geq 0}}$ of sheaves of rings or modules on a fixed space X, the inverse limit is computed on open sets:

$$\Gamma\big(U, \varprojlim_\mu \mathcal{F}_\mu\big) = \varprojlim_\mu \Gamma(U, \mathcal{F}_\mu) \qquad (U \subset X \text{ open}).$$

This is the sectionwise calculation in Remark II.3.119, (87); the same argument for abelian sheaves respects multiplication and module operations.

Objects, examples, and morphisms

Let $\mathfrak{X} = (X, \mathcal{O}_\mathfrak{X})$ be an object of $(\mathsf{LocRingSp}_\mathbb{C})$. The value $s(x)$ of a local section is its image in the residue field $\mathcal{O}_{\mathfrak{X},x}/\mathfrak{m}_x \cong \mathbb{C}$, as in Remark II.3.95. Define an ideal sheaf by

$$\mathfrak{n}_\mathfrak{X}(U) := \{s \in \mathcal{O}_\mathfrak{X}(U) \mid s(x) = 0 \text{ for every } x \in U\}.$$

Restrictions and gluing preserve this condition. It is a radical ideal sheaf: if a power of a section has value zero everywhere, then the section has value zero everywhere.

Definition IV.2.7 ([Bi, §1, p. 256]). A *formal complex space* is an object $\mathfrak{X} = (X, \mathcal{O}_\mathfrak{X})$ of $(\mathsf{LocRingSp}_\mathbb{C})$ satisfying the following conditions:

- for every integer $\mu \geq 0$, the locally ringed space $\mathfrak{X}^{(\mu)} := (X, \mathcal{O}_\mathfrak{X}/\mathfrak{n}_\mathfrak{X}^{\mu+1})$ is a complex space;
- the natural homomorphism
$$\mathcal{O}_\mathfrak{X} \longrightarrow \varprojlim_{\mu\geq 0} \mathcal{O}_\mathfrak{X}/\mathfrak{n}_\mathfrak{X}^{\mu+1}$$
is an isomorphism of sheaves.

We call $\mathfrak{X}^{(\mu)}$ the μ-th *intrinsic thickening* of $\mathfrak{X}$. Here X denotes the underlying topological space. □

The space $\mathfrak{X}^{(0)}$ is reduced, since $\mathfrak{n}_\mathfrak{X}$ is radical. The second condition expresses completeness at the level of sheaves.

Notation IV.2.8. Let $\mathfrak{X} = (X, \mathcal{O}_\mathfrak{X})$ be a formal complex space. For $\mu \in \mathbb{Z}_{\geq 0}$ and an $\mathcal{O}_\mathfrak{X}$-module $\mathcal{F}$, write

$$\mathcal{O}_{\mathfrak{X}^{(\mu)}} := \mathcal{O}_\mathfrak{X}/\mathfrak{n}_\mathfrak{X}^{\mu+1}, \qquad \mathcal{F}^{(\mu)} := \mathcal{F}/\mathfrak{n}_\mathfrak{X}^{\mu+1}\mathcal{F}.$$

This is the structure sheaf of $\mathfrak{X}^{(\mu)}$; $\mathcal{F}^{(\mu)}$ is a module on that object. We call $\mathfrak{X}$ *Stein* when $\mathfrak{X}^{(0)}$ is Stein. □

Example IV.2.9. Every complex space $(X, \mathcal{O}_X)$ is a formal complex space. In this case $\mathfrak{n}_\mathfrak{X}$ is the nilradical of $\mathcal{O}_X$, by Rückert's Nullstellensatz (Theorem III.2.37). This ideal is coherent by Oka's second coherence theorem (Theorem III.2.163). Locally choose finitely many generators. Each is nilpotent after shrinking, so a sufficiently high power of their ideal is zero on one common neighborhood. Thus the inverse system in the definition is locally eventually constant, with limit $\mathcal{O}_X$. Its finite terms are complex spaces by Proposition III.2.36. □

EXAMPLE IV.2.10. The one-point locally ringed space $(\{0\}, \mathbb{C}[[w_1, w_2, \dots, w_r]])$, where $r \geq 1$, is a formal complex space. Its ideal of sections with value zero is $(w_1, w_2, \dots, w_r)$, and its finite quotients are

$$\mathbb{C}[[w_1, w_2, \dots, w_r]]/(w_1, w_2, \dots, w_r)^{\mu+1} \cong \mathbb{C}\{w_1, w_2, \dots, w_r\}/(w_1, w_2, \dots, w_r)^{\mu+1}.$$

Truncation of power series gives the isomorphism and identifies their inverse limit with the formal power series ring, as in Chapter I, §5.5. For $r = 1$, the ideal $\mathfrak{n}_{\mathfrak{X}} = (w)$ is nonzero, whereas the nilradical of $\mathbb{C}[[w]]$ is zero. Thus vanishing at every point does not mean nilpotence in a formal space.

More generally, let $d \geq 0$, let $U \subset \mathbb{C}^d$ be open, and define a sheaf $\mathcal{O}_{\mathfrak{X}}$ on U by

$$\mathcal{O}_{\mathfrak{X}}(V) := \mathcal{O}_U(V)[[w_1, w_2, \dots, w_r]] \qquad (V \subset U \text{ open}).$$

A series whose constant coefficient is nonzero at x has an inverse on a neighborhood of x, obtained coefficient by coefficient. Hence the stalk at x is local, with residue field $\mathbb{C}$. Here $\mathfrak{n} = (w_1, w_2, \dots, w_r)$, and the finite quotients are the complex spaces defined by $(w_1, w_2, \dots, w_r)^{\mu+1}$ in $U \times \mathbb{C}^r$. Their inverse limit is $\mathcal{O}_{\mathfrak{X}}$, so $(U, \mathcal{O}_{\mathfrak{X}})$ is a formal complex space. The coefficients must be holomorphic on the same V; no convergence in the variables $w_1, w_2, \dots, w_r$ is required. □

DEFINITION IV.2.11. A *morphism of formal complex spaces* $(f, f^\sharp)\colon (X, \mathcal{O}_{\mathfrak{X}}) \to (Y, \mathcal{O}_{\mathfrak{Y}})$ is a morphism in $(\mathsf{LocRingSp}_{\mathbb{C}})$. Thus $f\colon X \to Y$ is continuous, $f^\sharp\colon \mathcal{O}_{\mathfrak{Y}} \to f_*\mathcal{O}_{\mathfrak{X}}$ is a morphism of sheaves of $\mathbb{C}$-algebras, and the induced homomorphism $\mathcal{O}_{\mathfrak{Y},f(x)} \to \mathcal{O}_{\mathfrak{X},x}$ is local for every $x \in X$. The category of formal complex spaces and these morphisms is denoted by (FCS). □

This is a full subcategory of $(\mathsf{LocRingSp}_{\mathbb{C}})$, and Example IV.2.9 identifies (CpxSp) with a full subcategory of (FCS). As in Remark II.3.93, we usually abbreviate $(f, f^\sharp)$ to f and use $f^\sharp$ also for the adjoint homomorphism $f^{-1}\mathcal{O}_{\mathfrak{Y}} \to \mathcal{O}_{\mathfrak{X}}$.

Values are preserved, so the image of $f^{-1}\mathfrak{n}_{\mathfrak{Y}}$ generates an ideal contained in $\mathfrak{n}_{\mathfrak{X}}$. The same holds for their powers. Consequently, f induces compatible holomorphic maps $f^{(\mu)}\colon \mathfrak{X}^{(\mu)} \to \mathfrak{Y}^{(\mu)}$. For each $\mu \in \mathbb{Z}_{\geq 0}$ these assignments give the functor $(-)^{(\mu)}\colon (\mathsf{FCS}) \xrightarrow{\text{functor}} (\mathsf{CpxSp})$, since the quotient homomorphisms respect identities and compositions. For the one-point formal disk, $w \mapsto w^2$ defines a morphism; it has the same underlying map as the identity, but a different homomorphism of structure sheaves.

Coherence and finite thickenings

The following analytic input lets us pass from finite algebra to relations on a common neighborhood. A subset is *semianalytic* if, in a local embedding, it is locally a finite union of sets defined by finitely many real analytic equalities and strict inequalities, of the form $\{f_1 = \cdots = f_r = 0,\ g_1 > 0, \dots, g_s > 0\}$. For a formal complex space $\mathfrak{X}$, a *Stein compactum* here means a compact semianalytic subset of $\mathfrak{X}^{(0)}$ having a basis of Stein open neighborhoods in each $\mathfrak{X}^{(\mu)}$. Such compact neighborhoods can be chosen arbitrarily small (See [Bi, §1, pp. 256–257]). For a sheaf $\mathcal{G}$, $\Gamma(K, \mathcal{G})$ means sections of its inverse image on K.

Remark IV.2.12. We describe the analytic facts behind the compact-set argument. Let Y be a complex space, let $K \subset Y$ be a compact subset having a basis of Stein open neighborhoods, and let $\mathcal{F}, \mathcal{G}$ and $\mathcal{J}$ be coherent analytic sheaves on a common neighborhood of K, with $\mathcal{J} \subset \mathcal{O}_Y$ an ideal sheaf. Write $A_K := \Gamma(K, \mathcal{O}_Y)$, $M_K := \Gamma(K, \mathcal{F})$ and $\mathfrak{a}_K := \Gamma(K, \mathcal{J})$. These are sections of inverse-image sheaves, not merely values of functions on K. The Oka–Cartan theorem (Theorem III.2.165), the compact-set comparison in Theorem II.3.112 and Remark II.3.113, and exactness of filtered direct limits (Proposition II.3.43) show that $\Gamma(K, -)$ is exact on coherent analytic sheaves represented near K. Here the Stein neighborhoods form a cofinal system and their positive-degree cohomology vanishes.

Cartan's theorem A (Theorem III.2.165 (A)) and compactness give finitely many sections generating $\mathcal{F}$ on a neighborhood of K. Apply the same argument to their coherent relation sheaf and shrink the neighborhood once more. Thus there is a finite presentation $\mathcal{O}^{\oplus a} \to \mathcal{O}^{\oplus b} \to \mathcal{F} \to 0$ near K, and exactness gives

$$A_K^{\oplus a} \longrightarrow A_K^{\oplus b} \longrightarrow M_K \longrightarrow 0.$$

In particular, M_K is finitely presented over A_K. Tensor the sheaf presentation with $\mathcal{G}$ and take sections. Comparing its cokernel with the tensor product of the displayed module presentation gives the natural isomorphism

$$\Gamma(K, \mathcal{F}) \otimes_{A_K} \Gamma(K, \mathcal{G}) \cong \Gamma(K, \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}).$$

Choose finitely many generators of $\mathcal{J}$ near K in the same way. The resulting surjection $\mathcal{F}^{\oplus c} \to \mathcal{J}\mathcal{F}$ shows that $\Gamma(K, \mathcal{J}\mathcal{F}) = \mathfrak{a}_K M_K$. Here $\mathcal{J}\mathcal{F}$ is the image of multiplication, not necessarily $\mathcal{J} \otimes \mathcal{F}$. Iteration and exactness therefore give, for every $\mu \in \mathbb{Z}_{\geq 0}$,

$$\Gamma(K, \mathcal{J}^{\mu+1}\mathcal{F}) = \mathfrak{a}_K^{\mu+1} M_K, \qquad \Gamma(K, \mathcal{F}/\mathcal{J}^{\mu+1}\mathcal{F}) \cong M_K/\mathfrak{a}_K^{\mu+1} M_K.$$

These conclusions do not imply that A_K is Noetherian (See Mini-Exercise IV.2.13). The additional analytic finiteness theorem is due to Frisch: if K is also semianalytic, then A_K is Noetherian [Fri]. More precisely, Siu's criterion says that A_K is Noetherian if and only if $K \cap A$ has finitely many topological connected components for every analytic subset A defined on an open neighborhood of K. The ambient complex space need not be reduced (For this statement and its proof, see [Siu1, Theorem 1, pp. 483 and 486–488, with Lemmata 1–4]).

The quotient formula above translates completion of a coherent analytic sheaf into ideal-adic completion of a finite module. Noetherianity then permits the use of the Artin–Rees and completion results of Chapter I, §5.5, in particular Proposition I.5.14. For ordinary analytic sheaves, restriction between such compact sets is flat by Oka–Cartan and the finite-relation criterion for flatness. At the formal level, one must also check compatibility of the ideals and their powers, separatedness and completeness before applying the Noetherian completion arguments; flatness is not deduced from an arbitrary inverse limit of flat maps. The next lemma records the resulting formal compact-set facts, with their proof referred to Bingener. The subsequent proposition uses Noetherianity to choose finitely many relations and flatness to make those same relations generate on a common open neighborhood. □

MINI-EXERCISE IV.2.13. Let $K := \{0\} \cup \{1/n \mid n \in \mathbb{Z}_{>0}\} \subset \mathbb{C}$. Show that K has a basis of Stein open neighborhoods, but that $\Gamma(K, \mathcal{O}_{\mathbb{C}})$ is not Noetherian.

LEMMA IV.2.14. *Let $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}})$ be a formal complex space and let $K \subset X$ be a Stein compactum. Put*

$$B_K := \varprojlim_{\mu} \Gamma(K, \mathcal{O}_{\mathfrak{X}^{(\mu)}}), \qquad \mathfrak{b}_K := \operatorname{Ker}\big(B_K \to \Gamma(K, \mathcal{O}_{\mathfrak{X}^{(0)}})\big).$$

Then B_K is Noetherian, separated and complete for the $\mathfrak{b}_K$-adic topology, and $B_K/\mathfrak{b}_K^{\mu+1} \cong \Gamma(K, \mathcal{O}_{\mathfrak{X}^{(\mu)}})$. If $K \subset L$ are Stein compacta, the restriction $B_L \to B_K$ is flat. Moreover,

$$\mathcal{O}_{\mathfrak{X},x} \cong \varinjlim_{K \ni x} B_K,$$

where K ranges over Stein compact neighborhoods of x, ordered by shrinking. In particular, $B_K \to \mathcal{O}_{\mathfrak{X},x}$ is flat for x in the interior of K.

PROOF. The Noetherianity, completeness, quotient formula, and flatness of restriction are the compact-set results of [Bi, Lemma (1.1)(1)–(2), p. 257]. They use analytic finiteness on Stein compacta, in addition to the completion arguments of Chapter I, §5.5. For the stalk formula, restrict a compatible family on K to its interior. Conversely, a section on an open neighborhood of x restricts to such a family on a smaller compact neighborhood. These operations are inverse after further shrinking. Finally, the filtered limit of the flat B_K-modules B_L, with x in the interior of $L \subset K$, is flat: tensor products and exactness commute with filtered limits of modules (See Proposition II.3.43). □

PROPOSITION IV.2.15. *Let $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}})$ be a formal complex space. Its structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is coherent, and every stalk $\mathcal{O}_{\mathfrak{X},x}$ is Noetherian.*

PROOF. The sheaf $\mathcal{O}_{\mathfrak{X}}$ is generated by 1 as a module over itself. Take finitely many sections $s_1, s_2, \ldots, s_r$ on an open set U and a Stein compact neighborhood $K \subset U$ of a point x. The sections give elements of B_K. Since B_K is Noetherian, the module of their relations is finitely generated. Choose a matrix A whose columns generate these relations; then

$$B_K^{\oplus m} \xrightarrow{A} B_K^{\oplus r} \xrightarrow{(s_1, s_2, \ldots, s_r)} B_K$$

is exact at the middle term. The entries of A give sections of $\mathcal{O}_{\mathfrak{X}}$ on the interior of K. For every point y there, flatness of $B_K \to \mathcal{O}_{\mathfrak{X},y}$ preserves this exactness. Thus these same finitely many columns generate the sheaf of relations throughout that interior. This proves coherence, exactly as required by Definition III.1.36 (Compare [Bi, Lemma (1.1)(3), pp. 257–258]).

For Noetherianity of the stalks, use [Bi, (1.4)(1), pp. 259–260], applied to the Stein compactum $\{x\}$: its ring of sections is $\mathcal{O}_{\mathfrak{X},x}$. □

The finite presentation and kernel, image, and cokernel properties for coherent modules now apply as in Chapter III. The next result describes what the completeness condition means for these modules.

THEOREM IV.2.16. *Let $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}})$ be a formal complex space.*

(*i*) *If $\mathcal{F}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module, then $\mathcal{F}^{(\mu)}$ is a coherent $\mathcal{O}_{\mathfrak{X}^{(\mu)}}$-module and $\mathcal{F} \to \varprojlim_{\mu} \mathcal{F}^{(\mu)}$ is an isomorphism. Conversely, if $\{\mathcal{F}_\mu\}_{\mu \in \mathbb{Z}_{\geq 0}}$ is an inverse system of coherent $\mathcal{O}_{\mathfrak{X}^{(\mu)}}$-modules whose transition maps induce isomorphisms $\mathcal{O}_{\mathfrak{X}^{(j)}} \otimes_{\mathcal{O}_{\mathfrak{X}^{(\mu)}}} \mathcal{F}_\mu \cong \mathcal{F}_j$ for $0 \leq j \leq \mu$, then $\varprojlim_{\mu} \mathcal{F}_\mu$ is coherent and its quotients recover the given system.*

(*ii*) *If $\mathfrak{X}$ is Stein, then every $\mathfrak{X}^{(\mu)}$ is Stein, and, for every coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$, its global sections generate all stalks and $H^q(X, \mathcal{F}) = 0$ for $q \geq 1$.*

PROOF. See [Bi, Lemma (1.2) and (1.3), pp. 258–259]. For (i), the compact-set description in Lemma IV.2.14 reduces the assertion to finite modules over the complete Noetherian rings B_K. The isomorphism of a finite module with its completion is the completion–tensor-product formula of Chapter I, §5.5, applied to a complete ring. Bingener also proves that compatible finite-stage modules give a finite B_K-module, and that these descriptions agree under restriction; these are needed for the converse. Part (ii) is the formal version of Cartan's Theorems A and B. □

In particular, $\mathfrak{n}_{\mathfrak{X}}$ is coherent. Indeed, the constant system $\mathcal{F}_\mu = \mathcal{O}_{\mathfrak{X}^{(0)}}$ satisfies (i), so $\mathcal{O}_{\mathfrak{X}^{(0)}}$ is coherent over $\mathcal{O}_{\mathfrak{X}}$; take the kernel of $\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}^{(0)}}$.

PROPOSITION IV.2.17. *Let $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}})$ be a formal complex space and let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ be a coherent ideal sheaf. Put $Y := \{x \in X \mid s(x) = 0 \text{ for all } s_x \in \mathcal{J}_x\}$ and let $j \colon Y \hookrightarrow X$ be the inclusion. Then $(Y, j^{-1}(\mathcal{O}_{\mathfrak{X}}/\mathcal{J}))$ is a formal complex space, called the* closed formal subspace defined by $\mathcal{J}$.

PROOF. This is [Bi, (1.6)(2), p. 261]. Locally, finite generators of $\mathcal{J}$ give holomorphic equations on $\mathfrak{X}^{(0)}$, so Y is a closed analytic subset of that reduced space. The quotient stalks are local with residue field $\mathbb{C}$, as for the quotient models in Chapter III. The quoted result supplies the remaining completeness and finite-stage conditions for this quotient. □

Completion along an analytic subset

Let $(X, \mathcal{O}_X)$ be a complex space, and let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf. Put $Z := \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ and let $j \colon Z \hookrightarrow X$ be the inclusion. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, set

$$\widehat{\mathcal{F}} := \varprojlim_{\mu \geq 0} j^{-1}(\mathcal{F}/\mathcal{I}^{\mu+1}\mathcal{F}), \qquad \widehat{\mathcal{O}}_{X,\mathcal{I}} := \varprojlim_{\mu \geq 0} j^{-1}(\mathcal{O}_X/\mathcal{I}^{\mu+1}).$$

The latter is a sheaf of rings, and the former is a module over it. There is a natural homomorphism $j^{-1}\mathcal{O}_X \to \widehat{\mathcal{O}}_{X,\mathcal{I}}$. For formal completions of complex spaces and coherent sheaves on them, see also [BS, Chapter VI, §2].

PROPOSITION IV.2.18. *Let $(X, \mathcal{O}_X)$ be a complex space, let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf, and put $Z := \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$. The locally ringed space $\mathfrak{X} := (Z, \widehat{\mathcal{O}}_{X,\mathcal{I}})$ is a formal complex space. The inclusion of Z and the natural structure homomorphism give a morphism $i \colon \mathfrak{X} \to (X, \mathcal{O}_X)$. For $x \in Z$, the homomorphism $\mathcal{O}_{X,x} \to \widehat{\mathcal{O}}_{X,\mathcal{I},x}$ is injective.*

PROOF. Write $\mathcal{O}_{\mathfrak{X}} = \widehat{\mathcal{O}}_{X,\mathcal{I}}$. A section whose image in $j^{-1}(\mathcal{O}_X/\mathcal{I})$ is nonzero at x is a unit near x at every finite stage: the kernels of reduction to the zeroth stage are nilpotent. On one neighborhood where that zeroth image is a unit, the inverses at all stages exist and are compatible. They give an inverse in $\mathcal{O}_{\mathfrak{X}}$. Thus $\mathcal{O}_{\mathfrak{X},x}$ is local, with residue field $\mathbb{C}$, and the natural homomorphism from $\mathcal{O}_{X,x}$ is local. That $\mathfrak{X}$ satisfies the definition of a formal complex space is the completion result of [Bi, (1.6)(1), p. 261].

If a germ in $\mathcal{O}_{X,x}$ has zero image in $\mathcal{O}_{\mathfrak{X},x}$, its image in every $\mathcal{O}_{X,x}/\mathcal{I}_x^{\mu+1}$ is zero. It therefore lies in $\bigcap_\mu \mathcal{I}_x^{\mu+1}$. Since $\mathcal{O}_{X,x}$ is Noetherian and $\mathcal{I}_x \subset \mathfrak{m}_{X,x}$, Krull's intersection theorem (Theorem I.5.15) makes this intersection zero (See also [Bani, Lemma 2.1(i), pp. 232–233]). □

For the module properties, use the compact-set description of [Bani, Lemma 2.2, p. 234]. For sufficiently small semianalytic Stein compact neighborhoods K of x in X, put

$$A_K := \Gamma(K, \mathcal{O}_X), \qquad \mathfrak{a}_K := \Gamma(K, \mathcal{I}), \qquad M_K := \Gamma(K, \mathcal{F}).$$

Then A_K is Noetherian, M_K is finite over A_K, and

$$\widehat{\mathcal{F}}_x \cong \varinjlim_{K \ni x} \widehat{M_K}, \qquad \widehat{M_K} := \varprojlim_{\mu} M_K/\mathfrak{a}_K^{\mu+1} M_K.$$

Here the compact neighborhoods are taken in the ambient analytic space. The finite quotients are computed by Cartan's theorem B (Theorem III.2.165 (B)) on these compacta; Noetherianity is the analytic finiteness input used in [Bani, pp. 210 and 234]. The order of the two limits in this formula is important.

PROPOSITION IV.2.19. *Let $(X, \mathcal{O}_X)$ be a complex space, let $\mathcal{I} \subset \mathcal{O}_X$ be a coherent ideal sheaf, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Put $Z := \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$, let $j\colon Z \hookrightarrow X$ be the inclusion, and write $\mathfrak{X} := (Z, \widehat{\mathcal{O}}_{X,\mathcal{I}})$. Completion is exact on coherent $\mathcal{O}_X$-modules, and there is a natural isomorphism*

$$\widehat{\mathcal{O}}_{X,\mathcal{I}} \otimes_{j^{-1}\mathcal{O}_X} j^{-1}\mathcal{F} \cong \widehat{\mathcal{F}}.$$

In particular, $\widehat{\mathcal{F}}$ is coherent over $\widehat{\mathcal{O}}_{X,\mathcal{I}}$. If $\widehat{\mathcal{I}}$ denotes the ideal generated by the image of $j^{-1}\mathcal{I}$, then

$$\widehat{\mathcal{F}}/\widehat{\mathcal{I}}^{\mu+1}\widehat{\mathcal{F}} \cong j^{-1}(\mathcal{F}/\mathcal{I}^{\mu+1}\mathcal{F}) \qquad (\mu \geq 0).$$

Moreover, $i\colon \mathfrak{X} \to X$ is flat, meaning that $\widehat{\mathcal{O}}_{X,\mathcal{I},x}$ is flat over $\mathcal{O}_{X,x}$ for every $x \in Z$.

PROOF. For a short exact sequence of coherent sheaves, taking sections on the compacta just described gives a short exact sequence of finite A_K-modules. Completion preserves this exactness by the result recalled after Theorem I.5.15 in Chapter I, §5.5. Passing to the filtered limit over K proves exactness on stalks, hence exactness of the completed sheaves. This is the argument of [Bani, Lemma 2.3(i), pp. 234–235].

The natural tensor-product homomorphism in the statement sends a local tensor $a \otimes s$ to the product of a with the image of s at every finite stage. It is an isomorphism for finite free sheaves. Choose a finite presentation of $\mathcal{F}$ locally. Right exactness of tensor products and

of completion then identifies the two cokernels, proving the formula. Since the completed structure sheaf is coherent by Proposition IV.2.15, this presentation also proves coherence of $\widehat{\mathcal{F}}$.

Apply exactness to $0 \to \mathcal{I}^{\mu+1}\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I}^{\mu+1}\mathcal{F} \to 0$. The completion of the last term is itself, since its $\mathcal{I}$-adic filtration is eventually zero. By the tensor-product formula, the image of the first completed term is $\widehat{\mathcal{I}}^{\mu+1}\widehat{\mathcal{F}}$. This proves the quotient formula.

Finally, fix $x \in Z$. A short exact sequence of finite $\mathcal{O}_{X,x}$-modules can be represented by an exact sequence of coherent sheaves near x, by finite presentations and coherence. The preceding exactness and tensor-product formula show that tensoring it with $\widehat{\mathcal{O}}_{X,\mathcal{I},x}$ is exact. Every short exact sequence over the Noetherian ring $\mathcal{O}_{X,x}$ is the filtered union of such sequences, obtained by taking finite submodules of its middle term. Tensor products commute with these filtered limits, so tensoring is exact for arbitrary modules. This proves flatness, also asserted in [Bi, (1.6)(1), p. 261]. □

Remark IV.2.20. These arguments do not identify $\widehat{\mathcal{O}}_{X,\mathcal{I},x}$ with the $\mathcal{I}_x$-adic completion of $\mathcal{O}_{X,x}$. For example, for $X = \mathbb{C}^2$ with coordinates (z, w) and $\mathcal{I} = (w)$, the two rings are

$$\varinjlim_{U \in \mathcal{N}_{\mathbb{C}}(\{0\})} \mathcal{O}_{\mathbb{C}}(U)[[w]] \quad \text{and} \quad \mathbb{C}\{z\}[[w]],$$

respectively. Here $\mathcal{N}_{\mathbb{C}}(\{0\})$ denotes the open neighborhoods of 0 in the z-axis, with the ordering of Example II.3.20. The series $\sum_{k\geq 1}(1-kz)^{-1}w^k$ belongs to the second ring but not to the first: its coefficients cannot all be holomorphic on one common neighborhood of 0. For a center consisting of one point, however, the stalk is taken on a one-point space and this distinction disappears. □

Formal jets and their functoriality

Definition IV.2.21. Let $\mathcal{X}$ be an analytic neighborhood space, choose a model $(X, \mathcal{I})$, put $Z := Z(X, \mathcal{I})$, and let $j_Z \colon Z \hookrightarrow X$ be the inclusion. Its *formal jet*, or *formal completion*, has representative

$$\mathscr{J}_\infty(X, \mathcal{I}) := \big(Z, \widehat{\mathcal{O}}_{X,\mathcal{I}}\big), \qquad \widehat{\mathcal{O}}_{X,\mathcal{I}} = \varprojlim_{\mu \geq 0} j_Z^{-1}(\mathcal{O}_X/\mathcal{I}^{\mu+1}).$$

This is an object of (FCS) by Proposition IV.2.18. Its isomorphism class is denoted by $\mathcal{X}^{(\infty)}$, or $\mathscr{J}_\infty(\mathcal{X})$; functoriality below makes it independent of the model. As for finite jets, we also use $\mathcal{X}^{(\infty)}$ for the chosen representative and $\mathcal{O}_{\mathcal{X}^{(\infty)}}$ for its structure sheaf. □

Remark IV.2.22. The finite and formal jets are related by

$$\mathcal{O}_{\mathcal{X}^{(\infty)}} = \varprojlim_{\mu \geq 0} \mathcal{O}_{\mathcal{X}^{(\mu)}}.$$

The jet $\mathcal{X}^{(\mu)}$ uses the specified ideal of the model, whereas $\mathfrak{X}^{(k)}$ uses the intrinsic ideal of a formal complex space. The parentheses indicate a finite stage in both cases. With the

extended ideal $\widehat{\mathcal{I}}$ specified, Proposition IV.2.19 recovers those finite quotients. The formal complex space alone need not determine this specified ideal or the resulting numbered finite jets. If the model ideal is radical, its extended ideal equals the intrinsic ideal, so the same quotient formula identifies $(\mathcal{X}^{(\infty)})^{(\mu)}$ with $\mathcal{X}^{(\mu)}$. It should also be distinguished from an actual analytic neighborhood: formal sections need not converge on such a neighborhood. For example, $\sum_{k\ge 0} k!w^k \in \mathbb{C}[[w]]$ does not define a holomorphic germ at 0. □

Proposition IV.2.23. *Let $(X,\mathcal{O}_X)$ be a complex space. If coherent ideals $\mathcal{I},\mathcal{J}\subset\mathcal{O}_X$ have the same zero set Z, their formal completions are canonically isomorphic over X.*

Proof. The analytic Nullstellensatz (Theorem III.2.37) gives $\sqrt{\mathcal{I}}=\sqrt{\mathcal{J}}$. Near each point choose finite generators of the two ideals. A power of each generator of either ideal belongs to the other ideal after shrinking a common neighborhood. By taking a sufficiently high total degree in the generators, we obtain integers $a,b\ge 1$ with $\mathcal{I}^a\subset\mathcal{J}$ and $\mathcal{J}^b\subset\mathcal{I}$ there. Hence $\mathcal{I}^{a\mu}\subset\mathcal{J}^{\mu}$ and $\mathcal{J}^{b\mu}\subset\mathcal{I}^{\mu}$ for every $\mu\ge 1$. The quotient maps between these two cofinal systems induce inverse isomorphisms of their inverse limits. They do not depend on the chosen exponents and glue on overlaps, giving the asserted isomorphism over X (Compare [Bani, Lemma 2.1(ii), pp. 232–233]). □

Example IV.2.24. On $\mathbb{C}$ with coordinate w, take $\mathcal{I}=(w)$ and $\mathcal{J}=(w^2)$. The zeroth jets have rings $\mathbb{C}$ and $\mathbb{C}\{w\}/(w^2)$, so they are not isomorphic. Both formal jets have ring $\mathbb{C}[[w]]$. Their extended ideals are (w) and (w^2), whereas the intrinsic ideal $\mathfrak{n}$ of that formal space is (w) in either case. Thus $(\mathcal{X}^{(\infty)})^{(\mu)}$ and $\mathcal{X}^{(\mu)}$ need not coincide: for $\mathcal{I}=(w^2)$ their rings are respectively $\mathbb{C}[[w]]/(w^{\mu+1})$ and $\mathbb{C}[[w]]/(w^{2\mu+2})$. We retain the original ideal in the definition of a model. □

Proposition IV.2.25. *The assignment $(X,\mathcal{I})\mapsto\mathscr{J}_\infty(X,\mathcal{I})$ extends to a functor*

$$\mathscr{J}_\infty\colon(\mathsf{ANM})\xrightarrow{\text{functor}}(\mathsf{FCS}).$$

Proof. Let $[f]\colon(X,\mathcal{I})\to(Y,\mathcal{J})$ be a morphism represented near the reduced center by f. Put $\mathcal{X}:=[(X,\mathcal{I})]$ and $\mathcal{Y}:=[(Y,\mathcal{J})]$. Write $Z_X:=Z(X,\mathcal{I})$ and $Z_Y:=Z(Y,\mathcal{J})$, and let $f_0\colon Z_X\to Z_Y$ be the induced map. For their inclusions j_X,j_Y, put $\mathcal{F}_{X,\mu}:=j_X^{-1}(\mathcal{O}_X/\mathcal{I}^{\mu+1})$ and $\mathcal{F}_{Y,\mu}:=j_Y^{-1}(\mathcal{O}_Y/\mathcal{J}^{\mu+1})$. Proposition IV.2.3 gives compatible homomorphisms $\mathcal{F}_{Y,\mu}\to(f_0)_*\mathcal{F}_{X,\mu}$. Taking inverse limits on Z_Y yields

$$\widehat{\mathcal{O}}_{Y,\mathcal{J}}\longrightarrow\varprojlim_{\mu}(f_0)_*\mathcal{F}_{X,\mu}\cong(f_0)_*\widehat{\mathcal{O}}_{X,\mathcal{I}}.$$

The isomorphism follows by evaluating on each open subset of Z_Y: both sides take the inverse limit of sections on its preimage. Reduction to the zeroth stage shows that the resulting stalk homomorphisms preserve values and maximal ideals, so they are local. This gives the morphism $f^{(\infty)}\colon\mathcal{X}^{(\infty)}\to\mathcal{Y}^{(\infty)}$ in (FCS). It depends only on $[f]$, and preserves identities and compositions, because the same assertions hold at every finite stage. □

PROPOSITION IV.2.26. *Let $\mathcal{X}$ be an analytic neighborhood space. The classes $\mathcal{X}^{(\mu)}$ and $\mathcal{X}^{(\infty)}$ are independent of the model of $\mathcal{X}$. A specified isomorphism of models induces compatible isomorphisms of their finite and formal jets.*

PROOF. Apply the functors of Propositions IV.2.3 and IV.2.25 to the isomorphism and its inverse. □

For $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, completion on a model similarly defines $\widehat{\mathcal{F}}$ on $\mathcal{X}^{(\infty)}$. It is independent of shrinking the model, and is an exact functor on coherent analytic sheaves by Proposition IV.2.19.

For the diagrams of Definition IV.1.15, applying $\mathscr{J}_\mu$ to every object and arrow gives diagrams of finite jets. A comparison there consists of automorphisms $H_i^{(\mu)}$ of the finite jets, preserving the specified ideal filtration and other prescribed data, with

$$\mathscr{J}_\mu(F(a)) \circ H_i^{(\mu)} = H_j^{(\mu)} \circ \mathscr{J}_\mu(F_{\mathrm{std}}(a)).$$

This notation does not presume analytic representatives of $H_i^{(\mu)}$. A separate choice at every order is different from a compatible family under restriction. A compatible family gives an isomorphism of the corresponding formal diagrams by taking inverse limits; it need not be analytically realizable. Keeping the chosen $\mathcal{I}$-adic filtration is essential when $\mathcal{I}$ is not radical, as Example IV.2.24 shows.

2.3 Cohomology and restriction to finite and formal jets

Finite neighborhoods and formal data enter two related questions: how much of the complex structure of a neighborhood germ they determine, and how they control extension and uniqueness of analytic objects on a fixed neighborhood germ. This viewpoint appears in Griffiths' treatment of the extension problem [Gri, §I.2, §I.5 (c), and §II.1]. The first question will be considered in § 4.2 and, for holomorphic tubular neighborhoods, in § 5.1. Here we fix the analytic neighborhood space and examine the cohomological restriction maps, especially their kernels.

Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center Z, and let $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$. Choose a model $(X, \mathcal{I})$ and a coherent representative, again denoted by $\mathcal{F}$, on a neighborhood of Z in X. Write $j_Z \colon Z \hookrightarrow X$ for the inclusion, and put

$$\mathcal{F}_\mu := j_Z^{-1}(\mathcal{F}/\mathcal{I}^{\mu+1}\mathcal{F}) \quad (\mu \in \mathbb{Z}_{\geq 0}), \qquad \widehat{\mathcal{F}} := \varprojlim_\mu \mathcal{F}_\mu.$$

Thus $\mathcal{F}_\mu$ is coherent on $\mathcal{X}^{(\mu)}$ and $\widehat{\mathcal{F}}$ is coherent on $\mathcal{X}^{(\infty)}$, by Proposition IV.2.19. The finite jets here are those of the specified ideal $\mathcal{I}$, not the intrinsic thickenings of a formal complex space (Remark IV.2.22). For $q \in \mathbb{Z}_{\geq 0}$, quotient and completion induce maps

$$\begin{aligned} \rho_\mu^q &\colon H^q(\mathcal{X}, \mathcal{F}) \longrightarrow H^q(\mathcal{X}^{(\mu)}, \mathcal{F}_\mu), \\ \widehat{\rho}^{\,q} &\colon H^q(\mathcal{X}, \mathcal{F}) \longrightarrow H^q(\mathcal{X}^{(\infty)}, \widehat{\mathcal{F}}). \end{aligned} \tag{282}$$

More explicitly, the shrinking-neighborhood comparison in Remark II.3.113 identifies the source with $H^q(Z, j_Z^{-1}\mathcal{F})$. Apply cohomology to the sheaf maps $j_Z^{-1}\mathcal{F} \to \mathcal{F}_\mu$ and $j_Z^{-1}\mathcal{F} \to \widehat{\mathcal{F}}$ on Z. These constructions commute with shrinking the model, and ρ_μ^q is the composition of $\widehat{\rho}^q$ with the map induced by $\widehat{\mathcal{F}} \to \mathcal{F}_\mu$.

Proposition IV.2.27. *Let $\mathcal{X}$ have compact reduced center, let $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, and let $\mu, q \in \mathbb{Z}_{\geq 0}$. The space $H^q(\mathcal{X}^{(\mu)}, \mathcal{F}_\mu)$ is finite-dimensional and Hausdorff with its natural topology. The map ρ_μ^q in (282) is continuous for the natural topology on $H^q(\mathcal{X}, \mathcal{F})$.*

Proof. For an empty center all the groups are zero, so assume the center is non-empty. The finite jet is a compact complex space, possibly non-reduced. Apply the Cartan–Serre finiteness theorem (Theorem III.2.174). For continuity, choose a model and a finite Stein shrinking family as in Lemma IV.1.38. On each representative neighborhood the quotient map $\mathcal{F} \to \mathcal{F}/\mathcal{I}^{\mu+1}\mathcal{F}$ induces continuous maps of Fréchet Čech cochains, and hence of their cohomology quotients (Proposition III.2.166). The closed-embedding comparison (216) identifies the target with the cohomology on the finite jet, with the same Čech quotient topology. Passing to the locally convex final topology on the source proves continuity of ρ_μ^q. □

Proposition IV.2.28. *Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center, let $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, and fix $q \in \mathbb{Z}_{\geq 0}$. The natural comparison map is an isomorphism*

$$H^q(\mathcal{X}^{(\infty)}, \widehat{\mathcal{F}}) \;\cong\; \varprojlim_{\mu} H^q(\mathcal{X}^{(\mu)}, \mathcal{F}_\mu). \tag{283}$$

Proof. The assertion is immediate for an empty center. Otherwise, choose a model $(X, \mathcal{I})$ and a coherent representative of $\mathcal{F}$. Take a finite Stein open covering of a neighborhood of the compact center Z. Its intersections with Z form a finite open covering $\mathfrak{U}$ of Z. Every finite intersection of the ambient Stein open sets is Stein. Its closed subspace in each finite jet is therefore Stein, and the corresponding open formal subspace is Stein because its reduced space is Stein. Thus $\mathfrak{U}$ is a Leray covering both for each $\mathcal{F}_\mu$ and for $\widehat{\mathcal{F}}$, by the analytic Cartan theorem B (Theorem III.2.165) and the formal Cartan theorem B (Theorem IV.2.16 (ii)).

Consider the projective sequence of Čech complexes $\check{C}^\bullet(\mathfrak{U}, \mathcal{F}_\mu)$. On every finite intersection, the map on sections $\mathcal{F}_{\mu+1} \to \mathcal{F}_\mu$ is surjective: its kernel is coherent on the $(\mu+1)$-st finite jet, and Cartan theorem B annihilates its first cohomology on that Stein intersection. Consequently the transition maps on cochains are surjective in each degree, so they satisfy the Mittag–Leffler condition. Sectionwise computation of projective limits also gives

$$\check{C}^\bullet(\mathfrak{U}, \widehat{\mathcal{F}}) = \varprojlim_{\mu} \check{C}^\bullet(\mathfrak{U}, \mathcal{F}_\mu).$$

For $q \geq 1$, the groups $H^{q-1}(\mathcal{X}^{(\mu)}, \mathcal{F}_\mu)$ are finite-dimensional by the Cartan–Serre finiteness theorem (Theorem III.2.174). At each fixed stage the images from later stages form a decreasing sequence of finite-dimensional subspaces and hence stabilize. This is the second Mittag–Leffler condition required in Remark II.3.119. The exact sequence (90) therefore gives (283), using the Leray comparisons (Theorem II.3.79). For $q = 0$ the assertion is the sectionwise description of the projective limit. □

For $\mathcal{X}$, $\mathcal{F}$, and q as in Proposition IV.2.28, we endow the left-hand side of (283) with the topology transported from the projective limit on the right. That limit is a closed subspace of the countable product of the finite-dimensional Hausdorff cohomology spaces of the finite jets. It is therefore a Hausdorff Fréchet space (Mini-Exercise I.6.42 (*iii*)–(iv)). With this topology, $\widehat{\rho}^{\,q}$ is continuous by Proposition IV.2.27 and the defining property of the projective topology.

Definition IV.2.29. Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center, let $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, and fix $q \in \mathbb{Z}_{\geq 0}$. We set

$$K^q_{\mathrm{for}}(\mathcal{X},\mathcal{F}) := \operatorname{Ker} \widehat{\rho}^{\,q} = \bigcap_{\mu \geq 0} \operatorname{Ker} \rho^q_\mu. \tag{284}$$

The last equality follows from Proposition IV.2.28. □

For the data in Definition IV.2.29, $K^q_{\mathrm{for}}(\mathcal{X},\mathcal{F})$ is a closed vector subspace of $H^q(\mathcal{X},\mathcal{F})$, and

$$\overline{\{0\}} \;\subset\; K^q_{\mathrm{for}}(\mathcal{X},\mathcal{F}), \tag{285}$$

where the closure is taken in the natural topology on $H^q(\mathcal{X},\mathcal{F})$. Indeed, each ρ^q_μ is continuous with Hausdorff target. In particular, injectivity of $\widehat{\rho}^{\,q}$ requires $H^q(\mathcal{X},\mathcal{F})$ to be Hausdorff. The reverse implication does not follow from these topological considerations.

One can also describe each finite-order kernel on a model $(X,\mathcal{I})$ with a coherent representative $\mathcal{F}$. The exact sequence with first term $\mathcal{I}^{\mu+1}\mathcal{F}$ and quotient $\mathcal{F}/\mathcal{I}^{\mu+1}\mathcal{F}$ gives

$$\operatorname{Ker} \rho^q_\mu = \operatorname{Image}\big(H^q(\mathcal{X},\mathcal{I}^{\mu+1}\mathcal{F}) \longrightarrow H^q(\mathcal{X},\mathcal{F})\big).$$

Here the source group uses the germ of the coherent sheaf $\mathcal{I}^{\mu+1}\mathcal{F}$ on the chosen model.

Proposition IV.2.30. *Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center, let $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, and fix $q \in \mathbb{Z}_{\geq 0}$. Give $H^q(\mathcal{X},\mathcal{F})$ its natural topology and use the restriction maps (282).*

(i) If $H^q(\mathcal{X},\mathcal{F})$ is finite-dimensional, there exists $\mu_0 \in \mathbb{Z}_{\geq 0}$ such that

$$K^q_{\mathrm{for}}(\mathcal{X},\mathcal{F}) = \operatorname{Ker} \rho^q_\mu \qquad (\mu \geq \mu_0).$$

For each $\mu \geq \mu_0$, ρ^q_μ induces a TVS isomorphism from $H^q(\mathcal{X},\mathcal{F})/K^q_{\mathrm{for}}(\mathcal{X},\mathcal{F})$ onto $\operatorname{Image}\rho^q_\mu$, with the quotient and subspace topologies, respectively.

(ii) If $H^q(\mathcal{X},\mathcal{F})$ is infinite-dimensional and Hausdorff, every ρ^q_μ has a non-zero kernel. If $\widehat{\rho}^{\,q}$ is injective, these non-zero kernels have zero intersection.

(iii) If $H^q(\mathcal{X},\mathcal{F})$ is non-Hausdorff, then $K^q_{\mathrm{for}}(\mathcal{X},\mathcal{F}) \neq 0$.

Proof. In (i), the kernels form a decreasing sequence of subspaces in a finite-dimensional vector space, so they stabilize. Their intersection is (284). The source is Hausdorff by Corollary IV.1.42, and its quotient by the closed kernel is finite-dimensional and Hausdorff. The target image is a subspace of a finite-dimensional Hausdorff space. Thus the induced continuous bijection is a TVS isomorphism. In (ii), a linear map from an infinite-dimensional vector space to a finite-dimensional one cannot be injective; the intersection assertion is (284). Finally, in (iii) the closure of zero is non-zero, so (285) applies. □

Comparison with shrinking stages

Let $\mathcal{X}$, $\mathcal{F}$, and q be as in Definition IV.2.29, and choose a model and a finite Stein shrinking family as in Proposition IV.1.43. Retain its maps h^q_λ and $h^q_{\lambda,\mu}$. Here μ indexes a shrinking stage, so use $n \in \mathbb{Z}_{\geq 0}$ for the independent jet degree and put $\rho^q_{\lambda,n} := \rho^q_n \circ h^q_\lambda$. Then

$$\bigcap_{n\geq 0} \operatorname{Ker} \rho^q_{\lambda,n} = (h^q_\lambda)^{-1}\big(K^q_{\mathrm{for}}(\mathcal{X},\mathcal{F})\big).$$

In particular, $\operatorname{Ker} h^q_\lambda$ is contained in this intersection. If $K^q_{\mathrm{for}}(\mathcal{X},\mathcal{F}) = 0$ and (λ,μ) is q-admissible, the intersection equals $\operatorname{Ker} h^q_{\lambda,\mu}$. If the analytic cohomology is also finite-dimensional, (i) of Proposition IV.2.30 gives a single n_0 such that this equality already holds with $\operatorname{Ker} \rho^q_{\lambda,n_0}$. In contrast, a class which is non-zero in analytic cohomology but belongs to K^q_{for} cannot be made zero by choosing an admissible pair: admissibility detects vanishing in the analytic limit, not vanishing only after completion.

Extension and uniqueness

For $\mathcal{X}$, $\mathcal{F}$, and q as above, two classes in $H^q(\mathcal{X},\mathcal{F})$ have the same formal restriction precisely when their difference belongs to $K^q_{\mathrm{for}}(\mathcal{X},\mathcal{F})$. Thus $K^q_{\mathrm{for}} = 0$ expresses uniqueness of an analytic cohomology class with a given formal restriction.

There is also an extension consequence. Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

be an exact sequence of coherent sheaves on an analytic neighborhood space $\mathcal{X}$ with compact reduced center. For $s \in H^0(\mathcal{X},\mathcal{G})$, denote its connecting class by $\partial s \in H^1(\mathcal{X},\mathcal{F})$. Completion is exact on these sheaves (Proposition IV.2.19), and the connecting maps commute with completion. If the formal restriction $\widehat{s}$ lifts to a section of $\widehat{\mathcal{E}}$, then $\partial s \in K^1_{\mathrm{for}}(\mathcal{X},\mathcal{F})$. Consequently, when $K^1_{\mathrm{for}}(\mathcal{X},\mathcal{F}) = 0$, s has an analytic lift after shrinking a representative neighborhood of the whole center.

3 Cohomology of linear bundle neighborhoods

We now study linear bundle neighborhoods using the finite and formal spaces introduced above. The line-bundle examples of Chapter III, § 2.15, are the starting point; here the base is a compact reduced complex space and the bundle may have higher rank. Homogeneous degrees separate in this model, relating the natural topology of cohomology to the best primitive estimates before we turn to comparison problems.

Let Y be a non-empty compact reduced complex space, let $E \to Y$ be a holomorphic vector bundle of rank $r \geq 1$, and let $p\colon X \to Y$ be the total space of the dual bundle E^*. Write $Z := (\text{Zero Section})_{E^*}$ and identify Z with Y by $p|_Z$. In this section we consider the analytic neighborhood space

$$\mathcal{X}_E := [(X,\mathcal{I}_Z)], \qquad \mathcal{F}_\mu := \mathcal{O}_Y(S^\mu E) \quad (\mu \in \mathbb{Z}_{\geq 0}),$$

where S^0E is the trivial line bundle and $\mathcal{I}_Z$ is the reduced ideal of the zero section. The positive powers of E in $\mathcal{F}_\mu$ arise because the ambient space is the total space of E^*.

3.1 Homogeneous coefficients and their norms

Fix a C^∞ Hermitian metric h on E, and let h^* be its dual metric on E^*. For $y \in Y$, a symmetric tensor $s \in S^\mu E_y$ determines the homogeneous polynomial

$$P_s(\xi) := \langle s, \xi^{\otimes\mu} \rangle \quad (\xi \in E_y^*).$$

We identify $S^\mu E_y$ with the symmetric tensors in $E_y^{\otimes\mu}$ by averaging over the permutations, and give it the induced tensor metric $S^\mu h$. These choices specify the normalization in the following formula.

LEMMA IV.3.1. *For $t > 0$ and $s \in S^\mu E_y$, one has*

$$\int_{|\xi|_{h^*}=t} |P_s(\xi)|^2 \, d\sigma_{h^*,t} = \frac{2\pi^r \mu!}{(\mu + r - 1)!} \, t^{2\mu+2r-1} |s|^2_{S^\mu h}, \tag{286}$$

where $d\sigma_{h^*,t}$ is Euclidean surface measure for the Hermitian vector space (E_y^*, h^*). Moreover $|P_s(\xi)| \le |s|_{S^\mu h} |\xi|^\mu_{h^*}$.

PROOF. Choose an orthonormal basis $e_1, e_2, \ldots, e_r$ of E_y. For $\alpha \in (\mathbb{Z}_{\ge 0})^r$ with $|\alpha| = \mu$, the averaged symmetric tensor $e^{\odot\alpha}$ has squared norm $\alpha!/\mu!$ and corresponds to the monomial ξ^α. Distinct monomials are orthogonal on a sphere by its invariance under coordinatewise unitary rotations. The monomial integral is

$$\int_{|\xi|=t} |\xi^\alpha|^2 \, d\sigma_t = \frac{2\pi^r \alpha!}{(\mu + r - 1)!} \, t^{2\mu+2r-1}.$$

For example, this follows by computing the Gaussian integral on $\mathbb{C}^r$ first as a product and then in polar coordinates. Summing the monomial identities gives (286). The last inequality is the Schwarz inequality in the tensor product. For $r = 1$, the factor in (286) is $2\pi t^{2\mu+1}$, as for the circle norm used in Chapter III. □

Let $U \subset Y$ be open and let f be holomorphic on a neighborhood of the fiber balls $\{|\xi|_{h^*} \le t\}$ over U. Its fiberwise Taylor expansion is written $f = \sum_{\mu \ge 0} f_\mu$, with $f_\mu = P_{s_\mu}$ and $s_\mu \in \Gamma(U, \mathcal{F}_\mu)$. We use the notation $P_{<N} f := \sum_{\mu<N} f_\mu$ and $P_{\ge N} f := f - P_{<N} f$, as in Remark III.1.24. The bundle version of the selected-variable norm is

$$\|f\|_{U,t} := \sup_{y \in U} \left(\int_{|\xi|_{h^*}=t} |f(\xi)|^2 \, d\sigma_{h^*,t} \right)^{1/2}.$$

In particular, if the expansion converges on a slightly larger fiber ball, orthogonality gives

$$\|f\|^2_{U,t} = \sup_{y \in U} \sum_{\mu \ge 0} \frac{2\pi^r \mu!}{(\mu + r - 1)!} t^{2\mu+2r-1} |s_\mu(y)|^2_{S^\mu h}.$$

For a product one uses the mixed estimate $\|fg\|_{U,t} \le \|f\|_{L^\infty}\|g\|_{U,t}$, with the supremum in the first factor taken on the fiber spheres (Compare (118)).

If $|f| \le M$ on the fiber balls of radius R, the orthogonal projection onto degree μ and (286) imply

$$|s_\mu(y)|_{S^\mu h} \le M\binom{\mu+r-1}{r-1}^{1/2} R^{-\mu}. \tag{287}$$

Conversely, a bound $\sup_U |s_\mu| \le MB^\mu$ implies normal convergence of $\sum_\mu P_{s_\mu}$ on every smaller fiber ball of radius $t < B^{-1}$, by the last assertion of the lemma. The same statements hold on compact subsets of U when the bounds are local. The coefficients are defined in holomorphic bundle frames and measured by the Hermitian metric.

3.2 Finite jets, formal cohomology, and the zero class

Choose finite Stein coverings $\mathcal{W} = \{W_j\}_{j\in J}$, $\mathcal{U} = \{U_j\}_{j\in J}$, and $\mathcal{U}' = \{U'_j\}_{j\in J}$ of Y such that $\overline{U'_j}$ is a compact subset of U_j and $\overline{U_j}$ is a compact subset of W_j, and so that E is trivial on an open neighborhood of $\overline{W_j}$. Here all closures are in Y. Such choices follow by taking small Stein coordinate neighborhoods, a finite shrinking, and then smaller Stein neighborhoods of the resulting compact pieces. If necessary the finite index set is enlarged when making these choices. In a bundle trivialization the sets $U_j \times B^r_t$ are Stein. Their finite intersections in the Hausdorff complex space X are Stein as well: the intersection of two Stein open subspaces is a closed subspace of their product, by the diagonal embedding. The Oka–Cartan fundamental theorem B (Theorem III.2.165) therefore makes these coverings Leray for the structure sheaf. The same observation applies to the base coverings and every $\mathcal{F}_\mu$.

The unions of these product tubes, with sufficiently small common $t > 0$, form a cofinal family of neighborhoods of Z. Indeed, $\overline{U_j}$ is compact and the zero section over it lies in any prescribed neighborhood of Z. Finitely many trivializations then give a uniform positive radius. The same argument works with W_j in place of U_j. On these fixed compact base pieces a metric ball and a product ball have radii comparable by fixed positive factors.

For every $q, \mu \ge 0$, $H^q(Y, \mathcal{F}_\mu)$ is finite-dimensional and Hausdorff. In the present reduced setting, apply Lemma III.2.122 to the two finite Stein coverings: the restriction map in that lemma is an isomorphism by Leray comparison. For a non-singular base one may also use Lemma III.2.121. Cohomology of $\mathcal{X}_E$ will always have the natural topology of §1 of this chapter. In using its compact restriction description, we shrink the base patches as well as the fiber radii, as in Lemma IV.1.38 and Proposition IV.1.40.

Proposition IV.3.2. *For the analytic neighborhood space $\mathcal{X}_E$ above and every $q, \mu \ge 0$, coefficient extraction gives isomorphisms*

$$H^q(\mathcal{X}_E^{(\mu)}, \mathcal{O}_{\mathcal{X}_E^{(\mu)}}) \cong \bigoplus_{\nu=0}^{\mu} H^q(Y, \mathcal{F}_\nu),$$
$$H^q(\mathcal{X}_E^{(\infty)}, \mathcal{O}_{\mathcal{X}_E^{(\infty)}}) \cong \prod_{\nu\ge 0} H^q(Y, \mathcal{F}_\nu).$$

The topology on the last product is the projective topology. The analytic-to-formal map $\widehat{\rho}^q$ is given by all homogeneous coefficient classes.

PROOF. The projection p and polynomial evaluation identify $\mathcal{O}_{\mathcal{X}_E^{(\mu)}}$, as an $\mathcal{O}_Y$-module, with $\bigoplus_{\nu=0}^{\mu}\mathcal{F}_\nu$. Multiplication is the polynomial multiplication with terms of degree greater than μ discarded. Transition matrices are linear in the fiber, so that they preserve each homogeneous degree. Finite direct sums commute with the cohomology computation on the fixed Leray covering. The transition maps of finite jets discard the last summand. Consequently, the cohomology systems in degrees q and $q-1$ are split surjective (with the negative cohomology group interpreted as zero). Both Mittag–Leffler conditions required in Proposition IV.2.28 are satisfied. That proposition gives the second isomorphism and its topology. Finally, Taylor expansion commutes with restriction and with the Čech differential, proving the assertion about $\widehat{\rho}^q$. □

PROPOSITION IV.3.3. *For every $q \geq 0$ one has*

$$K^q_{\mathrm{for}}(\mathcal{X}_E, \mathcal{O}_{\mathcal{X}_E}) = \overline{\{0\}} \quad \text{in } H^q(\mathcal{X}_E, \mathcal{O}_{\mathcal{X}_E}).$$

In particular this cohomology space is Hausdorff if and only if $\widehat{\rho}^q$ is injective.

PROOF. The finite coefficient maps are continuous and have Hausdorff targets. Hence the closure of zero is contained in their common kernel. For the converse, represent a formal-zero class by a holomorphic Čech cocycle g on a finite product-tube covering. After passing from $\mathcal{W}$ to $\mathcal{U}$ and decreasing the radius, its homogeneous expansion converges uniformly on compact subsets of every intersection. Write g_μ for the corresponding $\mathcal{F}_\mu$-valued cocycle on the base. Each g_μ is a coboundary on the fixed Leray covering, since its cohomology class vanishes. Choose a primitive for each coefficient. For any N, the finite sum of these polynomial primitives shows that $P_{<N}g$ is a coboundary on the same smaller tube covering. Since $P_{<N}g \to g$ in the Fréchet cochain topology, the class of g belongs to the closure of zero in its quotient topology, and hence in the neighborhood cohomology. When $q = 0$, coefficientwise vanishing simply says that $g = 0$. □

EXAMPLE IV.3.4. Let Y be an elliptic curve and let $F \to Y$ be a unitary flat holomorphic line bundle. Let X_F be its total space, put $Z := (\text{Zero Section})_F$, and consider the analytic neighborhood space $\mathcal{X}_F := [(X_F, \mathcal{I}_Z)]$ along its zero section. This is the preceding construction with $E = F^*$, so that $S^\nu E = F^{-\nu}$. If F is non-torsion, then $H^1(Y, \mathcal{O}_Y(F^{-\nu})) = 0$ for every fiber degree $\nu \geq 1$, by (241). Thus every finite and the formal first cohomology are $H^1(Y, \mathcal{O}_Y) \cong \mathbb{C}$. The map from analytic to formal cohomology is degree-zero coefficient extraction, and pullback from Y is a continuous section of it. When F is asymptotically positive it is a TVS isomorphism by (280). When F is asymptotically zero it is still surjective, but its kernel is the non-zero closure of zero by Example IV.1.44 and Proposition IV.3.3. In the latter case, the explicit cocycle (245) is formally zero, but remains non-zero on every analytic neighborhood of the whole zero section.

If F has order $m \geq 1$, choose a unitary trivialization of F^m and let u be the resulting global holomorphic function on X_F, locally $u = w_j^m$ in compatible unitary frames (Proposition III.2.205). Only the coefficient degrees divisible by m survive in first cohomology. Hence

the analytic-to-formal map is

$$H^1(Y,\mathcal{O}_Y)\otimes\mathbb{C}\{u\}\longrightarrow H^1(Y,\mathcal{O}_Y)\otimes\mathbb{C}[[u]], \tag{288}$$

induced by the inclusion of convergent power series into formal power series. Its restriction to the μ-jet truncates after $u^{\lfloor\mu/m\rfloor}$. After choosing a non-zero element of the one-dimensional $H^1(Y,\mathcal{O}_Y)$, this becomes $\mathbb{C}\{u\}\hookrightarrow\mathbb{C}[[u]]$. It is injective but not surjective, as the formal series $\sum_{k\geq 0}k!u^k$ is not convergent on any disc of positive radius. In particular, the formal kernel is zero even though the analytic cohomology is infinite-dimensional.

The topology of $\mathbb{C}\{u\}$ here is the locally convex final germ topology of (279), not the subspace topology from the product $\mathbb{C}[[u]]$. For example, $N!u^N$ tends to zero coefficientwise in the latter space, but not in the germ topology: the linear map $\sum a_k u^k\mapsto\sum a_k/k!$ is continuous on every $\mathcal{O}_{\Delta_\rho}(\Delta_\rho)$ by Cauchy's estimate, and hence on the germ space, whereas its value on $N!u^N$ is one. This also distinguishes injectivity of (288) from a topological embedding. □

3.3 The best bounded primitive constants

Retain the fixed coverings $\mathcal{U}$ and $\mathcal{U}'$. For $\mathcal{F}_\mu$-valued cochains put

$$\|a\|_\infty:=\max_j\sup_{U_j}|a_j|_{S^\mu h},\qquad \|g\|_\infty:=\max_{j,k}\sup_{U_j\cap U_k}|g_{jk}|_{S^\mu h}.$$

We use the Čech sign convention $(\check{\delta}a)_{jk}=a_k-a_j$, with both sections interpreted in the same bundle fiber. Let $\check{B}^1_\infty(\mathcal{U},\mathcal{F}_\mu)$ be the space of bounded holomorphic 1-cochains which are coboundaries in the ordinary holomorphic Čech complex. Define

$$\kappa_\mu:=\sup_{0\neq g\in\check{B}^1_\infty(\mathcal{U},\mathcal{F}_\mu)}\frac{\inf_{\check{\delta}a=g,\ \|a\|_\infty<\infty}\|a\|_\infty}{\|g\|_\infty},\qquad C_\mu:=\max\{1,\kappa_\mu\}. \tag{289}$$

In this definition the supremum over an empty set is defined to be 0. The infimum is taken modulo the freedom to add a global holomorphic section. Its attainment, or the existence of a continuous linear choice of primitives, is not assumed.

Lemma IV.3.5. *Each C_μ is finite. Every bounded exact 1-cochain g admits a bounded primitive a with $\|a\|_\infty\leq 2C_\mu\|g\|_\infty$.*

Proof. Fix μ. The bounded holomorphic cochain spaces, with the norms just defined, are Banach spaces. This follows from uniform convergence and the completeness of bounded holomorphic sections on each patch; fixed bundle frames on neighborhoods of the compact closures compare the metric norms with coefficient norms. The ordinary Fréchet space of holomorphic 1-coboundaries is closed in the space of 1-cochains: the cocycle space is closed, and its quotient $H^1(Y,\mathcal{F}_\mu)$ is Hausdorff. The inclusion of bounded cochains into Fréchet cochains is continuous. Its inverse image $\check{B}^1_\infty$ is therefore a closed Banach subspace.

If g is in this subspace, choose an ordinary holomorphic primitive a. It is bounded on every U'_k. For $y \in U_j$, choose k with $y \in U'_k$; the cochain equation gives $a_j(y) = a_k(y) - g_{jk}(y)$. Thus

$$\|a\|_{L^\infty(\mathcal{U})} \leq \|a\|_{L^\infty(\mathcal{U}')} + \|g\|_\infty < \infty.$$

The bounded differential from bounded 0-cochains onto $\check{B}^1_\infty$ is consequently surjective. The open mapping theorem for Banach spaces (Theorem I.6.46) gives a finite quotient estimate, which is precisely the finiteness of (289). Choosing within a fixed factor of the infimum gives $2C_\mu$; for $g = 0$ choose $a = 0$. □

THEOREM IV.3.6. *Let $Y, E, h, \mathcal{U}, \mathcal{U}'$ and C_μ be as above. The following are equivalent:*

(*i*) *$H^1(\mathcal{X}_E, \mathcal{O}_{\mathcal{X}_E})$ is Hausdorff.*

(*ii*) *The formal restriction map $\widehat{\rho}^1$ is injective.*

(*iii*) *There are constants $A, B > 0$ such that $C_\mu \leq AB^\mu$ for every $\mu \geq 1$.*

PROOF. The equivalence of (*i*) and (*ii*) is Proposition IV.3.3. Suppose (*iii*) holds and let a formal-zero class be represented on product tubes over the larger fixed covering $\mathcal{W}$. We may use this covering for any neighborhood germ: compactness of its closures gives a common tube radius contained in a representative neighborhood, and the tubes are Leray. On the middle covering $\mathcal{U}$, compactness and (287) bound its coefficient cocycles by

$$\|g_\mu\|_\infty \leq M \binom{\mu + r - 1}{r - 1}^{1/2} R^{-\mu}$$

for some $M, R > 0$, independent of μ. Each is exact. Choose primitives by Lemma IV.3.5. After increasing A to handle $\mu = 0$, their norms are bounded by $2AM\binom{\mu+r-1}{r-1}^{1/2}(B/R)^\mu$. The polynomial series of primitives therefore converges on a common smaller metric ball with radius less than R/B. Product tubes inside that ball give an actual open neighborhood on which the series is holomorphic. Compact convergence permits applying $\check{\delta}$ term by term, and its coboundary is the original cocycle. This proves (*ii*).

Conversely suppose (*iii*) fails. Since each C_μ is finite, there is a strictly increasing sequence μ_k for which $C_{\mu_k}^{1/\mu_k} \to \infty$ and $C_{\mu_k} > 2$. By the definition of the best constants, choose exact cochains g_{μ_k} of norm 1 for which every bounded primitive has norm greater than $C_{\mu_k}/4$. The homogeneous polynomial series

$$g := \sum_{k \geq 1} P_{g_{\mu_k}}$$

converges normally on the intersections of sufficiently small product tubes over $\mathcal{U}$: polynomial evaluation bounds its k-th term by $|\xi|_{h^*}^{\mu_k}$. It defines a cocycle and every coefficient class is zero.

Assume for a contradiction that its class becomes zero on a neighborhood of Z. Choose smaller product tubes T_j over *the same* U_j contained in that neighborhood and in the initial domain of g. This is possible since $\overline{U_j}$ lies in a fixed larger trivializing patch. The restricted

class is zero on $\bigcup_j T_j$, and its Stein Leray covering $\{T_j\}$ computes cohomology. Hence it has a holomorphic primitive a_j on T_j, even if the original vanishing was first witnessed on a refinement. Cauchy estimates on the compactly contained base patches U_j' and on smaller fiber spheres give constants $M_1, B_1 > 0$ such that

$$\|a_\mu\|_{L^\infty(\mathcal{U}')} \le M_1 B_1^\mu \quad (\mu \ge 0),$$

after absorbing the fixed polynomial factor in (287) into B_1^μ. The coefficient equation on $U_j \cap U_k$ and the fact that $\mathcal{U}'$ covers Y now give

$$\|a_{\mu_k}\|_{L^\infty(\mathcal{U})} \le M_1 B_1^{\mu_k} + 1.$$

These are bounded primitives for the chosen g_{μ_k}, in contradiction with $C_{\mu_k}^{1/\mu_k} \to \infty$. Thus the analytic class of g is non-zero, while its formal restriction vanishes; (ii) fails. □

The use of best quotient constants matters in the converse. An arbitrarily enlarged sequence of permissible constants could have superexponential growth even when the cohomology is Hausdorff. Also, the exponential bound above is a criterion for a *linear* coefficient equation on a neighborhood germ. It does not by itself prove convergence of a nonlinear coordinate change.

PROPOSITION IV.3.7. *For every $q \ge 0$, polynomial representatives define an injection*

$$\bigoplus_{\mu \ge 0} H^q(Y, \mathcal{F}_\mu) \longrightarrow H^q(\mathcal{X}_E, \mathcal{O}_{\mathcal{X}_E}).$$

If the latter space is finite-dimensional, then $H^q(Y, \mathcal{F}_\mu) = 0$ for all sufficiently large μ, and for all sufficiently large integers N the coefficient maps give TVS isomorphisms

$$H^q(\mathcal{X}_E, \mathcal{O}_{\mathcal{X}_E}) \cong \bigoplus_{\mu=0}^{N} H^q(Y, \mathcal{F}_\mu) \cong H^q(\mathcal{X}_E^{(N)}, \mathcal{O}_{\mathcal{X}_E^{(N)}}).$$

For $q = 1$, finite-dimensionality is equivalent to eventual vanishing of the coefficient groups together with the exponential estimate in Theorem IV.3.6.

PROOF. A finite sum of coefficient cocycles is polynomial in the fiber and hence represents an analytic class. Extracting its coefficients recovers the original classes, proving injectivity. Finite-dimensionality therefore forces eventual vanishing. Moreover, Corollary IV.1.42 makes this finite-dimensional analytic cohomology Hausdorff. Proposition IV.3.3 gives injectivity of its formal restriction. The remaining finitely many coefficient classes all have polynomial representatives, so that this restriction is also surjective onto their finite sum. The maps are continuous between finite-dimensional Hausdorff spaces, and are consequently TVS isomorphisms. Conversely, in degree 1 the exponential bound makes the formal restriction injective, and eventual vanishing makes its target finite-dimensional. The first part then applies. □

Remark IV.3.8. Eventual vanishing alone does not give the last conclusion. Example IV.3.4 provides a non-torsion unitary flat line bundle F over an elliptic curve. Its positive-degree coefficient groups vanish, but its neighborhood cohomology is not Hausdorff. In the present notation take $E = F^*$: the coefficients E^μ are exactly $F^{-\mu}$ in that example.

The same identification $E = F^*$ applies to a non-torsion unitary flat line bundle F over any compact Riemann surface Y, with coefficient sheaves $\mathcal{O}_Y(F^{-\mu})$. For this comparison, choose unitary trivializing coverings and give E the dual of a flat metric on F. Proposition IV.3.2 identifies $\widehat{\rho}^1([g])$ with the sequence of coefficient classes $([s_\mu])_{\mu\geq 0}$. If these classes vanish and $\alpha(F) > 0$, Lemma III.2.206 gives a primitive after shrinking: from a disc subbundle of radius r one may pass to any radius $0 < s < \alpha(F)r$. When $\alpha(F) = 0$, the cocycle of Proposition III.2.207 gives a non-zero class with all coefficient classes zero. Thus Proposition IV.3.3 shows that $H^1(\mathcal{X}_E, \mathcal{O}_{\mathcal{X}_E})$ is Hausdorff if and only if $\alpha(F) > 0$. On fixed unitary trivializing coverings, Ueda's estimate (Theorem III.2.204), applied only to exact coefficient cocycles, also gives $C_\mu \leq \max\{1, K/d(\mathbb{I}_Y, F^\mu)\}$ for $\mu \geq 1$ and a constant K independent of μ. For $\alpha(F) > 0$ this is an exponential bound. □

Remark IV.3.9. The criterion is unchanged if h is replaced by another metric h_1 satisfying $D^{-1}h \leq h_1 \leq Dh$ for some $D \geq 1$. Indeed, the symmetric-power norms differ by factors between $D^{-\mu/2}$ and $D^{\mu/2}$, so the quotient constants differ by at most D^μ. On a compact base any two C^∞ positive-definite metrics satisfy such a comparison. The same observation permits a specified pointwise measurable positive-definite metric with these two-sided bounds. We then use the pointwise supremum and take open geometric neighborhoods using an auxiliary smooth metric. For a line bundle this includes bounded real weights; in higher rank both the metric and its inverse must be controlled. □

3.4 Comparison with estimates for the Dolbeault equation

We now restrict to a non-singular compact connected base Y, with a fixed Hermitian metric ω, and to a holomorphic line bundle L with a C^∞ Hermitian metric h_L. For a smooth L-valued $(0,1)$-form v whose class in $H^{0,1}(Y, L)$ is zero, consider the least constant $K_{\overline{\partial}}(L, h_L, \omega)$ in the estimate

$$\|u\|_{L^2(h_L,\omega)} \leq K_{\overline{\partial}}(L, h_L, \omega)\, \|v\|_{L^2(h_L,\omega)}, \qquad \overline{\partial} u = v,$$

where u is chosen perpendicular to the holomorphic sections. These L^2 norms are taken over the base Y.

Here is the elementary direction of the comparison. Choose a fixed smooth partition of unity $\{\chi_k\}$ subordinate to a finite covering and let $g = \{(U_j \cap U_k, g_{jk})\}$ be a holomorphic exact cocycle. The smooth local sections $a_j := -\sum_k \chi_k g_{jk}$ satisfy $a_k - a_j = g_{jk}$. Their $\overline{\partial}$-derivatives agree on overlaps, so define a global exact form v. The partition and its first derivatives give bounds for a_j and v in terms of the Čech base L^2 norm of g. Subtracting a global solution u of $\overline{\partial} u = v$ gives the holomorphic primitive $a_j - u$. Consequently its Čech L^2 quotient constant is bounded by $\alpha + \beta K_{\overline{\partial}}(L, h_L, \omega)$. The constants depend on the fixed covering, partition, and base metric, not on the bundle metric: multiplication by the

fixed scalar cutoffs has the same pointwise bound for every h_L. For the precise nested-cover formulation on a compact Riemann surface, see [Og, Proposition 3.1].

If L is unitary flat and h_L is flat, holomorphic unitary frames allow the mean value inequality to convert local base L^2 bounds to supremum bounds on the smaller patches, with constants independent of L. The identity $a_j = a_k - g_{jk}$ then propagates bounds from a shrinking covering to the original one. In the other direction, local solutions of $\overline{\partial}$ on the fixed Stein patches give holomorphic differences; when the original form is exact, those differences are Čech coboundaries. Solving their bounded cochain equation and subtracting the resulting holomorphic sections glues the local solutions. This is the comparison used in [HKM, §2]; its uniform theorem states that, for every nontrivial unitary flat L, there is a constant K, independent of L, such that

$$K_{\overline{\partial}}(L, h_L, \omega) \leq \frac{K}{d(\mathbb{I}_Y, L)}$$

([HKM, Theorem 1.3]). Here d is the invariant distance on the group of unitary flat line bundles used in Theorem III.2.204, and $\mathbb{I}_Y$ is the trivial bundle. The solution is unique in this nontrivial flat case by Lemma III.2.197. For the trivial bundle one instead takes the quotient by constant sections and does not divide by $d(\mathbb{I}_Y, L) = 0$. The corresponding supremum-norm estimates on compact Riemann surfaces, including the dependence on the gap between nested patches, are given in [Og, §3.4, Propositions 3.10–3.11].

3.5 Positive line bundles and strongly 1-convex neighborhoods

Let Y be a compact reduced complex space and let $L \to Y$ be a holomorphic line bundle. Suppose that L has a C^∞ Hermitian metric h whose weights $|e_j|_h^2 = e^{-\varphi_j}$ admit, in local embeddings of Y, extensions which are simultaneously C^∞ and strictly plurisubharmonic. This is the metric meaning of positivity used here on reduced spaces (Definition III.2.87); positivity only on the regular locus would not be the same assumption. Apply the preceding construction with $E = L$.

Proposition IV.3.10. *On the total space $X = L^*$, put $Z = (\text{Zero Section})_{L^*}$ and $\rho(\xi) := |\xi|_{h^*}^2$. Then ρ is a proper nonnegative C^∞ plurisubharmonic function, is strictly plurisubharmonic off Z, and has $\rho^{-1}(0) = Z$. For every $c > 0$, the set $W_c := \{\rho < c\}$ is a relatively compact strongly 1-convex neighborhood of Z, and these sets form a neighborhood basis as $c \searrow 0$.*

Proof. In a dual frame write $\xi = w_j e_j^*$. Then $\rho = e^{\varphi_j}|w_j|^2$; the sign in the exponent is positive because this is the *dual* metric. On a smooth local ambient extension, the Levi form in a base direction a and a fiber direction b is

$$\mathcal{L}_\rho(a, b) = e^{\varphi_j}\left(|b + w_j \partial\varphi_j(a)|^2 + |w_j|^2 \mathcal{L}_{\varphi_j}(a)\right). \tag{290}$$

This proves semi-positivity everywhere and strict positivity when $w_j \neq 0$. At $w_j = 0$ the base directions remain in the kernel. The simultaneous ambient extensions make the same argument valid when Y is singular. Compactness of Y and comparison with product balls

show that ρ is proper and that W_c has compact closure in X. On W_c the function $\rho/(c-\rho)$ is an exhaustion, is plurisubharmonic everywhere, and is strictly plurisubharmonic off Z. This gives strong 1-convexity in the sense of Definition III.2.151. The radial derivative on a positive level is $2\rho \neq 0$. In particular, when Y is non-singular the boundary of W_c is smooth and strongly pseudoconvex. □

Remark IV.3.11. The positive-dimensional irreducible compact analytic subsets of X are contained in Z: otherwise the strictly plurisubharmonic function ρ has a maximum on such a subset away from Z. Thus the zero section is the locus relevant to the exceptional set and contraction results of Chapter III.

The relation between homogeneous coefficients and cohomological finiteness already appears in Grauert's [Gr3, §3, pp. 341–344, especially Hilfssatz 1 on p. 344]. In the line bundle setting above, the Andreotti–Grauert finiteness theorem for W_c and coherent coefficients, followed by the injection of polynomial coefficient classes, gives

$$H^q(Y, \mathcal{O}_Y(L^\mu)) = 0 \quad \text{for } \mu \gg 0 \quad (q \geq 1).$$

Indeed, each such group injects as a distinct homogeneous summand in the finite-dimensional $H^q(W_c, \mathcal{O}_{W_c})$. This is the same finite-dimensionality mechanism as in Proposition IV.3.7. Grauert treats the vector bundle case with coherent twists in the cited section. □

4 Comparison problems and finite-order obstructions

We compare the whole transition of a neighborhood, only its transverse values, or a bundle over a fixed neighborhood. These problems differ both in their target and in the changes which are allowed. The linear models in § IV.3 will provide coefficient spaces and estimates; first we specify the comparison whose coefficients are to be corrected.

4.1 Comparison problems and admissible changes

Let $\mathfrak{I}$ be a small category, and let $F, F_{\mathrm{std}} \colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{ANM})$ be two fixed diagrams. Write $F(j) = (V_j, \mathcal{I}_j)$, and denote its reduced center by Z_j. For a morphism $a \colon i \to j$, put $\phi_a := F(a)$. We seek morphisms $H_j \colon F(j) \to F_{\mathrm{std}}(j)$ making the following square commute:

$$\begin{array}{ccc} F(i) & \xrightarrow{\ \phi_a\ } & F(j) \\ {\scriptstyle H_i}\downarrow & & \downarrow{\scriptstyle H_j} \\ F_{\mathrm{std}}(i) & \xrightarrow[F_{\mathrm{std}}(a)]{} & F_{\mathrm{std}}(j). \end{array}$$

On representatives, start at the same point of the source: first apply ϕ_a and read its value by H_j, or first read its value by H_i and then use the prescribed standard arrow. The two values are required to agree. In terms of morphisms of models, the condition is

$$H_j \circ F(a) = F_{\mathrm{std}}(a) \circ H_i \qquad (a \colon i \to j). \tag{291}$$

Thus $H\colon F \xrightarrow{\text{nat}} F_{\text{std}}$ is a natural transformation. The objects $F(j)$ and $F_{\text{std}}(j)$ need not be equal, and the components of H need not be isomorphisms. All equalities are equalities of germs along the source centers; for a covering diagram, the restricted chart models and inclusion arrows are those in Example IV.1.16 (*iii*).

Comparison of entire neighborhoods, transverse comparison, and bundle comparison fit this framework as follows:

Problem	Prescribed target	Permitted change
Comparison of neighborhoods	All transitions of a fixed neighborhood model	Local biholomorphisms, matching all components
Transverse comparison	Linear transitions for the defining maps	Changes of the source which realize the permitted defining maps
Bundle comparison	A specified bundle over the same neighborhood	Frame changes over the identity of the base

The second target can have point centers even when Z_j has positive dimension. The linear arrows still record the transition of the normal data. In the third problem, the base is fixed and both bundles are specified.

We keep F_{std} fixed and now specify the permitted components. Choose initial comparison morphisms $q_j\colon F(j) \to F_{\text{std}}(j)$ and subgroups $G_j \subset \operatorname{Aut}_{(\mathsf{ANM})}(F(j))$. In a comparison realized by these source groups, the vertical arrow is a composite

$$F(j) \xrightarrow{h_j^{-1}} F(j) \xrightarrow{q_j} F_{\text{std}}(j), \qquad H_j = q_j \circ h_j^{-1}, \quad h_j \in G_j.$$

The group specifies which changes of the original geometry are permitted.

For a comparison of neighborhoods with chosen identifications $F(j) = F_{\text{std}}(j)$, take $q_j = \mathrm{id}$. Then $H_j = h_j^{-1}$, and (291) becomes $h_j^{-1} \circ F(a) \circ h_i = F_{\text{std}}(a)$. The inverses h_j are exactly the components in Definition IV.1.15; the direction of (267) is unchanged. For a transverse comparison, q_j is the initial defining map w_j and only the values $w_j \circ h_j^{-1}$ are compared. For a bundle comparison, local trivializations realize the source changes as $(x, v) \mapsto (x, A_j(x)v)$ with $A_j(x)$ invertible. These maps leave the base point x fixed.

A source change may depend on a base variable. For example, $(z, w) \mapsto (z, w + a(z)w^2)$ is locally biholomorphic along $w = 0$ for a holomorphic function a. A change on the point-centered target alone cannot encode a nonconstant $a(z)$. In the coefficient constructions below, we shall specify how the corrections are realized by these source changes.

Automorphisms and finite jets

We record the automorphism groups from which the source groups are chosen. Their local expressions distinguish the jet of a source change from the change induced in the comparison component after composition with q_j.

Definition IV.4.1. Let $\mathcal{X}$ be an analytic neighborhood space and choose a model $(X, \mathcal{I})$. The *automorphism group* of $\mathcal{X}$ is the group of self-isomorphisms of this model, with composition as group law:

$$\mathrm{Aut}(\mathcal{X}) := \mathrm{Aut}_{(\mathsf{ANM})}(X, \mathcal{I}).$$

Put $Z := Z(X, \mathcal{I})$. We write

$$\mathrm{Aut}_{\mathrm{center}}(\mathcal{X}) := \{f \in \mathrm{Aut}_{(\mathsf{ANM})}(X, \mathcal{I}) \mid f_{\mathrm{center}} = \mathrm{id}_Z\}.$$

Thus the latter group consists of the automorphisms fixing the reduced center pointwise. For $\mu \in \mathbb{Z}_{\geq 0}$ we also put

$$\mathrm{Aut}_{(\mu)}(\mathcal{X}) := \{a \in \mathrm{Aut}(\mathcal{X}) \mid a^{(\mu)} = \mathrm{id}_{\mathcal{X}^{(\mu)}}\}.$$

This is a subgroup of analytic automorphisms, not the automorphism group of the finite jet itself. □

By Lemma IV.1.7, an automorphism is represented by a biholomorphism $f\colon U \to V$ between two open neighborhoods of Z in X such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_U = \mathcal{I}|_U$. The neighborhoods U and V need not coincide. Composition and inversion are understood as operations on germs, so a smaller neighborhood may be used whenever they are represented by actual maps. The map $f \mapsto f_{\mathrm{center}}$ is a group homomorphism to $\mathrm{Aut}_{(\mathsf{CpxSp})}(Z)$, and $\mathrm{Aut}_{\mathrm{center}}(\mathcal{X})$ is its kernel.

If $h\colon (X, \mathcal{I}) \to (X', \mathcal{I}')$ is an isomorphism of models, then

$$f \longmapsto h \circ f \circ h^{-1}$$

is an isomorphism of their automorphism groups. Replacing h changes this identification by an inner automorphism of the target group. Thus $\mathrm{Aut}(\mathcal{X})$ is independent of the model up to the specified conjugation.

Proposition IV.4.2. *Let $\mathcal{X}$ be an analytic neighborhood space. For each $\mu \in \mathbb{Z}_{\geq 0}$, the finite and formal jet functors induce group homomorphisms*

$$\rho_\mu\colon \mathrm{Aut}(\mathcal{X}) \longrightarrow \mathrm{Aut}_{(\mathsf{CpxSp})}(\mathcal{X}^{(\mu)}),$$
$$\rho_\infty\colon \mathrm{Aut}(\mathcal{X}) \longrightarrow \mathrm{Aut}_{(\mathsf{FCS})}(\mathcal{X}^{(\infty)}).$$

These automorphisms are compatible with the closed embeddings of finite jets and with the homomorphisms from the formal structure sheaf to its finite quotients.

Proof. Apply Propositions IV.2.3 and IV.2.25 to an automorphism and its inverse. The compatibility follows from the quotient maps used to construct these functors. □

For $\mu \in \mathbb{Z}_{\geq 0}$, we say that f is *the identity to order μ* when $\rho_\mu(f) = \mathrm{id}_{\mathcal{X}^{(\mu)}}$. Thus $\mathrm{Aut}_{(\mu)}(\mathcal{X}) = \mathrm{Ker}\,\rho_\mu$; these groups form a decreasing sequence of normal subgroups of $\mathrm{Aut}(\mathcal{X})$. Always $\mathrm{Aut}_{(0)}(\mathcal{X}) \subset \mathrm{Aut}_{\mathrm{center}}(\mathcal{X})$, and equality holds if $\mathcal{I}$ is radical. For a non-radical ideal, the zeroth jet already contains nilpotent information, and fixing it is stronger than fixing the reduced center. Likewise, an arbitrary automorphism of the formal complex space need not preserve the specified extended ideal $\widehat{\mathcal{I}}$. The automorphisms obtained from a model do preserve it, and hence act on its numbered finite jets (See Remark IV.2.22).

Definition IV.4.3. Let G be a discrete group and let $\mathcal{X}$ be an analytic neighborhood space with a chosen model. A *G-action* on $\mathcal{X}$ is a group homomorphism $\alpha_{\mathcal{X}}\colon G \to \mathrm{Aut}(\mathcal{X})$. Let $\mathcal{X}$ and $\mathcal{Y}$ be analytic neighborhood spaces with chosen models and actions $\alpha_{\mathcal{X}}$ and $\alpha_{\mathcal{Y}}$ of G, respectively. A morphism $h\colon \mathcal{X} \to \mathcal{Y}$, represented on these models, is *G-equivariant* if

$$h \circ \alpha_{\mathcal{X}}(g) = \alpha_{\mathcal{Y}}(g) \circ h \qquad (g \in G). \tag{292}$$

□

Viewing G as the one-object category BG, its actions are diagrams and equivariant comparisons are natural transformations; invertible comparisons are natural isomorphisms, as in Example IV.1.16 (*ii*). Thus these are the group-action instances of (291). A specified isomorphism of models transports the action by conjugation and preserves equivariance. Finite jets and formal completion inherit actions through their functors, and equivariance passes to the induced morphisms.

Example IV.4.4. For $G = \{1\}$, equivariance imposes no extra condition, so we recover ordinary comparisons of neighborhoods. If X is a reduced complex space and $\mathcal{X} = [(X, 0)]$, its center is all of X and there is no shrinking to perform: actions and morphisms are the usual ones on X. In this case $\mathrm{Aut}_{\mathrm{center}}(\mathcal{X})$ and every $\mathrm{Aut}_{(\mu)}(\mathcal{X})$ are trivial, whereas $\mathrm{Aut}(\mathcal{X})$ can act non-trivially on X. □

Example IV.4.5 (A point center and one-variable dynamics). Put $\mathcal{X} := \underline{\mathbb{C}}_0$, represented by $(\mathbb{C}, \mathcal{I}_{\{0\}})$, where $\mathcal{I}_{\{0\}}$ is the defining ideal sheaf of the origin. Its stalk at 0 is $\mathfrak{m}_{\mathbb{C},0} = (w)$. The elements of $\mathrm{Aut}(\mathcal{X})$ are exactly the convergent germs

$$f(w) = \lambda w + a_2 w^2 + \cdots, \qquad \lambda \neq 0.$$

Indeed, the nonzero derivative is precisely the condition for a holomorphic local inverse. In contrast, $w \mapsto w^2$ defines an endomorphism of $\mathcal{X}$ but not an automorphism. All these maps fix the point center; their higher-order information is contained in the induced homomorphisms of structure sheaves. Here $\mathrm{Aut}_{(0)}(\mathcal{X}) = \mathrm{Aut}(\mathcal{X})$, while $\mathrm{Aut}_{(1)}(\mathcal{X})$ consists of the germs with multiplier 1.

Use the chosen representatives $\mathcal{X}^{(\mu)}$ and $\mathcal{X}^{(\infty)}$. For $\mu \in \mathbb{Z}_{\geq 1}$, an automorphism of $\mathcal{X}^{(\mu)}$ is represented by

$$w \longmapsto \lambda w + a_2 w^2 + \cdots + a_\mu w^\mu \pmod{w^{\mu+1}}, \qquad \lambda \neq 0.$$

Every such finite jet is induced by a convergent germ: the displayed polynomial itself has a local inverse. An automorphism of $\mathcal{X}^{(\infty)}$ is instead a formal series of this form with no convergence requirement. For example, $w + \sum_{k\geq 2} k! w^k$ has a formal inverse and determines compatible automorphisms of every finite jet, but is not a convergent germ. Here composition means composition of the displayed maps; the corresponding pullbacks satisfy $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$.

The linearization problem of Chapter III, §3.3 asks whether this f is conjugate in $\mathrm{Aut}(\mathcal{X})$ to $R_\lambda(w) = \lambda w$. We use the same direction $f \circ h = h \circ R_\lambda$ and normalization $h(0) = 0$,

$h'(0) = 1$ as in (251). Equivalently, put $G = \mathbb{Z}$ and endow two copies $\mathcal{X}_+$ and $\mathcal{X}_-$ of this space with actions $\alpha_+(1) = f$ and $\alpha_-(1) = R_\lambda$, respectively. Then $h\colon \mathcal{X}_- \to \mathcal{X}_+$ is a G-equivariant isomorphism. The reverse direction is the Schröder coordinate h^{-1}. For a general non-zero λ, R_λ is a linear map, and is a rotation only when $|\lambda| = 1$. The multiplier is unchanged by conjugacy. For $\lambda = e^{2\pi\sqrt{-1}\theta}$, Siegel's condition on θ gives a convergent conjugacy, whereas the strong Cremer examples give a formal but divergent one (Theorem III.3.10 and Proposition III.3.14). Equivalently, let F_f and F_{std} be the diagrams indexed by $B\mathbb{Z}$ which send the chosen generator to f and R_λ, respectively. Then $h\colon F_{\mathrm{std}} \xrightarrow{\mathrm{nat}} F_f$ is precisely the natural isomorphism in (267). Naturality for the generator implies naturality for every iterate as a germ. □

Coordinates along a non-singular center

Let $\mathcal{X}$ be an analytic neighborhood space with a model $(X, \mathcal{I}_Z)$, where Z is a closed complex submanifold of the complex manifold X. Put $\mathcal{I} := \mathcal{I}_Z$. Locally let $d = \dim Z$ and $r = \operatorname{codim}_X Z$, and choose adapted coordinates

$$(z, w) = (z^1, z^2, \dots, z^d, w^1, w^2, \dots, w^r)$$

so that $Z = \{w = 0\}$ and $\mathcal{I}_Z = (w^1, w^2, \dots, w^r)$ on the coordinate neighborhood. We use the partial Taylor expansion (112), writing its z' simply as z in the present setting. A germ f of a biholomorphism preserving Z has, in source and target adapted coordinates, the expression $f(z, w) = (\varphi(z, w), \psi(z, w))$, where φ takes values in $\mathbb{C}^d$ and ψ in $\mathbb{C}^r$:

$$\begin{aligned} \varphi(z, w) &= f_0(z) + A(z)w + \sum_{|\alpha| \geq 2} a_\alpha(z) w^\alpha, \\ \psi(z, w) &= B(z)w + \sum_{|\alpha| \geq 2} b_\alpha(z) w^\alpha. \end{aligned} \tag{293}$$

Here f_0 is the coordinate expression of $f|_Z$; A and B are holomorphic matrices of sizes $d \times r$ and $r \times r$, respectively, and a_α and b_α are holomorphic vector-valued functions of z. The multi-index α ranges over $(\mathbb{Z}_{\geq 0})^r$, with w^α as in the cited partial Taylor expansion. Both series converge on a sufficiently small coordinate neighborhood. When $d = 0$, the z-component is omitted.

At a point $(z, 0)$, the derivative has block form

$$\begin{pmatrix} df_0(z) & A(z) \\ 0 & B(z) \end{pmatrix}.$$

Consequently f_0 is locally biholomorphic and $B(z)$ is invertible. Conversely, these conditions on a convergent expression (293) give a local biholomorphism preserving Z, by the holomorphic inverse function theorem recalled at the end of Chapter III, §1.2. The matrix $B(z)$ represents the induced map on the normal bundle $N_{Z/X} = T_X|_Z / T_Z$ in these coordinates.

For $\mu \in \mathbb{Z}_{\geq 0}$, the expression of $\rho_\mu(f)$ is obtained by retaining the terms of total w-degree at most μ.

If f fixes Z pointwise and the same coordinates are used on source and target, then $f_0(z) = z$. If it also induces the identity on $N_{Z/X}$, then $B(z)$ is the identity matrix, but the term $A(z)w$ may still be nonzero. Thus these two conditions do not say that $\rho_1(f)$ is the identity. For instance, in two coordinates the map $(z, w) \mapsto (z + w, w)$ fixes $\{w = 0\}$ and its normal direction but is not the identity on the first jet. In general, for $\mu \in \mathbb{Z}_{\geq 1}$, being the identity to order μ means that every component of $\varphi - z$ and $\psi - w$ belongs to $(w^1, w^2, \ldots, w^r)^{\mu+1}$.

Remark IV.4.6. The coordinate description assumes a non-singular center and its reduced defining ideal; the automorphism groups and their actions on jets were defined for all analytic neighborhood models. For a general coherent ideal one retains the quotient sheaves $\mathcal{O}_X/\mathcal{I}^{\mu+1}$, without presupposing coordinates which generate $\mathcal{I}$ in this form. Even in adapted coordinates, making the w-component linear does not remove the w-dependence of the z-component. These are different requirements on the coordinate changes.

If the center is compact, the finite-data argument of Chapter II, Lemma II.3.115, allows finite collections of holomorphic sections and their identities to be represented after refinement and shrinking of neighborhoods. For coherent coefficients the corresponding cohomology comparison is Proposition IV.1.22. In a formal coordinate neighborhood the coefficients of a section must all be holomorphic on one common open set in the center, as in Example IV.2.10. □

Source jets and comparison-level corrections

Retain a source model $F(j) = (V_j, \mathcal{I}_j)$ and its chosen subgroup G_j. For $\mu \in \mathbb{Z}_{\geq 0}$, write $G_j^{(\mu)} := \rho_\mu(G_j)$ for the image in the automorphism group of its μ-th jet. The usual one-step kernel is

$$K_j^{(\mu)} := \operatorname{Ker}\big(G_j^{(\mu+1)} \longrightarrow G_j^{(\mu)}\big) \qquad (\mu \geq 0).$$

The truncation between these images is surjective: use the next jet of an analytic representative in G_j. This says nothing about surjectivity after naturality with the other arrows is imposed. For $\mu \in \mathbb{Z}_{\geq 1}$, a change which preserves the $(\mu-1)$-jet of $q_j \circ h_j^{-1}$ need not preserve the $(\mu-1)$-jet of h_j itself. Thus this ambient kernel must be distinguished from the space of degree-μ comparison-level corrections.

Remark IV.4.7. The distinction already occurs at a node. Work near the origin in $\mathbb{C}^2$, put $w(x, y) := xy$ and $\mathcal{I} := (w)$, and consider $g(x, y) := (x, y(1 + xy))$. This is a local biholomorphism preserving $\mathcal{I}$, and $w \circ g = w + w^2$. Consequently it preserves the first jet of the defining map w, but g is not the identity modulo $\mathcal{I}^2$. It is an allowed inverse source change when $g = h^{-1}$ belongs to the chosen source group.

In fact, a source change which is the identity modulo $\mathcal{I}^2$ has the form $(x + Aw^2, y + Bw^2)$, with A, B holomorphic, and gives

$$w(x + Aw^2, y + Bw^2) = w + (yA + xB)w^2 + ABw^4.$$

To obtain $w + w^2$ modulo $\mathcal{I}^3$, it would be necessary that $yA + xB = 1$ modulo (w), which is impossible at the origin. Thus w^2 is a correction to the comparison map w, but is not realized by the ambient kernel which fixes the first jet. □

4.2 Finite-order discrepancies and obstructions

The first question is whether a comparison already fixed at one jet order extends to the next. We compute its discrepancy before making any choice of norms or claiming convergence. The calculation is particularly explicit for a linear target, and its additive form also describes the full and bundle comparisons below when their permitted lifts have the required translation structure.

A linear transverse target

Fix again a small category $\mathfrak{I}$, a diagram $F\colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{ANM})$ with $F(j) = (V_j, \mathcal{I}_j)$, and a standard diagram $F_{\text{std}}\colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{ANM})$ with $F_{\text{std}}(j) = (\mathbb{C}^r, \mathcal{I}_{\{0\}})$, where $r \geq 1$. For $a\colon i \to j$ write $\phi_a := F(a)$ and $F_{\text{std}}(a)(v) = T_a v$, where $T_a \in \mathrm{GL}_r(\mathbb{C})$. Thus $T_{b\circ a} = T_b T_a$ for composable arrows and $T_{\mathrm{id}_j} = I_r$. Fix initial comparison maps $q_j = w_j$ and source groups G_j as in the preceding subsection. Here w_j is a tuple of sections of $\mathcal{I}_j$, not necessarily part of a coordinate system. The reduced centers Z_j can be singular and the ideals need not be radical.

Write the comparison components as tuples u_j. In these representatives, their unnormalized discrepancy is the holomorphic map $u_j \circ \phi_a - T_a u_i$ on a neighborhood of the source center. Fix $\mu \in \mathbb{Z}_{\geq 1}$ and a natural transformation $u^{(\mu-1)}\colon F^{(\mu-1)} \xrightarrow{\text{nat}} F_{\text{std}}^{(\mu-1)}$, where the superscript denotes application of the finite-jet functor. Let $\mathscr{L}_{j,\mu}$ be the set of its component lifts of the form $(w_j \circ h_j^{-1})^{(\mu)}$, $h_j \in G_j$. Assume these sets are non-empty, and choose $\widetilde{u}_j \in \mathscr{L}_{j,\mu}$ without requiring the lifted squares to commute.

Let $\jmath_j\colon Z_j \hookrightarrow V_j$ be the inclusion, and put $\mathcal{S}_j^\mu := \jmath_j^{-1}(\mathcal{I}_j^\mu / \mathcal{I}_j^{\mu+1})$, the degree sheaf of Notation IV.2.5. In particular, $\mathcal{S}_j^\mu$ is an ideal with square zero in the μ-th jet. Let $E_{j,\mu} := \Gamma(Z_j, (\mathcal{S}_j^\mu)^{\oplus r})$. To compare values in the same transverse frame, bring the j-side value back by the fixed matrix T_a^{-1} and set

$$c_{a,\mu} := T_a^{-1}\widetilde{u}_j \circ \phi_a - \widetilde{u}_i \pmod{\mathcal{I}_i^{\mu+1}} \in E_{i,\mu} \qquad (a\colon i \to j). \tag{294}$$

The fixed lower-order naturality puts the difference in this degree sheaf. Composition and congruence in this formula use finite-jet morphisms; chosen representatives give the same section.

The ideal condition on ϕ_a induces a pullback on degree sheaves. Together with the matrix above it gives the linear map

$$\mathcal{T}_a^{(\mu)}\colon E_{j,\mu} \longrightarrow E_{i,\mu}, \qquad b \longmapsto T_a^{-1} b \circ \phi_a.$$

Its composition rule is $\mathcal{T}_{b\circ a}^{(\mu)} = \mathcal{T}_a^{(\mu)} \mathcal{T}_b^{(\mu)}$. The matrices T_a and the induced maps $\mathcal{T}_a^{(\mu)}$ act on different spaces.

If another set of component lifts is $\widetilde{u}'_j = \widetilde{u}_j + b_j$ at this order, the first path changes by $b_j \circ \phi_a$, and the second by $T_a b_i$. Normalizing the frames therefore gives

$$c'_{a,\mu} = c_{a,\mu} + \mathcal{T}_a^{(\mu)} b_j - b_i. \tag{295}$$

For composable $a\colon i \to j$ and $b\colon j \to k$, add and subtract $T_a^{-1}\widetilde{u}_j \circ \phi_a$ to obtain

$$\begin{aligned} T_a^{-1}T_b^{-1}\widetilde{u}_k \circ \phi_b \circ \phi_a - \widetilde{u}_i = {} & (T_a^{-1}\widetilde{u}_j \circ \phi_a - \widetilde{u}_i) \\ & + T_a^{-1}(T_b^{-1}\widetilde{u}_k \circ \phi_b - \widetilde{u}_j) \circ \phi_a. \end{aligned}$$

Thus the discrepancies satisfy the cocycle identities

$$c_{b\circ a,\mu} = c_{a,\mu} + \mathcal{T}_a^{(\mu)} c_{b,\mu}, \qquad c_{\mathrm{id}_j,\mu} = 0. \tag{296}$$

If two paths represent the same morphism, they give the same value, since F and F_{std} are functors. For a finite presentation as in Definition II.3.24, one may therefore use the generating arrows and these path relations. For a covering, the values are sections on the actual intersections.

Keep the lower-order comparison just fixed. To use an additive obstruction, suppose that each $\mathscr{L}_{j,\mu}$ is an affine space under a specified complex vector subspace $M_{j,\mu} \subset E_{j,\mu}$: any two of its lifts differ by a unique element of $M_{j,\mu}$, and adding any such element to one lift gives another permissible lift. Each addition used here must be realizable by the specified source group at this finite order. The space $M_{j,\mu}$ need not be the ambient kernel $K_j^{(\mu-1)}$; Remark IV.4.7 shows why the two must be checked separately.

Write $c_\mu := (c_{a,\mu})_a$, and put

$$C_\mu^0 := \prod_{j \in \mathrm{ob}(\mathfrak{I})} M_{j,\mu}, \qquad C_\mu^1 := \prod_{a:i\to j \text{ in } \mathfrak{I}} E_{i,\mu}, \qquad (\delta_\mu b)_a := \mathcal{T}_a^{(\mu)} b_j - b_i.$$

Here $\delta_\mu\colon C_\mu^0 \to C_\mu^1$ is linear, and $Z_\mu^1 \subset C_\mu^1$ denotes the subspace satisfying (296). Substitution and the composition rule for $\mathcal{T}$ give $\operatorname{Image}\delta_\mu \subset Z_\mu^1$. The resulting quotient is denoted by $H_\mu^1 := Z_\mu^1/\operatorname{Image}\delta_\mu$. This is the cocycle–coboundary construction of Chapter I, §5.2, for the specified diagram and permitted corrections.

Proposition IV.4.8 (One-step comparison). *Let $\mathfrak{I}$ be a small category and let F, F_{std} be (ANM)-valued diagrams indexed by $\mathfrak{I}$. Fix $\mu \in \mathbb{Z}_{\geq 1}$ and a comparison $u^{(\mu-1)}$ of their $(\mu-1)$-jets. For each object j, let $E_{j,\mu}$ be a complex vector space and $M_{j,\mu} \subset E_{j,\mu}$ a vector subspace. Suppose that the set $\mathscr{L}_{j,\mu}$ of permissible component lifts to degree μ is non-empty and is an affine space under $M_{j,\mu}$. Suppose that linear transports $\mathcal{T}_a^{(\mu)}\colon E_{j,\mu} \to E_{i,\mu}$ for $a\colon i \to j$ respect identities and composition as above. Assume that each family of component lifts has a discrepancy $c_\mu = (c_{a,\mu})_a$ with $c_{a,\mu} \in E_{i,\mu}$, which is zero exactly when the lifts form a comparison, and that these discrepancies satisfy (295) and (296). Define C_μ^0, C_μ^1, δ_μ, Z_μ^1, and $H_\mu^1 = Z_\mu^1/\operatorname{Image}\delta_\mu$ by the formulas above. Then*

$$\mathrm{Obs}_\mu(u^{(\mu-1)}) := [c_\mu] \in H_\mu^1$$

is independent of the component lifts. It vanishes if and only if the fixed comparison extends to a permissible degree-μ comparison. Once one extension is fixed, all others are obtained from it by the elements of $\operatorname{Ker}\delta_\mu$. *The same assertion holds for local comparisons on an open covering, with C^0_μ, Z^1_μ, and δ_μ replaced by the Čech cochains, cocycles, and coboundary of the additive correction sheaf, provided that local lifts and their corrections respect restriction and that compatible local lifts glue.*

Proof. Any other component lifts are obtained by a unique family $b \in C^0_\mu$. Formula (295) changes c_μ to $c_\mu + \delta_\mu b$, so its class is unchanged. The class is zero exactly when $\delta_\mu b = -c_\mu$ can be solved. By the affine-space hypothesis, these corrections are permissible lifts, and their zero discrepancy is precisely naturality at order μ. The difference of two solutions is in $\operatorname{Ker}\delta_\mu$, and adding such a family to an extension again gives an extension. For the covering version, the same argument is applied to the Čech complex; a zero discrepancy means precisely that the corrected local lifts agree and glue. □

Remark IV.4.9 (Notation and scope of the one-step comparison). For convenience, we summarize the notation in the linear transverse comparison above.

Symbol	Definition	Role
Z_j	Reduced center of $(V_j, \mathcal{I}_j)$	Support of degree sheaves
ϕ_a	$F(a)$, for $a\colon i \to j$	Original diagram morphism
T_a	$F_{\mathrm{std}}(a)(v) = T_a v$	Prescribed linear map
w_j, G_j	Initial comparison map and permitted source group	Specify permitted changes
$u^{(\mu-1)}$	Fixed natural $(\mu-1)$-jet comparison	Starting point of a lifting step
$\mathscr{L}_{j,\mu}$	Permissible degree-μ component lifts	Local choices to be compared
$\mathcal{S}^\mu_j$	$\jmath_j^{-1}(\mathcal{I}^\mu_j/\mathcal{I}^{\mu+1}_j)$	Degree-μ coefficient sheaf
$E_{j,\mu}$	$\Gamma(Z_j, (\mathcal{S}^\mu_j)^{\oplus r})$	Ambient coefficient space
$M_{j,\mu}$	Translation space of $\mathscr{L}_{j,\mu}$ in $E_{j,\mu}$	Permissible comparison-level corrections
$\mathcal{T}^{(\mu)}_a$	$b \mapsto T_a^{-1} b \circ \phi_a$	Degree-μ transport
$c_{a,\mu}$	Formula (294)	Discrepancy
C^0_μ, C^1_μ, δ_μ	The products above, with $(\delta_\mu b)_a = \mathcal{T}^{(\mu)}_a b_j - b_i$	Encode corrections and their discrepancies
Z^1_μ, H^1_μ	Cocycles and $Z^1_\mu/\operatorname{Image}\delta_\mu$	Locate the obstruction class
$\mathrm{Obs}_\mu(u^{(\mu-1)})$	$[c_\mu] \in H^1_\mu$	One-step obstruction

Proposition IV.4.8 isolates the one-step argument: the coefficient spaces, permissible corrections, and transports may vary with the comparison problem, subject to the affine-space and discrepancy identities in its hypotheses. For the linear target, the path calculations above verify the required identities.

The subsequent sections apply this principle to entire neighborhoods, invertible sheaves, and Ueda-type defining-function systems. In each case the permissible corrections and their realization by source changes must be verified; the local version also requires compatibility with restriction and gluing. In particular, $M_{j,\mu}$ need not be the ambient jet kernel $K_j^{(\mu-1)}$ (See Remark IV.4.7). □

5 Comparison of neighborhoods and Grauert's theorem

We now specialize the comparison problem of § 4.2 to the holomorphic structure of entire neighborhoods of complex submanifolds. In particular, we discuss Grauert's holomorphic tubular neighborhood theorem from this viewpoint.

5.1 Holomorphic tubular neighborhoods and formal comparison

Definition IV.5.1 (Holomorphic tubular neighborhood). Let Z be a closed complex submanifold of a complex manifold X, with positive codimension. Write $N_{Z/X} = T_X|_Z/T_Z$ and $\widetilde{Z} := (\text{Zero Section})_{N_{Z/X}}$. A *holomorphic tubular neighborhood* of Z in X is a biholomorphism from an open neighborhood of Z in X to an open neighborhood of $\widetilde{Z}$ in $N_{Z/X}$, restricting to the zero-section identification $Z \to \widetilde{Z}$. Equivalently, it is an isomorphism $(X, \mathcal{I}_Z) \to (N_{Z/X}, \mathcal{I}_{\widetilde{Z}})$ in (ANM) with this specified map on the reduced centers. □

The identification in Definition IV.5.1 is biholomorphic, whereas the tubular neighborhood theorem of Chapter I (Theorem I.8.4) gives a C^∞ diffeomorphism.

For the local comparison, let also $Z' \subset X'$ be a closed complex submanifold with the same dimension and codimension as $Z \subset X$. Put $\mathcal{X} := [(X, \mathcal{I}_Z)]$ and $\mathcal{X}' := [(X', \mathcal{I}_{Z'})]$, and fix a center isomorphism $\phi_0 \colon Z \to Z'$ and a conormal isomorphism $\theta \colon \phi_0^* \mathcal{N}_{\mathcal{X}'}^* \to \mathcal{N}_{\mathcal{X}}^*$. Choose an adapted coordinate covering $\{V_j\}_{j \in J}$ of a neighborhood of Z, put $U_j := V_j \cap Z$, and choose corresponding adapted neighborhoods V_j' with $V_j' \cap Z' = \phi_0(U_j)$. The charts may be refined and shrunk when making local lifts.

Use the covering category of Example IV.1.16 (*iii*) as follows. For a finite non-empty $\sigma \subset J$ with $U_\sigma := \bigcap_{j \in \sigma} U_j \neq \emptyset$, its objects (σ, j), $j \in \sigma$, record the chart germs along U_σ. Its generating arrows are inclusions of restricted charts and the changes from one chart to another on the same intersection. Impose the composition relations for inclusions, the transition identities on triple intersections, and compatibility of transitions with restriction. Denote this category by $\mathfrak{I}$, and abbreviate $(\{j\}, j)$ by j. The two atlases define $F, F_{\mathrm{std}} \colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{ANM})$: the original objects are $(V_j, \mathcal{I}_Z|_{V_j})$ and $(V_j', \mathcal{I}_{Z'}|_{V_j'})$, respectively, and the other objects and arrows are their restrictions and transitions.

Choose local initial comparison isomorphisms q_j inducing ϕ_0 and θ. Such maps exist in adapted coordinates, as explained in Lemma IV.5.2 below. Let $G_j \subset \mathrm{Aut}_{(\mathsf{ANM})}(F(j))$ consist of the source changes fixing U_j pointwise and inducing the identity on its conormal sheaf. Apply the same definition to the restricted chart models. The permissible comparison

components are $H_j = q_j \circ h_j^{-1}$, with $h_j \in G_j$. Naturality for inclusions makes their other components restrictions of these H_j; naturality for transitions makes them glue. For $\mu \geq 0$, use exactly the jet images and kernels of § 4.1:

$$G_j^{(\mu)} := \rho_\mu(G_j), \qquad K_j^{(\mu)} := \operatorname{Ker}\big(G_j^{(\mu+1)} \longrightarrow G_j^{(\mu)}\big).$$

Thus $K_j^{(\mu-1)}$ describes the source corrections at the reached degree μ. At degree one, fixing the center and conormal permits the tangential term $A(z)w$ in $(z,w) \mapsto (z + A(z)w, w)$ modulo $\mathcal{I}_Z^2$.

For the tubular-neighborhood problem take $X' = N_{Z/X}$, with Z' its zero section and ϕ_0 the zero-section identification. If the source transitions have center part $\varphi_{jk}(z_k, 0)$ and normal part $T_{jk}(z_k)$, the standard transitions are

$$(z_k, w_k) \longmapsto \big(\varphi_{jk}(z_k, 0), T_{jk}(z_k)w_k\big).$$

One may prescribe θ when constructing a comparison; the definition of a holomorphic tubular neighborhood itself fixes only ϕ_0.

LEMMA IV.5.2. *Let $Z \subset X$ and $Z' \subset X'$ be non-singular closed submanifolds of complex manifolds X and X', with the same dimension and codimension. Consider the analytic neighborhood spaces $\mathcal{X} := [(X, \mathcal{I}_Z)]$ and $\mathcal{X}' := [(X', \mathcal{I}_{Z'})]$. For $\mu \in \mathbb{Z}_{\geq 1}$, every isomorphism $\phi_\mu \colon \mathcal{X}^{(\mu)} \to \mathcal{X}'^{(\mu)}$ is locally along Z induced by a biholomorphism between open subsets of X and X' preserving the submanifolds. Given an isomorphism $\phi_0 \colon Z \to Z'$, such local biholomorphisms can be chosen to induce any specified isomorphism $\phi_0^* \mathcal{N}_{\mathcal{X}'}^* \to \mathcal{N}_{\mathcal{X}}^*$ of conormal sheaves.*

PROOF. Choose adapted coordinates (z, w) and $(\widetilde{z}, \widetilde{w})$ near corresponding points, using tildes to distinguish the target coordinates. The pullbacks under ϕ_μ of the target coordinate functions have holomorphic representatives on a small source neighborhood. The representatives of $\widetilde{w}$ can be chosen in $\mathcal{I}_Z$. For $\mu \geq 1$, the isomorphism of finite jets induces isomorphisms both on Z and on $\mathcal{I}_Z/\mathcal{I}_Z^2$; the latter is obtained from the nilradical and its square. The representatives thus have the invertible block derivative in (293) along Z. The holomorphic inverse function theorem recalled at the end of Chapter III, §1.2 gives the required local biholomorphism, whose finite jet is ϕ_μ. For $\mu = 0$, extend the center map independently of w in these coordinates and use a linear expression $B(z)w$ for the transverse component, with $B(z)$ representing the prescribed conormal map. Its derivative is again invertible. □

PROPOSITION IV.5.3. *Let $\mathcal{X} = [(X, \mathcal{I}_Z)]$ have a non-singular center Z in a complex manifold X. Choose adapted product charts V_j with $U_j = V_j \cap Z$, and the groups G_j fixing the center and conormal as above. The groups $K_j^{(\mu-1)}$ are abelian. In these charts they are identified with the additive groups*

$$\begin{aligned} K_j^{(0)} &\cong \Gamma\big(U_j, \mathcal{O}_Z(T_Z \otimes N_{Z/X}^*)\big), \\ K_j^{(\mu-1)} &\cong \Gamma\big(U_j, \mathcal{O}_Z(T_X|_Z \otimes S^\mu N_{Z/X}^*)\big) \qquad (\mu \geq 2). \end{aligned} \tag{297}$$

These identifications commute with restriction and changes of adapted coordinates. The displayed locally free sheaves therefore also describe the corrections on the intersections.

PROOF. We use Grauert's infinitesimal automorphism calculation ([Gr3, §4, pp. 357–359]) in our jet convention. Let $\mu \geq 1$ and $g \in K_j^{(\mu-1)}$. As an automorphism of the μ-jet over U_j, it induces a structure homomorphism $g^\sharp \colon \mathcal{O}_{\mathcal{X}^{(\mu)}}|_{U_j} \to \mathcal{O}_{\mathcal{X}^{(\mu)}}|_{U_j}$. Put $D := g^\sharp - \mathrm{id}$, where id is the identity of this structure sheaf. Since g is the identity on the preceding jet, D takes values in $\mathcal{S}^\mu_{\mathcal{X}}|_{U_j}$. For $\mu \geq 2$, multiplicativity of $g^\sharp$ gives

$$D(ab) = aD(b) + bD(a).$$

This degree sheaf is annihilated by $\mathcal{I}_Z$ and is $\mathcal{O}_Z(S^\mu N^*_{Z/X})$ by Example IV.2.6. In adapted coordinates, the values of D on (z, w) are homogeneous polynomials of degree μ in w. The chain rule identifies their coefficients with a section of $\mathcal{O}_Z(T_X|_Z \otimes S^\mu N^*_{Z/X})$. The composition of two correction operators has degree at least $2\mu - 1 \geq \mu + 1$, and hence vanishes at this jet order. Thus composition is addition, with inverse correction $-D$.

Conversely, on a product chart extend the coefficients of such a section by the projection to U_j and add the corresponding homogeneous polynomials to (z, w). The resulting map fixes the center and conormal and has invertible derivative along U_j. After shrinking about U_j it represents an element of G_j, by the same local inverse construction as in Lemma IV.5.2. Its jet realizes the given section.

For $\mu = 1$, the conormal condition makes D zero on $\mathcal{I}_Z/\mathcal{I}_Z^2$. Its remaining part is a derivation from $\mathcal{O}_Z$ to $\mathcal{N}^*_{\mathcal{X}}$ and is represented by the tangential term $A(z)w$. This gives $\mathcal{O}_Z(T_Z \otimes N^*_{Z/X})$, again with addition as group law. The derivation descriptions are intrinsic, so both identifications commute with restrictions and coordinate changes. □

PROPOSITION IV.5.4. *Let $Z \subset X$ and $Z' \subset X'$ be non-singular closed submanifolds of complex manifolds X and X', with the same dimension and codimension, and consider the analytic neighborhood spaces $\mathcal{X} := [(X, \mathcal{I}_Z)]$ and $\mathcal{X}' := [(X', \mathcal{I}_{Z'})]$. Fix an isomorphism $\phi_0 \colon Z \to Z'$ and an isomorphism $\theta \colon \phi_0^* \mathcal{N}^*_{\mathcal{X}'} \to \mathcal{N}^*_{\mathcal{X}}$ of conormal sheaves. We identify the underlying spaces by ϕ_0 when using open coverings. For $\mu \in \mathbb{Z}_{\geq 2}$, fix an isomorphism $\phi_{\mu-1} \colon \mathcal{X}^{(\mu-1)} \to \mathcal{X}'^{(\mu-1)}$ inducing these data. There is an obstruction*

$$\mathrm{Obs}_\mu(\phi_{\mu-1}) \in H^1\big(Z, \mathcal{O}_Z(T_X|_Z \otimes S^\mu N^*_{Z/X})\big),$$

independent of the local lifts used to define it, whose vanishing is equivalent to an extension $\phi_\mu \colon \mathcal{X}^{(\mu)} \to \mathcal{X}'^{(\mu)}$. At the first step the same construction gives

$$\mathrm{Obs}_1(\phi_0, \theta) \in H^1\big(Z, \mathcal{O}_Z(T_Z \otimes N^*_{Z/X})\big),$$

whose vanishing is equivalent to a first-jet isomorphism inducing ϕ_0 and θ. Once one extension is fixed, all the others are obtained uniquely by adding global sections of the corresponding coefficient sheaf in (297).

PROOF. Use the covering diagram and source groups fixed above. Lemma IV.5.2 gives local degree-μ lifts Φ_j after refinement and shrinking. Every other lift is uniquely $\Phi_j \circ a_j$, with $a_j \in K_j^{(\mu-1)}$. On an overlap the discrepancy is

$$g_{jk} := \Phi_j^{-1} \circ \Phi_k. \tag{298}$$

It belongs to the kernel for the restricted source model along $U_j \cap U_k$. Identify it with a section c_{jk} of the coefficient sheaf in (297). Composition on triple overlaps gives the cocycle identity. If b_j is the additive section representing a_j, the change $\Phi'_j = \Phi_j \circ a_j$ replaces c_{jk} by $c_{jk} + b_k - b_j$, with restrictions understood. Thus the permissible lifts and their discrepancies give precisely the covering form of Proposition IV.4.8. Compatible local isomorphisms and their inverses glue.

Choose a Stein refinement of the local lifting cover. Its finite intersections are Stein, as explained before (218). Cartan's theorem B (Theorem III.2.165) and Leray comparison (Theorem II.3.79) identify the resulting Čech group with the stated sheaf cohomology group. The one-step proposition gives independence of lifts, the vanishing criterion, and the freedom of adding global sections. A common refinement gives independence of the covering. □

Retain the embeddings and initial data of Proposition IV.5.4. If

$$H^1\big(Z, \mathcal{O}_Z(T_Z \otimes N^*_{Z/X})\big) = 0, \qquad H^1\big(Z, \mathcal{O}_Z(T_X|_Z \otimes S^\mu N^*_{Z/X})\big) = 0 \quad (\mu \geq 2),$$

each chosen comparison can be extended to the next degree. Choose the extensions successively, starting with the prescribed center and conormal identifications. The inverse limits of the structure homomorphisms and their inverses give

$$\phi_\infty \colon \mathcal{X}^{(\infty)} \xrightarrow{\sim} \mathcal{X}'^{(\infty)} \tag{299}$$

in (FCS), by Definition IV.2.21. The maps are local on stalks and preserve the specified filtrations. In particular, taking the second embedding to be the zero section in $N_{Z/X}$ gives a formal comparison with the normal-bundle model. We next give geometric hypotheses under which an analytic comparison with the normal-bundle model exists.

5.2 Grauert's tubular neighborhood theorem

The preceding obstruction calculation leads to a holomorphic tubular neighborhood when the normal bundle is negative and the following cohomology groups vanish.

Theorem IV.5.5 (Grauert's tubular neighborhood theorem). *Let Z be a connected compact non-singular hypersurface in a complex manifold X, with $\dim Z \geq 1$. Assume that its normal line bundle $N_{Z/X}$ is negative, meaning that $N^*_{Z/X}$ is positive in the sense used in the Kodaira embedding theorem (Theorem III.2.136). If, for every $\mu \in \mathbb{Z}_{\geq 1}$,*

$$H^1\big(Z, \mathcal{O}_Z(T_Z \otimes N^{-\mu}_{Z/X})\big) = 0, \qquad H^1\big(Z, \mathcal{O}_Z(N^{-\mu}_{Z/X})\big) = 0, \tag{300}$$

then Z has a holomorphic tubular neighborhood in X.

This is the corollary to [Gr3, §4, Satz 7, p. 363]. We first explain the formal comparison and the analytic input in the general statement. For $\mu \geq 2$, tensoring the tangent-normal sequence by $N^{-\mu}_{Z/X}$ gives

$$0 \longrightarrow \mathcal{O}_Z(T_Z \otimes N^{-\mu}_{Z/X}) \longrightarrow \mathcal{O}_Z(T_X|_Z \otimes N^{-\mu}_{Z/X}) \longrightarrow \mathcal{O}_Z(N^{1-\mu}_{Z/X}) \longrightarrow 0.$$

The two outer H^1 groups vanish by (300), with indices μ and $\mu-1$. For $\mu = 1$, its first condition directly gives $H^1(Z, \mathcal{O}_Z(T_Z \otimes N^*_{Z/X})) = 0$. Thus § 5.1 constructs compatible comparisons with the normal-bundle model at every finite order, and hence a formal comparison.

The analytic input is the following form of Grauert's result: for a compact non-singular hypersurface with negative normal bundle, an isomorphism with a sufficiently high finite jet of another non-singular hypersurface is induced by an isomorphism of analytic neighborhood germs ([Gr3, §4, Satz 6, pp. 360–363, and Zusatz zur Korrektur, p. 367]). This supplies the required tubular neighborhood. Only a finite-order comparison is prescribed in this assertion; it does not claim convergence of each specified infinite formal isomorphism.

For a curve in a surface, we give a concrete proof using Rossi's theorem and a holomorphic Euler vector field. We first construct the strongly 1-convex neighborhood on the original surface and then prove the flow lemma which realizes the neighborhood comparison.

LEMMA IV.5.6. *Let Z be a compact Riemann surface embedded as a closed complex submanifold of a non-singular complex surface X. If $N_{Z/X}$ is negative, there are a connected open neighborhood W of Z and a nonnegative C^∞ plurisubharmonic exhaustion φ on W such that $\varphi^{-1}(0) = Z$ and φ is strictly plurisubharmonic on $W \setminus Z$. In particular, W is strongly 1-convex and its maximal compact analytic subset is Z.*

PROOF. Extend a negative-curvature Hermitian metric on $N_{Z/X} \cong [Z]|_Z$ to a C^∞ metric h_0 on $[Z]$ near Z, and put $r_0 := |\sigma_Z|^2_{h_0}$. Along Z, $\sqrt{-1}\partial\overline{\partial} r_0$ has kernel T_Z and is positive in the normal direction, whereas $-\sqrt{-1}\Theta_{h_0}$ is positive on T_Z. For $h := h_0 e^{Ar_0}$ one has

$$-\sqrt{-1}\Theta_h = -\sqrt{-1}\Theta_{h_0} + A\sqrt{-1}\partial\overline{\partial} r_0.$$

The tangential block is positive and the mixed block is bounded, so the Schur complement is positive for sufficiently large A. Compactness makes this choice uniform; after shrinking, $-\sqrt{-1}\Theta_h > 0$ near Z. For $r := |\sigma_Z|^2_h$, therefore, $\sqrt{-1}\partial\overline{\partial}\log r = -\sqrt{-1}\Theta_h > 0$ off Z. The identity

$$\sqrt{-1}\partial\overline{\partial} r = r\sqrt{-1}\partial\overline{\partial}\log r + r\sqrt{-1}\partial\log r \wedge \overline{\partial}\log r$$

shows that r is strictly plurisubharmonic off Z and, by continuity of its Levi form, plurisubharmonic on Z as well. This is the metric construction of § 3.5, now performed on the divisor bundle over the original surface, without a tubular neighborhood.

Choose an open neighborhood V with compact closure in this construction domain. The minimum of r on ∂V is positive. For a smaller $\varepsilon > 0$, let W be the component containing Z of $\{x \in V \mid r(x) < \varepsilon\}$. Then $\varphi := r/(\varepsilon - r)$ has relatively compact sublevels in W, vanishes exactly on Z, and has the required Levi properties. By Lemma III.2.91, its restriction to every compact irreducible positive-dimensional analytic subset is constant. Such a subset cannot meet $W \setminus Z$, where the restriction would be strictly plurisubharmonic at a regular point. Thus the maximal compact analytic subset is Z. □

LEMMA IV.5.7. *Let Z be a compact Riemann surface embedded as a closed complex submanifold of a non-singular complex surface X. Suppose that the tangent-normal sequence has a holomorphic splitting $s\colon N_{Z/X} \to T_X|_Z$ and that a holomorphic vector field E is defined near*

Z, vanishes on Z, and has differential $0\oplus\mathrm{id}_{N_{Z/X}}$ along Z with respect to $T_X|_Z = T_Z\oplus s(N_{Z/X})$. Then Z has a holomorphic tubular neighborhood in X, whose induced map on the normal bundle is the identity.

Proof. The differential of a vector field at a zero is an intrinsic endomorphism of the tangent space. Choose adapted holomorphic coordinates (z,w) with $Z=\{w=0\}$ and with $\partial/\partial w$ along Z spanning $s(N_{Z/X})$. Such coordinates exist: if in initial coordinates (ζ,w) the lift of the normal frame is $\partial/\partial w + a_0(\zeta)\partial/\partial\zeta$, replace ζ by $z=\zeta - a_0(\zeta)w$ and shrink. The first-order assumption then gives holomorphic functions a,b with

$$E = w^2 a(z,w)\frac{\partial}{\partial z} + \big(w + w^2 b(z,w)\big)\frac{\partial}{\partial w}.$$

We use the real-time flow Φ_t of $E+\overline{E}$. Its local existence and holomorphic dependence on the initial point were explained in Chapter III, §1.2, in the discussion of (123). Here we must also ensure existence for all negative times on one neighborhood of Z.

Take finitely many such charts covering Z, with smaller coordinate bidiscs whose closures lie in the original charts. In each chart, choose an inner base disc with positive distance from the boundary of the smaller base disc. The inner base discs still cover Z. Let $M>0$ bound $|a|$ and $|b|$ on all the smaller bidiscs, and fix $1/2<c<1$. For an initial point (z_0,w_0) over an inner disc, write $(z(\tau),w(\tau)) := \Phi_{-\tau}(z_0,w_0)$ for $\tau\ge 0$. The equations are

$$\frac{dz}{d\tau} = -w^2 a(z,w), \qquad \frac{dw}{d\tau} = -w - w^2 b(z,w).$$

Choose $\delta>0$ smaller than the normal radii of these bidiscs, so small that $M\delta<1-c$ and $M\delta^2/(2c)$ is smaller than every chosen base-disc margin. As long as the orbit stays in the smaller bidisc and $|w|<\delta$, the second equation gives $d|w|/d\tau \le -c|w|$ (the zero solution is treated by uniqueness). Consequently,

$$|w(\tau)| \le e^{-c\tau}|w_0|, \qquad |z(\tau)-z_0| \le \frac{M}{2c}|w_0|^2.$$

For $|w_0|<\delta$, these inequalities prevent escape from the smaller bidisc. The continuation theorem for ordinary differential equations therefore extends the orbit to every $\tau\ge 0$. In particular, no change of chart during this argument is needed.

The first equation and the estimate for w show that $z(\tau)$ converges. Moreover,

$$\left|\frac{d}{d\tau}\big(e^{\tau}w(\tau)\big)\right| \le M e^{-(2c-1)\tau}|w_0|^2.$$

Since $2c-1>0$, the limits

$$z_\infty(x) := \lim_{\tau\to\infty} z(\Phi_{-\tau}(x)), \qquad \eta(x) := \lim_{\tau\to\infty} e^{\tau} w(\Phi_{-\tau}(x))$$

exist locally uniformly in the initial point x. They are holomorphic, and the same integral estimates give $z_\infty = z_0 + O(w_0^2)$ and $\eta = w_0 + O(w_0^2)$.

It remains to check that these maps agree as maps to $N_{Z/X}$. On an overlap, coordinates adapted to the same splitting satisfy

$$z_j = \varphi_{jk}(z_k) + O(w_k^2), \qquad w_j = t_{jk}(z_k)w_k + O(w_k^2),$$

where $(z_k, w_k) \mapsto (\varphi_{jk}(z_k), t_{jk}(z_k)w_k)$ is the normal-bundle transition. Uniqueness identifies the two descriptions of every common orbit. The orbit remains in compact subsets of both original charts, so its limit on Z belongs to their overlap. Taking limits in the first transition and multiplying the second by e^τ before taking limits gives

$$z_{\infty,j} = \varphi_{jk}(z_{\infty,k}), \qquad \eta_j = t_{jk}(z_{\infty,k})\eta_k.$$

Here $e^\tau O(w_k(\tau)^2)$ tends to zero by the same estimate with exponent $2c-1$. Thus the local maps glue to a holomorphic map F from a neighborhood of Z to $N_{Z/X}$. It fixes the zero section, and its differential there is the identity under the chosen splitting.

The holomorphic inverse function theorem makes F a local biholomorphism near Z. To obtain injectivity on one neighborhood of the whole curve, take a decreasing neighborhood basis of Z with compact closures inside this local-biholomorphism locus. If F were injective on none of these neighborhoods, there would be distinct pairs x_ν, y_ν approaching Z with $F(x_\nu) = F(y_\nu)$. Passing to subsequences gives limits $x, y \in Z$. Since $F|_Z$ is the zero-section identification, $x = y$; local injectivity near that point is a contradiction. After shrinking, F is therefore a biholomorphism onto an open neighborhood of the zero section, as required. This follows the construction of a map and the final shrinking in the proof of the C^∞ tubular neighborhood theorem (Theorem I.8.4); the convergent holomorphic flow limits supply the map here. □

Proof of Theorem IV.5.5 in the surface case. For simplicity, we give this proof only when X is a non-singular complex surface and Z is a compact Riemann surface. By Lemma IV.5.6, choose a strongly 1-convex neighborhood W with maximal compact analytic subset Z and an exhaustion φ with the properties required in Rossi's theorem in terms of ideal powers (Chapter III, Theorem III.2.188). Fix a sufficiently small positive regular sublevel $\Omega = \{x \in W \mid \varphi(x) < \varepsilon\}$. Put $\mathcal{I} := \mathcal{I}_Z|_W$ and $\mathcal{T} := \mathcal{O}_W(T_W)$, and use the same symbols for their restrictions to Ω. All the following cohomology groups are taken on this fixed Ω or on Z.

For $\mu \geq 2$, tensor the tangent-normal sequence by $N_{Z/X}^{-\mu}$:

$$0 \longrightarrow \mathcal{O}_Z(T_Z \otimes N_{Z/X}^{-\mu}) \longrightarrow \mathcal{O}_Z(T_X|_Z \otimes N_{Z/X}^{-\mu}) \longrightarrow \mathcal{O}_Z(N_{Z/X}^{-(\mu-1)}) \longrightarrow 0.$$

The two outer sheaves have zero H^1 by (300), with indices μ and $\mu-1$. Hence the middle sheaf has zero H^1. For the closed embedding $j_Z\colon Z \hookrightarrow \Omega$, one has

$$\mathcal{I}^\mu\mathcal{T}/\mathcal{I}^{\mu+1}\mathcal{T} \cong j_{Z*}\mathcal{O}_Z(T_X|_Z \otimes N_{Z/X}^{-\mu}).$$

Cohomology is unchanged under this closed direct image (See the discussion preceding (216)). Thus $H^1(\Omega, \mathcal{I}^\mu\mathcal{T}/\mathcal{I}^{\mu+1}\mathcal{T}) = 0$ for every $\mu \geq 2$. The finite filtration with these successive quotients, together with the cohomology exact sequences, gives

$$H^1\big(\Omega, \mathcal{I}^2\mathcal{T}/\mathcal{I}^{k+2}\mathcal{T}\big) = 0 \qquad (k \geq 1).$$

Apply Rossi's theorem to the coherent sheaf $\mathcal{F} := \mathcal{I}^2\mathcal{T}$ and $q = 1$. For sufficiently large k, its injection is precisely

$$H^1(\Omega, \mathcal{I}^2\mathcal{T}) \longrightarrow H^1\big(\Omega, \mathcal{I}^2\mathcal{T}/\mathcal{I}^{k+2}\mathcal{T}\big),$$

since $\mathcal{I}^k\mathcal{F} = \mathcal{I}^{k+2}\mathcal{T}$. The target is zero, so $H^1(\Omega, \mathcal{I}^2\mathcal{T}) = 0$. This is the passage from finite quotients to cohomology on an actual neighborhood.

The $\mu = 1$ vanishing $H^1(Z, \mathcal{O}_Z(T_Z \otimes N_{Z/X}^{-1})) = 0$ also supplies a holomorphic splitting $s\colon N_{Z/X} \to T_X|_Z$: apply $\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_Z(N_{Z/X}), -)$ to the tangent-normal sequence of section sheaves, and lift the identity section of $\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_Z(N_{Z/X}), \mathcal{O}_Z(N_{Z/X}))$. Regard s as a section of

$$\mathcal{I}\mathcal{T}/\mathcal{I}^2\mathcal{T} \cong j_{Z*}\mathcal{O}_Z(T_X|_Z \otimes N_{Z/X}^{-1}).$$

The cohomology sequence of $0 \to \mathcal{I}^2\mathcal{T} \to \mathcal{I}\mathcal{T} \to \mathcal{I}\mathcal{T}/\mathcal{I}^2\mathcal{T} \to 0$ and the vanishing just proved lift s to $E \in H^0(\Omega, \mathcal{I}\mathcal{T})$. This is an actual holomorphic vector field, not a formal one. It vanishes on Z, and its differential there is the composition $T_X|_Z \to N_{Z/X} \xrightarrow{s} T_X|_Z$, namely $0 \oplus \mathrm{id}_{N_{Z/X}}$ under the splitting. Lemma IV.5.7 now gives the required holomorphic tubular neighborhood. □

Corollary IV.5.8. *Let C be a compact Riemann surface of genus g, embedded as a closed complex submanifold of a non-singular complex surface X. If*

$$(C.C) < \min\{0, 4 - 4g\},$$

then C has a holomorphic tubular neighborhood in X.

Proof. Put $d := \deg N_{C/X} = (C.C)$, using Example III.2.145. Since $d < 0$, Theorem III.2.177 shows that $N_{C/X}^*$ is positive. Serre duality (Lemma III.2.121) identifies the duals of the two groups in (300) with

$$H^0\big(C, \mathcal{O}_C(K_C^2 \otimes N_{C/X}^{\mu})\big), \qquad H^0\big(C, \mathcal{O}_C(K_C \otimes N_{C/X}^{\mu})\big),$$

respectively, since $T_C^* = K_C$. Their degrees are $4g{-}4{+}\mu d$ and $2g{-}2{+}\mu d$, by Lemma III.2.178. Both are negative for every $\mu \geq 1$ under the stated inequality. A line bundle of negative degree has no nonzero holomorphic section: otherwise its zero divisor would have nonnegative degree, contrary to (197). Thus both H^1 groups vanish, and the surface case of Theorem IV.5.5 just proved applies. □

For $g = 0$ or $g = 1$, negativity of the self-intersection already satisfies this sufficient condition. For higher genus the displayed bound is stronger than negativity alone. The surface proof separates two inputs: cohomological vanishing handles the finite quotients, while negativity provides the strongly 1-convex neighborhood on which Rossi's injection can be applied.

Remark IV.5.9 (Stein centers and adapted neighborhoods). Let X be a complex space and $Z \subset X$ a closed analytic subset with its reduced structure. Although these notes mainly

concern compact centers, important results related to the neighborhood problems considered in this section are also available when Z is Stein, without any compactness assumption. Siu's Stein neighborhood theorem [Siu2, Main Theorem, p. 89] states that, if Z is Stein, it has a Stein open neighborhood in X. Applying the theorem inside any prescribed open neighborhood shows that these Stein neighborhoods form a neighborhood basis of Z. Neither non-singularity nor irreducibility of Z is required.

We recall the proof strategy rather than the full construction. Siu combines Richberg's extension result, recalled in Chapter III, § 2.7, with techniques from Grauert's work on exceptional sets. An auxiliary plurisubharmonic function obtained from the envelope of holomorphy of a Reinhardt domain enters the construction of the neighborhood. A continuous strictly plurisubharmonic exhaustion then proves its Steinness by Narasimhan's criterion (See [Siu2, p. 89] for this outline and the subsequent sections for the proof). The existence of an extension of a strictly plurisubharmonic function alone does not ensure that it exhausts the ambient neighborhood. This is a Stein-center result, not a consequence of the compact negative-center statement proved above.

Here is a consequence using the Oka–Cartan theorem B (Chapter III, Theorem III.2.165). Let V be such a Stein neighborhood and $i\colon (Z,\mathcal{R}) \hookrightarrow (V,\mathcal{O}_V)$ a closed complex subspace structure on the same underlying set Z, with coherent ideal $\mathcal{I}$. Here $\mathcal{R}$ distinguishes this possibly non-reduced structure sheaf from that of the reduced center. The exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_V \longrightarrow i_*\mathcal{R} \longrightarrow 0$$

and $H^1(V,\mathcal{I}) = 0$ give the surjection

$$\mathcal{O}_V(V) \longrightarrow \mathcal{R}(Z).$$

Thus a holomorphic section on the center extends to an actual ambient neighborhood, also when the section contains nilpotent data. When Z and X are non-singular, Siu's Corollary 1 gives a holomorphic tubular neighborhood and hence, after shrinking, a holomorphic retraction $p\colon V \to Z$ with $p|_Z = \mathrm{id}_Z$. Composition with p extends holomorphic functions. □

6 Comparison of invertible sheaves on analytic neighborhood spaces

We now fix an analytic neighborhood space $\mathcal{X}$ and compare invertible sheaves over it, rather than comparing the neighborhoods themselves. For two invertible sheaves $\mathcal{L}$ and $\mathcal{M}$, $\mathcal{L} \cong \mathcal{M}$ is equivalent to $\mathcal{L}\otimes\mathcal{M}^* \cong \mathcal{O}_\mathcal{X}$. A basic question is therefore whether formal triviality implies analytic triviality. Write $\mathrm{Pic}(\mathcal{X})$ for the group of isomorphism classes of invertible $\mathcal{O}_\mathcal{X}$-modules, with tensor product as its operation. For such a sheaf $\mathcal{L}$, $\widehat{\mathcal{L}}$ denotes its formal restriction by completion (Proposition IV.2.19).

Definition IV.6.1 (Formal principle for line bundles). Let $\mathcal{X}$ be an analytic neighborhood space with non-empty compact reduced center. We say that the *formal principle for*

line bundles, or *FPLB*, holds for $\mathcal{X}$ if every invertible $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{L}$ with $\widehat{\mathcal{L}} \cong \mathcal{O}_{\mathcal{X}^{(\infty)}}$ is trivial on $\mathcal{X}$. For a representative on a model, this means triviality after shrinking to a neighborhood of the whole center. Equivalently, formal restriction

$$\rho_{\mathcal{X}} \colon \operatorname{Pic}(\mathcal{X}) \longrightarrow \operatorname{Pic}(\mathcal{X}^{(\infty)}, \mathcal{O}_{\mathcal{X}^{(\infty)}}), \qquad [\mathcal{L}] \longmapsto [\widehat{\mathcal{L}}]$$

is injective. □

For a non-singular ambient space and its reduced defining ideal, this is [Koi13, Definition 1.1].

Remark IV.6.2. In Griffiths' terminology, the FPLB is a *formal uniqueness problem* for holomorphic line bundles (See [Gri, §I.2 and §I.4]). Indeed, two invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ defined near $Z(\mathcal{X})$ become isomorphic after shrinking if their formal restrictions are isomorphic; apply the definition to $\mathcal{L} \otimes \mathcal{M}^*$. Equivalently, this is a *formal extension problem* for a nowhere vanishing section of $(\mathcal{L} \otimes \mathcal{M}^*)|_{Z(\mathcal{X})}$, or for an isomorphism $\mathcal{M}|_{Z(\mathcal{X})} \to \mathcal{L}|_{Z(\mathcal{X})}$: one asks whether the existence of a compatible formal extension implies the existence of an analytic one on a representative neighborhood of $Z(\mathcal{X})$. The analytic isomorphism need not induce the prescribed formal isomorphism at every order. Nor does the FPLB assert that every invertible sheaf on $\mathcal{X}^{(\infty)}$ comes from an analytic neighborhood. □

Remark IV.6.3. The FPLB is invariant under isomorphisms of analytic neighborhood models: pullback identifies the analytic and formal Picard groups and commutes with $\rho_{\mathcal{X}}$. For a model $(X, \mathcal{I})$ with compact reduced center Z, it is also unchanged when $\mathcal{I}$ is replaced by a coherent ideal with the same reduced center. Indeed, Proposition IV.2.23 identifies the formal completions over X, while the analytic neighborhood Picard group depends only on the germ along Z. These identifications preserve the injectivity of $\rho_{\mathcal{X}}$. □

Example IV.6.4. Let $\mathcal{X}$ be an analytic neighborhood space whose reduced center $Z(\mathcal{X})$ consists of one point. Every invertible sheaf is already trivial on a sufficiently small neighborhood, so $\operatorname{Pic}(\mathcal{X}) = 0$ and the FPLB holds. This does not require a prescribed formal trivialization to converge. For example, $1 + \sum_{k \geq 1} k! w^k$ is a unit in $\mathbb{C}[[w]]$ and gives a formal trivialization of the trivial sheaf at the point-center model $(\mathbb{C}, \mathcal{I}_{\{0\}})$, but it is not a convergent holomorphic germ. Analytic triviality is witnessed instead by 1. □

The Ueda invertible sheaf

Let $\mathcal{X} = [(X, \mathcal{I})]$ be an analytic neighborhood space with compact reduced center Z. If a holomorphic line bundle L near Z has unitary flat restriction, Lemma III.2.130 gives uniqueness of its compatible unitary flat isomorphism class on Z. Apply the shrinking-neighborhood cohomology comparison (Remark II.3.113) to the sheaf of locally constant $\mathrm{U}(1)$-valued functions. Its inverse image on Z is the corresponding constant sheaf, so the flat class extends to a neighborhood and its extension germ is unique up to unitary flat isomorphism, as in Example II.3.117. The compactness of Z and the Hausdorff property of the ambient complex space give the hypotheses of that comparison.

Definition IV.6.5. Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center Z, choose a model $(X,\mathcal{I})$, and let L be a holomorphic line bundle on a representative neighborhood of Z. Assume that $L|_Z$ is unitary flat, and choose a unitary flat extension $\widetilde{L}$ to a sufficiently small neighborhood V. The invertible sheaf $\mathcal{P}_L$ on $\mathcal{X}$ induced by $\mathcal{O}_V(L\otimes\widetilde{L}^{-1})$ is called the *Ueda invertible sheaf of* L. Its isomorphism class is independent of the chosen unitary flat extension. We also write $\mathcal{P}$ when L is fixed. □

One has $\mathcal{P}_L|_Z\cong\mathcal{O}_Z$. Choosing an identification $\widetilde{L}|_Z\cong L|_Z$ specifies a global frame of this restriction; uniqueness of the isomorphism class does not specify such a frame. When a fixed-frame lifting problem is considered, that frame is chosen as part of its data.

A motivating comparison comes from Ueda's study of a compact non-singular curve in a non-singular complex surface ([U1, Introduction and §2]). More generally, let Z be a compact reduced hypersurface of a complex manifold X, and suppose that the invertible sheaf $\mathcal{N}:=\mathcal{O}_Z([Z]|_Z)$ is unitary flat in the sense of Definition III.2.123. Fix its unitary flat structure and take its extension $\widetilde{\mathcal{N}}$ to a neighborhood V by the extension argument above. Its comparison with $\mathcal{O}_V(Z)$ is encoded by

$$\mathcal{P}_{[Z]}:=\mathcal{O}_V(Z)\otimes_{\mathcal{O}_V}\widetilde{\mathcal{N}}^*,\qquad \mathcal{P}_{[Z]}|_Z\cong\mathcal{O}_Z. \tag{301}$$

This is the specialization of Definition IV.6.5 to $L=[Z]$. For the analytic neighborhood space $\mathcal{X}:=[(X,\mathcal{I}_Z)]$ we also write $\mathcal{P}:=\mathcal{P}_{[Z]}$ and call it the *Ueda invertible sheaf of* $\mathcal{X}$. When Z is non-singular, $[Z]|_Z$ is its usual normal line bundle. The restriction here is the pullback of an invertible sheaf to Z.

Invertible sheaves on finite and formal jets

Let $\mathcal{X}$ be an analytic neighborhood space with non-empty compact reduced center. Choose a model $(X,\mathcal{I})$ and put $Z:=Z(X,\mathcal{I})$. Use the finite and formal jets $\mathcal{X}^{(\mu)}$ and $\mathcal{X}^{(\infty)}$ with their chosen representatives for $\mu\in\mathbb{Z}_{\geq 0}$, and write $j_Z\colon Z\hookrightarrow X$. For an invertible sheaf $\mathcal{L}$ on an open neighborhood V of Z, set

$$\mathcal{L}_\mu:=j_Z^{-1}(\mathcal{L}/\mathcal{I}^{\mu+1}\mathcal{L}),\qquad \widehat{\mathcal{L}}:=\mathcal{O}_{\mathcal{X}^{(\infty)}}\otimes_{j_Z^{-1}\mathcal{O}_V}j_Z^{-1}\mathcal{L}\cong\varprojlim_{\mu}\mathcal{L}_\mu. \tag{302}$$

Here $\mathcal{I}$ is restricted to V, and j_Z also denotes the inclusion into V. These are invertible modules over $\mathcal{O}_{\mathcal{X}^{(\mu)}}$ and $\mathcal{O}_{\mathcal{X}^{(\infty)}}$, respectively; the displayed isomorphism is Proposition IV.2.19. It can also be seen directly by trivializing $\mathcal{L}$ locally. Denote its restriction to the reduced center by $\mathcal{L}_Z:=\mathcal{O}_Z\otimes_{j_Z^{-1}\mathcal{O}_V}j_Z^{-1}\mathcal{L}$. When $\mathcal{I}$ is not radical, $\mathcal{L}_0$ and $\mathcal{L}_Z$ are sheaves on different complex spaces.

For a formal complex space $\mathfrak{Y}=(Y,\mathcal{O}_{\mathfrak{Y}})$, we use the same unitary flat terminology as in Definition III.2.123: a finite locally free $\mathcal{O}_{\mathfrak{Y}}$-module is unitary flat if it admits local trivializations with locally constant unitary transition matrices, viewed as matrices over $\mathcal{O}_{\mathfrak{Y}}$. Consequently, a specified unitary flat sheaf on V gives unitary flat sheaves on every $\mathcal{X}^{(\mu)}$ and on $\mathcal{X}^{(\infty)}$, using the same transition matrices. This construction uses the maps to V and does not require a holomorphic retraction onto Z.

LEMMA IV.6.6 (Formal triviality and compatible generators). *Let $\mathcal{X}$ be an analytic neighborhood space with non-empty compact reduced center Z, and let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$. Use the restrictions and completion in (302). Then $\widehat{\mathcal{L}} \cong \mathcal{O}_{\mathcal{X}^{(\infty)}}$ if and only if there is a compatible system $\{s_\mu\}_{\mu\in\mathbb{Z}_{\geq 0}}$ representing an element of*

$$\varprojlim_{\mu} H^0(\mathcal{X}^{(\mu)}, \mathcal{L}_\mu)$$

whose image in $H^0(Z, \mathcal{L}_Z)$ is nowhere vanishing. In that case every s_μ generates $\mathcal{L}_\mu$.

PROOF. This is the criterion of [Koi13, Lemma 3.1], expressed using (302). Sections commute with inverse limits of sheaves, so the compatible system determines a section s of $\widehat{\mathcal{L}}$. In a local trivialization its image on Z has no zeros. Its coefficient is therefore a unit at every finite stage, since the kernel of reduction to Z is nilpotent locally. On the same open patch the inverses at the finite stages are unique and compatible. They define an inverse coefficient in $\mathcal{O}_{\mathcal{X}^{(\infty)}}$. Multiplication by s consequently gives an isomorphism $\mathcal{O}_{\mathcal{X}^{(\infty)}} \to \widehat{\mathcal{L}}$. The same reasoning shows that each s_μ generates $\mathcal{L}_\mu$. Conversely, the image of 1 under a formal trivialization supplies such a compatible system, with nowhere vanishing reduction. □

In the compact setting, finite-dimensionality allows compatible frames to be chosen from separate finite-order trivializations.

PROPOSITION IV.6.7. *Let $\mathcal{X}$ be an analytic neighborhood space with non-empty compact reduced center Z, choose a model $(X, \mathcal{I})$, and let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$. Write $\mathcal{L}_\mu$, $\widehat{\mathcal{L}}$, and $\mathcal{L}_Z$ as in (302). Consider the following three assertions.*

(*i*) *For every $\mu \in \mathbb{Z}_{\geq 0}$, the invertible sheaf $\mathcal{L}_\mu$ is trivial on $\mathcal{X}^{(\mu)}$.*

(*ii*) *The invertible sheaf $\widehat{\mathcal{L}}$ is trivial on $\mathcal{X}^{(\infty)}$.*

(*iii*) *Every $\mathcal{L}_\mu$ is trivial, and, for every $\mu \in \mathbb{Z}_{\geq 0}$, every global frame of $\mathcal{L}_\mu$ is the restriction of a global frame of $\mathcal{L}_{\mu+1}$.*

Then (i) and (ii) are equivalent, and (iii) implies both of them. If (ii) holds, assertion (iii) is equivalent to the surjectivity, for every $\mu \in \mathbb{Z}_{\geq 0}$, of

$$H^0(\mathcal{X}^{(\mu+1)}, \mathcal{O}^*_{\mathcal{X}^{(\mu+1)}}) \longrightarrow H^0(\mathcal{X}^{(\mu)}, \mathcal{O}^*_{\mathcal{X}^{(\mu)}}).$$

PROOF. The implication (*ii*) ⇒ (*i*) follows by restriction. Suppose that (*i*) holds. In particular $\mathcal{L}_Z$ is trivial; fix a global frame s_Z. Each $\mathcal{X}^{(\mu)}$ has a compact complex space as a representative. Cartan–Serre finiteness (Theorem III.2.174), applied in degree zero, therefore makes $H^0(\mathcal{X}^{(\mu)}, \mathcal{L}_\mu)$ finite-dimensional. The restriction to Z of any frame of $\mathcal{L}_\mu$ differs from s_Z by an invertible holomorphic function on Z. Such a function is constant on each connected component, by the maximum principle on a compact reduced complex space. There are only finitely many components, since Z is compact and locally connected. Multiplying on these components by the reciprocal constants thus gives a frame restricting to s_Z.

For $\mu \in \mathbb{Z}_{\geq 0}$ let S_μ be the set of sections of $\mathcal{L}_\mu$ restricting to s_Z. It is a non-empty finite-dimensional affine space. Every member is a frame: in a local trivialization its coefficient is a unit modulo the nilpotent ideal defining Z, and is therefore a unit. For fixed μ, the images of S_ν in S_μ, as $\nu \geq \mu$ increases, are non-empty affine subspaces forming a decreasing sequence. Their dimensions, and hence the subspaces themselves, eventually stabilize. Denote the stable image by T_μ. The restriction $T_{\mu+1} \to T_\mu$ is surjective. Indeed, choose ν large enough that the images from S_ν in both S_μ and $S_{\mu+1}$ have stabilized. A lift in S_ν of any element of T_μ restricts to the required element of $T_{\mu+1}$. Starting in T_0 and lifting successively gives compatible frames at all orders. Lemma IV.6.6 proves (*ii*).

Assertion (*iii*) implies (*i*). Finally, under (*ii*) fix a formal frame $\widehat{e}$ and denote its restrictions by e_μ. Every global frame of $\mathcal{L}_\mu$ is uniquely $a_\mu e_\mu$, for a global unit a_μ on $\mathcal{X}^{(\mu)}$. A frame at the next order restricts to $a_\mu e_\mu$ exactly when its coefficient relative to $e_{\mu+1}$ restricts to a_μ. This proves the last assertion. □

LEMMA IV.6.8. *Let $\mathcal{X} = [(X, \mathcal{I})]$ be an analytic neighborhood space with non-empty compact reduced center Z, and let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$, with finite restrictions $\mathcal{L}_\mu$ as in (302). Write $j_Z \colon Z \hookrightarrow X$. Fix $\mu \in \mathbb{Z}_{\geq 1}$ and a global frame $s_{\mu-1}$ of $\mathcal{L}_{\mu-1}$. With the degree sheaf $\mathcal{S}^\mu_{\mathcal{X}} = j_Z^{-1}(\mathcal{I}^\mu/\mathcal{I}^{\mu+1})$ introduced above, there is a class*

$$\mathrm{Obs}_\mu(\mathcal{L}, s_{\mu-1}) \in H^1(Z, \mathcal{S}^\mu_{\mathcal{X}})$$

which vanishes if and only if $s_{\mu-1}$ lifts to a global frame of $\mathcal{L}_\mu$.

The data for the covering form of Proposition IV.4.8 are as follows. The covering diagram includes chart restrictions and overlaps as in Example IV.1.16 (*iii*); here its comparisons are isomorphisms of rank-one modules over the fixed finite jet.

Comparison datum	Fixed-frame problem
Objects to compare	$\mathcal{L}_\mu$ and $\mathcal{O}_{\mathcal{X}^{(\mu)}}$ locally
Lower-order comparison	The specified frame $s_{\mu-1}$
Permissible lifts	Local frame lifts σ_j on U_j
Corrections	$1 + b_j$, with $b_j \in \mathcal{S}^\mu_{\mathcal{X}}(U_j)$
Additive sheaf	$1 + \mathcal{S}^\mu_{\mathcal{X}} \cong \mathcal{S}^\mu_{\mathcal{X}}$
Discrepancy	$\sigma_k = (1 + c_{jk})\sigma_j$ on $U_j \cap U_k$
Obstruction group	$H^1(Z, \mathcal{S}^\mu_{\mathcal{X}})$

Proof of Lemma IV.6.8. Local freeness of rank one gives local lifts σ_j of the fixed frame; these are frames because their reductions are units. For $\mu \geq 1$, the ideal $\mathcal{S}^\mu_{\mathcal{X}}$ in $\mathcal{O}_{\mathcal{X}^{(\mu)}}$ is square-zero, since $\mathcal{I}^{2\mu} \subset \mathcal{I}^{\mu+1}$. Consequently $(1+a)(1+b) = 1+a+b$ at this order, and any two local frame lifts differ by a unique multiplication by $1 + b$. Their ratios have the form $\sigma_k = (1 + c_{jk})\sigma_j$ used in the table. Frame changes and restriction therefore give the additive covering comparison of Proposition IV.4.8. Its obstruction is a Čech class, independent of local lifts. Passing over refinements and using Theorem II.3.79 gives the stated class in sheaf cohomology. Its vanishing means that corrected local frames glue, exactly as in that proposition; conversely a global frame lift has zero discrepancy. □

When $\mathcal{I}$ is not radical, $\mathcal{S}^{\mu}_{\mathcal{X}}$ need not be an $\mathcal{O}_Z$-module. Here H^1 is taken for its underlying abelian sheaf on Z, and the comparison with Čech cohomology is made after refinement.

The Picard group of a neighborhood germ

On a model, the Picard group introduced at the start of this section is given by

$$\operatorname{Pic}(\mathcal{X}) = \varinjlim_{V \in \mathcal{N}_X(Z)} \operatorname{Pic}(V, \mathcal{O}_V). \tag{303}$$

Thus an element is represented by an invertible sheaf on some neighborhood of Z. Two representatives are equal precisely when they become isomorphic on a common smaller neighborhood. In particular, zero means triviality on $\mathcal{X}$. We write $[\mathcal{L}]$ for its class in this group (The model notation $\operatorname{Pic}_Z(X)$ denotes the same group).

Proposition IV.6.9. *Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center. Choose a model $(X, \mathcal{I})$, put $Z := Z(X, \mathcal{I})$, and let $j_Z \colon Z \hookrightarrow X$ be the inclusion. There is a natural isomorphism $\operatorname{Pic}(\mathcal{X}) \cong H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})$, and there are natural identifications*

$$\operatorname{Pic}(\mathcal{X}) \cong H^1\big(Z, j_Z^{-1}\mathcal{O}^*_X\big) \cong \operatorname{Pic}\big(Z, j_Z^{-1}\mathcal{O}_X\big). \tag{304}$$

Completion induces a homomorphism

$$\rho_{\mathcal{X}} \colon \operatorname{Pic}(\mathcal{X}) \longrightarrow \operatorname{Pic}(\mathcal{X}^{(\infty)}, \mathcal{O}_{\mathcal{X}^{(\infty)}}), \qquad [\mathcal{L}] \longmapsto [\widehat{\mathcal{L}}]. \tag{305}$$

Under the cohomological descriptions of the Picard groups, this homomorphism is induced by $j_Z^{-1}\mathcal{O}^*_X \to \mathcal{O}^*_{\mathcal{X}^{(\infty)}}$.

Proof. On a neighborhood V, we have $\operatorname{Pic}(V, \mathcal{O}_V) \cong H^1(V, \mathcal{O}^*_V)$ (Remark II.3.104). Passing to the direct limit gives the isomorphism with $H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})$. The first identification in (304) follows from the shrinking-neighborhood comparison for abelian sheaves in Remark II.3.113. Finally, $(j_Z^{-1}\mathcal{O}_X)^* = j_Z^{-1}\mathcal{O}^*_X$: a germ and its inverse are both represented on a smaller neighborhood. The same transition-function description on Z gives the last isomorphism.

Send the transition functions of $\mathcal{L}$ to their images in $\mathcal{O}_{\mathcal{X}^{(\infty)}}$. They remain units and define $\widehat{\mathcal{L}}$. A change of trivializations remains a coboundary, and restricting to a smaller neighborhood changes neither completed sheaf nor its class. Tensor products multiply transition functions, proving all assertions about $\rho_{\mathcal{X}}$. □

The descriptions on Z in (304) retain convergent ambient germs in their coefficients. They are not the coefficients $\mathcal{O}^*_Z$ or $\mathcal{O}^*_{\mathcal{X}^{(\infty)}}$. This is where the shrinking-neighborhood limit differs from the formal completion.

Logarithmic Ueda-type obstructions

We retain an analytic neighborhood space $\mathcal{X}$, a model $(X, \mathcal{I})$ with X a complex manifold, and its non-empty compact reduced center Z. Write $j_Z \colon Z \hookrightarrow X$ and $\mathcal{I}_{\mathcal{X}} := j_Z^{-1}\mathcal{I}$. Let L be a holomorphic line bundle on a representative neighborhood, let $\mathcal{L}$ be its induced invertible sheaf on $\mathcal{X}$, and use $\mathcal{L}_\mu$ from (302).

Definition IV.6.10 ([Koi13, §3.2]). Fix $\mu \in \mathbb{Z}_{\geq 1}$ and suppose that $\mathcal{L}_{\mu-1}$ is trivial. The exponential morphism $f \mapsto \exp(2\pi\sqrt{-1}f)$ on $\mathcal{I}_{\mathcal{X}}^{\mu}$ induces a homomorphism

$$\alpha_{\mathcal{I},\mu} \colon H^1(\mathcal{X}, \mathcal{I}_{\mathcal{X}}^{\mu}) \longrightarrow \mathrm{Pic}(\mathcal{X}).$$

Choose $\xi \in \alpha_{\mathcal{I},\mu}^{-1}([\mathcal{L}])$. We denote its image under the quotient-induced map $H^1(\mathcal{X}, \mathcal{I}_{\mathcal{X}}^{\mu}) \to H^1(Z, \mathcal{S}_{\mathcal{X}}^{\mu})$ by $u_\mu(\mathcal{I}, L; \xi)$ and put

$$U_\mu(\mathcal{I}, L) := \{u_\mu(\mathcal{I}, L; \xi) \mid \xi \in \alpha_{\mathcal{I},\mu}^{-1}([\mathcal{L}])\}.$$

When this set consists of one element, we write that element as $u_\mu(\mathcal{I}, L)$ and say that this Ueda-type obstruction is *well-defined*. □

The choice of ξ is possible: triviality of $\mathcal{L}_{\mu-1}$ allows local frames whose transition functions belong to $1 + \mathcal{I}_{\mathcal{X}}^{\mu}$. The logarithm taking 1 to 0 identifies this sheaf with $\mathcal{I}_{\mathcal{X}}^{\mu}$ after division by $2\pi\sqrt{-1}$. The resulting additive cocycle gives ξ (See [Koi13, Lemma 3.2]). For this fixed ξ, vanishing of $u_\mu(\mathcal{I}, L; \xi)$ means that ξ lifts to $H^1(\mathcal{X}, \mathcal{I}_{\mathcal{X}}^{\mu+1})$, by the long exact sequence of $0 \to \mathcal{I}_{\mathcal{X}}^{\mu+1} \to \mathcal{I}_{\mathcal{X}}^{\mu} \to \mathcal{S}_{\mathcal{X}}^{\mu} \to 0$. In particular, $\mathcal{L}_\mu$ is trivial if and only if $0 \in U_\mu(\mathcal{I}, L)$ (See [Koi13, Lemma 3.4]).

If a global frame $s_{\mu-1}$ has been fixed as in Lemma IV.6.8, choose local lifts and write their transitions as $p_{jk} = 1 + c_{jk}$ modulo $\mathcal{I}^{\mu+1}$. For the logarithmic lift obtained from these same frames, the local representative of $u_\mu(\mathcal{I}, L; \xi)$ is $c_{jk}/(2\pi\sqrt{-1})$. With the data matched in this way, the logarithmic class is the fixed-frame obstruction multiplied by $(2\pi\sqrt{-1})^{-1}$. This equality does not identify an arbitrary ξ with a previously specified frame. Nor does it assert that the possible-value set is always a singleton. We use Obs_μ for the fixed-frame lifting problem and u_μ for the logarithmic convention just defined; the non-zero scalar does not change their vanishing.

The additive kernel and the Picard kernel

The logarithmic description in [Koi13, §3.2] can be organized using the kernels introduced in § 2.3. We first prove their relation by the exponential exact sequence, and then describe the resulting correspondence by transition functions.

Let $\mathcal{X}$ be an analytic neighborhood space with non-empty compact reduced center Z. In this paragraph the coefficient sheaf is $\mathcal{O}_{\mathcal{X}}$ and the cohomology degree is 1. The exponential morphism $f \mapsto \exp(2\pi\sqrt{-1}f)$ induces

$$\mathrm{Exp} \colon H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \longrightarrow \mathrm{Pic}(\mathcal{X}).$$

For $\mu \in \mathbb{Z}_{\geq 0}$, write

$$p_\mu \colon \operatorname{Pic}(\mathcal{X}) \longrightarrow \operatorname{Pic}(\mathcal{X}^{(\mu)}, \mathcal{O}_{\mathcal{X}^{(\mu)}})$$

for finite-jet restriction. Retain ρ^1_μ for the additive cohomology map in (282) with coefficient $\mathcal{O}_\mathcal{X}$.

PROPOSITION IV.6.11. *Let $\mathcal{X}$ be an analytic neighborhood space with non-empty compact reduced center, and let $\mu \in \mathbb{Z}_{\geq 0}$. For the maps just defined, restriction of* Exp *gives an isomorphism of abelian groups*

$$\operatorname{Exp} \colon \operatorname{Ker} \rho^1_\mu \xrightarrow{\cong} \operatorname{Ker} p_\mu. \tag{306}$$

PROOF. Choose a model $(X, \mathcal{I})$ with reduced center Z and inclusion $j_Z \colon Z \hookrightarrow X$. On Z the analytic germ sheaf and the finite-jet structure sheaf have exponential exact sequences, compatible with their quotient map, whose beginning and end have the form

$$0 \longrightarrow \underline{\mathbb{Z}}_Z \longrightarrow \mathcal{R} \xrightarrow{\exp(2\pi\sqrt{-1}\,\cdot)} \mathcal{R}^* \longrightarrow 1,$$

where $\mathcal{R}$ is $j_Z^{-1}\mathcal{O}_X$ or $\mathcal{O}_{\mathcal{X}^{(\mu)}}$. Local logarithms prove exactness, also for a non-reduced finite jet: a unit can be locally lifted to a holomorphic unit, and the nilpotent part is treated by its finite logarithmic series. The map between the integer sheaves is the identity. By (304) and the shrinking-neighborhood comparison, these sequences give compatible cohomology sequences

$$H^1(Z, \mathbb{Z}) \longrightarrow H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}) \xrightarrow{\operatorname{Exp}} \operatorname{Pic}(\mathcal{X}) \longrightarrow H^2(Z, \mathbb{Z}),$$

and the same sequence with $\mathcal{X}$ replaced by $\mathcal{X}^{(\mu)}$. The vertical maps on the integer cohomology groups are identities.

The map $H^1(Z, \mathbb{Z}) \to H^1(Z, \mathcal{O}_Z)$ is injective. Indeed, apply Lemma III.2.91 to the real and imaginary parts of a holomorphic function on Z. Such a function is constant on each connected component. Every holomorphic unit on Z consequently has a global logarithm. The exponential sequence on Z now proves the claimed injection.

Suppose $a \in \operatorname{Ker} \rho^1_\mu$ and $\operatorname{Exp}(a) = 0$. The analytic sequence writes a as the image of $\gamma \in H^1(Z, \mathbb{Z})$. Restricting first to the finite jet and then to the reduced center shows that the image of γ in $H^1(Z, \mathcal{O}_Z)$ is zero. Hence $\gamma = 0$ and $a = 0$.

Conversely, let $[\mathcal{L}] \in \operatorname{Ker} p_\mu$. Its integer class in $H^2(Z, \mathbb{Z})$ is zero, because its finite restriction is trivial and the integer comparison is the identity. Choose $a \in H^1(\mathcal{X}, \mathcal{O}_\mathcal{X})$ with $\operatorname{Exp}(a) = [\mathcal{L}]$. The finite exponential sequence expresses $\rho^1_\mu(a)$ as the image of some $\gamma \in H^1(Z, \mathbb{Z})$. Subtract the analytic image of this same γ from a. The result lies in $\operatorname{Ker} \rho^1_\mu$ and still exponentiates to $[\mathcal{L}]$. This proves surjectivity. □

COROLLARY IV.6.12. *Let $\mathcal{X}$ be an analytic neighborhood space with non-empty compact reduced center. For the formal Picard restriction $\rho_\mathcal{X}$ in (305), exponentiation induces an isomorphism of abelian groups*

$$\operatorname{Exp} \colon K^1_{\text{for}}(\mathcal{X}, \mathcal{O}_\mathcal{X}) \xrightarrow{\cong} \operatorname{Ker} \rho_\mathcal{X}. \tag{307}$$

In particular, $\mathcal{X}$ satisfies FPLB if and only if $K^1_{\text{for}}(\mathcal{X}, \mathcal{O}_\mathcal{X}) = 0$.

PROOF. Proposition IV.6.7 gives $\operatorname{Ker}\rho_{\mathcal{X}} = \bigcap_{\mu\geq 0}\operatorname{Ker} p_\mu$. If $[\mathcal{L}]$ belongs to this intersection, Proposition IV.6.11 gives, for each μ, a class $a_\mu \in \operatorname{Ker}\rho^1_\mu$ with $\operatorname{Exp}(a_\mu) = [\mathcal{L}]$. Every a_μ lies in $\operatorname{Ker}\rho^1_0$. Injectivity of (306) for $\mu = 0$ shows that all these classes are the same class a. Thus $a \in K^1_{\text{for}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. The converse and injectivity follow from the same finite-order isomorphisms. The last assertion is the definition of FPLB. □

A logarithmic description

Let $\mathcal{X}$ have non-empty compact reduced center Z, fix $\mu \in \mathbb{Z}_{\geq 0}$, and let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$ whose restriction $\mathcal{L}_\mu$ is trivial. Choose a model $(X, \mathcal{I})$, a representative of $\mathcal{L}$, and a global frame of $\mathcal{L}_\mu$. After refining a finite covering of Z and shrinking in the ambient space, lift that frame to local analytic frames e_j on ambient open sets V_j. Put $U_j := V_j \cap Z$, $\mathcal{U} := \{U_j\}_{j\in J}$, and $U_{jk} := U_j \cap U_k$. If $e_k = g_{jk}e_j$ on $V_j \cap V_k$, then $g_{jk} \equiv 1 \bmod \mathcal{I}^{\mu+1}$. The branch of the logarithm with value zero at 1 gives

$$f_{jk} := (2\pi\sqrt{-1})^{-1}\log g_{jk} \in \mathcal{I}^{\mu+1}$$

as a section of the ambient germ sheaf on each overlap. It belongs to this ideal because $\log(1+t)$ is divisible by t. On triple overlaps, $f_{jk}+f_{k\ell}-f_{j\ell}$ is integer-valued and vanishes on Z, so it vanishes as an ambient germ. These finite data can be represented on a common smaller neighborhood (Lemma II.3.115). The collection $f := \{(U_{jk}, f_{jk})\}_{j,k\in J}$ lies in $\check{Z}^1(\mathcal{U}, j_Z^{-1}\mathcal{I}^{\mu+1})$, where $j_Z\colon Z \hookrightarrow X$ is the inclusion and the sections f_{jk} are taken as ambient germs. Its Čech class $[f]$ therefore gives, by the shrinking-neighborhood comparison, a class $\xi \in H^1(\mathcal{X}, \mathcal{I}^{\mu+1})$, whose image $a \in H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ belongs to $\operatorname{Ker}\rho^1_\mu$ and satisfies $\operatorname{Exp}(a) = [\mathcal{L}]$. This is the unique class in (306), although its lift ξ need not be unique (Compare [Koi13, Lemmata 3.2 and 3.5], with the ideal power there equal to $\mu + 1$).

For these data, $a = 0$ precisely when, after refinement and shrinking, one can write $f_{jk} = b_k - b_j$ with holomorphic b_j; then the frames $e'_j := \exp(-2\pi\sqrt{-1}b_j)e_j$ agree on overlaps. Here the b_j are not required to lie in $\mathcal{I}^{\mu+1}$. Consequently this criterion gives analytic triviality of $\mathcal{L}$, not the extension of the particular finite frame chosen above (For that problem, see Lemma IV.6.8).

COROLLARY IV.6.13. *Let $\mathcal{X}$ be an analytic neighborhood space with non-empty compact reduced center, and give $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ its natural topology.*

(i) If this space is finite-dimensional and $\mathcal{X}$ satisfies FPLB, there exists $\mu_0 \in \mathbb{Z}_{\geq 0}$ such that the finite-jet restriction homomorphism $p_{\mu_0}\colon \operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(\mathcal{X}^{(\mu_0)}, \mathcal{O}_{\mathcal{X}^{(\mu_0)}})$, sending $[\mathcal{L}]$ to $[\mathcal{L}_{\mu_0}]$, is injective. In particular, every invertible sheaf which is trivial on $\mathcal{X}^{(\mu_0)}$ is trivial on $\mathcal{X}$.

(ii) If this space is infinite-dimensional and Hausdorff, then, for every $\mu \in \mathbb{Z}_{\geq 0}$, there exists an invertible sheaf $\mathcal{L}$ which is non-trivial on $\mathcal{X}$ but trivial on $\mathcal{X}^{(\mu)}$.

(iii) If this space is non-Hausdorff, $\mathcal{X}$ does not satisfy FPLB: some non-trivial invertible sheaf on $\mathcal{X}$ is trivial on $\mathcal{X}^{(\infty)}$.

PROOF. Apply Proposition IV.2.30 with $q = 1$ and $\mathcal{F} = \mathcal{O}_{\mathcal{X}}$. For (i), FPLB makes the stabilized additive kernel zero by Corollary IV.6.12; now use Proposition IV.6.11. For (ii), take a non-zero class in $\operatorname{Ker} \rho^1_\mu$ and use the injectivity of (306). For (iii), use the non-zero formal kernel and (307). □

In (ii) of Corollary IV.6.13, the invertible sheaf may depend on μ. This does not produce one non-trivial sheaf which is trivial at every finite order. Neither finite-dimensionality alone nor Hausdorffness alone annihilates the formal kernel. The Rossi criterion below supplies a geometric injectivity statement of the kind needed in (i).

COROLLARY IV.6.14. *Let Y be an elliptic curve and let $F \to Y$ be a unitary flat holomorphic line bundle. Let X_F be its total space, put $Z :=$ (Zero Section)$_F$, and let $\mathcal{I}_Z \subset \mathcal{O}_{X_F}$ be the reduced defining ideal of Z. For the analytic neighborhood space $\mathcal{X}_F := [(X_F, \mathcal{I}_Z)]$, FPLB holds if and only if F is torsion or is non-torsion and asymptotically positive. Equivalently, for these linear models, FPLB holds if and only if $H^1(\mathcal{X}_F, \mathcal{O}_{\mathcal{X}_F})$ is Hausdorff.*

PROOF. Corollary IV.6.12 identifies FPLB with the vanishing of $K^1_{\mathrm{for}}(\mathcal{X}_F, \mathcal{O}_{\mathcal{X}_F})$. For this model, Proposition IV.3.3 identifies that kernel with the closure of zero. The three cohomology cases are exactly those computed in Example IV.1.44. □

For a non-torsion asymptotically zero F, the analytic class of (245) therefore gives, through (307), an invertible sheaf on $\mathcal{X}_F$ which is formally trivial but analytically non-trivial. This happens even though the ambient neighborhood is already the total-space model of the normal line bundle. It is a failure of the formal principle for line bundles on this fixed neighborhood, not a failure to identify that neighborhood with its normal-bundle model. For torsion F, on the other hand, the injective but non-surjective map (288) shows why FPLB need not mean that every prescribed formal cohomology class converges. These conclusions use the computed kernel for $\mathcal{X}_F$.

An exceptional-center case from Rossi's theorem

For comparison with the tubular neighborhood result in § 5.2, we next obtain the FPLB in the setting of the contraction theorem of Chapter III. The exceptional-subspace result is due to Peternell [Pet] (See also [Koi13, Theorem 6.2]). In the following manifold setting we deduce it from Rossi's theorem and the already established comparison with the additive formal kernel.

THEOREM IV.6.15. *Let X be a connected strongly 1-convex complex manifold of complex dimension $n \geq 1$, and let $A \neq \emptyset$ be its reduced exceptional analytic set as in Theorem III.2.183. For every coherent ideal $\mathcal{I} \subset \mathcal{O}_X$ with $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}) = A$, the FPLB holds for $\mathcal{X} = [(X, \mathcal{I})]$.*

PROOF. By Remark IV.6.3, we may replace $\mathcal{I}$ by the reduced defining ideal $\mathcal{I}_A$. If X is compact, then $A = X$ and $\mathcal{I}_A = 0$, so the conclusion is immediate. Assume that X is non-compact, and take $a \in K^1_{\mathrm{for}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. The shrinking-neighborhood comparison

(Proposition IV.1.22) represents a by a class on some open neighborhood V of A. Choose a positive regular value ε of the exhaustion in (222) so small that $\overline{\Omega} \subset V$, where $\Omega := X_\varepsilon$, and denote the restricted class by $a_\varepsilon \in H^1(\Omega, \mathcal{O}_\Omega)$. These sublevels form a neighborhood basis of A, as explained before Theorem III.2.188.

Rossi's theorem (Theorem III.2.188), with $\mathcal{F} = \mathcal{O}_X$ and $q = 1$, gives an integer $\mu \geq 1$ for which

$$H^1(\Omega, \mathcal{O}_\Omega) \longrightarrow H^1(A^{(\mu-1)}, \mathcal{O}_{A^{(\mu-1)}}) \quad \text{is injective,} \qquad A^{(\mu-1)} := \mathscr{J}_{\mu-1}(X, \mathcal{I}_A). \tag{308}$$

Here the closed embedding of $A^{(\mu-1)}$ in Ω identifies Rossi's target with the displayed group, by (216). By Definition IV.2.29, the class a restricts to zero on every finite jet. Hence the image of a_ε in (308) is zero. Injectivity gives $a_\varepsilon = 0$, and therefore $a = 0$ in the cohomology of the neighborhood germ. Thus $K^1_{\mathrm{for}}(\mathcal{X}, \mathcal{O}_\mathcal{X}) = 0$; Corollary IV.6.12 now gives FPLB. □

The FPLB turns formal triviality of a particular $\mathcal{P}_L$ into analytic triviality, but it quantifies over all invertible sheaves, not only this comparison. We next use elliptic suspensions to bring together the action, neighborhood, and bundle comparisons of §§ IV.4–IV.6.

7 Elliptic suspensions and comparison problems

We now connect a point-centered action with a neighborhood of an elliptic curve. Recall the suspension construction preceding Lemma I.7.31 in Chapter I. Our convention for endpoint transport gives a homomorphism from $\pi_1(Y, *)^{\mathrm{op}}$. For an actual action on one fixed fiber, the construction uses the universal cover and deck transformations. An action on a neighborhood germ need not have representatives on one common invariant neighborhood, so that quotient construction cannot simply be applied to all its elements. We use instead a finite gluing in which one holonomy generator is the identity. It converts the one-object comparison of Example IV.1.16 (*ii*) into a local gluing problem. The specified retraction and foliation remain part of the comparison in Proposition IV.7.1.

Let $C := \mathbb{C}^*/\langle z \mapsto az \rangle$, where $a \in \mathbb{C}$ and $|a| > 1$, be an elliptic curve, and let $p\colon \mathbb{C}^* \to C$ be the quotient map. Let f be a holomorphic germ with $f(0) = 0$ and $f'(0) \neq 0$. Choose a small neighborhood U_0 of 0 on which f is injective, and put $V_1 := U_0 \cap f(U_0)$ and $U_1 := (f|_{U_0})^{-1}(V_1)$. Thus $U_1 \subset U_0$ and $f\colon U_1 \to V_1$ is biholomorphic. For sufficiently small $\varepsilon > 0$, the annuli

$$A := p\big(\{|a|^{-1} < |z| < 1\}\big), \qquad B := p\big(\{1 - \varepsilon < |z| < 1 + \varepsilon\}\big)$$

cover C, and $A \cap B$ has two components, denoted by B_- and B_+ according as the representative in $\{1 - \varepsilon < |z| < 1 + \varepsilon\}$ has $|z| < 1$ or $|z| > 1$. Glue $A \times U_0$ and $B \times U_1$ by

$$(z, w) \longmapsto (z, w) \quad \text{on } B_- \times U_1, \qquad (z, w) \longmapsto (z, f(w)) \quad \text{on } B_+ \times U_1, \tag{309}$$

where the images lie in $B_- \times U_1$ and $B_+ \times V_1$ inside the first chart. Here z denotes a base point of C. This is the construction of [Koi5, §2.1], with the reciprocal base generator; it gives a Hausdorff complex surface near the zero sections. Denote the resulting neighborhood

by X_f. The first projections glue to a holomorphic retraction $\pi_f\colon X_f \to C$, and the slices $w = $ constant glue to a non-singular holomorphic foliation $\mathcal{F}_f$ with compact leaf C. Write $\mathcal{U}_f := [(X_f, \mathcal{I}_C)]$.

Fix a base point and generators γ_1, γ_2 of $\pi_1(C, *) \cong \mathbb{Z}^2$, oriented so that the holonomy is (id, f). The first generator runs around an annulus and the second crosses its seam. This also specifies which germ is f rather than f^{-1}. In the notation of this chapter, [Koi5, Proposition 2.1] associates to these data an analytic neighborhood space $\mathcal{U}_f$ over the marked curve C, with its retraction and foliation, up to an isomorphism preserving both. The next proposition identifies the corresponding isomorphism classes with conjugacy classes of the transverse germs.

Proposition IV.7.1. *Let f and g be holomorphic germs fixing 0 with non-zero multipliers, and form $\mathcal{U}_f$ and $\mathcal{U}_g$ over the same marked elliptic curve as above. A biholomorphic germ h with $h(0) = 0$ and $f \circ h = h \circ g$ induces an isomorphism $\mathcal{U}_g \to \mathcal{U}_f$ over C preserving the foliations. Conversely, an isomorphism over C preserving these foliations induces such a conjugacy on a transverse germ at the chosen base point.*

Proof. In the two charts use $(z, w) \mapsto (z, h(w))$, restricting the transverse neighborhoods as necessary. It commutes with the identity transition, and compatibility on the other component is exactly $f \circ h = h \circ g$. The inverse maps glue in the same way. Conversely, restriction to the fiber over the base point is a biholomorphic transverse germ. Preservation of the foliation makes this germ commute with transport around each marked loop, which gives the conjugacy equation. □

Put $\lambda := f'(0)$ and $R_\lambda(w) := \lambda w$. For $g = R_\lambda$, the construction X_{R_λ} is a neighborhood of the zero section in the normal line bundle N_{C/X_f}: its linear transitions are precisely the derivatives of (309). Thus the equivariant linearization from Example IV.4.5 gives a tubular comparison preserving the retraction and foliation. The group action has been recorded in this gluing data, not discarded. Forgetting the additional structures does not in general preserve the classification by holonomy (See [KO2, §3.3.1–§3.3.2, pp. 431–433]).

For this particular construction there is also a converse without requiring the tubular comparison to preserve the foliation. Keep the elliptic curve C, the germ f, the retraction π_f, and the foliation $\mathcal{F}_f$ constructed above, and put $\lambda := f'(0)$ and $\mathcal{N} := \mathcal{O}_C(N_{C/X_f})$. The normal line bundle has degree zero, since its transition functions are locally constant non-zero numbers. It admits a unitary flat structure by Proposition III.2.138 and the degree description of $\mathrm{Pic}^0(C)$. Fix that structure and its unitary flat extension $\widetilde{\mathcal{N}}$ to a neighborhood V of C, with the specified identification on C. Define the Ueda invertible sheaf of $\mathcal{U}_f$ by

$$\mathcal{P}_f := \mathcal{O}_V(C) \otimes_{\mathcal{O}_V} \widetilde{\mathcal{N}}^*,$$

regarded as a sheaf on the neighborhood germ. When $|\lambda| \neq 1$, the original multiplier itself is not a unitary transition constant.

An elliptic curve with topologically trivial normal bundle has a neighborhood of *class* (β) when it admits local defining functions u_j with $u_j = t_{jk} u_k$ for locally constant $t_{jk} \in \mathrm{U}(1)$. This is the fixed unitary comparison in (301) ([U1]; [KO2, §2.2]). For the present suspension, the following proposition proves the required equivalence directly.

PROPOSITION IV.7.2. *Let $C \subset X_f$, $\mathcal{U}_f$, π_f, and $\mathcal{F}_f$ be the elliptic suspension in (309), with marked holonomy (id, f) and $\lambda = f'(0) \neq 0$. Fix the unitary flat comparison and the sheaf $\mathcal{P}_f$ just described. Then the following are equivalent.*

(*i*) *The germ f is linearizable.*

(*ii*) *The curve C has a holomorphic tubular neighborhood in X_f.*

(*iii*) *The invertible sheaf $\mathcal{P}_f$ is trivial on $\mathcal{U}_f$.*

PROOF. This is the special suspension case of the comparison in [KO2, Theorem 1.2(ii) and §4.4]. We give the proof in this setting, using the one-variable results of Chapter III for the non-linearizable irrational case. If (*i*) holds, Proposition IV.7.1 with $g(w) = \lambda w$ gives (*ii*). A tubular comparison pulls back unitary flat defining functions of the zero section of N_{C/X_f}. The induced unitary flat extension of $\mathcal{N}$ agrees with the fixed one after shrinking, by Example II.3.117. Thus (*ii*) implies (*iii*).

Assume (*iii*). Choose local defining functions w_j for C and flat frames of $\widetilde{\mathcal{N}}$, with transition constants t_{jk}. The corresponding frames e_j of $\mathcal{P}_f$ satisfy $e_k = p_{jk} e_j$, where $p_{jk} = w_j/(t_{jk} w_k)$. Write a global frame of $\mathcal{P}_f$ as $a_j e_j$ on the j-th patch. Then $a_j = p_{jk} a_k$, so the functions $u_j := w_j / a_j$ satisfy $u_j = t_{jk} u_k$. In particular, they vanish simply along C, and

$$\Phi = -\log |u_j|^2$$

is a well-defined pluriharmonic function on a smaller $V \setminus C$, tending to $+\infty$ as a point approaches C. We show that f is linearizable by considering three cases.

If $|\lambda| \neq 1$, the Kœnigs linearization theorem (Theorem III.3.10) already gives (*i*).

Suppose next that λ has finite order $q \geq 1$ in $\mathrm{U}(1)$, and set $F := f^{\circ q}$. Pull back the suspension by the finite covering of C corresponding to the subgroup generated by γ_1 and $q\gamma_2$. Its marked holonomy is (id, F), and its normal bundle is holomorphically trivial, since $F'(0) = 1$. The pulled-back unitary flat normal structure is consequently trivial: a nowhere vanishing holomorphic section on this compact curve has constant norm in a flat metric, and its coefficients in flat frames are constant. Its unitary flat extension is also trivial after shrinking. Thus (*iii*) on this finite cover gives a global holomorphic defining function H for the lifted curve, vanishing simply there.

Choose additive periods ω_1, ω_2 of the lifted elliptic curve in the directions of its marked generators. Analytic continuation of the transverse coordinate along its universal cover gives, for the finite Taylor jets of H,

$$\widetilde{H}(\zeta, w) = \sum_{\nu \geq 1} A_\nu(\zeta) w^\nu, \qquad A_\nu \in \mathcal{O}_{\mathbb{C}}(\mathbb{C}).$$

Here the series is understood near each point of the zero section; a radius uniform in ζ is not asserted. The endpoint-transport convention in the gluing gives

$$\widetilde{H}(\zeta + \omega_1, w) = \widetilde{H}(\zeta, w), \qquad \widetilde{H}(\zeta + \omega_2, w) = \widetilde{H}(\zeta, F(w)).$$

The first coefficient A_1 is doubly periodic and nowhere zero, hence is a non-zero constant. Rescale H so that $A_1 = 1$. If $F \neq \mathrm{id}$, write $F(w) = w + bw^m + O(w^{m+1})$ with $m \geq 2$ and $b \neq 0$. Comparing the coefficients of degree m gives

$$A_m(\zeta + \omega_1) = A_m(\zeta), \qquad A_m(\zeta + \omega_2) - A_m(\zeta) = b.$$

The entire function A_m' is then doubly periodic and constant. The first period forces that constant to be zero, whereas the second identity has non-zero right-hand side, a contradiction. It follows that $f^{\circ q} = \mathrm{id}$. The function

$$\Psi(w) := \frac{1}{q} \sum_{j=0}^{q-1} \lambda^{-j} f^{\circ j}(w)$$

satisfies $\Psi'(0) = 1$ and $\Psi \circ f = \lambda \Psi$. Its local inverse is a linearizing function, proving (i) in this case.

It remains to consider $\lambda = e^{2\pi\sqrt{-1}\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Suppose that f is not linearizable. Choose a sufficiently small admissible domain for f and a Siegel compactum K as in Theorem III.3.18, so that $f(K) = K = f^{-1}(K)$ and $0 \in \partial K$. By Pérez-Marco's boundary-orbit theorem (Theorem III.3.30), a non-zero point has a two-sided orbit dense in ∂K. Since $0 \in \partial K$, there are $w_0 \in \partial K \setminus \{0\}$ and integers n_ν with $|n_\nu| \to \infty$ and $f^{\circ n_\nu}(w_0) \to 0$. In the two suspension charts, glue the subsets $A \times K$ and $B \times K$ using (309), and denote the result by M_K. Complete invariance of K makes this well-defined. It is compact: a finite closed trivializing covering of the compact base expresses it as a finite union of images of products of compact sets with K. By choosing the admissible domain small enough, we have $M_K \subset V$.

Let L be the leaf of $\mathcal{F}_f$ through the point with transverse coordinate w_0, equipped with its intrinsic leaf topology. The map $\pi_f|_L \colon L \to C$ is a holomorphic covering. Indeed, on a simply connected base patch the horizontal lifts are sheets indexed by points of the orbit of w_0, and every base path can be continued: each seam crossing applies f or f^{-1}, which stays in K. In the leaf topology these sheets are separate even when their images accumulate in X_f. Also $L \subset M_K \setminus C$, since an iterate of a non-zero point cannot equal the fixed point 0. Let $\widetilde{L}$ be the universal covering Riemann surface of L, with covering map $p_L \colon \widetilde{L} \to L$, and write $\iota_L \colon L \to V \setminus C$ for the natural holomorphic immersion. Since $\pi_f|_L$ is a holomorphic covering and the universal cover of the elliptic curve C is $\mathbb{C}$, we have $\widetilde{L} \cong \mathbb{C}$.

The function $|u_j|$ is globally defined on V and bounded above on M_K. Thus $(\iota_L \circ p_L)^*\Phi$ is a harmonic function on $\widetilde{L} \cong \mathbb{C}$ bounded from below, and hence is constant by Mini-Exercise I.9.3. Consequently, $\Phi|_L$ is constant. On the other hand, the points with transverse coordinates $f^{\circ n_\nu}(w_0)$ lie on this same leaf over the chosen base point and tend to C. The values of Φ there tend to $+\infty$, a contradiction. This proves $(iii) \Rightarrow (i)$. □

The same suspension exhibits the distinction between formal and analytic triviality of one fixed invertible sheaf.

Proposition IV.7.3. *Let $\mathcal{U}_f = [(X_f, \mathcal{I}_C)]$ be the elliptic suspension above, and suppose that $f'(0) = \lambda = e^{2\pi\sqrt{-1}\theta}$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $\mathcal{P}_f$ be its Ueda invertible sheaf with the fixed*

unitary flat comparison. Then $\widehat{\mathcal{P}_f}$ *is trivial, and compatible global frames can be chosen on all finite jets. If* f *is not linearizable,* $[\mathcal{P}_f]$ *is a non-zero member of* $\operatorname{Ker}\rho_{\mathcal{U}_f}$*, and* $\mathcal{U}_f$ *does not satisfy the FPLB.*

PROOF. Since $\lambda^n \neq 1$ for every $n \geq 1$, the recursion (252) gives a formal series $h(w) = w + O(w^2)$ satisfying $f \circ h = h \circ R_\lambda$, where $R_\lambda(w) = \lambda w$. Its formal inverse H satisfies $H(f(w)) = \lambda H(w)$ and $H(w) = w + O(w^2)$. Use the original suspension coordinates w_j. Their linear transitions t_{jk} are 1 or λ (or the corresponding inverses), so they already describe the specified unitary flat normal bundle and its extension. Set $\widehat{u}_j := H(w_j)$. These formal functions satisfy $\widehat{u}_j = t_{jk}\widehat{u}_k$. In the local frames e_j with $e_k = [w_j/(t_{jk}w_k)]e_j$, the coefficients

$$\widehat{a}_j := \frac{w_j}{H(w_j)}$$

are formal units, have constant term 1, and satisfy $\widehat{a}_j = [w_j/(t_{jk}w_k)]\widehat{a}_k$. They define a formal frame of $\widehat{\mathcal{P}_f}$. Its restrictions give compatible frames at every finite order, as in Lemma IV.6.6. If f is not linearizable, Proposition IV.7.2 shows that $\mathcal{P}_f$ is not trivial on the neighborhood germ. The definition of $\rho_{\mathcal{U}_f}$ in (305) now gives the last assertions. □

REMARK IV.7.4. Choose a polynomial f as in Proposition III.3.14, with a multiplier furnished by Proposition III.3.15, and form the elliptic suspension $\mathcal{U}_f$ above. This is the elliptic-curve case of Ueda's example in [U1, §5.4, pp. 605–607]. Its small cycles give compact curves in the suspension approaching C, the geometric feature used in Ueda's construction.

The normal bundle is non-torsion and unitary flat. The formal linearizing coordinate H in the preceding proof changes the gluing by id and f to that by id and R_λ: the maps $(z, w) \mapsto (z, H(w))$ therefore give a formal isomorphism with the normal-bundle neighborhood. The equivalence of (i) and (ii) in Proposition IV.7.2 excludes a holomorphic tubular neighborhood, whereas Proposition IV.7.3 describes the associated invertible sheaf: $\widehat{\mathcal{P}_f}$ is trivial, but $\mathcal{P}_f$ is not trivial on $\mathcal{U}_f$ and witnesses the failure of FPLB.

Ueda credits Arnol'd's work for the suggestion of this example ([U1, Introduction, p. 584]). To the author's knowledge, this was the first explicit construction of an elliptic-curve neighborhood with non-torsion unitary flat normal bundle which is formally equivalent to its normal-bundle model but not analytically equivalent to it. □

8 Ueda theory on analytic neighborhood spaces

The preceding suspension relates a local dynamical obstruction to the triviality of one specified invertible sheaf on an analytic neighborhood space. We now describe the corresponding finite-order geometric constructions. We keep two settings in view from the outset. In both, the ambient space X is a complex manifold and the center Z is non-empty and compact.

(S) We consider the analytic neighborhood space $\mathcal{X} = [(X, \mathcal{I}_Z)]$, where Z is a non-singular compact complex submanifold of codimension $r \geq 1$ and $\mathcal{I}_Z$ is its reduced ideal. The normal bundle $N_{Z/X}$ has a specified unitary flat structure.

(*D*) An effective Cartier divisor D on X with reduced support Z is specified, and the invertible sheaf $\mathcal{N}_D := \mathcal{O}_Z([D]|_Z)$ has a specified unitary flat structure. The support may be singular or reducible, and the multiplicities of D are retained.

We use a model only to describe the local functions and their transition rules; the resulting constructions concern its analytic neighborhood space. The basic question is how far an ideal's local generators can be made to agree with a fixed flat comparison. In higher codimension this involves a system of functions, not just one invertible sheaf.

In setting (S), we seek a defining-function system satisfying $u_j = T_{jk}u_k$, for the specified locally constant unitary matrices. Transitions of the other coordinates may still have the form $z_j = \varphi'_{jk}(z_k, u_k)$, with φ'_{jk} unknown and dependent on u_k. Thus the desired normal form prescribes only part of the morphism data. It is weaker than the holomorphic tubular comparison in §5.1.

8.1 Fixing the unitary flat comparison

In setting (D), let $\mathcal{I}$ be a specified coherent ideal with radical $\mathcal{I}_Z$, and put $\mathcal{X} := [(X, \mathcal{I})]$. Choose the unitary flat extension $\widetilde{\mathcal{N}}_D$ to a neighborhood V of Z as in Definition IV.6.5, and fix its unitary flat identification with $\mathcal{N}_D$ on Z. The associated comparison is

$$\mathcal{P}_D := \mathcal{O}_V(D) \otimes_{\mathcal{O}_V} \widetilde{\mathcal{N}}_D^*, \qquad \mathcal{P}_D|_Z \cong \mathcal{O}_Z.$$

We use the same symbol $\mathcal{P}_D$ for the induced invertible $\mathcal{O}_{\mathcal{X}}$-module, and call it the *Ueda invertible sheaf associated with* D, for this specified comparison. Triviality of $\mathcal{P}_D$ on $\mathcal{X}$ means triviality of this representative after restriction to some neighborhood of Z. When $D = Z$ and $\mathcal{I} = \mathcal{I}_Z$, it is the sheaf $\mathcal{P}$ in (301). The chosen ideal is part of the data when finite jets are considered.

Triviality on a neighborhood, a finite jet, or the formal completion means an isomorphism of invertible modules over the corresponding structure sheaf. General holomorphic or formal units, including units congruent to 1 modulo a power of the ideal, are allowed in such an isomorphism. In contrast, replacing $\widetilde{\mathcal{N}}_D$ by a different $\mathbb{C}^*$-flat extension of the same holomorphic sheaf on Z changes $\mathcal{P}_D$ itself.

For the analytic neighborhood space $\mathcal{X}$ in setting (S), choose its model $(X, \mathcal{I}_Z)$ and use $N_{Z/X}$ with its specified unitary flat structure. We need a flat extension of the rank-r conormal bundle. A tubular neighborhood of the underlying real submanifold Z deformation retracts onto Z (Theorem I.8.4). On each connected component, transport of a unitary frame of $N^*_{Z/X}$ gives a representation of $\pi_1(Z, *)^{\mathrm{op}}$ in $\mathrm{U}(r)$, with the endpoint-transport convention of Chapter I. The retraction identifies this fundamental group with that of a sufficiently small neighborhood V. Use the same representation to glue constant frames on V. The resulting locally constant unitary transition matrices are holomorphic, and define a unitary flat locally free sheaf $\mathcal{E}$ on V, together with a flat identification $\mathcal{E}|_Z \cong \mathcal{O}_Z(N^*_{Z/X})$.

For precision, the monodromy classification used here is applied to the principal bundle of unitary frames, whose fiber is $\mathrm{U}(r)$ with its free and transitive action on itself (Lemma I.7.31). The associated vector bundle then has fiber $\mathbb{C}^r$. The same classification and the fundamental-group identification show uniqueness of the extension germ with its flat structure.

8.2 Defining-function systems and one-step obstructions

Let $\mathcal{X} = [(X, \mathcal{I}_Z)]$ be as in setting (S). Keep Z, $N_{Z/X}$, the ideal $\mathcal{J} := \mathcal{I}_Z$, and the extension $\mathcal{E}$ of $\mathcal{O}_Z(N^*_{Z/X})$ just constructed. The intrinsic conormal and degree sheaves are $\mathcal{N}^*_{\mathcal{X}} = \mathcal{J}/\mathcal{J}^2$ and $\mathcal{S}^{\mu+1}_{\mathcal{X}} = \mathcal{J}^{\mu+1}/\mathcal{J}^{\mu+2}$, regarded as sheaves on Z. Write $j_Z\colon Z \hookrightarrow V$ for the inclusion into a neighborhood where $\mathcal{E}$ is defined. For an integer $\mu \geq 1$, a *defining-function system of order* μ is a $j_Z^{-1}\mathcal{O}_V$-linear morphism

$$\varphi_\mu\colon j_Z^{-1}\mathcal{E} \longrightarrow j_Z^{-1}(\mathcal{J}/\mathcal{J}^{\mu+1})$$

whose reduction modulo $\mathcal{J}^2$ is the specified identification $\mathcal{E}|_Z \cong \mathcal{J}/\mathcal{J}^2$.

To see the local functions in this definition, choose sufficiently small adapted coordinate neighborhoods (z_j, w_j) along Z, where $w_j = (w_j^1, w_j^2, \dots, w_j^r)$ are defining functions and z_j are coordinates along Z. Choose them so that the classes of $w_j^1, w_j^2, \dots, w_j^r$ modulo $\mathcal{J}^2$ form a flat frame of the conormal sheaf. If e_j is the column of the corresponding frame sections of $\mathcal{E}$, write $e_j = T_{jk}e_k$ on overlaps, with locally constant $T_{jk} \in \mathrm{U}(r)$. The images of these frame sections under φ_μ have local holomorphic lifts, still denoted by w_j, satisfying

$$T_{jk}w_k - w_j \in (\mathcal{J}^{\mu+1})^{\oplus r}.$$

Conversely, such local lifts define φ_μ. Let V_j be these adapted neighborhoods and put $U_j = V_j \cap Z$. Use the chart-and-intersection category of Example IV.1.16 (iii), with overlap pieces on which the flat transition matrices are constant. The source diagram consists of the models $(V_j, \mathcal{J}|_{V_j})$ and their restricted transitions. The standard objects are $(\mathbb{C}^r, \mathcal{I}_{\{0\}})$; a transition from the j-chart to the k-chart has standard arrow $v \mapsto T_{kj}v$. Choose G_j to consist of the source changes preserving the product projection to U_j, fixing U_j, and inducing the identity on the conormal sheaf. The local expressions of φ_μ give a natural comparison $u^{(\mu)}$ of degree μ in § 4.2. Its component lifts form $\mathscr{L}_{j,\mu+1}$, with additive correction space

$$M_{j,\mu+1} \cong \Gamma\big(U_j, \mathcal{O}_Z(N_{Z/X} \otimes S^{\mu+1}N^*_{Z/X})\big).$$

Indeed, a homogeneous transverse correction is realized in product coordinates by leaving z_j unchanged and changing w_j by that polynomial. The next proposition identifies its obstruction.

Proposition IV.8.1. *In setting (S) with the data $\mathcal{J}$, $N_{Z/X}$, and $\mathcal{E}$ above, fix $\mu \geq 1$ and a defining-function system φ_μ of order μ. There is a class*

$$\begin{aligned} u_\mu(\varphi_\mu) &\in H^1\big(Z, \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{N}^*_{\mathcal{X}}, \mathcal{S}^{\mu+1}_{\mathcal{X}})\big) \\ &\cong H^1\big(Z, \mathcal{O}_Z(N_{Z/X} \otimes S^{\mu+1}N^*_{Z/X})\big) \end{aligned}$$

which vanishes if and only if φ_μ lifts to a defining-function system $\varphi_{\mu+1}$ of order $\mu+1$. It depends on the specified system, but not on the local lifts used to compute it.

PROOF. Local freeness of $\mathcal{E}$ gives lifts ψ_j of φ_μ with target $j_Z^{-1}(\mathcal{J}/\mathcal{J}^{\mu+2})$, on a sufficiently fine covering $\mathcal{U} = \{U_j\}_{j\in J}$ of Z. For $U_{jk} := U_j \cap U_k$, put $c_{jk} := (\psi_k - \psi_j)|_{U_{jk}}$. This difference takes values in $j_Z^{-1}(\mathcal{J}^{\mu+1}/\mathcal{J}^{\mu+2})$, which is annihilated by $\mathcal{J}$. Thus it is a section of

$$\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{N}_{\mathcal{X}}^*, \mathcal{S}_{\mathcal{X}}^{\mu+1}) \cong \mathcal{O}_Z(N_{Z/X} \otimes S^{\mu+1}N_{Z/X}^*),$$

by (281). These sections give the additive action on the local lifts, compatibly with restriction. In the j-frame, c_{jk} is $T_{jk}\widetilde{w}_k \circ \phi_a - \widetilde{w}_j$ for the arrow a from the j-chart to the k-chart, with $T_a = T_{kj}$. It is therefore exactly the discrepancy of (294) at reached degree $\mu + 1$. Proposition IV.4.8, in its covering form, gives a well-defined class and its vanishing criterion. Under the displayed coefficient identification this is $u_\mu(\varphi_\mu) = \mathrm{Obs}_{\mu+1}(u^{(\mu)})$. The passage from Čech classes to sheaf cohomology uses refinement and Theorem II.3.79. The corrected local lifts retain the specified reduction modulo $\mathcal{J}^2$ and hence give the required system. □

REMARK IV.8.2. In setting (S), let $\nu \geq 2$. Tensoring the tangent–normal exact sequence by the locally free sheaf of sections of $S^\nu N_{Z/X}^*$ gives

$$0 \to \mathcal{O}_Z(T_Z \otimes S^\nu N_{Z/X}^*) \to \mathcal{O}_Z(T_X|_Z \otimes S^\nu N_{Z/X}^*) \to \mathcal{O}_Z(N_{Z/X} \otimes S^\nu N_{Z/X}^*) \to 0. \tag{310}$$

The middle sheaf describes full coordinate corrections in Proposition IV.5.3. The last describes corrections to the transverse defining functions in Proposition IV.8.1. The coefficients thus distinguish comparison of whole neighborhoods from Ueda-type linearization. At the first full step the conormal action is fixed separately, and its kernel is instead $\mathcal{O}_Z(T_Z \otimes N_{Z/X}^*)$.

The projection records the transverse part of a coordinate correction. To compare obstruction classes through this projection, use the same fixed lower-order comparison data. Transverse changes are realized locally using the chosen product projections. □

In Proposition IV.8.1, the case $r = 1$ has coefficient sheaf $\mathcal{O}_Z(N_{Z/X}^{-\mu})$. This is the coefficient appearing in the codimension-one Ueda construction. In higher codimension the construction is the fixed-system form of the obstruction in [Koi6, §§2.2–3.1, Definition 3.3]. Following that reference, $u_\mu(\varphi_\mu)$ obstructs raising the order from μ to $\mu+1$. Its homogeneous coefficient has degree $\mu + 1$, the reached degree of $\mathrm{Obs}_{\mu+1}$ in § 4.2. Statements about independence under changing the lower-order system are distinct from the independence of local lifts just proved.

An order-one system always exists: it is the specified map to $\mathcal{J}/\mathcal{J}^2$. If systems of every order exist with this fixed first-order identification, a compatible family can be chosen. Indeed, the possible morphisms at any fixed order form a non-empty affine subspace of a finite-dimensional space of global sections of a coherent sheaf on a compact finite jet. Their restriction images stabilize, just as in the proof of Proposition IV.6.7. A compatible family gives formal generators with the prescribed linear transition matrices. It does not by itself provide convergent generators, or a retraction onto Z.

8.3 Reduced Cartier centers and the Ueda invertible sheaf

We now work in setting (D) with $D = Z$ reduced and connected, and choose $\mathcal{I} = \mathcal{I}_Z$. Thus Z is a compact connected reduced Cartier divisor in the complex manifold X,

possibly singular or reducible. Set $\mathcal{X} := [(X, \mathcal{I}_Z)]$ and $\mathcal{N} := \mathcal{O}_Z([Z]|_Z)$, fix its unitary flat structure and extension $\widetilde{\mathcal{N}}$, and use its Ueda invertible sheaf $\mathcal{P}$. This setting also includes the codimension-one case of (S) when Z is non-singular.

Choose open subsets V_j of X covering a neighborhood of Z, and put $U_j := V_j \cap Z$ and $U_{jk} := U_j \cap U_k$. Choose defining functions w_j for $Z \cap V_j$ on V_j and flat frames of $\widetilde{\mathcal{N}}$ there so that $w_j = t_{jk} w_k$ modulo $\mathcal{I}_Z^2$, with locally constant $t_{jk} \in \mathrm{U}(1)$. Such functions are obtained by lifting the prescribed generators of $\mathcal{I}_Z/\mathcal{I}_Z^2$ and shrinking the patches. They generate the Cartier ideal near Z. A system of order $\mu \geq 1$ in this divisor setting means a choice of such generators satisfying

$$t_{jk} w_k = w_j + f_{jk,\mu+1} w_j^{\mu+1} \pmod{\mathcal{I}_Z^{\mu+2}}, \tag{311}$$

where $f_{jk,\mu+1} \in \Gamma(U_{jk}, \mathcal{O}_Z)$. The congruence is computed with local holomorphic lifts of $f_{jk,\mu+1}$ to the ambient space, after refinement and shrinking as necessary; changing a lift by an element of $\mathcal{I}_Z$ does not change it modulo $\mathcal{I}_Z^{\mu+2}$. This description uses a Cartier generator, not a coordinate system on a possibly singular Z.

Lemma IV.8.3. *Let $\mathcal{X}$, Z, $\mathcal{N}$, and $\mathcal{P}$ be as in the preceding reduced Cartier setting, and fix an integer $\mu \geq 1$. Then $\mathcal{P}_{\mu-1}$ is trivial if and only if there is a defining-function system of order μ with the prescribed first-order unitary comparison. For a system as in (311), the transition functions of $\mathcal{P}$ satisfy*

$$\begin{aligned} p_{jk} &:= \frac{w_j}{t_{jk} w_k} = 1 - f_{jk,\mu+1} w_j^{\mu} \pmod{\mathcal{I}_Z^{\mu+1}}, \\ \frac{1}{2\pi\sqrt{-1}} \log p_{jk} &= -\frac{1}{2\pi\sqrt{-1}} f_{jk,\mu+1} w_j^{\mu} \pmod{\mathcal{I}_Z^{\mu+1}}. \end{aligned} \tag{312}$$

The class of $(-f_{jk,\mu+1} w_j^{\mu})$ is $\mathrm{Obs}_{\mu}(\mathcal{P}, s_{\mu-1})$ for the frame $s_{\mu-1}$ given by this system. Its coefficient sheaf is $\mathcal{I}_Z^{\mu}/\mathcal{I}_Z^{\mu+1} \cong \mathcal{N}^{-\mu}$ on Z.

Proof. Start with generators compatible with the prescribed flat normal frames to first order. The corresponding frames e_j of $\mathcal{P}$ have $e_k = p_{jk} e_j$ and give the same frame of $\mathcal{P}_0$. If $\mathcal{P}_{\mu-1}$ is trivial, choose its global frame so that its reduction is this frame of $\mathcal{P}_0$; the necessary adjustment is a non-zero constant on the compact connected reduced space Z. Write its coefficients as a_j and lift them locally to holomorphic units. Replacing w_j by w_j/a_j makes their transitions unitary linear modulo $\mathcal{I}_Z^{\mu+1}$, while retaining the same first-order identification. Conversely, such defining functions have $p_{jk} = 1$ modulo $\mathcal{I}_Z^{\mu}$, so the frames e_j agree on $\mathcal{X}^{(\mu-1)}$.

For the specified system, divide (311) by w_j and invert. Since $2\mu \geq \mu + 1$, the degree-μ term of the reciprocal is $-f_{jk,\mu+1} w_j^{\mu}$ and all its powers of order at least two vanish in the stated quotient. The same observation gives the logarithmic identity, using the logarithm with constant term zero. Lemma IV.6.8 now identifies the class. Finally, multiplication of Cartier generators identifies $\mathcal{I}_Z^{\mu}/\mathcal{I}_Z^{\mu+1}$ with $(\mathcal{I}_Z/\mathcal{I}_Z^2)^{\otimes\mu} = \mathcal{N}^{-\mu}$. □

For the same defining system and flat comparison, the three local representatives in the coefficient sheaf $\mathcal{I}_Z^{\mu}/\mathcal{I}_Z^{\mu+1}$ are

$$f_{jk,\mu+1} w_j^{\mu}, \qquad -f_{jk,\mu+1} w_j^{\mu}, \qquad -\frac{1}{2\pi\sqrt{-1}} f_{jk,\mu+1} w_j^{\mu}.$$

When Z is non-singular, the first represents the defining-system class $u_\mu(\varphi_\mu)$ under the identification of its coefficient sheaf with $\mathcal{N}^{-\mu}$. The second represents $\mathrm{Obs}_\mu(\mathcal{P}, s_{\mu-1})$ for the corresponding frame. The third represents $u_\mu(\mathcal{I}_Z, P; \xi)$ of Definition IV.6.10, where P is the line bundle inducing $\mathcal{P}$ and ξ is obtained from these same frames. The last two descriptions also apply to the reduced Cartier setting. They record the reciprocal in (312) and the chosen exponential normalization (Compare [U1, §2], [Koi7, Remark 3.3], and [Koi13, Remark 3.3]). Using $\mathcal{P}^*$ reverses the comparison sign. Changing local lifts of a fixed frame gives a coboundary; replacing the lower-order frame or the flat comparison is a different operation.

Definition IV.8.4. Let $\mathcal{X} = [(X, \mathcal{I}_Z)]$ be an analytic neighborhood space with compact connected reduced Cartier center Z as in Lemma IV.8.3. Fix the unitary normal sheaf $\mathcal{N}$, its unitary extension, and the resulting Ueda invertible sheaf $\mathcal{P}$. For $\mu \geq 0$, put $\mathcal{P}_\mu := \mathcal{P}|_{\mathcal{X}^{(\mu)}}$, where the finite jets are those of the $\mathcal{I}_Z$-adic filtration. Define the *Ueda type* of this comparison to be the least integer $\mu \geq 1$ for which $\mathcal{P}_\mu$ is nontrivial, if such an integer exists. The comparison is then of *finite type* μ. If $\mathcal{P}_\mu$ is trivial for every $\mu \geq 0$, we say that it is of *infinite type* (in the sense of Ueda). □

For these data, Lemma IV.8.3 says that finite type μ is the first failure to raise a defining-function system of order μ to one of order $\mu + 1$; equivalently, $\mathcal{P}_{\mu-1}$ is trivial and $\mathcal{P}_\mu$ is not. It is independent of the local lifts, since it is defined by the isomorphism classes of the fixed sheaves $\mathcal{P}_\mu$. Infinite type is equivalent to formal triviality by Proposition IV.6.7. The construction of defining functions and the codimension-one linearization theory originate in Ueda's work; the formal lifting calculation itself is not restricted to complex dimension two. The analytic convergence problem is different from that calculation: Proposition IV.7.3 supplies infinite-type comparisons for which analytic triviality fails.

Remark IV.8.5. Let X be a non-singular complex surface and let $Z \subset X$ be a non-singular compact connected complex curve with $c_1(N_{Z/X}) = 0$. Its normal bundle is unitary flat by Proposition III.2.138. Put $\mathcal{X} = [(X, \mathcal{I}_Z)]$ and let $\mathcal{P} = \mathcal{P}_{[Z]}$ be its Ueda invertible sheaf. Ueda's classification can be expressed in these terms as follows:

- The pair (Z, X) is of *class* (α) if $\widehat{\mathcal{P}} \ncong \mathcal{O}_{\mathcal{X}^{(\infty)}}$; this is precisely the finite-type case.
- It is of *class* (β) if $\mathcal{P} \cong \mathcal{O}_{\mathcal{X}}$.
- It is of *class* (γ) if $\widehat{\mathcal{P}} \cong \mathcal{O}_{\mathcal{X}^{(\infty)}}$ but $\mathcal{P} \ncong \mathcal{O}_{\mathcal{X}}$.

Here class (β) combines the classes (β') and (β'') in [U1, §5.1]. The finite and formal descriptions agree by Proposition IV.6.7; analytic triviality is the exact unitary transverse comparison discussed above.

Class (α) also yields important information about functions on punctured neighborhoods. Ueda's work yields both strictly plurisubharmonic functions with controlled growth and rigidity results for plurisubharmonic functions of sufficiently slow growth on punctured neighborhoods of Z. We shall discuss this important function-theoretic aspect of Ueda's work in Chapter V, § 4.2 (See [U1, Theorems 1–2 and §5]). □

REMARK IV.8.6. There exists an elliptic curve Z in a non-singular complex surface X, with holomorphically trivial normal bundle and Ueda type 1, which is a leaf of a non-singular holomorphic foliation with linear nonunitary holonomy (See [CLPT, Remark 2.2]). For this example put $\mathcal{X} = [(X, \mathcal{I}_Z)]$ and $\mathcal{N} = \mathcal{O}_Z(N_{Z/X})$, give $\mathcal{N}$ its unitary flat structure, and let $\widetilde{\mathcal{N}}$ be its specified unitary flat extension. Its Ueda invertible sheaf $\mathcal{P}$ satisfies

$$\mathcal{P}_0 \cong \mathcal{O}_Z, \qquad \mathcal{P}_1 \ncong \mathcal{O}_{\mathcal{X}^{(1)}}.$$

On the other hand, the linear holonomy gives local first integrals with locally constant nonzero transitions. Consequently $\mathcal{O}_V(Z)$ itself has a $\mathbb{C}^*$-flat structure on a neighborhood V, extending the same underlying holomorphic normal sheaf. Comparing it with this latter extension instead gives the trivial sheaf already on V. Thus the specified unitary comparison has nontrivial first jet, whereas the comparison with this other flat extension is trivial. □

8.4 Lifting every frame in the reduced Cartier case

The three assertions in Proposition IV.6.7 distinguish separate finite-order existence, formal existence, and extension of an arbitrary frame already chosen. For the following specific Ueda comparison, all three are equivalent.

PROPOSITION IV.8.7. *Let X be a complex manifold and Z a compact connected reduced Cartier divisor. Consider the analytic neighborhood space $\mathcal{X} := [(X, \mathcal{I}_Z)]$, and suppose that $\mathcal{N} = \mathcal{O}_Z([Z]|_Z)$ is unitary flat. Fix its unitary flat extension and let $\mathcal{P}$ be the resulting Ueda invertible sheaf. Then assertions (i), (ii), and (iii) of Proposition IV.6.7, applied to $\mathcal{L} = \mathcal{P}$, are equivalent. More precisely, if $\widehat{\mathcal{P}}$ is trivial, then for every $\mu \in \mathbb{Z}_{\geq 0}$ the map*

$$H^0(\mathcal{X}^{(\infty)}, \mathcal{O}_{\mathcal{X}^{(\infty)}}) \longrightarrow H^0(\mathcal{X}^{(\mu)}, \mathcal{O}_{\mathcal{X}^{(\mu)}})$$

is surjective, and every global unit on the finite jet has a global formal unit as a lift.

PROOF. Assume that $\widehat{\mathcal{P}}$ is trivial and choose a formal frame s. Take Cartier generators w_j and comparison frames e_j with $e_k = p_{jk} e_j$ and $p_{jk} = w_j/(t_{jk} w_k)$, where the t_{jk} are the fixed unitary constants. Write $s = a_j e_j$. Then the formal units a_j satisfy $a_j = p_{jk} a_k$, so $\widehat{w}_j := w_j/a_j$ are formal ideal generators satisfying $\widehat{w}_j = t_{jk} \widehat{w}_k$ exactly.

Fix $\mu \geq 0$ and a global function a_μ on $\mathcal{X}^{(\mu)}$. Its reduction on Z is a constant c_0, by compactness, connectedness, and reducedness. Subtract its constant formal lift. Suppose inductively that, after subtracting global formal functions, the remaining finite jet vanishes to order k, where $1 \leq k \leq \mu$. Its degree-k class is a global section of $\mathcal{I}_Z^k/\mathcal{I}_Z^{k+1} \cong \mathcal{N}^{-k}$. Write this section locally as $c_j \widehat{w}_j^k$ modulo the next power. The sheaf $\mathcal{N}^{-k}$ is unitary flat on the compact connected reduced complex space Z. The argument of Lemma III.2.197 shows that each c_j is locally constant: the squared norm of the section is plurisubharmonic and constant, and a holomorphic function of constant modulus on a reduced complex space is locally constant. This also includes the zero section.

View these locally constant coefficients as sections of the formal structure sheaf on the same open patches of Z. The expressions $c_j \widehat{w}_j^k$ now glue exactly, since both the coefficients

and the generators have the specified flat transition rules. They define a global formal function whose degree-k class is the required one. Subtracting its restriction leaves a remainder of order $k+1$. After the finitely many steps $k = 1, 2, \dots, \mu$, the sum of the functions constructed, together with c_0, is a formal lift of a_μ. If a_μ is a unit, then $c_0 \neq 0$; its formal lift is a unit, as follows by taking the compatible local inverses in the finite quotients. Thus the asserted restriction maps, including those for units, are surjective. Apply the last assertion of Proposition IV.6.7 to obtain $(ii) \Rightarrow (iii)$; the other implications were proved there. □

8.5 A specified divisor with multiplicities

Let $\mathcal{X} = [(X, \mathcal{I})]$ be an analytic neighborhood space represented by a complex manifold X and a coherent ideal $\mathcal{I}$. Suppose that its reduced center Z is compact, and let D be an effective Cartier divisor on X with support Z. Fix a unitary flat structure on $\mathcal{N}_D = \mathcal{O}_Z([D]|_Z)$ and its unitary extension, and denote the resulting comparison on $\mathcal{X}$ by $\mathcal{P}_D$. Keep distinct the model ideal $\mathcal{I}$, the radical ideal $\mathcal{I}_Z$, and the invertible ideal $\mathcal{J}_D := \mathcal{O}_X(-D)$. The cofinality of powers for ideals with the same support identifies their formal completions (Proposition IV.2.23). It does not identify their finite jets, associated graded sheaves, or order indices.

In particular, $\mathcal{P}_D|_Z \cong \mathcal{O}_Z$ does not assert $\mathcal{P}_D|_D \cong \mathcal{O}_D$. Write $\mathcal{K} := \mathcal{I}_Z/\mathcal{J}_D$ for the nilradical of $\mathcal{O}_D = \mathcal{O}_X/\mathcal{J}_D$. Its powers give successive thickenings from Z to D; some power is zero on this compact space. A frame at one stage has a one-step obstruction with coefficient $\mathcal{K}^m/\mathcal{K}^{m+1}$, by the square-zero computation in Lemma IV.6.8. This is an initial comparison on D, before any assertion about higher D-adic jets.

For the $\mathcal{J}_D$-adic filtration the degree sheaves are

$$\mathcal{J}_D^{\mu}/\mathcal{J}_D^{\mu+1} \cong \mathcal{O}_D(-\mu D).$$

Their coefficient ring is $\mathcal{O}_D$, which may be non-reduced; they are not in general a single power of a line bundle on the reduced space Z. The $\mathcal{I}_Z$-adic filtration instead records orders along the reduced support. Once the ideal and the lower-order frame are specified, Lemma IV.6.8 applies to either filtration. The reduced Cartier type in Definition IV.8.4 does not assign an unspecified choice of filtration, or a single choice-independent type, to an arbitrary divisor with multiplicities.

8.6 Analytic realization of a transverse comparison

We now give a quantitative criterion for the transverse comparison of § 4.2. The criterion isolates the coefficient construction in Ueda's method ([U1, §4]; [Koi6, §4]). We construct inverse expressions for the old defining maps w_j in terms of new maps u_j. On a product patch these have the form $w_j = F_j(z_j, u_j)$, where $F_j(z_j, X)$ has linear part X; the general construction uses specified extensions of its coefficients.

Fix a finitely presented category $\mathfrak{I}$ and two diagrams $\mathscr{F}, \mathscr{F}_{\mathrm{std}} \colon \mathfrak{I} \xrightarrow{\text{functor}} (\mathsf{ANM})$. The standard objects are $(\mathbb{C}^r, \mathcal{I}_{\{0\}})$, where $r \geq 1$, and its arrows are $X \mapsto T_a X$, with $T_a \in \mathrm{GL}_r(\mathbb{C})$. Here $X = (X_1, X_2, \dots, X_r)$ denotes independent transverse variables. Choose source models

$\mathscr{F}(j) = (V_j, \mathcal{I}_j)$ and initial comparison maps $w_j = (w_j^1, w_j^2, \dots, w_j^r)$ generating $\mathcal{I}_j$. Their first jets are assumed to form a natural transformation. Fix source groups $G_j \subset \mathrm{Aut}_{(\mathsf{ANM})}(\mathscr{F}(j))$, and seek components $u_j = w_j \circ h_j^{-1}$, $h_j \in G_j$, with this first-order part. For an arrow $a\colon j \to k$, the desired square is

$$\begin{array}{ccc} \mathscr{F}(j) & \xrightarrow{\phi_a} & \mathscr{F}(k) \\ {\scriptstyle u_j}\downarrow & & \downarrow{\scriptstyle u_k} \\ (\mathbb{C}^r, \mathcal{I}_{\{0\}}) & \xrightarrow[X \mapsto T_a X]{} & (\mathbb{C}^r, \mathcal{I}_{\{0\}}), \end{array} \qquad \phi_a := \mathscr{F}(a).$$

Thus $u_k \circ \phi_a = T_a u_j$. The discrepancy and its change are those of (294) and (295).

For a covering diagram, impose the restriction arrows first. The independent components are then indexed by the finitely many original patches, and the other components are their restrictions. Use J for this finite set of patches. More generally, suppose a finite set J of independent components has been chosen, with the remaining components obtained by specified restrictions. Choose a finite family of comparison equations which, together with these restrictions, implies naturality for the whole diagram. One equation for an invertible arrow also implies the equation for its inverse. All representatives, restrictions, and coefficient spaces in what follows refer to this fixed choice.

The formal lifting condition

For $\mu \geq 1$, let $\mathscr{T}_\mu$ be the set of permissible natural μ-jet comparisons extending the specified first-order comparison. Use the additive one-step description of Proposition IV.4.8 at each order. In particular, the permissible component lifts are non-empty affine spaces under the specified spaces $M_{j,\nu}$. For the induction below the required formal condition is

$$\mathscr{T}_1 \neq \emptyset, \qquad \mathscr{T}_{\mu+1} \longrightarrow \mathscr{T}_\mu \text{ is surjective for every } \mu \geq 1.$$

It concerns extension of the comparison currently chosen. The following finite-order criterion is useful for verifying it.

Lemma IV.8.8. *Keep the diagrams, first-order part, permissible comparisons, and one-step correction spaces just specified. For each $\nu \geq 2$, suppose that every family in $\operatorname{Ker}\delta_\nu$ is represented by homogeneous vector polynomials $Q_{j,\nu}(X)$ of degree ν, with constant coefficients, satisfying*

$$Q_{k,\nu}(T_a X) = T_a Q_{j,\nu}(X).$$

The identity is required after the prescribed restrictions for every $a\colon j \to k$. Suppose also that postcomposition by $X \mapsto X + Q_{j,\nu}(X)$ preserves permissibility at every finite order. If $\mathscr{T}_\mu \neq \emptyset$ for every $\mu \geq 1$, each map $\mathscr{T}_{\mu+1} \to \mathscr{T}_\mu$ is surjective.

Proof. Fix $u^{(\mu)} \in \mathscr{T}_\mu$ and choose $v^{(\mu+1)} \in \mathscr{T}_{\mu+1}$. Their first-order parts agree. If their parts below degree $\nu \leq \mu$ agree, their first difference at degree ν belongs to $\operatorname{Ker}\delta_\nu$, since

both are natural to that order. Represent the difference, with the sign which changes v to u, by the polynomials $Q_{j,\nu}$. Postcomposition changes this degree by $Q_{j,\nu}(w_j)$ and leaves the lower degrees unchanged. The displayed identity preserves naturality, and the permissibility hypothesis allows the change at order $\mu+1$. Apply this successively for $\nu = 2, 3, \dots, \mu$. The resulting $(\mu+1)$-jet is a lift of the prescribed $u^{(\mu)}$. □

Coefficient representatives and fixed norms

Write $U_j := Z(V_j, \mathcal{I}_j)$ with its reduced structure, and begin with the Banach algebra

$$B_j := \{b \in \mathcal{O}_{U_j}(U_j) \mid \sup_{U_j} |b| < \infty\}, \qquad \|b\|_{B_j} := \sup_{U_j} |b|.$$

A stronger coefficient norm may be used when needed. For example, on specified coordinate patches one may use the sum of the supremum of a function and the suprema of its first derivatives, restricting to the functions for which this sum is finite. The norms used below must dominate the supremum norm, give norm one to the constant function 1, and satisfy $\|bc\| \le \|b\|\|c\|$. This product inequality also holds for the indicated derivative norm, by the product rule. All choices of patches and derivatives are fixed independently of the degree.

For every $\nu \ge 2$, fix permitted subspaces of coefficient families $F_\nu = (F_{j,\alpha})_{j\in J,\,|\alpha|=\nu}$, where $F_{j,\alpha} = (F^1_{j,\alpha}, \dots, F^r_{j,\alpha})$ has components in B_j, and specified linear extensions $\mathsf{E}_{j,\nu}$ to holomorphic functions on the fixed V_j. Restrictions used in the diagram are understood. The inverse expression to be constructed is

$$\begin{aligned} F_j(p, X) &= X + \sum_{|\alpha|\ge 2} (\mathsf{E}_{j,|\alpha|} F_{j,\alpha})(p) X^\alpha, \\ w_j(p) &= F_j(p, u_j(p)). \end{aligned} \tag{313}$$

Here $p \in V_j$, and F_j is initially a formal series in X. On a product patch with extension by the fixed projection to z_j, we write $F_j(z_j, X) = X + \sum_{|\alpha|\ge 2} F_{j,\alpha}(z_j) X^\alpha$. For each homogeneous family set

$$\begin{aligned} \|F_\nu\|_{0,\nu} &:= \max_{j\in J} \sum_{|\alpha|=\nu} \max_{1\le\lambda\le r} \|F^\lambda_{j,\alpha}\|_{B_j}, \\ a_\nu &:= \max_{j\in J} \sum_{|\alpha|=\nu} \max_{1\le\lambda\le r} \|\mathsf{E}_{j,\nu} F^\lambda_{j,\alpha}\|_{V_j}. \end{aligned}$$

The chosen ambient norms $\|\cdot\|_{V_j}$ dominate the ambient supremum norms. Use $M_{j,\nu}$, C^0_ν, C^1_ν, and Z^1_ν from § 4.2, with the prescribed restrictions imposed. Require the permitted coefficient families to give elements of C^0_ν by the component formula below. It defines the linear map

$$\pi_\nu \colon \{\text{permitted degree-}\nu\text{ coefficient families}\} \longrightarrow C^0_\nu,$$

$$(\pi_\nu F_\nu)_j = \sum_{|\alpha|=\nu} (\mathsf{E}_{j,\nu} F_{j,\alpha}) w_j^\alpha \pmod{\mathcal{I}_j^{\nu+1}}.$$

Thus F_ν is a coefficient family, whereas $\pi_\nu F_\nu$ is its comparison-level correction. Fix a norm $\|\cdot\|_{1,\nu}$ on the discrepancy cocycles $Z^1_\nu \subset C^1_\nu$; the correction map is $\delta_\nu \colon C^0_\nu \to Z^1_\nu$.

We write g_ν for the discrepancy cocycle c_ν when using coefficient polynomials. Adding a homogeneous term with coefficients F_ν to an inverse expression changes its solution and its discrepancy at degree ν by

$$u'_j - u_j = -(\pi_\nu F_\nu)_j, \qquad g'_\nu = g_\nu - \delta_\nu(\pi_\nu F_\nu).$$

Indeed, the linear part of the inverse expression is X. Substituting the change into its terms of degree at least two affects only degrees at least $\nu+1$. Condition (A) below requires these operations to be realized by the specified source groups.

For a formal series in X, write $P_\nu[\cdot]$ for its homogeneous part of degree ν in X, leaving the other variables as parameters. This is the degree projection obtained from the partial Taylor expansion in Chapter III, and does not require convergence of the series. We use the sum of coefficient norms in the fixed frames for the estimates below. For a scalar homogeneous polynomial $P(z,X) = \sum_{|\alpha|=\nu} b_\alpha(z) X^\alpha$ and $\rho > 0$, Lemma IV.3.1 bounds the sphere norm $\|P\|_{U,\rho}$ by $(2\pi^r/(r-1)!)^{1/2}\rho^{\nu+r-1/2}\sum_\alpha \|b_\alpha\|_U$, using unnormalized sphere measure and taking the supremum over $z \in U$ outside the integral. For vector-valued polynomials an additional factor $\sqrt{r}$ suffices. The sphere norm is unitary invariant, whereas the coefficient sum is measured in the chosen frames. Matrix substitutions are included explicitly in condition (D).

The following guide records the roles of the coefficient data. The last three quantities are introduced in conditions (B)–(D) below.

Symbol	Definition / meaning	Role
$\mathscr{T}_\mu$	Permissible natural μ-jet comparisons	Formal lifting problem
B_j	Coefficient algebra with the chosen norm	Measures coefficients on U_j
F_ν	Permitted family $(F_{j,\alpha})_{j,\,\lvert\alpha\rvert=\nu}$	Unknown degree-ν coefficients
$\mathsf{E}_{j,\nu}$	Specified linear extension from U_j to V_j	Produces ambient coefficients
π_ν	Linear map to C^0_ν defined above	Produces comparison-level corrections
g_ν	Discrepancy from the chosen lower-degree coefficients	Known right-hand side at this step
$\|F_\nu\|_{0,\nu}$, a_ν	Norm sums before and after extension	Sizes of the same coefficient family
$\|g_\nu\|_{1,\nu}$	Chosen norm on Z^1_ν	Size of the discrepancy
D_ν	Constant in (B)	Cost of extension
C_ν	Constant in (C)	Cost of a bounded primitive
$A(t)$	Scalar majorant in (D)	Controls convergence

The equation to be solved is $\delta_\nu(\pi_\nu F_\nu) = g_\nu$. Conditions (C) and (B) control the choice and

extension of the same coefficient representative, respectively.

Four analytic conditions

Assume the following properties of these representatives and norms.

(A) Realization of permitted changes.
For every permitted finite coefficient system, the solutions of

$$w_j(p) = u_j(p) + \sum_{2\le|\alpha|\le\nu} (\mathsf{E}_{j,|\alpha|}F_{j,\alpha})(p)u_j(p)^\alpha$$

give permissible comparison jets. Adding a permitted degree-ν family preserves permissibility and realizes the component change $-(\pi_\nu F_\nu)_j$ computed above. The permitted subspaces are fixed before a primitive is chosen.

For every compatible sequence of permitted coefficient choices with $\sum_{\nu\ge2} a_\nu t^\nu$ convergent near zero, the holomorphic solutions of (313) are realized, on a sufficiently small neighborhood of the whole U_j, by

$$h_j = \iota_j(F_j - X) \in G_j, \qquad u_j = w_j \circ h_j^{-1}.$$

(B) Bounded extension of the same representatives.
For every $\nu \ge 2$, there is a constant $D_\nu \ge 1$ such that, for every j and every permitted coefficient b,

$$(\mathsf{E}_{j,\nu}b)|_{U_j} = b, \qquad \|\mathsf{E}_{j,\nu}b\|_{V_j} \le D_\nu\|b\|_{B_j}.$$

(C) A bounded primitive in permitted representatives.
For every $\nu \ge 2$, there is $C_\nu > 0$ such that each bounded $g_\nu \in \operatorname{Image}\delta_\nu \subset Z^1_\nu$ arising from a permitted lower coefficient system has a permitted representative F_ν satisfying

$$\delta_\nu(\pi_\nu F_\nu) = g_\nu, \qquad \|F_\nu\|_{0,\nu} \le C_\nu\|g_\nu\|_{1,\nu}.$$

The extension in (B) is applied to this same representative.

(D) Fixed Cauchy domains and the convergence cost.
For each of the finitely many testing equations associated with $a\colon j \to k$, choose ordinary source coordinates $(z,w) = (z_j, w_j)$. Use $\phi_a(z,w)$ also for the representative of the arrow in these coordinates. Reversing an equation is allowed only when the arrow is invertible. The source coefficients are extended by the fixed projection to z. There is a fixed transverse polydisc on which these expressions are defined for every z in a fixed testing base domain. The coefficient norm there has the product property specified above, and restriction of an extended source coefficient to this base does not increase its chosen ambient norm.

Put $f_a(z,w) := T_a^{-1}w_k(\phi_a(z,w)) - w$. For each permitted homogeneous target coefficient family of degree $\mu \ge 2$, define the coefficients $d_{a,\alpha,\gamma}$ by expanding the following difference,

with w and X independent:

$$\begin{aligned} f_a(z,w) &= \sum_{|\beta|\geq 2} f_{a,\beta}(z) w^{\beta}, \\ T_a^{-1} \sum_{|\alpha|=\mu} &\big[(\mathsf{E}_{k,\mu} F_{k,\alpha})(\phi_a(z,w)) \\ &\quad - (\mathsf{E}_{k,\mu} F_{k,\alpha})(\phi_a(z,0))\big](T_a X)^{\alpha} \\ &= \sum_{\substack{|\alpha|=\mu \\ |\gamma|\geq 1}} d_{a,\alpha,\gamma}(z) X^{\alpha} w^{\gamma}. \end{aligned} \tag{314}$$

There are constants $M \geq 1$ and $R > 0$, independent of μ and of the permitted coefficient family, such that

$$\max_{\lambda} \|f^{\lambda}_{a,\beta}\| \leq M R^{|\beta|}, \qquad \sum_{|\alpha|=\mu} \max_{\lambda} \|d^{\lambda}_{a,\alpha,\gamma}\| \leq M a_{\mu} R^{|\gamma|}.$$

The second bound is imposed after the substitution $X \mapsto T_a X$.

There is also a constant $L \geq 1$, independent of ν, such that

$$\|g\|_{1,\nu} \leq L \max_{a} \sum_{|\alpha|=\nu} \max_{\lambda} \|g^{\lambda}_{a,\alpha}\| \qquad (g \in Z^1_{\nu}).$$

Here $g_{a,\alpha}$ are the coefficients of the restriction of g to the chosen testing domain, expressed in the source frame. For a covering this estimate includes the passage from the testing overlaps to the cochain norm on the fixed covering.

Finally, put $R_0 := rR$, $M_0 := LM(R_0^2 + R_0)$, and define the formal series $A(t) = t + \sum_{\nu\geq 2} A_{\nu} t^{\nu}$ by

$$\sum_{\nu\geq 2} \frac{A_{\nu}}{C_{\nu} D_{\nu}} t^{\nu} = \frac{M_0 A(t)^2}{1 - R_0 A(t)}. \tag{315}$$

Require this series to have positive radius of convergence. Its coefficients are determined recursively and are nonnegative.

The testing coordinates in (D) are required only at the source of each testing equation. Its target may be a singular patch; there the coefficients use the specified extension and are evaluated at $\phi_a(z,w)$. For supremum norms the two coefficient bounds follow from the Cauchy formula on the fixed polydiscs once the corresponding uniform bounds hold. For derivative norms the same estimates must be checked in those norms. In (C), a constant obtained from a quotient norm includes the loss in choosing a representative near the infimum. Exactness is instead supplied by the formal lifting condition.

Theorem IV.8.9 (Analytic realization). *Fix the diagrams, permissible comparison jets, coefficient representatives, extensions, and norms described above. Suppose the formal lifting*

condition and (A)–(D) hold. Then the first-order comparison extends to a natural transformation $H\colon \mathscr{F} \xrightarrow{\mathrm{nat}} \mathscr{F}_{\mathrm{std}}$ *in* (ANM)*. Its components are holomorphic solutions* $H_j = u_j$ *of* (313)*, realized by the specified source groups, and satisfy*

$$(u_j^1, u_j^2, \dots, u_j^r) = \mathcal{I}_j, \qquad u_j = w_j \pmod{\mathcal{I}_j^2}.$$

There is one positive radius for the coefficient construction on all the fixed patches. If each source is a point-centered non-singular r*-dimensional model and* w_j *is a coordinate system,* H *is a natural isomorphism.*

Proof. We choose the coefficients successively, with $a_\mu \le A_\mu$. Suppose those of degrees $2, \dots, \nu-1$ have been chosen and give a natural $(\nu-1)$-jet comparison. For this step, let u_j denote the local solutions for these finite coefficient polynomials. On an ordinary testing source, substitute

$$w = X + \sum_{2 \le |\alpha| \le \nu-1} F_{j,\alpha}(z) X^\alpha$$

and put $u_k = T_a X$ in the target implicit equation. The sum is empty when $\nu = 2$. Before making this substitution for w, the residual in the source frame is

$$\begin{aligned} &\sum_{2 \le |\alpha| \le \nu-1} F_{j,\alpha}(z) X^\alpha + f_a(z, w) \\ &\quad - T_a^{-1} \sum_{2 \le |\alpha| \le \nu-1} (\mathsf{E}_{k,|\alpha|} F_{k,\alpha})(\phi_a(z,w)) (T_a X)^\alpha. \end{aligned}$$

After substitution, the source polynomial and the target terms of w-degree zero have no contribution at degree ν. Thus the degree-ν residual is

$$\begin{aligned} g_{a,\nu} &= H_{a,\nu}^{\mathrm{trans}} - H_{a,\nu}^{\mathrm{coeff}}, \\ H_{a,\nu}^{\mathrm{trans}} &:= P_\nu \left[\sum_{|\beta| \ge 2} f_{a,\beta}(z) \left(X + \sum_{2 \le |\delta| \le \nu-1} F_{j,\delta}(z) X^\delta \right)^\beta \right], \\ H_{a,\nu}^{\mathrm{coeff}} &:= P_\nu \left[\sum_{\substack{2 \le |\alpha| \le \nu-1 \\ |\gamma| \ge 1}} d_{a,\alpha,\gamma}(z) X^\alpha \left(X + \sum_{2 \le |\delta| \le \nu-1} F_{j,\delta}(z) X^\delta \right)^\gamma \right]. \end{aligned} \tag{316}$$

The projection is taken after the substitution for w. For extensions by local projections, the second term is the contribution of the base-coordinate change.

This residual is the discrepancy in (294). Indeed,

$$u_j \left(z, X + \sum_{2 \le |\alpha| \le \nu-1} F_{j,\alpha}(z) X^\alpha \right) = X.$$

The normalized target equation has linear part the identity in its unknown transverse value, and the two values being compared already agree below degree ν. Substituting their difference into terms of degree at least two changes only degrees at least $\nu+1$. The substitution for w is $X+O(|X|^2)$ and therefore also leaves the leading discrepancy unchanged. Consequently the differences in (316) are the restrictions of one cocycle g_ν; the two terms separately need not be cocycles. Adding the degree-ν family F_ν to the inverse expression changes the discrepancy by $-\delta_\nu(\pi_\nu F_\nu)$.

Use $\preceq$ for coefficientwise comparison of scalar series with nonnegative coefficients. The product inequality for the fixed coefficient norms bounds the sums of norms of the degree-ν coefficients of the two terms by the degree-ν coefficients of, respectively,

$$M\sum_{|\beta|\geq 2} R^{|\beta|}\left(t+\sum_{\mu=2}^{\nu-1}a_\mu t^\mu\right)^{|\beta|},$$
$$M\left(\sum_{\mu=2}^{\nu-1}a_\mu t^\mu\right)\sum_{|\gamma|\geq 1} R^{|\gamma|}\left(t+\sum_{\mu=2}^{\nu-1}a_\mu t^\mu\right)^{|\gamma|}.$$

For the second expression, first sum the target coefficients of each degree μ in the second estimate of (D). All factors are measured in the same coefficient algebra, also when its norm includes derivatives.

There are $\binom{r+\ell-1}{\ell}\leq r^\ell$ monomials of degree $\ell\geq 1$ in r variables. After the restriction estimate of (D), the two scalar bounds become

$$\frac{LMR_0^2\left(t+\sum_{\mu=2}^{\nu-1}a_\mu t^\mu\right)^2}{1-R_0\left(t+\sum_{\mu=2}^{\nu-1}a_\mu t^\mu\right)},$$
$$\frac{LMR_0\left(\sum_{\mu=2}^{\nu-1}a_\mu t^\mu\right)\left(t+\sum_{\mu=2}^{\nu-1}a_\mu t^\mu\right)}{1-R_0\left(t+\sum_{\mu=2}^{\nu-1}a_\mu t^\mu\right)}.$$

In the following display $P_\nu[\cdot]$ takes the degree-ν term in the scalar variable t. Since $a_\mu\leq A_\mu$ for $2\leq\mu\leq\nu-1$, summing the two bounds gives

$$\begin{aligned}\|g_\nu\|_{1,\nu}t^\nu &\preceq P_\nu\left[\frac{M_0\left(t+\sum_{\mu=2}^{\nu-1}a_\mu t^\mu\right)^2}{1-R_0\left(t+\sum_{\mu=2}^{\nu-1}a_\mu t^\mu\right)}\right]\\ &\preceq P_\nu\left[\frac{M_0A(t)^2}{1-R_0A(t)}\right]=\frac{A_\nu}{C_\nu D_\nu}t^\nu.\end{aligned}\tag{317}$$

The last equality is (315). Its degree-ν coefficient depends only on $A_2,A_3,\dots,A_{\nu-1}$, since the right side starts in degree two.

The formal lifting condition and Proposition IV.4.8 make g_ν exact for the comparison currently chosen. Apply (C), then (B) to the same representative, to obtain

$$\delta_\nu(\pi_\nu F_\nu) = g_\nu, \qquad \|F_\nu\|_{0,\nu} \le C_\nu \|g_\nu\|_{1,\nu},$$
$$a_\nu \le D_\nu \|F_\nu\|_{0,\nu} \le D_\nu C_\nu \|g_\nu\|_{1,\nu} \le A_\nu.$$

By (A), this cancels the discrepancy and gives a permissible natural comparison through degree ν. Starting with the prescribed first-order comparison, induction constructs all coefficients and proves $a_\nu \le A_\nu$.

Since $a_\nu \le A_\nu$ and $A(t)$ converges near zero, the inverse expressions $F_j(p, X)$ converge on a common transverse polydisc for all the fixed patches. At each $p_0 \in U_j$, we have $F_j(p_0, 0) - w_j(p_0) = 0$, and the derivative with respect to X is I_r. Apply the holomorphic implicit function theorem of Chapter III, § 1.2, to $F_j(p, X) - w_j(p)$. For a singular V_j, first extend this convergent holomorphic map locally in an embedding of V_j into a complex Euclidean space and then restrict the resulting solution to V_j. This gives a unique local holomorphic solution u_j near p_0, with $u_j(p_0) = 0$ and $F_j(p, u_j(p)) = w_j(p)$.

The coefficient bounds make the derivative of $F_j(p, X) - X$ with respect to X uniformly small on a smaller common transverse polydisc. Hence each $F_j(p, \cdot)$ is injective there. Taking the local solutions with values in this polydisc, they agree on overlaps for each fixed j and give u_j on a neighborhood of the whole U_j, after shrinking.

Since $F_j(p, X) - X$ starts in degree two, write $F_j(p, X) - X = A_j(p, X)X$, where $A_j(p, 0) = 0$. The matrix $I_r + A_j(p, u_j(p))$ is invertible after shrinking, so $w_j = (I_r + A_j(p, u_j))u_j$ proves the stated ideal equality and first-order identity. Condition (A) supplies the actual source realization $u_j = w_j \circ \iota_j (F_j - X)^{-1}$.

The convergent solution and the finite coefficient polynomials have the same comparison jet at every fixed order. Hence each component of $u_k \circ \phi_a - T_a u_j$ lies in every power of the source ideal at each point of its center. The analytic local ring is Noetherian and that ideal is contained in its maximal ideal. The Krull intersection theorem (Theorem I.5.15) therefore makes this germ zero. The finitely many testing equations hold as analytic germs, and the composition rules give naturality for the whole diagram. Finally, in the point-centered case the first derivative of each u_j is invertible, so the holomorphic inverse function theorem makes its component an isomorphism. □

Remark IV.8.10. The following applications use two methods to check the last convergence requirement in (D). If $C_\nu D_\nu \le K$ for every $\nu \ge 2$, the coefficients of A are bounded by those of the solution

$$B(t) - t = \frac{K M_0 B(t)^2}{1 - R_0 B(t)}, \qquad B(0) = 0.$$

The derivative with respect to B of the implicit equation is one at the origin. Thus B, and hence A, converges near zero. Alternatively, suppose a positive sequence e_k satisfies

$$C_\nu D_\nu \le e_{\nu-1}^{-1}, \qquad e_k \ge (2k)^{-\alpha}, \qquad e_{p-q} \le e_p + e_q \quad (p > q \ge 1)$$

for some $\alpha > 0$. The Siegel–majorant theorem (Theorem II.1.1) applied with M_0, R_0 gives a convergent majorant for (315). A fixed positive factor in the lower bound for e_k is absorbed by increasing α. The difference inequality is part of this test; a polynomial upper bound on $C_\nu D_\nu$ alone is not that test. □

8.7 Kœnigs–Siegel linearization as an application

Let f be a holomorphic germ fixing 0 with multiplier $\lambda \neq 0$, and write $f(w) = \lambda w + \sum_{k\geq 2} a_k w^k$ and $R_\lambda(w) = \lambda w$. We first recall the single-coefficient calculation, then check Theorem IV.8.9 for this one-object diagram. This is the smallest instance of the criterion: there is no base variable and no coefficient extension problem.

The next coefficient of a conjugacy

Return to the point-centered space $\mathcal{X}$ and the germ f of Example IV.4.5. Fix the reached degree $\mu \geq 2$. Suppose that $h_{\mu-1}(w) = w + \sum_{k=2}^{\mu-1} b_k w^k$ has been chosen so that $f \circ h_{\mu-1}$ and $h_{\mu-1} \circ R_\lambda$ have the same $(\mu-1)$-jet (The sum is empty for $\mu = 2$). Define $r_\mu \in \mathbb{C}$ by

$$f(h_{\mu-1}(w)) - h_{\mu-1}(\lambda w) = r_\mu w^\mu + O(w^{\mu+1}). \tag{318}$$

Replacing $h_{\mu-1}$ by $h_\mu(w) := h_{\mu-1}(w) + b_\mu w^\mu$ changes this coefficient to $r_\mu - (\lambda^\mu - \lambda)b_\mu$. Thus extension of the conjugacy to degree μ is the equation

$$(\lambda^\mu - \lambda)b_\mu = r_\mu. \tag{319}$$

Indeed, the two compositions change at this degree by $\lambda b_\mu w^\mu$ and $\lambda^\mu b_\mu w^\mu$, respectively. This is the next step of (252).

To identify the data of § 4.2, take $\mathfrak{I} = B\mathbb{Z}$ with generator γ, with $\mathscr{F}(\gamma) = f$ and $\mathscr{F}_{\mathrm{std}}(\gamma) = R_\lambda$. Let G_* be the group of convergent germs tangent to the identity and $q_*(w) = w$. Its jet images are $G_*^{(\ell)} = \rho_\ell(G_*)$. The kernel $K_*^{(\mu-1)}$ consists of

$$k_b(w) := w + bw^\mu \pmod{w^{\mu+1}}, \qquad b \in \mathbb{C}.$$

These jets are realized by the displayed polynomials, and $k_b \circ k_c = k_{b+c}$ at this order. Since $q_* = \mathrm{id}$, the comparison-level correction space is $M_{*,\mu} = \mathbb{C}w^\mu$, identified with this additive kernel. For the natural comparison $\Phi = h^{-1}$, the source-frame transport of § 4.2 is

$$\mathcal{T}_\gamma^{(\mu)} \colon M_{*,\mu} \longrightarrow M_{*,\mu}, \qquad bw^\mu \longmapsto \lambda^{\mu-1} b w^\mu.$$

This follows either by substitution in the degree sheaf or from $R_\lambda^{-1} \circ k_b \circ R_\lambda = k_{\lambda^{\mu-1}b}$. A cocycle for $B\mathbb{Z}$ is determined by its value on γ, as verified in Remark IV.8.12. Recording that value reduces the correction equation to

$$\delta_\mu = \mathcal{T}_\gamma^{(\mu)} - \mathrm{id}_{M_{*,\mu}} \colon M_{*,\mu} \longrightarrow M_{*,\mu}.$$

In particular, the obstruction space of § 4.2 is

$$H_\mu^1 = M_{*,\mu}/\mathrm{Image}\,\delta_\mu. \tag{320}$$

The normalized discrepancy of Φ has leading term $(r_\mu/\lambda)w^\mu$.

PROPOSITION IV.8.11. *Let f be a holomorphic germ fixing 0, with multiplier $\lambda \neq 0$, and put $R_\lambda(w) := \lambda w$. Fix $\mu \geq 2$ and a $(\mu-1)$-jet $h_{\mu-1}$ with $h_{\mu-1}(0) = 0$, $h'_{\mu-1}(0) = 1$, satisfying $f \circ h_{\mu-1} = h_{\mu-1} \circ R_\lambda$ to order $\mu - 1$. For the discrepancy and correction space above, the class*

$$\mathrm{Obs}_\mu(f, h_{\mu-1}) := \left[\frac{r_\mu}{\lambda} w^\mu\right] \in H^1_\mu$$

is independent of the lift of this jet to degree μ. It vanishes if and only if the conjugacy extends to degree μ. Once one extension is fixed, the differences between corrections are the elements of $\operatorname{Ker} \delta_\mu \subset M_{*,\mu}$.

PROOF. Any other lift is $h_{\mu-1}(w) + bw^\mu$ modulo $w^{\mu+1}$. By (318), its normalized discrepancy changes by $-(\lambda^{\mu-1} - 1)bw^\mu = -\delta_\mu(bw^\mu)$. This is the negative component change associated with the inverse comparison $\Phi = h^{-1}$. The assertion follows from Proposition IV.4.8, or directly from (319). □

If $\lambda^{\mu-1} \neq 1$, the next coefficient is uniquely $b_\mu = r_\mu/(\lambda^\mu - \lambda)$. If $\lambda^{\mu-1} = 1$, an extension exists precisely when $r_\mu = 0$, and then this coefficient is free. For example, $r_2 = a_2$, so $b_2 = a_2/(\lambda^2 - \lambda)$ when $\lambda \neq 1$. When λ is not a root of unity, these steps construct a compatible formal conjugacy. The estimates below control the small denominators in this construction.

Verification of (A)–(D)

Assume first that λ is not a root of unity. The coefficient calculation above extends every chosen finite conjugacy, so it verifies the formal lifting condition. The same is true for the inverse comparison $\Phi = h^{-1}$. The dictionary with Theorem IV.8.9 is

Diagram datum	Fixed-point problem
$\mathfrak{I}$	$B\mathbb{Z}$, with generator γ
$\mathscr{F}(*)$, $\mathscr{F}(\gamma)$	$(\mathbb{C}, \mathcal{I}_{\{0\}})$ and f
$\mathscr{F}_{\mathrm{std}}(*)$, T_γ	$(\mathbb{C}, \mathcal{I}_{\{0\}})$ and λ
G_*, q_*	Germs tangent to the identity, and $q_*(w) = w$
H, inverse representation	$\Phi = h^{-1}$, $F_*(X) = h(X) = X + \sum_{\nu \geq 2} b_\nu X^\nu$, and $w = F_*(u)$
B_*, $M_{*,\nu}$	$\mathbb{C}$ and $\mathbb{C}w^\nu$, for reached degree $\nu \geq 2$

Only the equation for γ needs an analytic estimate. The inverse equation follows by inverting the germs, and all other arrows follow by composition.

For (A), a finite polynomial $F_*(X) = X + \sum_{\mu=2}^{\nu} b_\mu X^\mu$ has an inverse germ, and the same holds for a convergent series tangent to the identity. Take $\iota_*(F_* - X) = h = F_*$ in G_*. Its inverse is the comparison component, and the degree-ν change of that inverse has the negative sign used in (A). For (B), extension of a complex number does nothing, and $D_\nu = 1$. For (C), measure the discrepancy on γ by the absolute value of its normalized scalar coefficient. The correction operator is $b \mapsto (\lambda^{\nu-1} - 1)b$, so one may take

$$C_\nu = |\lambda^{\nu-1} - 1|^{-1}.$$

For (D), the initial nonlinear term is $f_\gamma(w) = \lambda^{-1}f(w) - w$. Fix a disc on which f is holomorphic and bounded. The Cauchy formula gives its coefficient bounds MR^ν. The transported degree-μ correction is $\lambda^{-1}b_\mu(\lambda X)^\mu$, which has no dependence on the old source variable w. Its difference from its value at $w = 0$ is zero, so the second nonlinear term in (316) is absent. The restriction constant is $L = 1$.

If $0 < |\lambda| < 1$, then $|\lambda^{\nu-1} - 1| \geq 1 - |\lambda|$; if $|\lambda| > 1$, then $|\lambda^{\nu-1} - 1| \geq |\lambda| - 1$. In both cases $C_\nu D_\nu$ is uniformly bounded. The implicit majorant argument following Theorem IV.8.9 verifies the last part of (D), giving Kœnigs' linearization.

For Siegel's case write $\lambda = e^{2\pi\sqrt{-1}\theta}$, where θ satisfies the Diophantine condition of Chapter III, § 3.1. Put $e_n := |1 - \lambda^n|$. The comparison with $d(n\theta, \mathbb{Z})$ gives $e_n \geq cn^{-\tau}$ for fixed $c, \tau > 0$. For $p > q \geq 1$,

$$e_{p-q} = |\lambda^p - \lambda^q| \leq |\lambda^p - 1| + |\lambda^q - 1| = e_p + e_q.$$

Consequently the Siegel–majorant theorem (Theorem II.1.1) verifies (D), with $C_\nu D_\nu = e_{\nu-1}^{-1}$ after the fixed lower-bound constant has been absorbed into its exponent. The resulting Φ is invertible, and naturality says $\Phi \circ f = R_\lambda \circ \Phi$. Its inverse therefore satisfies $f \circ h = h \circ R_\lambda$, with $h(0) = 0$ and $h'(0) = 1$. These checks recover the two cases of Theorem III.3.10 from the coefficient estimates.

Remark IV.8.12 (The cohomology in the coefficient equation). The preceding quotient is the first cohomology of the infinite cyclic group. For an abelian group M with an action of $G = \langle\gamma\rangle \cong \mathbb{Z}$, let T denote the action of γ. A cocycle $c\colon G \to M$ satisfies $c(gg') = c(g)+g\cdot c(g')$ and is determined by $a := c(\gamma)$:

$$c(\gamma^k) = \sum_{j=0}^{k-1} T^j a \quad (k \geq 1), \qquad c(1) = 0, \qquad c(\gamma^{-k}) = -T^{-k}c(\gamma^k).$$

Conversely, these formulas define a cocycle for every $a \in M$. A coboundary has the form $c(g) = g \cdot b - b$, and its value at γ belongs to $\mathrm{Image}(T - \mathrm{id})$. Thus

$$H^0(G, M) = \mathrm{Ker}(T - \mathrm{id}), \qquad H^1(G, M) = M/\mathrm{Image}(T - \mathrm{id}). \tag{321}$$

In the coefficient equation take $M = M_{*,\mu}$ and $T = \mathcal{T}_\gamma^{(\mu)}$. This gives (320) and the freedom $\mathrm{Ker}\,\delta_\mu$ of the next coefficient. The interpretation by an affine space of lifts is the group-action case of Proposition IV.4.8 (For the derived-functor definition of group cohomology, see [Weib, Chapter 6]; compare also [Bro, §3.2, Example 3.1]). □

8.8 Ueda-type linearization in arbitrary codimension

Let $\mathcal{X}$ be an analytic neighborhood space with a model $(X, \mathcal{I}_Z)$, where X is a complex manifold and Z is a compact connected non-singular submanifold of codimension $r \geq 1$. Fix the unitary flat structure of $N_{Z/X}$, its conormal sheaf $\mathcal{N}_{\mathcal{X}}^* = \mathcal{I}_Z/\mathcal{I}_Z^2$, and its specified unitary flat extension to the neighborhood germ, as in § 8.1. Choose unitary flat conormal frames

and write their transition matrices as $T_{jk} \in \mathrm{U}(r)$, so that initial defining functions satisfy $w_j = T_{jk} w_k$ modulo $\mathcal{I}_Z^2$.

We say in this smooth setting that the system is of *infinite type* (in the sense of Ueda) if, with this prescribed first-order part, systems agreeing with the flat comparison exist to every finite order. This is the higher-codimensional form of the reduced Cartier definition.

For a correction of homogeneous degree $\nu \geq 2$, the coefficient sheaf is

$$\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{N}_{\mathcal{X}}^*, \mathcal{S}_{\mathcal{X}}^{\nu}) \cong \mathcal{O}_Z(N_{Z/X} \otimes S^{\nu} N_{Z/X}^*). \tag{322}$$

The output of the vector-valued correction accounts for the factor $N_{Z/X}$. In rank one the coefficient bundle is $N_{Z/X}^{1-\nu}$. The class $u_\mu(\varphi_\mu)$ in Proposition IV.8.1 is the μ-th Ueda obstruction, with homogeneous degree $\nu = \mu + 1$.

THEOREM IV.8.13 (Ueda-type linearization). *For $\mathcal{X}$, Z, $N_{Z/X}$, and the fixed unitary comparison as above, suppose that the system is of infinite type. Assume one of the following conditions.*

(*i*) *The monodromy representation of $N_{Z/X}$ has finite image.*

(*ii*) *On a finite normal unramified covering $\pi\colon \widetilde{Z} \to Z$, the pullback of $N_{Z/X}$ is a direct sum of unitary flat line bundles $N_1, N_2, \dots, N_r$. There is $\alpha > 0$ such that*

$$d\left(\mathbb{I}_{\widetilde{Z}}, \bigotimes_{\lambda=1}^{r} N_\lambda^{a_\lambda}\right) \geq (2\textstyle\sum_\lambda a_\lambda)^{-\alpha} \tag{323}$$

for every $a \in \mathbb{Z}^r$ with $\sum_\lambda a_\lambda \geq 1$. Here d is an invariant distance on the group of unitary flat line bundles and $\mathbb{I}_{\widetilde{Z}}$ is the trivial line bundle.

Then there are defining functions u_j on a common neighborhood of Z such that $u_j = T_{jk} u_k$ on all overlaps and $u_j = w_j$ modulo $\mathcal{I}_Z^2$. They define a non-singular holomorphic foliation of codimension r with Z as a leaf. In rank one, the assumptions include the torsion case and the Diophantine condition $d(\mathbb{I}_Z, N_{Z/X}^{\mu}) \geq c\mu^{-\alpha}$ for all $\mu \geq 1$, where $c > 0$ and $\mathbb{I}_Z$ is a holomorphically trivial line bundle on Z.

The rank-one theorem is Ueda's [U1, Theorem 3 and §4]. Although that paper states the result for a curve in a surface, its bounded-cochain and majorant argument applies to a smooth compact hypersurface in general dimension; that convergence mechanism is Ueda's. The higher-codimensional cases stated here are covered by [Koi6, Theorem 1.1, §§4.1–4.7]. The conclusion gives defining functions with the prescribed unitary linear transition rules.

We apply Theorem IV.8.9. The following correspondence fixes the diagrams and the permitted changes. As in Example IV.1.16, the chart models are restricted to intersections before their transition maps are used as arrows.

Data	Present comparison
Source	The models $(V_j, \mathcal{I}_Z\|_{V_j})$, their restrictions, and the given chart transitions.
Standard	$(\mathbb{C}^r, \mathcal{I}_{\{0\}})$, with $T_a = T_{kj}$ for a transition from the j-chart to the k-chart.
Components	The defining maps u_j, with $u_k \circ \phi_a = T_{kj} u_j$.
Permitted changes	Local biholomorphisms preserving the chosen projection p_j, fixing U_j, and inducing the identity on $\mathcal{I}_Z/\mathcal{I}_Z^2$.
Degree ν	Sections of $N_{Z/X} \otimes S^\nu N_{Z/X}^*$, represented by $F_{j,\alpha}^\lambda$ for $\|\alpha\| = \nu$.

The correction spaces in this table are those of § 8.2. Consequently $\operatorname{Ker} \delta_\nu$ consists of global sections of the unitary flat bundle $N_{Z/X} \otimes S^\nu N_{Z/X}^*$. Each such section is parallel: its squared norm is plurisubharmonic on compact Z, hence constant, and the local sum-of-squares formula makes its holomorphic components constant in flat frames. Its homogeneous polynomial expressions $Q_{j,\nu}(X)$ therefore satisfy $Q_{j,\nu}(T_{jk}X) = T_{jk}Q_{k,\nu}(X)$. Postcomposition by $X \mapsto X + Q_{j,\nu}(X)$ preserves the linear transitions and the first-order part, and is realized with z_j unchanged in an adapted chart. Lemma IV.8.8 therefore turns infinite type into the formal lifting condition for every comparison chosen in the induction (Compare [Koi6, Lemma 3.6, Proposition 3.7, and Lemma 4.4]).

Fixed projections and local inverses verify (A)–(B), scalar cochain estimates verify (C), and two Cauchy estimates with the arithmetic bounds verify (D). The cubic coefficient equation exhibits the contribution of the base-coordinate change.

Fixed domains, permitted changes, and coefficient estimates

We prove Theorem IV.8.13 by constructing the coefficients in (327) with bounds on one fixed covering. We first work when the normal bundle is trivial or is a direct sum of unitary flat line bundles as in condition (ii), without the finite covering. The passage to a finite covering will be treated at the end. The coefficient construction and estimates follow [Koi6, §§4.3–4.6], which develop the convergence argument of Ueda [U1, §4].

If Z is zero-dimensional, it is a point and one adapted coordinate chart already gives the conclusion. Suppose henceforth that $d := \dim Z \geq 1$, and use product coordinates (z_j, w_j), where z_j has d components and w_j has r components. Choose finite coverings $\mathcal{U} = \{U_j\}_{j\in J}$ and $\mathcal{U}^* = \{U_j^*\}_{j\in J}$ of Z, such that $\overline{U_j^*}$ is a compact subset of U_j for every $j \in J$. The coordinates are defined on larger patches containing the closures needed here. The local projection $p_j(z_j, w_j) = z_j$ is fixed throughout the argument. Thus a holomorphic function a on U_j is extended as $a \circ p_j$, without changing its supremum norm on a product patch.

For these patches take B_j with its supremum norm and $\mathsf{E}_{j,\nu} b = b \circ p_j$. Thus $D_\nu = 1$ in (B). For a finite inverse expression $F_j(z, X)$, or a convergent one, with linear part X, set

$$h_j(z, X) = (z, F_j(z, X)).$$

This is a local biholomorphism along U_j, preserves p_j, and fixes the center and the conormal frame. Its inverse has transverse component determined by $w = F_j(z, u)$, so $u_j = w_j \circ h_j^{-1}$.

This defines the realization $\iota_j(F_j - X) = h_j$ required in (A), both for finite polynomials and for the convergent coefficient choices. The same formula also realizes addition of a homogeneous coefficient with the sign in (A).

On an overlap write

$$T_{jk} w_k = w_j + f_{jk}(z_j, w_j), \qquad z_k = z_k(z_j, w_j), \qquad f_{jk}(z_j, w_j) = O(|w_j|^2).$$

The matrices T_{jk} are diagonal and unitary in the split case. We choose $R > 0$ large enough that, whenever $z_j \in U_j \cap U_k^*$, the polydisc in the w_j-variables with coordinate radii R^{-1} lies in the overlap and its image under z_k lies in U_k. The coordinate maps are defined on slightly larger polydiscs. All these choices are possible simultaneously: the smaller base patches have compact closures in the larger ones, the original coordinate patches have a fixed buffer, and there are only finitely many pairs (j, k).

Choose $M > 0$ so that every scalar component of f_{jk} has absolute value at most M on these polydiscs. Increasing R and M once is harmless. In particular, these constants will not depend on the degree of a coefficient constructed later.

We use polydiscs here to obtain estimates for individual monomials. The sphere norm of Chapter III will also be used below. A ball and a polydisc in the fixed finite-dimensional normal space contain fixed rescalings of one another; this comparison is made once, not anew at every degree.

Write

$$f_{jk}^{\lambda}(z_j, w_j) = \sum_{|\alpha| \geq 2} f_{jk,\alpha}^{\lambda}(z_j) w_j^{\alpha} \qquad (1 \leq \lambda \leq r).$$

These coefficient functions are holomorphic in z_j, by the expansion with respect to selected variables in (112). Applying the Cauchy integral formula only in the w_j-variables gives

$$\sup_{U_j \cap U_k^*} |f_{jk,\alpha}^{\lambda}| \leq M R^{|\alpha|}. \tag{324}$$

For example, this follows by writing the coefficient as the integral of $f_{jk}^{\lambda}(z_j, \zeta)\zeta^{-\alpha - \mathbf{1}}$ over the product of the circles $|\zeta^i| = R^{-1}$. One may first use slightly smaller circles and then let their radii increase to R^{-1}.

The second application is to the change of base coordinates. Let a be any bounded holomorphic function on U_k. There is an expansion

$$a(z_k(z_j, w_j)) = a(z_k(z_j, 0)) + \sum_{|\gamma| \geq 1} a_{kj,\gamma}(z_j) w_j^{\gamma}.$$

Since $z_k(z_j, w_j) \in U_k$ on the same fixed polydiscs, the same Cauchy formula yields

$$\sup_{U_j \cap U_k^*} |a_{kj,\gamma}| \leq R^{|\gamma|} \|a\|_{L^\infty(U_k)}. \tag{325}$$

The significance of the fixed domain is that (325) applies to every coefficient $F_{k,\alpha}^{\lambda}$ chosen in the induction, with the same R. For each coefficient $F_{k,\alpha}$ chosen below, write

$$F_{k,\alpha}(z_k(z_j, w_j)) = F_{k,\alpha}(z_k(z_j, 0)) + \sum_{|\gamma| \geq 1} F_{kj,\alpha,\gamma}(z_j) w_j^{\gamma}.$$

Thus $F_{kj,\alpha,\gamma}$ is a holomorphic vector-valued Taylor coefficient of this composite, and

$$\sup_{U_j\cap U_k^*} |F^\lambda_{kj,\alpha,\gamma}| \le R^{|\gamma|}\|F^\lambda_{k,\alpha}\|_{L^\infty(U_k)}.$$

For comparison with the existing norms, let $U \subset \mathbb{C}^d$ be a coordinate base domain, fix $t > 0$ and $\nu \in \mathbb{Z}_{\ge 0}$, and let $P(z,u) = \sum_{|\alpha|=\nu} a_\alpha(z)u^\alpha$ be scalar-valued. Formula (116) gives

$$\|P\|^2_{U,t} = \frac{2\pi^r t^{2\nu+2r-1}}{(\nu+r-1)!}\sup_{z\in U}\sum_{|\alpha|=\nu}\alpha!\,|a_\alpha(z)|^2. \tag{326}$$

The measure is the unnormalized sphere measure in (113). By Lemma IV.3.1, the squared symmetric-tensor norm of the coefficient section is

$$\sum_{|\alpha|=\nu}\frac{\alpha!}{\nu!}|a_\alpha|^2.$$

Thus (326) gives the precise conversion between the two norms. Ordinary monomials need not form an orthonormal frame of a symmetric power.

The homogeneous projection $P_\nu[\cdot]$ has norm at most one in the sphere norm, by the same orthogonality used in (117). Products are controlled by (118), with one factor in the supremum norm. For the nonlinear composition below we use individual coefficients in the split frames.

The successive coefficient equations

Use $X = (X_1, X_2, \dots, X_r)$ as independent transverse variables in the coefficient calculation. Seek a change of defining functions through the inverse expression

$$w_j = F_j(z_j, u_j), \qquad F_j(z_j, X) = X + \sum_{|\alpha|\ge 2} F_{j,\alpha}(z_j)X^\alpha. \tag{327}$$

This is (313) with $\mathsf{E}_{j,\nu}b = b\circ p_j$; dependence on p is through $z_j = p_j(p)$. At this stage F_j is a formal power series, with holomorphic $\mathbb{C}^r$-valued coefficients $F_{j,\alpha}$. For $\nu \ge 2$,

$$P_\nu\left[F_j(z_j,X)\right] = \sum_{|\alpha|=\nu} F_{j,\alpha}(z_j)X^\alpha.$$

A scalar component $F^\lambda_{j,\alpha}$ is expressed in the flat frame of $N_\lambda \otimes \bigotimes_{i=1}^r N_i^{-\alpha_i}$. Through the corresponding normal and conormal frames, the homogeneous polynomial represents a section of $\mathcal{O}_Z(N_{Z/X}\otimes S^\nu N^*_{Z/X})$ on U_j. For the coefficient family $F_\nu = (F_{j,\alpha})_{j,\,|\alpha|=\nu}$ of § 8.6, its correction is therefore identified with the 0-cochain

$$\pi_\nu F_\nu = \{(U_j, P_\nu[F_j])\}_{j\in J} \in \check{C}^0\big(\mathcal{U}, \mathcal{O}_Z(N_{Z/X}\otimes S^\nu N^*_{Z/X})\big)\,.$$

In rank one its scalar coefficient is $F_{j,\nu}(z_j)$ and its polynomial expression is $F_{j,\nu}(z_j)X^\nu$.

Suppose the coefficients of degrees less than ν have been chosen. Only the finite polynomial

$$X + \sum_{2\le|\alpha|\le\nu-1} F_{j,\alpha}(z_j)X^{\alpha}$$

is needed to compute the next equation. Its derivative at $X = 0$ is the identity, so it has a local holomorphic inverse. Convergence of the full series is not assumed in this step. The contribution from the nonlinear part of the transverse transition is

$$H^{\mathrm{trans}}_{jk,\nu} := P_\nu\left[f_{jk}\left(z_j, X + \sum_{2\le|\alpha|\le\nu-1} F_{j,\alpha}(z_j)X^{\alpha}\right)\right].$$

The contribution arising from the dependence of the base-coordinate change on the transverse variables is

$$H^{\mathrm{base}}_{jk,\nu} := P_\nu\Bigg[T_{jk}\sum_{\substack{2\le|\alpha|\le\nu-1\\ |\gamma|\ge1}} F_{kj,\alpha,\gamma}(z_j)(T_{kj}X)^{\alpha} \cdot\left(X + \sum_{2\le|\delta|\le\nu-1} F_{j,\delta}(z_j)X^{\delta}\right)^{\gamma}\Bigg].$$

Here both expressions are homogeneous vector polynomials in X. Their difference represents a coefficient section $g_{jk,\nu}$ on U_{jk}. Put $g_\nu := \{(U_{jk}, g_{jk,\nu})\}_{j,k\in J}$. Comparing the two expressions for $T_{jk}w_k$ gives

$$\check{\delta}(\pi_\nu F_\nu) = g_\nu, \qquad g_{jk,\nu} = H^{\mathrm{trans}}_{jk,\nu} - H^{\mathrm{base}}_{jk,\nu}. \tag{328}$$

The second identity uses the same polynomial identification. In the j-frame the left side is

$$T_{jk}\sum_{|\alpha|=\nu} F_{k,\alpha}(z_k(z_j,0))(T_{kj}X)^{\alpha} - \sum_{|\alpha|=\nu} F_{j,\alpha}(z_j)X^{\alpha}.$$

This is the convention $\check{\delta}a = a_k - a_j$ for sections written in one frame. It fixes the minus sign in (328). Triple-overlap compatibility modulo degree $\nu+1$ gives

$$g_\nu \in \check{Z}^1\big(\mathcal{U}, \mathcal{O}_Z(N_{Z/X}\otimes S^\nu N^*_{Z/X})\big).$$

This is the cocycle c_ν of § 4.2, written in the j-frame as in (316). The formal lifting condition verified above makes its class zero for the system currently chosen. Hence a primitive may be chosen with the bounds below, and the induction continues with that choice.

The two contributions can already be distinguished in degree three. For this calculation let $r = 1$, abbreviate $t = t_{jk}$, put $\zeta_{kj}(z_j) := z_k(z_j, 0)$, and write

$$tw_k = w_j + f_{jk,2}(z_j)w_j^2 + f_{jk,3}(z_j)w_j^3 + O(w_j^4).$$

Here ζ_{kj} is the change of base coordinates on Z, whereas $f_{jk,\nu}$ records the discrepancy in the j-frame. The transported quadratic coefficient has the expansion

$$F_{k,2}(z_k(z_j, w_j)) = F_{k,2}(\zeta_{kj}(z_j)) + F_{kj,2,1}(z_j)w_j + O(w_j^2).$$

Degree two gives

$$t^{-1}F_{k,2} \circ \zeta_{kj} - F_{j,2} = f_{jk,2}.$$

In degree three, the coefficient on the k-side is

$$t^{-2}F_{k,3} \circ \zeta_{kj} + t^{-1}F_{kj,2,1}(z_j),$$

whereas the coefficient on the j-side is $F_{j,3} + 2f_{jk,2}F_{j,2} + f_{jk,3}$. Hence

$$\begin{aligned} t^{-2}F_{k,3} \circ \zeta_{kj} - F_{j,3} = f_{jk,3} + 2f_{jk,2}F_{j,2} \\ - t^{-1}F_{kj,2,1}(z_j). \end{aligned} \tag{329}$$

This last term is the first contribution of H^{base}. The base-coordinate changes cannot be assumed independent of w_j in a general neighborhood.

Bounded primitives and the convergence condition

We verify (C) in the coefficient norms of § 8.6. The formal lifting condition makes the scalar coefficient equations exact. Let $\mathbb{I}_Z$ be the holomorphically trivial line bundle on Z. For a nontrivial unitary flat coefficient bundle F, Ueda's estimate (Theorem III.2.204) gives

$$\|a\|_{L^\infty(\mathcal{U})} \le \frac{K}{d(\mathbb{I}_Z, F)}\|g\|_{L^\infty(\mathcal{U})}, \qquad \check{\delta}_F a = g. \tag{330}$$

The constant K is independent of F, and the primitive is unique by Lemma III.2.197. For the trivial bundle, Lemma IV.3.5 gives a bounded primitive modulo global sections, with one constant $K_0 \ge 1$ on the fixed covering. The possible loss in choosing near the infimum is included in K_0.

Use

$$\|g_\nu\|_{1,\nu} := \max_{j,k} \sum_{|\alpha|=\nu} \max_{\lambda} \sup_{U_{jk}} |g^\lambda_{jk,\alpha}|.$$

The analogous one-patch expression gives $\|F_\nu\|_{0,\nu}$, which also equals the extended size a_ν, since the extensions are by the fixed projections. If every scalar equation of degree ν has primitive bound κ_ν, coefficientwise choices give

$$\|F_\nu\|_{0,\nu} \le r(\#J)^2\kappa_\nu\|g_\nu\|_{1,\nu}. \tag{331}$$

To see the fixed factor, replace the maximum over output components by their sum and the maximum over overlaps by their sum, and interchange these finite sums with the sum over monomials. Enlarge K and K_0 once to include this factor. For the numbers $e_k > 0$ specified in the proof below, this gives $C_\nu = e_{\nu-1}^{-1}$ in (C), with $D_\nu = 1$ in (B).

For (D), use the testing overlaps $U_j \cap U_k^*$ chosen above. The first Cauchy test is (324). For the second, apply (325) to each extended target coefficient of degree μ and sum over its monomials and output components. The matrices T_{jk} are diagonal and unitary, so substitution multiplies each scalar monomial by a number of absolute value one. The resulting bound is the second test of (314), with the same extended size a_μ.

To pass from testing overlaps to the full cochain norm, take a point of $U_j \cap U_k$ and choose ℓ with that point in U_ℓ^*. In a common flat frame, $g_{jk,\nu} = g_{j\ell,\nu} - g_{k\ell,\nu}$. Both terms on the right are measured on testing overlaps. Taking suprema, summing over the finitely many choices of ℓ, and then over monomials gives the restriction estimate in (D) with $L = 2\#J$. This completes the linear estimates required in (A)–(D). The two nonlinear contributions are those of (316); their common majorant and the analytic realization are supplied by Theorem IV.8.9. It remains to verify its last convergence condition.

Proof of Theorem IV.8.13. First suppose that $N_{Z/X}$ is trivial or is split as in (ii), without passing to a covering. The preceding preparations verify the formal lifting condition and (A)–(C), together with the Cauchy and restriction estimates in (D). If $N_{Z/X}$ is trivial, take $e_k = K_0^{-1}$ for every $k \geq 1$. Then $C_\nu D_\nu = K_0$ is bounded, so the bounded-cost criterion following Theorem IV.8.9 verifies the remaining convergence condition.

In the split arithmetic case, let K include the fixed factor in (331), and set

$$e_k := K^{-1} \inf_{\substack{a \in \mathbb{Z}^r \\ \sum_i a_i = k}} d\left(\mathbb{I}_Z, \bigotimes_{i=1}^{r} N_i^{a_i}\right) \qquad (k \geq 1). \tag{332}$$

The signed exponents have sum k. Assumption (323) gives $e_k \geq K^{-1}(2k)^{-\alpha} > 0$, hence the required polynomial lower bound after increasing the fixed exponent. The coefficient bundle $N_\lambda \otimes \bigotimes_i N_i^{-\alpha_i}$ in degree ν has exponent sum $1 - \nu$; its dual has sum $\nu - 1$. Since $d(\mathbb{I}_Z, F^{-1}) = d(\mathbb{I}_Z, F)$, (332) and (330) give $C_\nu D_\nu = e_{\nu-1}^{-1}$ as an allowable cost bound.

For $p > q \geq 1$, choose signed exponent vectors of sums p and q whose distances approach the two infima in (332). Their difference has sum $p - q$. Translation invariance and the triangle inequality, followed by passage to the infima, give

$$e_{p-q} \leq e_p + e_q.$$

The convergence criterion following Theorem IV.8.9, using the Siegel–majorant theorem (Theorem II.1.1), therefore verifies the last part of (D).

Apply Theorem IV.8.9. It supplies holomorphic defining maps with a common positive radius and $u_j = T_{jk} u_k$. Its coefficient construction has $a_\nu \leq A_\nu$, where $A(t) = t + \sum_{\nu \geq 2} A_\nu t^\nu$ is the convergent series in (315). In the present product coordinates the resulting bounds are

$$|(P_\nu[F_j])^\lambda(z_j, u)| \leq A_\nu t^\nu, \qquad \sum_{i=1}^{r} \left| \frac{\partial (P_\nu[F_j])^\lambda}{\partial u^i}(z_j, u) \right| \leq \nu A_\nu t^{\nu-1} \quad (|u^i| \leq t). \tag{333}$$

They also give convergence in the selected-variable sphere norm:

$$\|P_\nu[F_j]\|_{U_j, t} \leq \left(r \frac{2\pi^r}{(r-1)!} \right)^{1/2} t^{r-1/2} A_\nu t^\nu. \tag{334}$$

This is the norm comparison in (326), summed over the homogeneous terms. The prescribed first-order part makes du_j have rank r along Z, so, after shrinking, its level sets give the asserted non-singular foliation.

For condition (i), pass to the finite normal covering of Z which kills the monodromy. For condition (ii), use the covering in the statement. A sufficiently small smooth tubular neighborhood retracts onto Z (Theorem I.8.4); hence the covering extends to that neighborhood. The pulled-back complex charts make it a holomorphic unramified covering. The defining systems and their finite-order comparisons pull back. We carry out the preceding construction there in trivial or split unitary frames.

To descend, express the resulting defining maps in the pullbacks of the originally fixed flat frames. They are local expressions of one section of the pulled-back flat comparison bundle. Let Γ be the finite deck group. Transporting the value of a section by the bundle action is understood in the notation $\gamma \cdot u$. The average

$$u_{\mathrm{av}} := \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \gamma \cdot u \tag{335}$$

is invariant, obeys the same linear transition rule, and has the same prescribed first-order part. In the unitary metric, its norm on a deck-invariant neighborhood is at most the maximum of the norms of the summands. This averaging is performed on already convergent bundle-valued maps. A constant unitary change from the split frames preserves the Euclidean fiber norm; using polydiscs instead requires only one fixed radius reduction. Thus no degree-dependent coefficient loss is inserted at every step of the induction. The average descends and still has rank r along the center. Its expression on a downstairs adapted chart is realized by the inverse of $(z_j, w_j) \mapsto (z_j, u_j)$, as in (A).

This proves the theorem with the original flat comparison. In codimension one the resulting defining system also trivializes the specified Ueda invertible sheaf, by Lemma IV.8.3. □

A proper defining function in the torsion case. Let $\mathcal{X} = [(X, \mathcal{I})]$ be in setting (D) of §8.1, and suppose that the reduced support Z of D is connected. Retain the specified unitary flat comparison and its Ueda invertible sheaf $\mathcal{P}_D$. Assume that $\mathcal{P}_D \cong \mathcal{O}_{\mathcal{X}}$ and that $([D]|_Z)^{\otimes m}$ is holomorphically trivial for some positive integer m. Then an open neighborhood V of Z admits a proper surjective holomorphic map

$$p \colon V \longrightarrow \Delta_\varepsilon := \{t \in \mathbb{C} \mid |t| < \varepsilon\}$$

such that $p^{-1}(0) = Z$ and $\operatorname{div}(p) = mD$.

Indeed, the analytic triviality of $\mathcal{P}_D$ identifies $[D]$ with the specified unitary flat extension on a sufficiently small neighborhood of Z. By Lemma III.2.130, the unitary flat structure on $([D]|_Z)^{\otimes m}$ is trivial. The uniqueness of flat extensions after shrinking (Example II.3.117) therefore gives a nowhere vanishing holomorphic frame e of $[D]^{\otimes m}$ on a connected open neighborhood W of Z. The function $p := \sigma_D^m / e$ has divisor mD and zero set Z. It is nonconstant, hence is an open map (this follows locally by restricting to a complex line on which it is nonconstant and using the one-variable open mapping theorem). Apply Lemma I.3.17

to $p\colon W \to \mathbb{C}$ at 0. Its zero fiber is compact, and W is locally compact and Hausdorff. This gives V and Δ_ε as asserted. In the non-singular codimension-one case, the torsion case of Theorem IV.8.13 supplies the required analytic comparison. The construction retains the multiplicities of D: $p^{-1}(0) = Z$ is a statement about underlying sets, whereas the zero divisor is mD.

For related small-divisor criteria for transverse linearization in higher codimension and for equivalence of full neighborhoods, see Gong and Stolovitch [GS1]. Chapter V concentrates on applications of Ueda-type linearization to the semi-positivity of holomorphic line bundles. Other applications concern the Levi problem for domains with real analytic boundary in complex surfaces [DO1], and the realization of Levi-flat manifolds as C^∞ real hypersurfaces in complex manifolds [Bar, KO1].

8.9 Ueda-type linearization near singular curves

At a reduced Cartier center the one-step coefficient sheaf is still $\mathcal{O}_Z(N^{1-\nu})$, although a defining function need not be a coordinate. We now verify the analytic conditions using two different local extension procedures. The comparison remains the diagram of defining maps to $(\mathbb{C}, \mathcal{I}_{\{0\}})$ with fixed scalar transition factors.

Curves with nodes

Let $\mathcal{X} = [(X, \mathcal{I}_Z)]$, where X is a non-singular complex surface and Z is a compact connected reduced curve whose singularities are ordinary nodes. The curve may be reducible, and a component may meet itself at a node. Fix a unitary flat structure on $N := [Z]|_Z$, its comparison extension, and first-order defining functions as in § 8.1. Let $i\colon \widetilde{Z} \to Z$ be the normalization. For a unitary flat line bundle L on Z, use the notation $\mathbb{C}_Z(L)$ for its sheaf of local flat sections, as in Example II.3.65 and Corollary III.2.126. On the present reduced curve it consists of sections whose coefficients in holomorphic frames compatible with the fixed unitary flat structure are locally constant. In particular, it is a subsheaf of $\mathcal{O}_Z(L)$.

Theorem IV.8.14 (Koike). *Keep the nodal center and specified unitary comparison above, and assume that $\mathcal{X}$ is of infinite type. Suppose that N is torsion, or that there are $c, \alpha > 0$ with*

$$d(\mathbb{I}_Z, N^k) \geq ck^{-\alpha} \qquad (k \geq 1),$$

where d is the invariant distance on unitary flat line bundles specified in [Koi4, §4.1]. Suppose in addition that i^*N is torsion and

$$H^1(Z, \mathbb{C}_Z(N^{-k})) = 0 \qquad (k \geq 1). \tag{336}$$

Then there are defining functions u_j, on a common neighborhood of Z, satisfying $u_j = t_{jk}u_k$ and $u_j = w_j$ modulo $\mathcal{I}_Z^2$. Thus $[Z]$ is unitary flat on that neighborhood, with the prescribed comparison.

This is the connected case of [Koi4, Theorem 1.4]. Ueda's analysis of a rational nodal curve in [U2] is a foundational result for singular centers. In the proof below we use the adapted covering from [Koi4, Remarks 2.2 and 4.3] and the fixed-bundle estimates on the normalization used in its §4. We spell out the two primitive equations and the coefficient estimates on that covering.

Proof of Theorem IV.8.14. Choose the finite covering $\mathcal{U} = \{U_j\}_{j\in J}$ and ambient neighborhoods V_j as in [Koi4, Remarks 2.2 and 4.3]. For each Stein piece U_j, choose O_j with $O_j \cap Z = U_j$ and apply Remark IV.5.9 to this closed subspace of O_j (Compare [U2, §2, p. 684] and [Koi9, Remark 3.4]). The cited covering construction provides the further refinements and shrinkings for the estimates. Ordinary patches have coordinates (z_j, w_j) with $U_j = \{w_j = 0\}$ and a fixed projection p_j onto U_j. A patch containing a node, indexed by s, has coordinates (x_s, y_s) and $w_s = x_s y_s$. Such nodal patches are pairwise disjoint. The remaining intersections are in the regular part of Z, so every transition equation can be tested from an ordinary chart. Include the intersection models and restriction arrows as in Example IV.1.16; the standard arrow from the j-chart to the k-chart is multiplication by t_{kj}.

The permissible comparison components are defining functions with the specified first-order part. Take G_j to consist of local biholomorphisms fixing U_j pointwise and inducing the identity on its conormal sheaf. On ordinary patches we may restrict further to those preserving p_j. The one-step comparison-level corrections are sections of $N^{1-\nu}$. A global section of this unitary flat bundle is parallel: pull it to each compact component of $\widetilde{Z}$, apply the sum-of-squares maximum-principle argument, and use the equality of the two values over a node. Constant homogeneous postcompositions preserve the allowed defining functions and are realized by the local changes described below. Thus Lemma IV.8.8 gives the formal lifting condition. This is the fixed-comparison form of [Koi4, Remark 3.5 and Proposition 3.6].

Coefficient norms and extension: (B). Choose the ordinary coordinate discs and both discs at a node with radii less than one. A holomorphic function on U_s is a pair $b_+(x_s), b_-(y_s)$ with common value b_0 at zero. Use the coefficient norms

$$\begin{aligned} \|b_j\|_{B_j} &:= \sup_{U_j} |b_j| + \sup_{U_j} |db_j/dz_j|, \qquad j \text{ ordinary}, \\ \|b_s\|_{B_s} &:= \sup_{U_s} |b_s| + \sup |b'_+| + \sup |b'_-|, \qquad s \text{ nodal}. \end{aligned} \tag{337}$$

Take the holomorphic functions for which these norms are finite. They are Banach algebras with norm one for the constant 1; the product inequality follows by applying the product rule on each branch. On a nodal bidisc use the ambient norm $\sup|B| + \sup|\partial B/\partial x_s| + \sup|\partial B/\partial y_s|$. The extensions are

$$\mathsf{E}_j b = b \circ p_j, \qquad \mathsf{E}_s b = b_+(x_s) + b_-(y_s) - b_0. \tag{338}$$

The ordinary extension preserves its norm. The nodal extension has supremum at most $3\sup_{U_s}|b|$, and its two derivatives are b'_+ and b'_-. Consequently $\|\mathsf{E}_j b\|_{V_j} \le 3\|b\|_{B_j}$ on every patch, so $D_\nu = 3$ works for all ν. The same linear extensions will be used for the primitives, not only for their restrictions.

Source realization: (A). Use the inverse expression

$$F_j(p,X) = X + \sum_{\nu\geq 2} (\mathsf{E}_j F_{j,\nu})(p) X^\nu, \qquad w_j(p) = F_j(p, u_j(p)).$$

For any finite polynomial, or any series with convergent extended sizes, its implicit solution has the form $u_j = w_j v_j$, where v_j is holomorphic and $v_j = 1$ modulo (w_j). On an ordinary patch $k_j = (z_j, u_j)$ is a local biholomorphism. At a node use

$$k_s(x_s, y_s) := (x_s, y_s v_s(x_s, y_s)), \qquad w_s \circ k_s = u_s. \tag{339}$$

Since $v_s = 1 + x_s y_s a_s$ for a holomorphic a_s, the map fixes both axes pointwise and its Jacobian determinant is one on them. It is therefore a biholomorphism as a neighborhood germ. Its action on the conormal sheaf is the identity. Set $\iota_j(F_j - X) = k_j^{-1}$. Then $u_j = w_j \circ \iota_j (F_j - X)^{-1}$, as required in (A). This also shows that constant homogeneous postcompositions used in the formal lifting argument are permitted. The filtration here is the order of the change in u_j; it is exactly the comparison-level correction used in Proposition IV.4.8.

Two primitive equations: (C). Fix $\nu \geq 2$ and put $L_\nu := N^{1-\nu}$. Let $g = \{g_{jk}\}$ be the bounded exact discrepancy at this degree, written in the j-frame. Its cochain norm $\|g\|_{1,\nu}$ is the maximum of the supremum of g_{jk} plus the supremum of its derivative, measured in the fixed ordinary coordinate on each overlap. Equivalent choices of orientation have uniformly comparable norms on this finite covering. The coboundary convention is

$$(\check{\delta} b)_{jk} = t_{jk}^{1-\nu} b_k - b_j.$$

The transition factors are locally constant, so differentiation commutes with this coboundary.

The first equation is on the normalization:

$$\check{\delta}\eta = i^*(dg), \qquad \eta \in \check{C}^0(i^{-1}\mathcal{U}, \mathcal{O}_{\widetilde{Z}}(K_{\widetilde{Z}} \otimes i^* L_\nu)).$$

It is solvable because g is holomorphically exact. Since i^*N is torsion, only finitely many coefficient bundles $K_{\widetilde{Z}} \otimes i^* L_\nu$ occur up to their specified flat tensor factor. Apply the fixed-bundle bounded-primitive argument of Lemma IV.3.5 on the normalization, with fixed metrics and coverings. Taking the maximum of the finitely many constants gives $\|\eta\|_\infty \leq K_1 \|g\|_{1,\nu}$ with K_1 independent of ν (Compare [Koi4, p. 861]).

Integrate η on each ordinary disc from its center and on each nodal branch from the point over the node. Denote these primitives by b_j^0 and $b_{s,+}^0, b_{s,-}^0$; both latter values are zero at the node. They therefore give one function b_s^0 on U_s. The disc radii and the comparison between coordinate and metric norms are fixed. Hence

$$\max_j \|b_j^0\|_{B_j} \leq K_2 \|g\|_{1,\nu}.$$

The remaining discrepancy

$$\beta := g - \check{\delta} b^0$$

has zero derivative on every overlap, and is a cocycle with coefficients in $\mathbb{C}_Z(L_\nu)$. Its supremum is at most $K_3\|g\|_{1,\nu}$. The second equation is

$$\check{\delta}c = \beta, \qquad c \in \check{C}^0(\mathcal{U}, \mathbb{C}_Z(L_\nu)). \tag{340}$$

Hypothesis (336) ensures its solvability. Then $b = b^0 + c$ solves $\check{\delta}b = g$ on Z. Set $F_{j,\nu} = b_j$ in the chosen frames and apply (338) to these same representatives.

For a fixed unitary flat bundle the constant-cochain equation has a bounded primitive: it is a finite system of linear equations modulo its kernel. If N is torsion, only finitely many such systems occur, and their constants have a common bound ([Koi4, Lemma 4.1]). In the non-torsion case the estimate for constant cochains in [Koi4, Lemma 4.2] gives

$$\|c\|_\infty \le \frac{K_4}{d(\mathbb{I}_Z, L_\nu)}\|\beta\|_\infty.$$

Here exactness has already been supplied by (336). Thus (C) holds with constants independent of the chosen exact discrepancy and

$$C_\nu \le \begin{cases} K, & N \text{ torsion}, \\ K\big(1 + d(\mathbb{I}_Z, N^{\nu-1})^{-1}\big), & N \text{ non-torsion}, \end{cases} \tag{341}$$

for a fixed $K \ge 1$. The torsion hypothesis on i^*N was used for the differential equation; the constant-sheaf vanishing was used after integration.

Cauchy estimates in the same norms: (D). Use the buffered ordinary testing charts of [Koi4, §4.2.3 and Remark 4.3]. They are fixed before the induction. In an ordinary source coordinate (z, w), the transverse testing discs have a common positive radius, and the derivatives with respect to z of the target coordinates z_k, or x_s, y_s, have a fixed bound C_0. The smaller covering still covers Z. This geometric choice of testing domains is the quantitative covering input from that paper, beyond the Stein-neighborhood existence recalled in Remark IV.5.9.

Fix an arrow $a\colon j \to k$ with ordinary testing source, and write $\phi_a(z, w)$ for its representative. For one target coefficient $F_{k,\ell}$, put $a_\ell := \max_j \|\mathsf{E}_j F_{j,\ell}\|_{V_j}$ and expand

$$\begin{aligned}(\mathsf{E}_k F_{k,\ell})(\phi_a(z,w)) = (\mathsf{E}_k F_{k,\ell})(\phi_a(z,0)) \\ + \sum_{q\ge 1} d_{a,\ell,q}(z) w^q.\end{aligned}$$

If the target is ordinary, its z-derivative is $F'_{k,\ell}(z_k)\partial z_k/\partial z$. If it is nodal, write $F_{k,\ell,+}, F_{k,\ell,-}$ for its two branches; the derivative is

$$F'_{k,\ell,+}(x_s)\frac{\partial x_s}{\partial z} + F'_{k,\ell,-}(y_s)\frac{\partial y_s}{\partial z}.$$

Both derivatives are bounded by a fixed multiple of the extended norm. On the testing base use the norm B_a given by the supremum plus the supremum of the z-derivative. Apply the

Cauchy formula in w, and differentiate that integral with respect to z. After fixing $M, R > 0$ it gives

$$\|f_{a,q}\|_{B_a} \le MR^q, \qquad \|d_{a,\ell,q}\|_{B_a} \le Ma_\ell R^q. \tag{342}$$

The first estimate is for the initial nonlinear transverse transition ($q \ge 2$); the second is for the positive-degree transport of an individual coefficient ($q \ge 1$). Multiplication by $t_{jk}^{1-\ell}$ changes neither estimate. This calculation is the differentiated Cauchy step in [Koi4, Lemma 4.7].

For the passage to the whole cochain, choose at a point of an overlap a member of the smaller covering containing that point. The cocycle identity expresses g_{jk} as the difference of two discrepancies on overlaps with that smaller patch. Differentiate this identity in the ordinary source coordinate. The transition factors are constant; only the fixed coordinate chain rule occurs. Taking suprema and summing over the finite covering gives

$$\|g\|_{1,\nu} \le L \max_a \|g|_a\|_{B_a}$$

with L independent of ν. In using a nodal overlap the equation is read from its ordinary end, as in the testing construction (Compare [Koi4, Lemmata 4.6–4.7]). This proves the linear restriction estimate in (D).

The two nonlinear terms are now exactly (316) with $r = 1$. The product inequality for (337) allows their norms, including the derivatives, to be estimated by the same products as in that proof. Before combining them, the bounds for their degree-ν terms come from the two series

$$\frac{LMR^2\left(t + \sum_{\mu=2}^{\nu-1} a_\mu t^\mu\right)^2}{1 - R\left(t + \sum_{\mu=2}^{\nu-1} a_\mu t^\mu\right)},$$
$$\frac{LMR\left(\sum_{\mu=2}^{\nu-1} a_\mu t^\mu\right)\left(t + \sum_{\mu=2}^{\nu-1} a_\mu t^\mu\right)}{1 - R\left(t + \sum_{\mu=2}^{\nu-1} a_\mu t^\mu\right)}.$$

These are the bounds obtained from (342).

In the torsion case (341) and $D_\nu = 3$ give bounded convergence cost. Otherwise put $d_k := d(\mathbb{I}_Z, N^k)$ and, increasing K once, set

$$e_k := \frac{\min\{1, d_k\}}{6K} \qquad (k \ge 1).$$

Then $C_\nu D_\nu \le e_{\nu-1}^{-1}$. Translation invariance gives $d_{p-q} \le d_p + d_q$, and $\min\{1, a+b\} \le \min\{1,a\} + \min\{1,b\}$ for $a, b \ge 0$ gives $e_{p-q} \le e_p + e_q$. The stated Diophantine lower bound gives the polynomial lower bound required by Theorem II.1.1, after increasing its fixed exponent. Hence the last convergence condition of (D) holds in either case.

Theorem IV.8.9 now gives the natural defining maps. At a node they have the form $x_s y_s v_s$ with v_s a unit, so they generate the original reduced Cartier ideal. Lemma IV.8.3 identifies the resulting comparison with the trivialization of the specified Ueda invertible sheaf. This proves the theorem. □

A rational curve with an ordinary cusp

The local algebra is different at a cusp. With normalization parameter ζ, it is $\mathbb{C}\{\zeta^2, \zeta^3\}$: the coefficient of ζ must vanish (Mini-Exercise III.2.168). We use the particular covering and local coordinates of [KUeh2, §§2.1 and 4.1].

THEOREM IV.8.15 (Koike–Uehara). *Let $\mathcal{X} = [(X, \mathcal{I}_Z)]$, where X is a non-singular complex surface and Z is a compact reduced rational curve whose only singularity is an ordinary cusp. If $N_{Z/X}$ is holomorphically trivial and $\mathcal{X}$ is of infinite type, a neighborhood of Z admits a holomorphic defining function for the reduced divisor Z. After shrinking, Z is a fiber of a local elliptic fibration.*

PROOF. See [KUeh2, Theorems 4.1 and 1.1 (i)]. We give the connection of its coefficient construction with (A)–(D), using the special covering and bounded primitive of that paper; the final fibration assertion is the cited Theorem 1.1 (i). Fix a trivial conormal frame, so the standard arrows are the identity. The allowed source changes fix the center and this frame; on ordinary patches they also preserve the chosen local projection.

On the cusp patch put $w_{\mathrm{sing}} = y^2 - x^3$, $|x| < \varepsilon^2$, $|y| < 2\varepsilon^3$, as in [KUeh2, §2.1]. The reference generator is $w_0 = e_0 w_{\mathrm{sing}}$, where the holomorphic unit e_0 is chosen to match the global conormal frame. A bounded holomorphic function on $U_0 = \{w_{\mathrm{sing}} = 0\}$ pulls back to

$$b(\zeta) = a(\zeta^2) + \zeta^3 c(\zeta^2), \qquad a(\zeta^2) = \frac{b(\zeta) + b(-\zeta)}{2}.$$

The absence of a linear term makes c holomorphic. The odd part has supremum at most $\|b\|_\infty$ on $|\zeta| < \varepsilon$, and division by ζ^3 and the maximum principle give $\sup_{|x|<\varepsilon^2} |c(x)| \le \varepsilon^{-3}\|b\|_\infty$. Therefore the linear extension

$$\mathsf{E}_0 b(x, y) := a(x) + y c(x), \qquad \|\mathsf{E}_0 b\|_{V_0} \le 3\|b\|_{U_0} \tag{343}$$

is bounded. This is [KUeh2, Lemma 2.6]. On the ordinary patches extend by their fixed projections. Thus (B) holds with $D_\nu = 3$ and supremum norms.

For (A), the equation $w_0(p) = F_0(p, u_0(p))$ in (313) gives $u_0 = w_0 v_0$, with $v_0 = 1$ modulo (w_0). For this realization calculation only, use coordinates (ξ, η) with $w_0 = \eta^2 - \xi^3$. Such coordinates are obtained from $(x, y) \mapsto (x e_0^{1/3}, y e_0^{1/2})$ on a sufficiently small cusp patch. The extension in (343) remains defined in the original (x, y)-coordinates. In the new coordinates choose the roots of v_0 which equal one on U_0, and set

$$k_0(\xi, \eta) := (\xi v_0^{1/3}, \eta v_0^{1/2}), \qquad w_0 \circ k_0 = v_0 w_0 = u_0. \tag{344}$$

This fixes the cusp curve pointwise. At its singular point the derivative is the identity; on its regular part the induced tangent and normal maps are the identity. Hence it is a biholomorphism along the center after shrinking. Transport it back to the original chart and use $h_0 = k_0^{-1}$; the ordinary patches use (z_j, u_j) as before. These formulas work for finite and convergent coefficient systems and also permit the constant homogeneous changes in Lemma IV.8.8. Here the kernel consists of global holomorphic functions on compact reduced

Z, hence constants. Infinite type gives the formal lifting condition, consistently with [KUeh2, Lemmata 3.2–3.3].

All coefficient bundles $N^{1-\nu}$ are trivial in the chosen comparison. The bounded primitive estimate on the fixed cusp covering is [KUeh2, Lemma 2.4]: a bounded exact cocycle has a primitive of supremum at most K_0 times its norm. Applying (343) to this same primitive gives the ambient estimate of its Lemma 4.2. Thus (C) and (B) have degree-independent costs K_0 and 3, respectively.

For (D), the special covering has ordinary charts along the boundary of the cusp patch. There the chosen coordinates are $(z_j, w_j) = (\sqrt{x}, w_0)$, with a fixed square-root branch. The functions x, y are holomorphic in these coordinates on the testing domains. The smaller annular base domains stay away from the cusp, and the transverse discs in [KUeh2, §4.1, (5)–(6)] have a fixed radius. Consequently the initial transitions and every transported extended coefficient admit the individual Cauchy estimates of (D). In particular a target coefficient $\mathsf{E}_0 b$ is bounded on V_0 by (343); applying the Cauchy integral in the ordinary w_j-coordinate bounds its positive transverse coefficients by a fixed geometric factor times that norm. The initial transverse error is zero on these cusp overlaps, since $w_j = w_0$. The same Cauchy reasoning applies to an ordinary target. The cocycle identity on the smaller covering transfers these bounds to all overlaps with a fixed finite factor, as in [KUeh2, Lemma 4.4].

Using the two expansions (316), the product estimate for the supremum norm now gives the quadratic majorant in (317). Since $C_\nu D_\nu \le 3K_0$, its convergence is the bounded-cost case of Theorem IV.8.9. The obtained defining functions have identity transition factors and glue to a holomorphic function with divisor Z. The cited fibration result completes the assertion. □

The two singular calculations keep the Cartier equation and its ideal, not just the normalization of the support. The extension procedure and the testing domains depend on that local algebra. For a general ideal the associated graded algebra need not be the symmetric algebra of its conormal sheaf, so these coefficient representations must be specified for the problem in question. For the use of [Koi4] we retain the corrections in [KUeh2, Appendix A]; its corrections to the contracted pair and to divisor growth scales concern the subsequent finite-type applications. In the reduced hypersurface cases just treated, the exact unitary defining system trivializes the specified Ueda invertible sheaf and makes $[Z]$ unitary flat near the center. This supplies the divisor-side input for the metric constructions of Chapter V.

Problem IV.8.16. *Find sufficient conditions for Ueda-type linearization near a compact connected reduced curve in a non-singular complex surface, allowing more general curve singularities.* □

Remark IV.8.17. Ueda's nodal cubic example gives a rational curve Z with one node in a non-singular projective surface X, such that $[Z]|_Z$ is topologically trivial but not unitary flat ([U2, Example 2]). Put $\mathcal{X} := [(X, \mathcal{I}_Z)]$, and let F be the extension of the specified $\mathbb{C}^*$-flat structure on $[Z]|_Z$ to a neighborhood V of Z, as in [U2, Remark 2]. Let $\mathcal{P}$ be the invertible sheaf on $\mathcal{X}$ induced by $\mathcal{O}_V([Z]|_V \otimes F^{-1})$. This is the same construction as the Ueda invertible sheaf in Definition IV.6.5, with the specified $\mathbb{C}^*$-flat structure in place of a

unitary flat structure. In this example,

$$\widehat{\mathcal{P}} \cong \mathcal{O}_{\mathcal{X}^{(\infty)}}, \qquad \mathcal{P} \ncong \mathcal{O}_{\mathcal{X}}$$

([U2, Remarks 2–3]). Thus the transverse defining functions can be made to have the prescribed locally constant transitions formally, but not holomorphically on any neighborhood of Z. In particular, $\mathcal{X}$ does not satisfy the FPLB. □

9 From transverse linearization to standard neighborhoods

We now discuss Arnol'd's sufficient condition for a holomorphic tubular neighborhood of an elliptic curve with unitary flat normal bundle. Unitary flatness alone does not suffice, as Remark IV.7.4 shows.

THEOREM IV.9.1 (Arnol'd). *Let $\mathcal{X}$ be an analytic neighborhood space with a model $(S, \mathcal{I}_C)$, where S is a non-singular complex surface and $C \subset S$ is an elliptic curve. Assume that $N_{C/S}$ is unitary flat, fix its unitary flat structure, and let d be an invariant distance on $\mathrm{Pic}^0(C)$ as in Chapter III, § 2.15. Assume that there are $c, \tau > 0$ such that*

$$d(\mathbb{I}_C, N_{C/S}^n) \geq cn^{-\tau} \qquad (n \geq 1),$$

where $\mathbb{I}_C$ is a holomorphically trivial line bundle on C. Then C has a holomorphic tubular neighborhood in S. Thus $\mathcal{X}$ is isomorphic to the analytic neighborhood space of the zero section in $N_{C/S}$, with the zero-section identification on reduced centers.

We use the proof through Ueda's theorem and a correction of the coordinate along C, following [KUeh3, Theorem 1.7 and §8.1, Lemmata 8.2–8.3]. Write $C = \mathbb{C}/\Lambda$ and choose a finite coordinate covering $\mathcal{U} = \{U_j\}_{j \in J}$ induced by local inverses of this quotient. Its coordinates satisfy $z_k = z_j + A_{kj}$, where $A_{kj} \in \mathbb{C}$ is constant on each connected component of $U_{jk} := U_j \cap U_k$. Split the overlaps into their connected components when necessary. The following lemma treats the second step, after the defining functions have already been chosen.

LEMMA IV.9.2. *Use the elliptic curve, normal bundle, and Diophantine condition of Theorem IV.9.1, and the covering just chosen. Suppose that V_j is a neighborhood of U_j in S and that (ζ_j, w_j) are coordinates on V_j, with $\zeta_j|_{U_j} = z_j$, $U_j = \{w_j = 0\}$, and $w_j = t_{jk} w_k$ on $V_{jk} := V_j \cap V_k$, where $t_{jk} \in \mathrm{U}(1)$ are the specified flat transition factors. After shrinking around C, there are holomorphic functions Z_j with $Z_j|_{U_j} = z_j$ and*

$$Z_k = Z_j + A_{kj} \quad \text{on } V_{jk}.$$

The pairs (Z_j, w_j) are coordinates along C.

PROOF. Choose an open covering $\mathcal{U}^* = \{U_j^*\}_{j\in J}$ of C such that the closure of U_j^* is a compact subset of U_j for each j. Use fixed transverse discs on the testing overlaps, as in (324) and (325). The coordinates are taken on larger patches before making these choices. The projection $(\zeta_j, w_j) \mapsto \zeta_j$ is used only to extend coefficient functions from U_j to V_j. Write $t_{kj} := t_{jk}^{-1}$ and expand

$$\zeta_k = \zeta_j + A_{kj} + \sum_{n\geq 1} f_{kj,n}(\zeta_j) w_j^n, \qquad w_k = t_{kj} w_j. \tag{345}$$

We seek

$$Z_j = \zeta_j - \sum_{n\geq 1} b_{j,n}(\zeta_j) w_j^n. \tag{346}$$

The coefficients $b_{j,n}$ will be holomorphic on U_j. The first power in this correction is w_j, rather than w_j^2.

Suppose that the coefficients of degrees less than n have been chosen so that the corresponding truncated Z_j satisfy the required identity modulo $\mathcal{I}_C^n$. For $m < n$, write

$$b_{k,m}(\zeta_k(\zeta_j, w_j)) = b_{k,m}(\zeta_j + A_{kj}) + \sum_{\ell\geq 1} b_{k,m;\ell}(\zeta_j) w_j^\ell.$$

The degree-n discrepancy of the truncated functions has coefficient

$$\begin{aligned} g_{kj,n} &:= f_{kj,n} - \sum_{m=1}^{n-1} t_{kj}^m b_{k,m;n-m}, \\ t_{kj}^n b_{k,n}(z_j + A_{kj}) - b_{j,n}(z_j) &= g_{kj,n}(z_j). \end{aligned} \tag{347}$$

Indeed, adding the n-th terms in (346) changes that discrepancy by $b_{j,n} - t_{kj}^n b_{k,n}(z_j + A_{kj})$. On triple overlaps the truncated differences telescope. Taking the first remaining coefficient shows that g_n is a Čech cocycle with values in $\mathcal{O}_C(N_{C/S}^{-n})$. In the j-frame the coboundary is precisely the left-hand side of (347), consistently with $\check{\delta} b_{jk} = b_k - b_j$.

The Diophantine condition makes $N_{C/S}$ non-torsion. The form dz on $\mathbb{C}$ descends to a nowhere vanishing form on C, so K_C is trivial. Serre duality and Lemma III.2.197 give

$$H^1(C, \mathcal{O}_C(N_{C/S}^{-n})) = 0 \qquad (n \geq 1)$$

(as in (241)). Hence (347) is solvable at every step. The vanishing of $H^0(C, \mathcal{O}_C(N_{C/S}^{-n}))$ also makes the solution unique.

We next control these solutions on the fixed covering. Choose $M, Q > 0$ large enough for the two Cauchy estimates

$$|f_{kj,n}| \leq MQ^n, \qquad |b_{k,m;\ell}| \leq Q^\ell \|b_{k,m}\|_{L^\infty(U_k)}$$

on $U_j \cap U_k^*$. The second inequality follows by applying the Cauchy integral in w_j to the single function $b_{k,m}(\zeta_k(\zeta_j, w_j))$; its argument lies in U_k on the chosen transverse disc. Thus,

if $\max_j \|b_{j,m}\|_{L^\infty(U_j)} \le a_m$ for $m < n$, then

$$|g_{kj,n}| \le MQ^n + \sum_{m=1}^{n-1} a_m Q^{n-m}$$

on those overlaps. At a point of $U_j \cap U_k$, choose U_ℓ^* containing the point. The cocycle identity expresses $g_{kj,n}$ as the difference of the discrepancies from j and k to ℓ, transported to the same frame. Since the transition factors have absolute value one, this transfers the estimate to the full overlap with a factor 2. Ueda's estimate (Theorem III.2.204) therefore gives, with a constant K independent of n,

$$d(\mathbb{I}_C, N^n_{C/S}) \max_j \|b_{j,n}\|_{L^\infty(U_j)} \le 2K \left(MQ^n + \sum_{m=1}^{n-1} a_m Q^{n-m} \right). \tag{348}$$

Here $d(\mathbb{I}_C, N^{-n}_{C/S}) = d(\mathbb{I}_C, N^n_{C/S})$ by invariance and symmetry of the distance.

Put $e_n := d(\mathbb{I}_C, N^n_{C/S})$ and define a formal series $A(s) = \sum_{n\ge1} a_n s^n$ by

$$\sum_{n\ge1} e_n a_n s^n = 2KQ \, \frac{(M + A(s))s}{1 - Qs}. \tag{349}$$

Its coefficients are non-negative and recursively determined. The inequality (348) gives the required coefficient domination by induction. To check convergence, put $B(s) := s + sA(s)$ and let $\widehat{B}(s) = s + \sum_{n\ge2} \widehat{B}_n s^n$ solve

$$\sum_{n\ge2} e_{n-1} \widehat{B}_n s^n = 2KQ(M+1) \frac{\widehat{B}(s)^2}{1 - Q\widehat{B}(s)}.$$

The lower bound on e_n has the form required by Theorem II.1.1, after increasing its exponent. For $p > q \ge 1$, the triangle inequality gives

$$e_{p-q} = d(N^q_{C/S}, N^p_{C/S}) \le e_p + e_q.$$

That theorem makes $\widehat{B}$ convergent. Multiplying (349) by s shows that the right-hand side for B is $2KQ(Ms + B(s) - s)s/(1 - Qs)$. For non-negative coefficients and $B(s) \succeq s$, this is bounded coefficientwise by $2KQ(M+1)B(s)^2/(1 - QB(s))$. Induction on the coefficients therefore gives $B \preceq \widehat{B}$. Consequently A has positive radius of convergence.

The series in (346) now converge on one smaller neighborhood of C. Every coefficient of $Z_k - Z_j - A_{kj}$ is zero, so these functions satisfy the exact transition identity. On C, $Z_j = z_j$, and the derivative of $(\zeta_j, w_j) \mapsto (Z_j, w_j)$ is triangular with diagonal entries $1, 1$. The inverse function theorem proves the last assertion. □

Proof of Theorem IV.9.1. The vanishing $H^1(C, \mathcal{O}_C(N^{-n}_{C/S})) = 0$ proved above, together with the one-step obstruction calculation of Proposition IV.8.1, gives infinite type. Apply the

rank-one case of Ueda-type linearization (Theorem IV.8.13) to obtain $w_j = t_{jk} w_k$ with the prescribed first-order part. Extend the additive coordinates z_j to ζ_j and apply Lemma IV.9.2. The resulting transitions are

$$(Z_k, w_k) = (Z_j + A_{kj}, t_{kj} w_j), \tag{350}$$

which are the transitions of $N_{C/S}$. They define a holomorphic map from a neighborhood of C to $N_{C/S}$, restricting to the zero-section identification and locally biholomorphic along C. It is injective on a smaller neighborhood: otherwise two distinct points with the same image could approach C; compactness gives limits on C, which must coincide, in contradiction to local injectivity there. Its image is open, so it gives the required isomorphism of analytic neighborhood spaces. □

For elliptic curves, Ogawa weakens the Diophantine condition in Theorem IV.9.1 to a Brjuno-type condition by combining the convergence criterion of Gong and Stolovitch with Ueda-type estimates for Čech coboundary equations (See [GS1] and [Og, Corollary 1.3 and §3]). Related linearization results for higher-dimensional complex tori and for projective space bundles over elliptic curves are obtained in [GS2] and [KoiS], respectively.

Remark IV.9.3 (Koike). A *cycle of rational curves* here is a connected compact reduced curve whose singularities are nodes, whose normalization is a disjoint union of projective lines, and whose dual graph is a cycle. This includes one rational curve with one node. Fix an orientation of the cycle. Trivializing a topologically trivial line bundle on the normalization and recording the identifications of its fibers at the nodes identifies $\mathrm{Pic}^0(C)$ with $\mathbb{C}^*$; write $t(L)$ for the parameter of a bundle L. The parameter is the product of the gluing constants in the chosen orientation ([Koi9, §2.1.2 and §2.3]).

Let C be a cycle of rational curves and let $i\colon C \hookrightarrow S$ and $i'\colon C \hookrightarrow S'$ be holomorphic embeddings in non-singular complex surfaces. Suppose that their normal line bundles are topologically trivial and, pulled back to the same curve C, have the same parameter

$$t(N_{i(C)/S}) = t(N_{i'(C)/S'}) = e^{2\pi\sqrt{-1}\theta},$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ satisfies $|n\theta - m| \geq c n^{-\tau}$ for some $c, \tau > 0$, all $n \geq 1$, and all $m \in \mathbb{Z}$. Then there is an isomorphism

$$[(S, \mathcal{I}_{i(C)})] \cong [(S', \mathcal{I}_{i'(C)})]$$

inducing $i' \circ i^{-1}$ on the centers. For the proof and the standard neighborhood construction, see [Koi9, Theorem 1.1 and §§2–4]. □

Remark IV.9.4 (Koike–Uehara). Let $C_0^+, C_0^- \subset \mathbb{P}^2$ be non-singular plane cubics. For each sign choose nine distinct points on $C_0^\pm$, let $S^\pm$ be the corresponding blow-up, and let $C^\pm \subset S^\pm$ be the strict transform. Suppose that an isomorphism $g\colon C^+ \to C^-$ identifies the normal bundles $N_\pm := N_{C^\pm/S^\pm}$ by $g^* N_- \cong N_+^*$, and that they satisfy the Diophantine condition in Theorem IV.9.1. That theorem provides standard tubular neighborhoods on both sides. Choose paired base charts $U_j^- = g(U_j^+)$ with $z_j^- = z_j^+ \circ g^{-1}$ and dual flat fiber

coordinates. On suitable annular parts, after a constant rescaling of the fiber coordinates, the gluing is

$$(z_j^+, w_j^+) \longmapsto (z_j^+, (w_j^+)^{-1}).$$

The first coordinate is read in the paired minus chart, so the induced map on the base is g. The duality of the transition factors makes these local maps compatible. Gluing the complements of smaller tubular neighborhoods in this way produces a K3 surface ([KUeh3, Theorem 1.5 and Proposition 2.1]). The construction also yields projective K3 surfaces ([KUeh1]). Moreover, for almost every period in the Lebesgue measure class on the K3 period domain, every marked K3 surface with that period is biholomorphic to a surface obtained by the Koike–Uehara construction ([Leq, Remark 22 and Corollary 26]). The use of neighborhood models in gluing constructions also appears in Tsuji's construction of complex structures on $S^3 \times S^3$ ([Ts1]). □

Chapter V

Line bundle semi-positivity and the geometry and dynamics of analytic neighborhood spaces

The positivity of line bundles on compact complex spaces has a rich theory, developed in close interaction with birational geometry. This is particularly apparent on projective manifolds and compact Kähler manifolds, where global sections, intersection numbers, and curvature provide different ways of studying a line bundle. We begin with a selective overview of this background. For a comprehensive treatment from the analytic point of view, see Demailly's *Analytic Methods in Algebraic Geometry* [De3]. Our main interest is the existence of a C^∞ Hermitian metric of semi-positive curvature on a specified line bundle. We then turn to effective divisors and to the geometry and function theory of their neighborhoods and complements, using the analytic neighborhood spaces of Chapter IV.

Effective divisors connect this question with flat neighborhoods, fibrations, and extension of holomorphic functions. We then study formal obstructions, finite-type growth, and criteria for the FPLB. The final section brings these viewpoints together for nine-point blow-ups of the projective plane.

1 Global positivity and numerical dimension

Let X be a compact Riemann surface of genus g. Lemma III.2.178 gives $\deg K_X = 2g-2$. Together with its proof in the case $g = 1$ and Theorem III.2.177, this shows that K_X^{-1} is positive for $g = 0$, K_X is holomorphically trivial for $g = 1$, and K_X is positive for $g \geq 2$. In higher dimensions, the classification of surfaces and the minimal model program place the canonical bundle and its powers at the center of birational classification. The questions concern positivity on suitable birational models, or positivity of its dual along fibers of contractions; these models may be singular. This motivates the section-growth invariants introduced next (See [De3]).

1.1 Positivity of line bundles and growth of sections

Growth of spaces of sections

Let Y be a non-empty compact reduced complex space and let $L \to Y$ be a holomorphic line bundle. We use L^m for its m-th tensor power, where $m \in \mathbb{Z}_{\geq 0}$ and L^0 is holomorphically trivial.

LEMMA V.1.1 (Serre–Siegel). *Let Y be a non-empty compact reduced complex space, let $L \to Y$ be a holomorphic line bundle, and put $n := \dim Y$. There is a constant $C > 0$ such that*

$$\dim H^0(Y, \mathcal{O}_Y(L^m)) \leq C(m+1)^n \qquad (m \in \mathbb{Z}_{\geq 0}).$$

PROOF. By desingularization (Remark III.2.72), there is a proper modification $\rho\colon \widetilde{Y} \to Y$, with $\widetilde{Y}$ non-singular, which is biholomorphic over Y_{reg}. Pullback of sections is injective: a section whose pullback is zero vanishes on the dense set Y_{reg}, and then on Y, since Y is reduced. The space $\widetilde{Y}$ is compact and has finitely many connected components, each of dimension at most n. It suffices to prove the estimate on each such component and add the bounds. A zero-dimensional component is a point and contributes one dimension for every m.

Assume now that Y is a compact complex manifold of pure dimension $n \geq 1$. Choose finitely many coordinate maps $\varphi_j\colon V_j \to \Delta^n_{r_{j,3}}$, $j = 1, 2, \ldots, N$, and holomorphic frames e_j of L on V_j, where $\Delta_r := \{\zeta \in \mathbb{C} \mid |\zeta| < r\}$. Mini-Exercise I.3.10 (*ii*) gives a covering by open sets A_j with $\overline{A_j} \subset V_j$. Choose $0 < r_{j,1} < r_{j,2} < r_{j,3}$ such that $\varphi_j(\overline{A_j}) \subset \Delta^n_{r_{j,1}}$, and put $U_{j,\ell} := \varphi_j^{-1}(\Delta^n_{r_{j,\ell}})$ for $\ell = 1, 2$ and $p_j := \varphi_j^{-1}(0)$. Thus the $U_{j,1}$ cover Y and $\overline{U_{j,1}}$ is a compact subset of $U_{j,2}$, while $\overline{U_{j,2}}$ is a compact subset of V_j. These choices are independent of m.

We first record a local estimate. For a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, put $|\alpha| := \sum_i \alpha_i$ and write $\partial^\alpha_z f$ for the corresponding holomorphic partial derivative of order $|\alpha|$, with $\partial^0_z f = f$. If f is holomorphic on a neighborhood of $\overline{\Delta^n_R}$ and $(\partial^\alpha_z f)(0) = 0$ for $|\alpha| < q$, where $q \geq 1$, then

$$\max_{\overline{\Delta^n_r}} |f| \leq (r/R)^q \max_{\overline{\Delta^n_R}} |f| \qquad (0 < r < R).$$

Indeed, for $z \in \overline{\Delta^n_r}$ the function $\zeta \mapsto f(\zeta z)$ is holomorphic on a neighborhood of $\overline{\Delta_{R/r}}$ and vanishes to order at least q at zero. Divide it by ζ^q and apply the one-variable maximum modulus principle at $\zeta = 1$.

Fix a C^∞ Hermitian metric h on L. There are $0 < a \leq b$ with $a \leq |e_j|_h \leq b$ on $\overline{U_{j,2}}$, simultaneously for all j. For $s \in H^0(Y, \mathcal{O}_Y(L^m))$, write $s|_{V_j} = (f_j \circ \varphi_j) e_j^{\otimes m}$ and put $M := \max_Y |s|_{h^m}$. Then $\max_{\overline{\Delta^n_{r_{j,2}}}} |f_j| \leq a^{-m} M$. Set $\theta := \max_j (r_{j,1}/r_{j,2}) \in (0,1)$. If all the f_j vanish to order at least q at zero, the local estimate gives

$$M \leq (b/a)^m \theta^q M.$$

Choose a positive integer c with $(b/a)\theta^c < 1$ and put $q_m := cm + 1$. Then $(b/a)^m \theta^{q_m} < 1$, also when $m = 0$. Thus simultaneous vanishing to order q_m at these fixed points forces $s = 0$.

Let $\mathfrak{m}_{Y,p_j}$ be the maximal ideal of $\mathcal{O}_{Y,p_j}$. The jet-evaluation map

$$H^0(Y,\mathcal{O}_Y(L^m)) \longrightarrow \bigoplus_{j=1}^{N} \mathcal{O}_Y(L^m)_{p_j}/\mathfrak{m}_{Y,p_j}^{q_m}\mathcal{O}_Y(L^m)_{p_j}$$

is therefore injective. In a coordinate system and a frame, the j-th quotient has the monomials z^α with $|\alpha| < q_m$ as a basis. Their number is $\binom{n+q_m-1}{n}$, by the monomial count of Chapter I, §1. Consequently

$$\dim H^0(Y,\mathcal{O}_Y(L^m)) \le N\binom{n+cm}{n} \le C(m+1)^n.$$

The initial reduction proves the assertion in the stated generality. □

Ample and big line bundles

Let X be a connected non-singular projective manifold and let $L \to X$ be a holomorphic line bundle. The bundle is *globally generated* if for every $x \in X$ some global holomorphic section is nonzero at x. A basis of $H^0(X,\mathcal{O}_X(L))$ then defines a holomorphic map $\Phi_1\colon X \to \mathbb{P}^{N_1}$, where $N_1 := \dim H^0(X,\mathcal{O}_X(L)) - 1$. We call L *very ample* if Φ_1 is an embedding, *ample* if some positive tensor power is very ample, and *semi-ample* if some positive tensor power is globally generated.

Definition V.1.2. Let X be a connected non-singular projective manifold of dimension $n \ge 1$ and let $L \to X$ be a holomorphic line bundle. We call L *big* if

$$\limsup_{m\to\infty} \frac{\dim H^0(X,\mathcal{O}_X(L^m))}{m^n} > 0.$$

The limsup is finite by Lemma V.1.1. □

Bigness describes maximal-order growth of sections, without requiring the sections of any positive power L^m to define an embedding of X into a projective space. For a line bundle L on a compact complex manifold, Kodaira's embedding theorem (Theorem III.2.136) identifies ampleness with positivity of L. For the converse, pull back the Fubini–Study metric from an embedding defined by L^m and take its m-th root. This is a criterion for the specified bundle; the existence of some positive line bundle is a criterion for projectivity of X.

Theorem V.1.3 (Nakai–Moishezon). *Let X be a connected non-singular projective manifold and let $L \to X$ be a holomorphic line bundle. Then L is ample if and only if*

$$\int_V c_1(L)^{\wedge \dim V} > 0$$

for every positive-dimensional irreducible closed subvariety $V \subset X$, including $V = X$.

Proof. See [Laz, Theorem 1.2.23 and its proof]. □

The integral is the pairing with the integration current T_V, as in the discussion following (204); the subvariety V need not be non-singular. On a curve this is the positive-degree criterion of Theorem III.2.177.

Section maps and Kodaira dimension

Let X be a connected non-singular projective manifold and let $L \to X$ be a holomorphic line bundle. For a positive integer m with $H^0(X, \mathcal{O}_X(L^m)) \neq 0$, let B_m be the common zero set of its sections. Choose a basis $s_0, s_1, \ldots, s_{N_m}$, and write $s_j = f_j e$ in a local frame e of L^m. The map

$$\Phi_m \colon X \setminus B_m \longrightarrow \mathbb{P}^{N_m}, \qquad x \longmapsto [f_0(x) : f_1(x) : \cdots : f_{N_m}(x)]$$

is independent of that frame. It is the map defined by the complete linear system of L^m, and is a rational map in this projective setting. The formula does not in general give a morphism on B_m.

Definition V.1.4. Let X be a connected non-singular projective manifold and let $L \to X$ be a holomorphic line bundle. If $H^0(X, \mathcal{O}_X(L^m)) = 0$ for all positive integers m, set $\kappa(L) := -\infty$. Otherwise put

$$\kappa(L) := \max_{m \colon H^0(X, \mathcal{O}_X(L^m)) \neq 0} \dim \overline{\Phi_m(X \setminus B_m)},$$

where B_m and Φ_m are the base locus and section map just defined and the closure is taken in the corresponding projective space. The *Kodaira dimension* of X is $\kappa(X) := \kappa(K_X)$. □

Changing the basis composes Φ_m with a projective linear automorphism. The standard comparison between section growth and section maps shows that L is big if and only if $\kappa(L) = \dim X$ (See [De3] and [Laz, Chapter 2]).

The Kodaira dimension $\kappa(X)$ is one of the basic invariants in the birational classification mentioned at the beginning of this section. Here we are interested also in line bundles other than K_X. Since determining the spaces of sections of their powers can be difficult, we next consider numerical information obtained from intersection numbers.

1.2 Nefness and numerical dimension

Nakai–Moishezon's criterion (Theorem V.1.3) expresses ampleness through intersection numbers. Nefness is the boundary condition below.

Definition V.1.5. Let X be a connected non-singular projective manifold and let $L \to X$ be a holomorphic line bundle. We call L *nef* if $(L.C) \geq 0$ for every irreducible curve $C \subset X$, where $(L.C) := \deg \nu_C^*(L|_C)$ and $\nu_C \colon \widetilde{C} \to C$ is its normalization (Theorem III.2.167). □

The normalization $\widetilde{C}$ is a compact Riemann surface, since a normal complex curve is non-singular (See [St, Tag 0B8Y and its proof]). Thus the degree on the right is the one defined in (196). Changing variables off the finite singular set shows that this agrees with the intersection pairing of Chapter III, §2.9.

Let X be a connected non-singular projective manifold. Define the *Néron–Severi group* by $\mathrm{NS}(X) := \mathrm{Image}\big(c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})\big)$. By Definition III.2.137, it is naturally isomorphic to $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. Put $\mathrm{NS}(X)_{\mathbb{R}} := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, and regard it as the real linear span of

the classes $c_1^{\mathbb{R}}(L)$ in $H^2(X,\mathbb{R})$. In particular, it is finite-dimensional. Two holomorphic line bundles L_1, L_2 are *numerically equivalent* if $(L_1.C) = (L_2.C)$ for every irreducible curve $C \subset X$. In the present projective setting, it is known that this is equivalent to $c_1^{\mathbb{R}}(L_1) = c_1^{\mathbb{R}}(L_2)$ (See [CJLO, Introduction]). The *ample cone* $\mathrm{Amp}(X)$ is the cone of positive real combinations of the classes $c_1^{\mathbb{R}}(L)$ of ample line bundles. The *nef cone* $\mathrm{Nef}(X)$ consists of the classes having nonnegative intersection with every irreducible curve. The *pseudoeffective cone* $\mathrm{PE}(X)$ is the closure of the cone generated by classes of bundles associated with effective divisors, and the *big cone* $\mathrm{Big}(X)$ is its interior. Bigness in Definition V.1.2 is equivalent to membership of the numerical class in this interior (See [De3]). The basic relations, with closure and interior in $\mathrm{NS}(X)_{\mathbb{R}}$, are

$$\overline{\mathrm{Amp}(X)} = \mathrm{Nef}(X), \qquad \mathrm{Int}\,\mathrm{Nef}(X) = \mathrm{Amp}(X),$$

$$\overline{\mathrm{Big}(X)} = \mathrm{PE}(X), \qquad \mathrm{Int}\,\mathrm{PE}(X) = \mathrm{Big}(X).$$

Moreover, $\mathrm{Amp}(X) \subset \mathrm{Big}(X)$ and $\mathrm{Nef}(X) \subset \mathrm{PE}(X)$. They are not four terms of a single inclusion chain: a nef bundle need not be big, and a big bundle need not be nef (See [Laz, §1.4] and [De3]).

Recall singular Hermitian metrics from Definition III.2.112. On a projective manifold, the description of the pseudoeffective cone also has an analytic interpretation. Positive real combinations of effective divisors give closed semi-positive currents. If their classes tend to $c_1^{\mathbb{R}}(L)$, their masses with respect to a fixed Kähler form remain bounded. A weakly convergent subsequence gives a closed semi-positive current in that class. The $\partial\overline{\partial}$-lemma then expresses it as the normalized curvature of a singular metric on L. The converse, and this equivalence with pseudoeffectivity, are discussed in [DPS2, §1, Definition 1.1 and Proposition 1.2].

Definition V.1.6. Let L be a holomorphic line bundle on a complex manifold. For singular Hermitian metrics h_1, h_2 with semi-positive curvature current, with plurisubharmonic weights φ_1, φ_2 in the same local frame, we say that h_1 is *less singular* than h_2 if $\varphi_2 \le \varphi_1 + C$ locally for some constant C. They have *equivalent singularities* if both inequalities hold. A singular Hermitian metric h on L is called a *minimal singular metric* if its curvature current is semi-positive and it is less singular than every singular Hermitian metric on L with semi-positive curvature current. □

The comparison is unchanged by a change of frame. It also makes sense on a reduced complex space for metrics specified by plurisubharmonic local weights.

Proposition V.1.7 ([DPS2, Theorem 1.5]). *Let X be a connected compact complex manifold and let $L \to X$ admit a singular Hermitian metric with semi-positive curvature current. Then L has a minimal singular metric, unique up to equivalence of singularities.*

Proof. Fix a C^∞ Hermitian metric h_∞ and put $\theta := \sqrt{-1}\Theta_{h_\infty}/(2\pi)$. Write another metric as $h_\infty e^{-\chi}$; its normalized curvature is $\theta + dd^c\chi$. Consider

$$\chi_{\min} := \left(\sup\{\chi \mid \theta + dd^c\chi \ge 0,\ \chi \le 0\}\right)^*, \qquad h_{\min} := h_\infty e^{-\chi_{\min}},$$

where the functions in the supremum are upper semi-continuous local sums of a plurisubharmonic function and a smooth function, are not identically $-\infty$, and the star denotes upper semi-continuous regularization. The family is nonempty by the hypothesis and normalization by an additive constant. Locally add a smooth potential of θ. The upper-envelope theorem (Proposition III.2.86) then shows that $\theta + dd^c\chi_{\min} \geq 0$; the bound $\chi \leq 0$ and one member of the family ensure that this is a singular metric. For any other weight ψ, compactness gives $\sup_X \psi < \infty$. The normalized weight $\psi - \sup_X \psi$ belongs to the family, so $\psi \leq \chi_{\min} + \sup_X \psi$. This proves minimality. Two minimal singular metrics are less singular than one another, proving the last assertion. □

Remark V.1.8. A singular Hermitian metric is said to have *analytic singularities* if, near each point, its local weight has the form $\alpha \log \sum_{j=1}^N |f_j|^2 + O(1)$, where $\alpha > 0$, $f_1, f_2, \ldots, f_N$ are holomorphic functions on that neighborhood, and the remainder is locally bounded. Even for a big line bundle on a non-singular projective manifold, a minimal singular metric need not have analytic singularities (See [Koi1, Theorem 1.1]; see also [HoK, Theorem 1.3 and §6.1] for descriptions using holomorphic tubular neighborhoods).

Non-finite generation of the section ring does not force minimal singularities to be non-analytic. Zariski's example in [Koi2, Theorem 1.1 and Example 4.3] is nef and big and has a non-finitely generated section ring, but admits a C^∞ Hermitian metric with semi-positive curvature. This is a minimal singular metric, with locally bounded weights, which have this form with $f_1 = 1$. □

Definition V.1.9. Let X be a connected non-singular projective manifold of dimension $n \geq 1$, let $L \to X$ be nef, and fix a Kähler form ω on X. Its *numerical dimension* is

$$\mathrm{nd}(L) := \max\left\{ k \in \{0, 1, \ldots, n\} \;\middle|\; \int_X c_1(L)^{\wedge k} \wedge \omega^{\wedge(n-k)} > 0 \right\}.$$

For nef L, this number is independent of ω. A nef bundle is *abundant* if $\kappa(L) = \mathrm{nd}(L)$. □

For nef L one has $\kappa(L) \leq \mathrm{nd}(L) \leq n$, and bigness is equivalent to $\mathrm{nd}(L) = n$. If L is semi-ample, both dimensions equal the dimension of the image of the map defined by a globally generated positive power (See [De3]; for the numerical and abundance terminology, see also [Fuj, Definition 2.2]).

Remark V.1.10. For a connected non-singular projective manifold X, the classical *abundance conjecture* predicts that nefness of K_X implies its semi-ampleness. Equivalently, it predicts $\kappa(K_X) = \mathrm{nd}(K_X)$ when K_X is nef. The implication from this equality to semi-ampleness is a theorem about the canonical bundle, not an equivalence built into the definition of an arbitrary abundant line bundle (See [Fuj, Corollary 2.5], with zero boundary and base a point). □

Remark V.1.11. Let now X be a connected compact Kähler manifold and fix a Kähler form ω. A holomorphic line bundle L is *nef* if, for every $\varepsilon > 0$, there is a C^∞ Hermitian metric h_ε on L with

$$\frac{\sqrt{-1}}{2\pi}\Theta_{h_\varepsilon} \geq -\varepsilon\omega.$$

This does not depend on ω, and agrees with the curve-intersection definition when X is projective. Pseudoeffectivity means that $c_1^{\mathbb{R}}(L)$ has a closed semi-positive current representative (See [DPS2, §1]). For a nef line bundle in this compact Kähler setting, the same intersection formula in Definition V.1.9 defines $\mathrm{nd}(L)$; its values are nonnegative and independent of ω (See [De3]). The theorem of Demailly–Păun states that the Kähler cone is one connected component of the set of real $(1,1)$-cohomology classes α with $\int_V \alpha^{\wedge \dim V} > 0$ for every positive-dimensional irreducible analytic subset $V \subset X$ (See [DP, Main Theorem 0.1]). □

1.3 From numerical dimension to semi-positivity

Let X be a connected non-singular projective manifold and let $L \to X$ be a holomorphic line bundle. Recall that L is *semi-positive* if there is a C^∞ Hermitian metric h with $\sqrt{-1}\Theta_h \geq 0$.

LEMMA V.1.12. *Let X be a connected non-singular projective manifold and let $L \to X$ be a holomorphic line bundle. If L is semi-ample, then it is semi-positive. If L is semi-positive, then it is nef.*

PROOF. Suppose first that L^m is globally generated for a positive integer m. A basis of $H^0(X, \mathcal{O}_X(L^m))$ defines a holomorphic map $\Phi\colon X \to \mathbb{P}^N$. Evaluation of sections identifies L^m with $\Phi^* L_{\mathrm{taut}}^{-1}$, as in the proof of Theorem III.2.136. The Fubini–Study quotient metric on L_{taut}^{-1} has positive curvature when $N \geq 1$ (Mini-Exercise III.2.26); when $N = 0$, use the constant metric on the quotient line. Its pullback gives a C^∞ Hermitian metric h_m of semi-positive curvature on L^m. Taking its positive m-th root in local frames of L gives a metric h with $h^m = h_m$ and $\sqrt{-1}\Theta_h = m^{-1}\sqrt{-1}\Theta_{h_m} \geq 0$. This proves the first assertion.

Now suppose that h is a C^∞ Hermitian metric of semi-positive curvature on L. For an irreducible curve $C \subset X$, let $\nu_C\colon \widetilde{C} \to C$ be its normalization and $\iota_C\colon C \hookrightarrow X$ its inclusion. The degree formula (196) gives

$$(L.C) = \int_{\widetilde{C}} (\iota_C \circ \nu_C)^* \left(\frac{\sqrt{-1}}{2\pi} \Theta_h \right) \geq 0.$$

Thus L is nef by Definition V.1.5. □

EXAMPLE V.1.13. Let X be an elliptic curve and let L be a unitary flat line bundle for which no positive tensor power is trivial. It is semi-positive and $\mathrm{nd}(L) = 0$, but no positive power has a nonzero section (Lemma III.2.197). Thus $\kappa(L) = -\infty$ and L is not semi-ample. □

EXAMPLE V.1.14. Let B be an elliptic curve and let $E \to B$ correspond to a non-split extension

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \mathcal{O}_B(E) \longrightarrow \mathcal{O}_B \longrightarrow 0.$$

Such an extension is specified by a nonzero class in $\mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_B, \mathcal{O}_B) \cong H^1(B, \mathcal{O}_B)$. Indeed, $\mathrm{Hom}_{\mathcal{O}_B}(\mathcal{O}_B, -) = \Gamma(B, -)$, so their right derived functors agree (Definition II.3.61). The Dolbeault isomorphism (Theorem III.2.116) and Example III.2.127 show that $\dim H^1(B, \mathcal{O}_B) =$

1. All nonzero classes differ by a scalar, which can be absorbed by changing the identification of the subbundle $\mathcal{O}_B$. Thus the middle bundle E is determined up to holomorphic isomorphism.

Put $X := \mathbb{P}(E)$. Here $\mathbb{P}(E) \to B$ is the holomorphic projective space bundle obtained by taking the hyperplanes in each fiber of E and gluing through its holomorphic local trivializations. The kernels of the quotient $E \to \mathbb{I}_B$ define a section $Z \subset X$. This is the classical example of Serre: $X \setminus Z$ is biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ and hence Stein, but is not affine with the algebraic structure induced from the projective surface X. The normal bundle $N_{Z/X}$ is trivial, so $(Z.Z) = 0$. The bundle $[Z]$ is nef: its degree on Z is zero and its intersection with every other irreducible curve is nonnegative. Nevertheless $[Z]$ is not semi-positive. Its non-semi-positivity was proved by Yau [Yau, p. 228] and later rediscovered by Demailly–Peternell–Schneider [DPS1, Example 1.7] (See also [Tos, Theorem 2.1 and the following discussion]). The latter authors also determine its singular Hermitian metrics with semi-positive curvature current; in particular, the canonically attached singular Hermitian metric $h_{\mathrm{sing},Z}$ of Example III.2.140 is a minimal singular metric. An alternative proof using Ueda theory is given in [Koi3, Theorem 1.1 and Example 3.5]; it is the application of Theorem V.4.6 and Lemma V.4.4 discussed below. □

Example V.1.15. Let B be a compact Riemann surface of genus two and choose a nonsplit extension

$$0 \longrightarrow \mathcal{O}_B(K_B) \longrightarrow \mathcal{O}_B(E) \longrightarrow \mathcal{O}_B \longrightarrow 0.$$

For $\pi\colon X := \mathbb{P}(E) \to B$, let Z be the section defined by the quotient $E \to \mathbb{I}_B$. Then $N_{Z/X} \cong K_B^{-1}$, using $\pi|_Z$ to identify Z with B. The bundle $L := [Z] \otimes \pi^* K_B$ is nef: $L|_Z$ is trivial, whereas its intersection with every other irreducible curve is nonnegative. Moreover $(L.L) = 2$, so L is big. It is not semi-positive (See [Koi7, Example 4.1 and the proof of Theorem 1.2], where the first obstruction of Theorem 1.4 is computed). □

Remark V.1.16. Kim constructed a nef and big line bundle which is not semi-positive on a projective threefold ([Kim, Example 2.14]). For surfaces, Filip and Tosatti proposed Grauert's construction as a candidate ([FT1, Problem 2.2]). The preceding Example V.1.15 is Koike's variant of Grauert's construction ([Koi7, Theorem 1.2 and Example 4.1]). Zhang subsequently proved non-semi-positivity for the original Grauert example ([Zha, Theorem 1.2]). Both proofs use a nonzero first Ueda-type obstruction class of the line bundle along the curve, contrary to the vanishing under semi-positivity in Theorem V.3.2. □

For a nef line bundle L on a connected compact Kähler manifold X of dimension $n \geq 1$, fix a Kähler form ω. The two extreme numerical dimensions have particularly concrete meanings. If $\mathrm{nd}(L) = 0$, then $\int_X c_1(L) \wedge \omega^{\wedge(n-1)} = 0$. A nef class has a closed semi-positive current representative (See [DPS2, §1]); this current has zero mass and is therefore zero. Thus $c_1^{\mathbb{R}}(L) = 0$, and the unitary-flatness criterion from the $\partial\overline{\partial}$-lemma (Proposition III.2.131) makes L unitary flat. Conversely a unitary flat bundle has numerical dimension zero. At the other extreme, $\mathrm{nd}(L) = n$ means $\int_X c_1(L)^{\wedge n} > 0$. The theorem of Demailly–Păun gives a Kähler current in this nef class, and the resulting line bundle is big, with bigness understood by the same section-growth condition as in Definition V.1.2 (See [DP, Theorem 0.4 and the

discussion following it] and [De3]). We next study the intermediate case $\mathrm{nd}(L) = 1$ when L is associated with an effective divisor.

2 Semi-positivity of effective line bundles and neighborhood geometry

2.1 Effective divisors and plurisubharmonic functions

Let D be an effective Cartier divisor on a complex manifold X, and put $Y := \operatorname{Supp} D$. Its canonical section turns the metric question into a question about functions on $X \setminus Y$. We retain the multiplicities of D throughout. Let σ_D be the canonical holomorphic section of $[D]$ (Proposition III.2.65). Choose local defining functions f_j for D on open subsets V_j and frames e_j of $[D]$ with $\sigma_D = f_j e_j$. For a C^∞ Hermitian metric h on $[D]$, set

$$\psi_h := -\log |\sigma_D|_h^2 \quad \text{on } X \setminus Y, \qquad \varphi_j := \psi_h + \log |f_j|^2 = -\log |e_j|_h^2. \tag{351}$$

The last expression shows that φ_j extends as a real-valued C^∞ function across $Y \cap V_j$. The curvature formula (149) gives

$$\sqrt{-1}\partial\overline{\partial}\psi_h = \sqrt{-1}\Theta_h \quad \text{on } X \setminus Y, \qquad dd^c\varphi_j = \frac{\sqrt{-1}}{2\pi}\Theta_h \quad \text{on } V_j. \tag{352}$$

Thus h has semi-positive curvature if and only if ψ_h is plurisubharmonic on $X \setminus Y$. In this implication across Y, use the C^∞ extension φ_j: its Levi form is semi-positive on the dense complement of Y, hence also on Y by continuity.

Conversely, let ψ be a real-valued C^∞ function on $X \setminus Y$ such that each $\psi + \log |f_j|^2$ extends to a C^∞ function φ_j on V_j. Then $|e_j|_h^2 := e^{-\varphi_j}$ defines a metric on $[D]$. In fact, writing $g_{jk} := f_j/f_k$, one has $e_k = g_{jk} e_j$ and $\varphi_k = \varphi_j - \log |g_{jk}|^2$, which is exactly the metric transformation rule. This gives a bijection between such functions ψ and C^∞ Hermitian metrics on $[D]$.

Proposition V.2.1 (Locality of semi-positivity for an effective divisor). *Let X be a complex manifold and let D be an effective Cartier divisor with compact support Y. Then $[D]$ is semi-positive on X if and only if $[D]|_V$ is semi-positive on some open neighborhood V of Y. More precisely, a C^∞ Hermitian metric h of semi-positive curvature on $[D]|_V$ extends, after restriction to a smaller neighborhood of Y, to a C^∞ Hermitian metric of semi-positive curvature on $[D]$ over X.*

Proof. The restriction implication is immediate. We prove the converse (See also [Koi14, Lemma 2.8 and Remark 2.10]). If Y is empty, σ_D is a global frame and the metric for which it has norm 1 has zero curvature. Assume $Y \neq \emptyset$. Fix a metric h as in the statement and let $\psi = -\log |\sigma_D|_h^2$ on $V \setminus Y$. Choose an open set W containing Y whose closure is a compact subset of V. The boundary ∂W is compact and disjoint from Y. Consequently there is a

relatively compact open neighborhood A of ∂W in $V \setminus Y$ on which ψ is bounded above. Fix $\varepsilon > 0$ and choose $C \in \mathbb{R}$ so large that $\psi < C - 2\varepsilon$ on A. On $X \setminus Y$ define

$$\Psi := \begin{cases} \mathrm{RegMax}_\varepsilon(\psi, C) & \text{on } W \setminus Y, \\ C & \text{outside } W. \end{cases}$$

Here $\mathrm{RegMax}_\varepsilon$ is the function of Example I.4.9, with a fixed smoothing kernel. The two expressions agree near ∂W. Convexity and monotonicity in each variable show that Ψ is C^∞ and plurisubharmonic on $X \setminus Y$.

The function $|\sigma_D|_h^2$ extends continuously to V, vanishes on Y, and is positive off Y. Compactness of Y therefore gives a smaller open neighborhood $W' \subset W$ of Y on which $\psi > C + 2\varepsilon$ away from Y. Thus $\Psi = \psi$ on $W' \setminus Y$. In particular, $\Psi + \log|f_j|^2$ has the required C^∞ extension near Y. Away from Y, σ_D itself is a frame. The metric–function correspondence above constructs a global C^∞ Hermitian metric of semi-positive curvature on $[D]$, which agrees with h on W'. □

For the neighborhood W chosen in the proof, the constructed metric has zero curvature outside W. Its logarithmic terms retain the multiplicities of D.

A Hermitian metric of semi-positive curvature on $[D]$ gives the plurisubharmonic function $-\log|\sigma_D|_h^2$ off its support. Near D it is analogous to $-\log(-r)$ for a defining function $r < 0$ of a domain. Nearby regular levels of this function are real hypersurfaces, on which the Levi form can be studied with the methods of Chapter III.

2.2 Semi-positivity on analytic neighborhood spaces

Let $\mathcal{X}$ be an analytic neighborhood space and let $\mathcal{L}$ be an invertible $\mathcal{O}_\mathcal{X}$-module. By a C^∞ Hermitian metric on $\mathcal{L}$ we mean a germ, along $Z(\mathcal{X})$, of a C^∞ Hermitian metric on a representative line bundle in an analytic neighborhood model. Two representatives define the same metric if they agree after restriction to a smaller neighborhood, using the specified identification of their invertible sheaves.

For a reduced complex space, the meaning of a C^∞ metric can be given in local frames: its weights $-\log|e_j|^2$ are real-valued C^∞ functions in the sense of Definition III.2.87. We say that this metric has *semi-positive* (resp. *positive*) curvature when its weights are C^∞ plurisubharmonic (resp. strictly plurisubharmonic) in that same sense. In particular, a weight must locally have an extension to an ambient complex Euclidean open set which is simultaneously C^∞ and plurisubharmonic (resp. strictly plurisubharmonic). Changing a frame adds the logarithm of the squared modulus of a holomorphic unit, hence a pluriharmonic function. These conditions therefore do not depend on the frames. For a non-singular model they are exactly the curvature conditions used above.

Definition V.2.2. Let $\mathcal{X}$ be an analytic neighborhood space and let $\mathcal{L}$ be an invertible $\mathcal{O}_\mathcal{X}$-module. We call $\mathcal{L}$ *semi-positive* (resp. *positive*) if, on some open neighborhood of the center in a model of $\mathcal{X}$, a representative of $\mathcal{L}$ has a C^∞ Hermitian metric of semi-positive (resp. positive) curvature as just described. □

This definition is independent of the model. Indeed, Lemma IV.1.7 represents a model isomorphism by a biholomorphism after shrinking near the center. Pulling back a local metric preserves the stated conditions on its weights, and the inverse biholomorphism gives the converse. The sheaf has a representative by Definition IV.1.19; Proposition III.2.58 realizes an invertible representative as the sheaf of sections of a holomorphic line bundle.

For an effective divisor with support Z, the corresponding ψ is a germ of a real-valued C^∞ function on punctured neighborhoods $V \setminus Z$. In a local Cartier equation f_j, the function $\psi + \log|f_j|^2$ must extend as a C^∞ weight.

Definition V.2.3. Let $\mathcal{X}$ be an analytic neighborhood space. An *effective Cartier divisor* on $\mathcal{X}$ is represented by an effective Cartier divisor D on an analytic neighborhood model $(X, \mathcal{I})$ of $\mathcal{X}$. Two representatives are identified by model isomorphisms after restriction near the center. It is *supported on the center* if $\operatorname{Supp} D = Z(\mathcal{X})$ after this identification. The associated invertible sheaf is induced by $\mathcal{O}_X(D)$ and is denoted by $\mathcal{O}_{\mathcal{X}}(D)$. □

Equivalently, the divisor is specified by an invertible ideal locally generated by a holomorphic non-zero-divisor. Choose a model $(X, \mathcal{I})$ and a representative D supported on $Z := Z(X, \mathcal{I})$. Its invertible ideal $\mathcal{J}_D := \mathcal{O}_X(-D)$ and the model ideal $\mathcal{I}$ satisfy $\sqrt{\mathcal{J}_D} = \sqrt{\mathcal{I}}$ near Z. Equality of these radicals does not require $\mathcal{J}_D = \mathcal{I}$, nor does it remove the multiplicities of D. Semi-positivity of this divisor means semi-positivity of its associated invertible sheaf $\mathcal{O}_{\mathcal{X}}(D)$ in Definition V.2.2.

Problem V.2.4. *Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center, and let D be an effective Cartier divisor on $\mathcal{X}$ supported on $Z(\mathcal{X})$. Characterize the semi-positivity of $\mathcal{O}_{\mathcal{X}}(D)$ by its neighborhood geometry, or by the existence and behavior of holomorphic and plurisubharmonic functions there.* □

The elliptic suspensions of Chapter IV give a concrete comparison problem: Proposition IV.7.2 relates the local dynamics to triviality of the Ueda invertible sheaf. Here the additional question is how metrics of semi-positive curvature determine such neighborhood data.

In particular, for a complex manifold X and an effective Cartier divisor D with compact support Z, the natural model gives

$$\mathcal{X} = [(X, \mathcal{O}_X(-D))], \qquad \mathcal{O}_{\mathcal{X}}(D).$$

Proposition V.2.1 and Definitions V.2.2–V.2.3 identify semi-positivity of $[D]$ on X with semi-positivity of this invertible sheaf on $\mathcal{X}$. Thus even a metric question on all of X can, for such a divisor, be answered by information on its analytic neighborhood space.

2.3 Effective nef divisors of numerical dimension one

Let X be a connected compact Kähler manifold of dimension n, and fix a Kähler form ω on X. Nefness and numerical dimension are as in §V.1, with the compact Kähler interpretation of Definition V.1.9.

THEOREM V.2.5 ([Koi14, Theorem 1.1]). *Let X be a connected compact Kähler manifold and let D be a non-zero effective divisor on X. Assume that $[D]$ is nef and $\mathrm{nd}([D]) = 1$, and put $Y := \operatorname{Supp} D$. The following conditions are equivalent.*

(*i*) *The line bundle $[D]$ is semi-positive.*

(*ii*) *There is an open neighborhood V of Y such that $[D]|_V$ is unitary flat.*

(*iii*) *There are open subsets V_j covering Y and holomorphic functions s_j on V_j such that $s_j^{-1}(0) = Y \cap V_j$, and each quotient s_j/s_k extends to a holomorphic unit of modulus 1 on $V_j \cap V_k$.*

The multiplicities in (*iii*) are not prescribed to be those of D. We first prepare the numerical comparison needed to recover them, and the fibration result used in the proof.

LEMMA V.2.6. *Let X be a connected compact Kähler manifold of dimension $n \geq 2$, fix a Kähler form ω, and let D, D' be non-zero effective divisors with the same connected support Y. Suppose that*

$$\int_{Y_\nu} c_1([D]) \wedge \omega^{\wedge(n-2)} = \int_{Y_\nu} c_1([D']) \wedge \omega^{\wedge(n-2)} = 0$$

for every irreducible component Y_ν of Y. Then $D' = aD$ for some $a \in \mathbb{Q}_{>0}$.

PROOF. Write $D = \sum_{\nu=1}^N m_\nu Y_\nu$ and $D' = \sum_{\nu=1}^N m'_\nu Y_\nu$, and set

$$A = (a_{\nu\mu})_{1\leq\nu,\mu\leq N}, \qquad a_{\nu\mu} := \int_X c_1([Y_\nu]) \wedge c_1([Y_\mu]) \wedge \omega^{\wedge(n-2)}.$$

For $\nu \neq \mu$, positivity of the intersection multiplicities and of ω gives $a_{\nu\mu} \geq 0$, with strict inequality exactly when $Y_\nu \cap Y_\mu \neq \emptyset$. Thus A is real symmetric, with nonnegative off-diagonal entries and connected incidence graph. The hypotheses say $Am = Am' = 0$ for the positive multiplicity vectors m, m'. Proposition II.2.2 (*ii*) makes zero the largest eigenvalue. Lemma II.2.1 makes its eigenspace one-dimensional: two independent eigenvectors would have a nonzero linear combination with a zero coordinate. Thus $m' = am$, where $a > 0$ and integral multiplicities give $a \in \mathbb{Q}$. □

LEMMA V.2.7. *Let X be a connected compact Kähler manifold of dimension at least two. Suppose that Y_1, Y_2, Y_3 are pairwise disjoint, connected non-singular compact hypersurfaces whose associated line bundles have the same integral first Chern class. Then there are a compact Riemann surface R and a surjective holomorphic map $\pi\colon X \to R$ with connected fibers such that at least one Y_j is the reduced support of a fiber.*

PROOF. We use Neeman's argument separating the injective and noninjective cases of $H^1(X, \mathcal{O}_X) \to H^1(Y_1, \mathcal{O}_{Y_1})$ (See [Nee, pp. 109–110] for the surface case, and [Tot, Theorem 2.1] for the projective formulation. See also the compact Kähler argument in the proof of [Koi11, Lemma 3.6, p. 1220]). In the first case the restriction homomorphism on Pic^0 has finite kernel. The class of $[Y_2 - Y_3]$ belongs to this kernel: its integral first Chern class is

zero, and its restriction to Y_1 is trivial. Thus some positive integer k gives linearly equivalent divisors kY_2 and kY_3. Their canonical sections define a pencil without base points, hence a holomorphic map to $\mathbb{P}^1$. In the second case, disjointness and the equality of Chern classes give $c_1(N_{Y_1/X}) = c_1([Y_2]|_{Y_1}) = 0$. The Albanese-quotient argument therefore supplies a map to a curve having Y_1 as a fiber support (See [CLPT, Proposition 2.7]). Stein factorization gives connected fibers; the connectedness of the hypersurface already obtained as a fiber support ensures that it remains a whole fiber support. □

Proof of Theorem V.2.5. We follow [Koi14, §§3–4], using the two Lemmata above for the global fibration and multiplicity arguments. If $\dim X = 1$, choose disjoint coordinate discs around the finitely many points of Y. They give (ii) and (iii), and Proposition V.2.1 gives (i). Assume henceforth $n := \dim X \geq 2$ and fix a Kähler form ω.

Write $D = \sum_\nu m_\nu Y_\nu$. Nefness gives $\int_{Y_\nu} c_1([D]) \wedge \omega^{\wedge(n-2)} \geq 0$, and their weighted sum is $\int_X c_1([D])^{\wedge 2} \wedge \omega^{\wedge(n-2)} = 0$. Hence each term is zero. This calculation applies for every Kähler form. Assume (iii). The quotients s_j/s_k glue the zero divisors of s_j to an effective divisor D' supported on Y, with a unitary flat bundle near Y. On each connected component, both divisors satisfy Lemma V.2.6. Choose positive integers a, b with $aD = bD'$ there. A flat metric on $[D']^b$ transfers to $[D]^a$, and its positive a-th root gives a flat metric on $[D]$. This proves (ii). Condition (ii) implies (i) by Proposition V.2.1; it implies (iii) by writing σ_D in holomorphic frames compatible with the unitary flat structure in (ii).

It remains to prove $(i) \Rightarrow (ii)$. We can work with one connected component of Y. Indeed, let D_0 be the corresponding summand of D. Near that component the other summands are canonically trivial, so $[D_0]$ inherits a Hermitian metric of semi-positive curvature. Proposition V.2.1 extends it to X. The numerical calculation just made gives $\int_{Y_\nu} c_1([D_0]) \wedge \omega^{\wedge(n-2)} = 0$ on its components, and thus $\int_X c_1([D_0])^{\wedge 2} \wedge \omega^{\wedge(n-2)} = 0$. We replace D by D_0, write its connected support again as Y, and fix a Hermitian metric h of semi-positive curvature on $[D]$ over X. Put

$$\eta := |\sigma_D|_h^2, \qquad \psi := -\log \eta \quad \text{on } X \setminus Y.$$

All sufficiently small sublevels of η considered below are contained in one fixed relatively compact neighborhood of Y.

The following elementary features also hold when Y is singular and D has multiplicities. Near a point of Y, write $\eta = |s_j|^2 e^{-\varphi}$, where s_j is holomorphic and φ is C^∞. Apply the Łojasiewicz gradient inequality (Theorem III.2.57) to $|s_j|^2$ at that point. Since $\|\partial |s_j|^2\| = |s_j| \, \|\partial s_j\|$ in the coordinate norm, it gives $\|\partial s_j/s_j\| \geq c|s_j|^{2\theta-2}$ off Y, for some $c > 0$ and $0 < \theta < 1$ after shrinking. The right-hand side tends uniformly to infinity as s_j tends to zero, whereas $\partial\varphi$ stays bounded on a smaller coordinate neighborhood with compact closure. Hence $\partial\psi = -\partial s_j/s_j + \partial\varphi$ does not vanish there. Consequently $d\eta \neq 0$ sufficiently near Y, off Y (This is the argument in [Koi14, Lemma 3.2], applied to $|s_j|^2$). Compactness gives a common neighborhood on which this holds. The sublevel neighborhoods of η are relatively compact and form a neighborhood basis of Y. They are connected when sufficiently small: every component must meet Y, since otherwise η would attain a positive interior minimum there; connectedness of Y then excludes more than one component. Removing a complex

analytic hypersurface from a connected complex manifold does not disconnect it. The proper submersion of this punctured neighborhood to a sufficiently small interval $(0, \varepsilon)$, given by η, is a locally trivial fibration by Ehresmann's fibration theorem (Theorem I.8.7). Since the interval is contractible, its compact fibers are connected as well. In particular,

$$M_s := \{\psi = s\}, \qquad \Omega_s := Y \cup \{\psi > s\}$$

are, for large s, a connected compact smooth real hypersurface and a relatively compact neighborhood of Y, respectively. All sets in these formulas are taken in the fixed small neighborhood when necessary.

We now compute the boundary integral. Orient M_s as the boundary of the sublevel side $\{\psi < s\}$. Set

$$F_\omega(s) := \int_{M_s} d^c\psi \wedge dd^c\psi \wedge \omega^{\wedge(n-2)}. \tag{353}$$

Semi-positivity of $dd^c\psi$ gives $F_\omega(s) \geq 0$. For $s_1 < s_2$, Stokes' theorem gives

$$F_\omega(s_2) - F_\omega(s_1) = \int_{\{s_1<\psi<s_2\}} (dd^c\psi)^{\wedge 2} \wedge \omega^{\wedge(n-2)} \geq 0. \tag{354}$$

We use the proof of Nemirovski's theorem (Theorem III.2.191), including the factor $\omega^{\wedge(n-2)}$ for the present dimension.

To determine the terminal value, choose a C^∞ metric which agrees with h near Y and for which σ_D has norm 1 outside a slightly larger neighborhood. This is the compact-support curvature representative of Lemma III.2.146. Write α for its normalized curvature and $\widetilde{\psi}$ for the associated function on $X \setminus Y$. Then $\alpha = dd^c\widetilde{\psi}$ on $X \setminus Y$, and $\widetilde{\psi} = \psi$ near M_s for large s. Another application of Stokes gives

$$F_\omega(s) = \int_{X\setminus\Omega_s} \alpha^{\wedge 2} \wedge \omega^{\wedge(n-2)} = \int_X \alpha^{\wedge 2} \wedge \omega^{\wedge(n-2)} - \int_{\Omega_s} \alpha^{\wedge 2} \wedge \omega^{\wedge(n-2)}.$$

The second integral tends to zero: α is smooth, the neighborhoods decrease to the analytic hypersurface Y, and Y has zero volume. Since α represents $c_1^{\mathbb{R}}([D])$, we obtain

$$\lim_{s\to\infty} F_\omega(s) = \int_X c_1([D])^{\wedge 2} \wedge \omega^{\wedge(n-2)}. \tag{355}$$

If this number is positive, every sufficiently large level has a positive tangential Levi direction at some point. We shall use that case in §2.5. For the divisor under consideration, we have

$$\int_X c_1([D])^{\wedge 2} \wedge \omega^{\wedge(n-2)} = 0. \tag{356}$$

Then non-negativity, (354), and the zero terminal value give $F_\omega(s) = 0$ for every sufficiently large s.

The integrand in (353) is a non-negative density. Its vanishing says that the trace of the Levi form of ψ on the complex tangent space of M_s is zero. Since that restricted form is semi-positive, it is zero. The hypersurface M_s is therefore Levi-flat. Moreover, semi-positivity of

the full Levi matrix makes its mixed entries with this null complex tangent space zero. Since $d\psi \neq 0$, there is a C^∞ function $f \geq 0$ on the punctured neighborhood such that

$$\partial\overline{\partial}\psi = f\,\partial\psi \wedge \overline{\partial}\psi. \tag{357}$$

Differentiating this identity shows that f is constant along every leaf of the Levi foliation of every M_s. Indeed, $d(\partial\overline{\partial}\psi) = 0$ implies $df \wedge \partial\psi \wedge \overline{\partial}\psi = 0$; the other terms vanish by (357). This is precisely the required leafwise constancy.

Suppose first that there are arbitrarily large values of s for which $f|_{M_s}$ is nonconstant. Fix one such s. Sard's theorem, recalled in Chapter I, §8.1, and compactness of M_s give a nonempty open interval of regular values of $f|_{M_s}$. On its preimage the map is a proper submersion. Ehresmann's fibration theorem (Theorem I.8.7) allows us to follow one connected component of its fibers. Choose three distinct values in this interval. The corresponding components are pairwise disjoint connected compact hypersurfaces Y_1, Y_2, Y_3 of X. They are complex hypersurfaces: their tangent spaces coincide with the complex tangent spaces of the Levi leaves, since f is leafwise constant and the values are regular. The resulting smooth isotopy in X shows that the three associated line bundles have the same integral first Chern class. Lemma V.2.7 therefore gives $\pi\colon X \to R$ with connected fibers and with one of these hypersurfaces as a fiber support.

By Remmert's proper mapping theorem (Chapter III, §2.11), $\pi(Y)$ is an analytic subset of R. This image is connected and misses the point corresponding to that fiber. Thus $\pi(Y) = \{p\}$ for some $p \in R$. To see that Y is the whole fiber support, write the fiber divisor as $\sum_i b_i E_i$. Since a different fiber is disjoint from Y,

$$0 = \int_X c_1([D]) \wedge c_1([\pi^*\{p\}]) \wedge \omega^{\wedge(n-2)} = \sum_i b_i \int_{E_i} c_1([D]) \wedge \omega^{\wedge(n-2)}.$$

The summands for $E_i \subset Y$ vanish by the numerical calculation. All other summands are nonnegative, and are strictly positive if E_i meets Y. Thus no additional component meets Y; connectedness of the fiber excludes additional components. The restriction of $[\pi^*\{p\}]$ to each component of this fiber is trivial. Lemma V.2.6 now gives $\pi^*\{p\} = aD$ for some $a \in \mathbb{Q}_{>0}$. A local coordinate at p trivializes the fiber divisor bundle near Y. Taking positive tensor powers and a root of its flat metric, as in the implication $(iii) \Rightarrow (ii)$, gives a flat metric on $[D]$ there.

In the remaining case, f is constant on every sufficiently large level. Since η is a submersion with connected fibers, we have $f = B \circ \eta$ for a C^∞ function B on a small positive interval. Fix a non-singular point of one component of Y, away from the other components, with multiplicity m in D. Choose coordinates (z, w) centered at that point, with $Y = \{w = 0\}$ locally, and write $\eta = |w|^{2m} e^{-\varphi}$. The $(w, \bar{w})$ coefficient of (357) gives

$$f = \frac{|w|^2 \frac{\partial^2 \varphi}{\partial w\, \partial \overline{w}}}{\left|-m + w \frac{\partial \varphi}{\partial w}\right|^2}.$$

The denominator is nonzero after shrinking. Along the real arc $z = 0$, $w = r \geq 0$, the function $t^{1/(2m)} = r e^{-\varphi(0,r)/(2m)}$, where $t = \eta(0, r)$, has a positive derivative at $r = 0$. Substituting its

local inverse in the displayed formula gives

$$B(t) = t^{1/m} A(t)$$

with A continuous at zero. Since f depends only on η, this one function describes every sufficiently small level. The identity $\psi = -\log \eta$ gives

$$\partial\overline{\partial}\eta = G(\eta)\, \partial\eta \wedge \overline{\partial}\eta, \qquad G(t) = \frac{1}{t} - t^{1/m-1} A(t). \tag{358}$$

We solve $\chi'' + G\chi' = 0$ with logarithmic leading term, as in Mini-Exercise I.4.15 with $a = 1/m$ and $F = -A$ (See also the arguments in [MST, §5] and [MT, §5.1]). Explicitly, put

$$Q(t) := \int_0^t s^{1/m-1} A(s)\, ds, \qquad H(t) := \int_0^t \frac{e^{Q(s)} - 1}{s}\, ds, \qquad \chi(t) := \log t + H(t).$$

Since $Q(t) = O(t^{1/m})$, the integrand defining H is $O(t^{1/m-1})$. Thus $H(t) = O(t^{1/m})$ and H extends continuously to zero with $H(0) = 0$. Moreover $\chi'(t) = e^{Q(t)}/t$ and $(\log \chi')' = -G$, so the required differential equation holds. Consequently

$$\widehat{\psi} := -\chi(\eta) = \psi - H(\eta), \qquad \partial\overline{\partial}\widehat{\psi} = 0.$$

For each local equation s_j of D, let φ_j be the C^∞ weight of h defined by $\eta = |s_j|^2 e^{-\varphi_j}$. The function $\widehat{\psi} + \log|s_j|^2 = \varphi_j - H(\eta)$ is locally bounded near every point of Y and is pluriharmonic off Y. Since the analytic hypersurface Y is locally pluripolar, the removable singularity theorem (Lemma III.2.94) extends this function pluriharmonically across Y. The extensions satisfy the metric transformation rule on the dense complement, hence everywhere by continuity. They are the weights of a C^∞ Hermitian metric on $[D]$ with zero curvature, proving (ii). Finally combine the disjoint neighborhoods of the original connected components of the support. □

For the analytic neighborhood space $\mathcal{X} = [(X, \mathcal{O}_X(-D))]$, the theorem identifies semi-positivity of $\mathcal{O}_{\mathcal{X}}(D)$ with the flat local defining functions in (iii), under the stated compact Kähler and numerical hypotheses.

2.4 Flat neighborhoods and transverse dynamics

Lemma V.2.8 ([Koi14, Lemmata 3.4–3.5]). *Let X be a complex manifold and let $D \neq 0$ be an effective divisor with compact connected support Z. Suppose that $[D]$ has a specified unitary flat structure on a neighborhood of Z. Choose $* \in Z$, and let $\rho\colon \pi_1(Z, *) \to \mathrm{U}(1)$ be the monodromy of its restriction to Z.*

(*i*) *If X is Kähler and the image of ρ is finite, some neighborhood V of Z admits a proper surjective holomorphic map $p\colon V \to \Delta_\varepsilon := \{t \in \mathbb{C} \mid |t| < \varepsilon\}$ with $p^{-1}(0) = Z$. It can be chosen with $\mathrm{div}(p) = kD$ for some positive integer k.*

(*ii*) *If the image of ρ is infinite, then $\Gamma(V \setminus Z, \mathcal{O}_X) = \mathbb{C}$ for every sufficiently small connected neighborhood V of Z.*

PROOF. For (*i*), choose $k \geq 1$ which kills the finite image. The unitary flat bundle $[D]^k|_Z$ is trivial. The uniqueness of flat extensions after shrinking (Example II.3.117) gives a nowhere vanishing holomorphic frame e of $[D]^k$ on a neighborhood W, compatible with that unitary flat structure. After shrinking, take W to be connected. The function $p := \sigma_D^k/e$ is holomorphic there and has divisor kD, so $p^{-1}(0) = Z$. It is nonconstant, hence open. Applying Lemma I.3.17 to $p\colon W \to \mathbb{C}$ at 0 gives an open neighborhood V of Z and $\varepsilon > 0$ such that $p\colon V \to \Delta_\varepsilon$ is proper and surjective. The neighborhood V is connected: each connected component is open and closed in V, so its image is non-empty, open, and closed in the disc (Mini-Exercise I.3.16 (*i*)), and hence meets 0. Since Z is connected, only one such component can occur. The divisor of the restriction is still kD.

For (*ii*), choose holomorphic frames e_j compatible with the fixed unitary flat structure and write $\sigma_D = f_j e_j$. The local functions f_j have constant unitary transition factors, so $|f_j|$ agrees on overlaps. Sufficiently small positive levels are connected compact smooth Levi-flat hypersurfaces, by the regular-level argument in the proof of Theorem V.2.5. The local functions f_j give their transverse first integrals. The holonomy calculation in [Koi14, Lemma 3.4 and the proof of Lemma 3.5 (*ii*)] identifies their transverse rotations with the flat monodromy and shows that every Levi leaf is dense in a level when the image is infinite. This is a calculation on the punctured neighborhood, where the levels are non-singular. An infinite subgroup of U(1) is dense. Apply Lemma III.2.189 to one of these compact hypersurfaces in the connected punctured neighborhood. Every holomorphic function there is constant. □

Thus the same constant transition factors describe both a flat line bundle and transverse rotations. This is the dynamical information used here: finite and infinite monodromy have different consequences for holomorphic functions.

2.5 Fibrations and Hartogs extension

The boundary calculation also applies when the self-intersection is positive. Together with the flat case it gives a dichotomy for the function theory of the complement.

THEOREM V.2.9 ([Koi14, Theorem 1.4]). *Let X be a connected compact Kähler manifold and let $D \neq 0$ be an effective divisor with connected support Z. Suppose that $[D]$ is semi-positive. Exactly one of the following holds.*

(*i*) *There are a compact Riemann surface R, a proper surjective holomorphic map $\pi\colon X \to R$ with connected fibers, a point $p \in R$, and $a \in \mathbb{Q}_{>0}$ such that $\pi^*\{p\} = aD$ as divisors.*

(*ii*) *For every connected open neighborhood V of Z in X, every holomorphic function on $V \setminus Z$ extends uniquely to $X \setminus Z$.*

For $\mathcal{X}_D = [(X, \mathcal{O}_X(-D))]$ and $M = X \setminus Z$, condition (*ii*) says precisely that restriction is an isomorphism

$$\Gamma(M, \mathcal{O}_M) \xrightarrow{\cong} \Gamma(\mathcal{X}_D^*, \mathcal{O}_{\mathcal{X}_D}). \tag{359}$$

The meaning of the right side is unchanged by shrinking V. Connectedness of $V \setminus Z$ and the identity theorem recover the assertion for the original V from an extension on a smaller neighborhood. We prepare the flat case before proving the theorem.

LEMMA V.2.10. *Let X be a connected compact Kähler manifold of dimension $n \geq 2$, and let $D \neq 0$ be an effective divisor with connected support Z. If $[D]$ is unitary flat near Z and its monodromy on Z has finite image, then condition (i) of Theorem V.2.9 holds.*

PROOF. Lemma V.2.8 gives a proper holomorphic function p near Z with $\operatorname{div}(p) = kD$. Choose a small disc of nonzero regular values. Over that disc, Ehresmann's fibration theorem (Theorem I.8.7) allows us to follow a connected component of the fibers. Three distinct fibers in this family give disjoint connected non-singular compact hypersurfaces with equal integral first Chern classes. Lemma V.2.7 supplies a global fibration $\pi\colon X \to R$ with connected fibers and one of these hypersurfaces as a fiber support. The connected set Z is disjoint from it, so $\pi(Z)$ is a point. The numerical support argument in the proof of Theorem V.2.5, using flatness of $[D]$ along every component of Z, shows that the fiber over that point has support exactly Z.

Write its divisor as $\pi^*\{p_0\}$. Its associated line bundle is trivial near Z, since a coordinate on the base pulls back to a defining function. Both this divisor and D therefore have zero intersection with each component of Z, against any Kähler form to the power $n-2$. Lemma V.2.6 gives $\pi^*\{p_0\} = aD$ with $a \in \mathbb{Q}_{>0}$. □

Proof of Theorem V.2.9. For $\dim X = 1$, connectedness of the support gives $D = m\{p\}$; take $R = X$, $\pi = \mathrm{id}_X$, and $a = 1/m$. Suppose henceforth that $n := \dim X \geq 2$. Choose a Hermitian metric h of semi-positive curvature and a Kähler form ω, and set $\psi = -\log|\sigma_D|_h^2$ on $M = X \setminus Z$. Put

$$S := \int_X c_1([D])^{\wedge 2} \wedge \omega^{\wedge(n-2)} \geq 0.$$

We use the levels M_s and the divisor-side neighborhoods Ω_s of the proof of Theorem V.2.5.

If $S > 0$, (355) makes $F_\omega(s) > 0$ for all sufficiently large s. The sublevel $W_s := \{x \in M \mid \psi(x) < s\}$ is then a relatively compact domain in X, with smooth boundary M_s. Its boundary Levi form is semi-positive and has a positive eigenvalue somewhere, since the integral of its nonnegative trace density is positive. These domains are connected for large s: the connected high levels form a product with an interval under the proper submersion. The adjacent inner levels lie in one component of W_s; a different component of W_s would have empty boundary in M, contradicting connectedness of M.

Let f be holomorphic on a connected neighborhood V of Z minus Z. Choose sufficiently large $s_0 < s$ with $\Omega_{s_0} \subset V$ and with the above boundary property at s. The compact set $K := \{\psi \leq s_0\} \subset W_s$ has complement $\{s_0 < \psi < s\}$, a connected level annulus with no relatively compact component in W_s. Ohsawa's Hartogs theorem (Theorem III.2.192) extends the restriction of f from this annulus to W_s. This extension glues with f on $\Omega_{s_0} \setminus Z$, giving a holomorphic function on M. The identity theorem on $V \setminus Z$ gives agreement with the originally given f. This proves (ii) when $S > 0$.

If $S = 0$, the semi-positive curvature form has square identically zero: its wedge square with $\omega^{\wedge(n-2)}$ is a nonnegative density of integral zero. Higher powers vanish as well, whereas $\int_X c_1([D]) \wedge \omega^{\wedge(n-1)} > 0$ because D is nonzero and effective. Hence $[D]$ is nef of numerical dimension one. Theorem V.2.5 makes it unitary flat near Z. If the resulting monodromy has finite image, Lemma V.2.10 gives (i). If it has infinite image, Lemma V.2.8 makes every function on a sufficiently small punctured neighborhood constant. The identity theorem again proves (ii) for every connected V.

Finally the alternatives are exclusive. Suppose (i) holds, and choose a coordinate v on a disc $U \subset R$ with $v(p) = 0$. Choose $c \in v(U)\backslash\{0\}$. On the inverse image of a smaller disc about p, the function $(v \circ \pi - c)^{-1}$ is holomorphic. If it extended to M, the identity $(v \circ \pi - c)f = 1$ would hold on $\pi^{-1}(U) \setminus Z$ by the identity theorem. This set is connected because the proper map has connected fibers and $U \setminus \{p\}$ is connected. At a point of the nonempty fiber over $v^{-1}(c)$ the identity is impossible. Thus (ii) fails, completing the proof. □

3 Semi-positivity and formal obstructions

3.1 A general line bundle and its obstructions

For an effective divisor, its canonical section turns a metric into a function on the complement. A general line bundle has no such distinguished section. The following conjecture extends the setting considered in [Koi7, Conjecture 1.1].

Conjecture V.3.1. *Let X be a complex manifold, let $Y \subset X$ be a compact analytic subset endowed with its reduced complex structure, and let L be a semi-positive holomorphic line bundle on X. If $L|_Y$ is unitary flat, then there is an open neighborhood V of Y in X such that $L|_V$ is unitary flat.*

The cited formulation assumes that Y is a compact Kähler submanifold and that $L|_Y$ is topologically trivial. In that setting the $\partial\overline{\partial}$-lemma supplies its unitary flat structure (Proposition III.2.131). The formulation above instead assumes unitary flatness directly and allows a reduced analytic subset.

A possible approach has two distinct steps: show that the Ueda invertible sheaf relative to the unitary flat comparison is formally trivial, and then use an additional formal principle to obtain analytic triviality. The results below address the first step in specified settings; the criteria in §V.5 address the second.

The first obstruction for a semi-positive line bundle

Let $\mathcal{X} = [(X, \mathcal{I}_Z)]$ be an analytic neighborhood space, where X is a non-singular complex manifold and $Z \subset X$ is a compact non-singular connected submanifold. Let L be a holomorphic line bundle on a neighborhood of Z with unitary flat restriction. Fix its unitary flat structure, choose a unitary flat extension $\widetilde{L}$ as in Example II.3.117, and choose an identification $\widetilde{L}|_Z \cong_{\mathrm{U}(1)-\mathrm{flat}} L|_Z$. This identification gives a global frame s_0 of $\mathcal{P}_L|_Z$, where $\mathcal{P}_L$

is the Ueda invertible sheaf of Definition IV.6.5. For $L = [Z]$ in codimension one, it is the comparison $\mathcal{P}_{[Z]}$ in (301).

Lemma IV.6.8 gives the obstruction to extending s_0 to a frame on $\mathcal{X}^{(1)}$. With the logarithmic normalization of Definition IV.6.10, write

$$u_1(\mathcal{P}_L) := \frac{1}{2\pi\sqrt{-1}}\,\mathrm{Obs}_1(\mathcal{P}_L, s_0) \in H^1(Z, \mathcal{S}^1_{\mathcal{X}}).$$

Here $\mathcal{S}^1_{\mathcal{X}} \cong \mathcal{O}_Z(N^*_{Z/X})$ by Example IV.2.6. To see its local representative, choose frames e_j of L and parallel frames $\widetilde{e}_j$ of $\widetilde{L}$ on open sets V_j, so that $e_j \otimes \widetilde{e}_j^{-1}$ restricts to s_0. Write $e_k = g_{jk}e_j$ and $\widetilde{e}_k = t_{jk}\widetilde{e}_j$, with locally constant $t_{jk} \in \mathrm{U}(1)$. The classes $f_{jk} := g_{jk}/t_{jk} - 1 \bmod \mathcal{I}_Z^2$ belong to $\mathcal{S}^1_{\mathcal{X}}(U_{jk})$, where $U_{jk} := Z \cap V_j \cap V_k$, and represent $\mathrm{Obs}_1(\mathcal{P}_L, s_0)$. Thus $u_1(\mathcal{P}_L)$ is represented by $\{(U_{jk}, f_{jk}/(2\pi\sqrt{-1}))\}$. This is the class of [Koi7, §3.2] multiplied by $(2\pi\sqrt{-1})^{-1}$. Any other global frame on Z differs from s_0 by a nonzero constant, which lifts to the neighborhood. Hence this first-order class is independent of that choice, and its vanishing is equivalent to a frame lift (Compare [Koi13, Lemma 3.4 (*iii*)]).

THEOREM V.3.2 ([Koi7, Theorem 1.4]). *Let X be a non-singular complex manifold, let $Z \subset X$ be a compact connected Kähler submanifold, and let L be a holomorphic line bundle on a neighborhood of Z with unitary flat restriction. Put $\mathcal{X} := [(X, \mathcal{I}_Z)]$ and form $\mathcal{P}_L$ and $u_1(\mathcal{P}_L)$ as above. If L has a C^∞ Hermitian metric of semi-positive curvature near Z, then $u_1(\mathcal{P}_L) = 0$.*

PROOF. Fix a C^∞ Hermitian metric h of semi-positive curvature on L as in the statement. If $d := \dim Z$ is zero, the obstruction group vanishes. Assume $d > 0$. The restriction of the curvature to Z is semi-positive and has zero integral against a Kähler form to the power $d-1$, because $c_1(L|_Z)$ vanishes in real cohomology. Its trace is a nonnegative continuous function with integral zero, so the restricted curvature is zero. The logarithm of the ratio of $h|_Z$ to a flat metric for the fixed unitary structure is pluriharmonic, hence constant on the compact connected manifold Z. Choose local parallel frames $\widetilde{e}_j$ of $\widetilde{L}$ whose images under the chosen identification on Z have h-norm one, and extend these images to holomorphic frames e_j of L. Their quotients with $\widetilde{e}_j$ restrict to s_0. Write $|e_j|_h^2 = e^{-\varphi_j}$, so that $\varphi_j|_Z = 0$.

For coordinates (z_j, w_j) with $w_j = (w_j^1, w_j^2, \dots, w_j^r)$ defining Z, Taylor's theorem gives

$$\varphi_j(z_j, w_j) = \sum_{\lambda=1}^{r} a_{j,\lambda}(z_j) w_j^\lambda + \overline{\sum_{\lambda=1}^{r} a_{j,\lambda}(z_j) w_j^\lambda} + O(|w_j|^2).$$

The tangential block of the Levi matrix is zero at $w_j = 0$. For a semi-positive Hermitian matrix, a zero diagonal entry forces the corresponding row and column to vanish (Mini-Exercise I.2.1; equivalently, take the two-by-two principal minors). Consequently the mixed coefficients at $w_j = 0$ vanish: $\overline{\partial}_z a_{j,\lambda} = 0$ for every λ. The functions $a_{j,\lambda}$ are holomorphic.

Replace e_j by $e'_j = e_j(1 + \sum_\lambda a_{j,\lambda} w_j^\lambda)$, shrinking the chart if necessary. The new weight is $O(|w_j|^2)$. The identity $\varphi'_k - \varphi'_j = -\log|g'_{jk}|^2$ then forces the terms of degree one in $g'_{jk}/t_{jk} - 1$ to vanish. Since $e'_j|_Z = e_j|_Z$, the quotients $e'_j \otimes \widetilde{e}_j^{-1}$ glue on $\mathcal{X}^{(1)}$ to a frame of $\mathcal{P}_L|_{\mathcal{X}^{(1)}}$ extending s_0. Thus $u_1(\mathcal{P}_L) = 0$. □

Higher-order obstructions and an open problem

We ask whether a semi-positive line bundle agrees formally with a specified unitary flat extension of its restriction to the center.

PROBLEM V.3.3. *Let X be a complex manifold, let $Z \subset X$ be a compact reduced analytic subset, and let L be a semi-positive holomorphic line bundle on a neighborhood of Z whose restriction to Z is unitary flat. Fix this unitary flat structure, a unitary flat extension $\widetilde{L}$, and an identification $\widetilde{L}|_Z \cong_{\mathrm{U}(1)-\mathrm{flat}} L|_Z$. Let $\mathcal{X} := [(X, \mathcal{I}_Z)]$ be the corresponding analytic neighborhood space, and let $\mathcal{P}_L$ be the Ueda invertible sheaf of L on $\mathcal{X}$. Must $\widehat{\mathcal{P}_L}$ be trivial on $\mathcal{X}^{(\infty)}$?* □

Equivalently, can one choose global frames s_μ of $\mathcal{P}_L|_{\mathcal{X}^{(\mu)}}$ for $\mu \in \mathbb{Z}_{\geq 0}$, compatible under restriction and inducing the prescribed frame on Z? This is the equivalence in Proposition IV.6.7. At a stage $\mu \geq 1$, Lemma IV.6.8 tests extension of the already chosen frame by $\mathrm{Obs}_\mu(\mathcal{P}_L, s_{\mu-1})$. An affirmative answer, together with FPLB on $\mathcal{X}$, would give the analytic comparison in Conjecture V.3.1.

3.2 All-order vanishing for divisor bundles

For a non-singular hypersurface Z with unitary flat normal bundle, the formal-comparison question in Problem V.3.3 has an affirmative answer for the divisor bundle $[Z]$.

THEOREM V.3.4 ([Koi8, Theorem 1.2]). *Let $\mathcal{X}$ be an analytic neighborhood space with a model $(X, \mathcal{I}_Z)$, where X is a non-singular complex manifold and $Z \subset X$ is a compact non-singular connected hypersurface with unitary flat normal bundle. If $[Z]$ has a C^∞ Hermitian metric of semi-positive curvature on a neighborhood of Z, then $\mathcal{X}$ is of infinite type (in the sense of Ueda).*

PROOF. Fix a C^∞ Hermitian metric h of semi-positive curvature on $[Z]$ as in the statement. We prove the C^∞ assertion by the coefficient comparison in [Koi8, §3.1]. If Z is zero-dimensional, a local defining coordinate already proves the assertion. Otherwise choose the unitary constants t_{jk}, base coordinates z_j, and defining functions w_j as in Chapter IV, §8.2. Set $\psi := -\log|\sigma_Z|_h^2$ and $\varphi_j := \psi + \log|w_j|^2$. The functions φ_j are C^∞ and plurisubharmonic. On Z they glue to a global plurisubharmonic function, because $|t_{jk}| = 1$. Compactness and the maximum principle make this function constant, and rescaling the metric makes it zero.

For a finite integer N, use Taylor's theorem in the normal variable in the form

$$\varphi_j(z_j, w_j) = \sum_{p+q \leq N} \varphi_j^{p,q}(z_j) w_j^p \overline{w_j}^q + O(|w_j|^{N+1}), \qquad \varphi_j^{q,p} = \overline{\varphi_j^{p,q}}. \tag{360}$$

All coefficients are C^∞ in z_j. We always take enough additional Taylor terms before differentiating, so the estimates below hold uniformly with the stated derivatives on smaller base charts.

We prove the following assertion for every $\mu \geq 0$: there is a system of order $\mu + 1$, and in such a system all coefficients $\varphi_j^{p,q}$ with $0 \leq p, q \leq \mu$ are locally constant. For $\mu = 0$ this

follows from $\varphi_j^{0,0} = 0$. Suppose the assertion has been established for $\mu - 1$. Start with a system of order μ, so that

$$t_{jk} w_k = w_j + f_{jk,\mu+1}(z_j) w_j^{\mu+1} + O(w_j^{\mu+2}). \tag{361}$$

The coefficients in the rectangle $0 \le p, q \le \mu - 1$ are constant.

First consider the tangential Levi form, evaluated at a fixed base vector ξ. It is a nonnegative real function of w_j. Its homogeneous terms of degrees $1, 2, \ldots, 2\mu - 2$ all vanish. Indeed, proceed in order of degree. For an odd degree the first potentially nonzero homogeneous term vanishes by Mini-Exercise I.4.4 (i). For degree $2l$ with $l \le \mu - 1$, its middle coefficient is the tangential Levi form of the constant $\varphi_j^{l,l}$, hence is zero. Part (ii) of the same exercise makes all its coefficients zero. Polarization in ξ shows that $\varphi_j^{p,q}$ is pluriharmonic for $p+q \le 2\mu-2$. When $\mu = 1$, apply the odd-degree argument at degree one instead. It follows that the tangential Levi block is $o(|w_j|^{2\mu-2})$, also when $\mu = 1$.

For the following coefficient calculation, coordinate subscripts denote partial derivatives: $a_{\overline{z}_j^\lambda} := \partial a / \partial \overline{z}_j^\lambda$ and $a_{w_j \overline{z}_j^\lambda} := \partial^2 a / (\partial w_j \, \partial \overline{z}_j^\lambda)$ for a smooth function a on the coordinate patch. Next differentiate once in w_j and once in $\overline{z}_j^\lambda$. The resulting mixed derivative has leading term

$$(\varphi_j)_{w_j \overline{z}_j^\lambda} = \mu (\varphi_j^{\mu,0})_{\overline{z}_j^\lambda} w_j^{\mu-1} + o(|w_j|^{\mu-1}).$$

The normal–normal derivative is bounded. Positivity of the corresponding two-by-two principal minor therefore gives

$$0 \le -\mu^2 |(\varphi_j^{\mu,0})_{\overline{z}_j^\lambda}|^2 |w_j|^{2\mu-2} + o(|w_j|^{2\mu-2}).$$

Thus $\varphi_j^{\mu,0}$ is holomorphic.

We now compute the difference of weights in two ways. The first uses the defining functions:

$$\varphi_k - \varphi_j = \log |w_k / w_j|^2 = f_{jk,\mu+1} w_j^\mu + \overline{f_{jk,\mu+1}} \, \overline{w_j}^\mu + O(|w_j|^{\mu+1}). \tag{362}$$

For the second, substitute $w_k = t_{jk}^{-1}(w_j + f_{jk,\mu+1} w_j^{\mu+1} + O(w_j^{\mu+2}))$ and the base change $z_k = z_k(z_j, w_j)$ into (360). Terms of lower degree have constant coefficients. Comparing successively the degrees below μ shows that their differences are zero. The degree-μ part is consequently

$$\sum_{p+q=\mu} \big(t_{jk}^{-p+q} \varphi_k^{p,q}(z_k(z_j, 0)) - \varphi_j^{p,q}(z_j) \big) w_j^p \overline{w_j}^q. \tag{363}$$

In particular,

$$t_{jk}^{-\mu} \varphi_k^{\mu,0} - \varphi_j^{\mu,0} = f_{jk,\mu+1}. \tag{364}$$

This is the required holomorphic coboundary with coefficients in $N_{Z/X}^{-\mu}$, using the transition convention in (311).

Define $\widetilde{w}_j := w_j - \varphi_j^{\mu,0}(z_j) w_j^{\mu+1}$. Its first derivative in w_j on Z is one. Substitution into (361), using (364), gives $t_{jk} \widetilde{w}_k = \widetilde{w}_j + O(\widetilde{w}_j^{\mu+2})$. Thus the new system has order $\mu + 1$. The

old constant rectangle is unchanged: this coordinate change, and the change of weights by $\log|\widetilde{w}_j/w_j|^2$, can only add pure terms or increase one normal exponent by at least μ.

It remains to provide the constant rectangle needed at the next step. Use the new defining functions, dropping the tildes. They satisfy $t_{jk}w_k = w_j + O(w_j^{\mu+2})$. The right-hand side of $\varphi_k - \varphi_j = \log|w_k/w_j|^2$ consists of a holomorphic function of (z_j, w_j) of order at least $\mu+1$, and its conjugate. In particular it has no term $w_j^p\overline{w_j}^q$ with $p, q > 0$.

The degree argument for the tangential Levi form, now also at the odd degree $2\mu-1$, shows that all coefficients of total degree at most $2\mu-1$ are pluriharmonic. The pure degree-μ coefficients glue by (363), since $f_{jk,\mu+1} = 0$ in the new system. They are global pluriharmonic sections of unitary flat line bundles, hence locally constant. For clarity, this last fact follows by applying the maximum principle to the global function given by the squared norm of such a section: in parallel frames its Levi form is the sum of the squares of the holomorphic and anti-holomorphic derivatives of the local section. Both derivatives must vanish.

Next compare the coefficient of $w_j^\mu\overline{w_j}^\nu$, successively for $\nu = 1, 2, \ldots, \mu$, and its conjugate. The nonlinear change of w_k increases one exponent by at least $\mu+1$, so cannot contribute in this rectangle. A nonconstant contribution from the Taylor expansion of $\varphi_k^{p,q}(z_k(z_j, w_j))$ could only come from a coefficient with $p \le \mu$, $q \le \nu$ and smaller total degree. Such a coefficient lies in the old constant rectangle or on an edge which has already been treated. Its derivatives are zero. Thus the edge coefficients glue as sections of $N_{Z/X}^{-\mu+\nu}$ and $N_{Z/X}^{\mu-\nu}$. If $\nu < \mu$, they are pluriharmonic by the preceding degree argument, and therefore locally constant.

Finally $\varphi_j^{\mu,\mu}$ glues to a global real-valued function. At total degree 2μ the middle coefficient of the tangential Levi form is nonnegative by Mini-Exercise I.4.4 (*ii*). Hence this global function is plurisubharmonic on Z, and compactness makes it constant. The same exercise then makes the other degree-2μ tangential Levi coefficients zero. This closes the rectangle and the induction. Since μ was arbitrary, systems of every finite order exist, proving infinite type. □

Let $\mathcal{X} = [(X, \mathcal{I}_Z)]$ satisfy the hypotheses of Theorem V.3.4. Use the Ueda invertible sheaf $\mathcal{P}_{[Z]}$ of (301), formed with the specified unitary flat extension of $[Z]|_Z = N_{Z/X}$. The theorem and Lemma IV.8.3 make $\mathcal{P}_{[Z]}|_{\mathcal{X}^{(\mu)}}$ trivial for every $\mu \in \mathbb{Z}_{\ge 0}$. Proposition IV.6.7 then gives $\widehat{\mathcal{P}_{[Z]}} \cong \mathcal{O}_{\mathcal{X}^{(\infty)}}$. If $\mathcal{X}$ also satisfies FPLB (Definition IV.6.1), then $\mathcal{P}_{[Z]} \cong \mathcal{O}_{\mathcal{X}}$, so $[Z]$ agrees with its unitary flat comparison bundle on a neighborhood of Z.

3.3 Pseudoconvex neighborhoods and linearization

The preceding results use plurisubharmonic metric weights to constrain formal neighborhood data. One can also ask what restrictions a fundamental system of locally pseudoconvex neighborhoods of a compact analytic subset places on its formal and analytic neighborhood structure, and under what additional hypotheses it forces linearization or a formal principle. Here a fundamental system is cofinal in the open neighborhoods ordered by reverse inclusion (Chapter II, Example II.3.20); local pseudoconvexity is tested at boundary points in holomorphic charts, as in Chapter III, §2.10. The following theorem answers the linearization

question for a hypersurface which is a holomorphic section.

THEOREM V.3.5 (Koike–Takakura, [KTa, Theorem 1.1]). *Let $p\colon X \to B$ be a holomorphic surjective submersion with one-dimensional fibers, where X is a complex manifold and B is a connected compact complex manifold. Let $s\colon B \to X$ be a holomorphic section, put $Z := s(B)$, and assume that $N_{Z/X}$ is unitary flat. The following conditions are equivalent.*

(*i*) *Z has a fundamental system of locally pseudoconvex open neighborhoods in X.*

(*ii*) *Z has such a fundamental system $\{\Omega_\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ for which every fiber of $p|_{\Omega_\nu}$ is connected and simply connected.*

(*iii*) *Z has a holomorphic tubular neighborhood in X in the sense of Definition IV.5.1.*

PROOF. See [KTa, §§2–3] for the proof. It uses the sharp Ohsawa–Takegoshi extension theorem and plurisubharmonic variation of fiberwise Bergman kernels. Unitary flatness and compactness make the diagonal kernel along the section constant in unitary normal coordinates; the equality case of the extension estimates then yields holomorphic fiber coordinates with unitary linear transition functions. □

We apply this to the elliptic suspensions already constructed in Chapter IV, §IV.7.

COROLLARY V.3.6 ([KTa, Corollary 4.1]). *Let $C \subset X_f$ be the elliptic suspension defined by (309), with holomorphic retraction $\pi_f\colon X_f \to C$ and holonomy (id, f), where $f(0) = 0$ and $f'(0) \neq 0$. The following conditions are equivalent.*

(*i*) *The germ f is linearizable.*

(*ii*) *The line bundle $[C]$ on X_f is semi-positive.*

(*iii*) *C has a holomorphic tubular neighborhood in X_f.*

(*iv*) *C has a fundamental system of locally pseudoconvex open neighborhoods in X_f.*

PROOF. The equivalence of (*i*)–(*iii*) follows from the results of Koike and Ogawa [KO2]; the tubular comparison is also proved in Proposition IV.7.2. For the metric condition, semi-positivity near C is equivalent to semi-positivity on X_f by Proposition V.2.1. As recalled in Chapter IV, the normal bundle has degree zero on the elliptic curve C and hence admits a unitary flat structure. The retraction π_f is a holomorphic submersion with the inclusion of C as a section, so Theorem V.3.5 gives (*iii*) $\Leftrightarrow$ (*iv*). □

In particular, Ueda's example in Remark IV.7.4 is of class (γ) in the sense of Remark IV.8.5: its Ueda invertible sheaf $\mathcal{P}_f$ is formally trivial but is not trivial on the analytic neighborhood space $\mathcal{U}_f = [(X_f, \mathcal{I}_C)]$. By Corollary V.3.6, its center has no fundamental system of locally pseudoconvex neighborhoods. Thus formal triviality alone does not ensure the existence of such a system.

PROBLEM V.3.7. *Let X be a non-singular complex surface and let $Z \subset X$ be a non-singular compact connected complex curve with $c_1(N_{Z/X}) = 0$. If (Z, X) is of class (γ), can Z admit a fundamental system of locally pseudoconvex neighborhoods in X?* □

PROBLEM V.3.8. *Let $\mathcal{X} = [(X, \mathcal{I})]$ be an analytic neighborhood space with compact connected reduced center $Z = Z(\mathcal{X})$. Assume that X is non-singular, that Z is a Cartier divisor in X, and that $[Z]|_Z$ is unitary flat. Put $\mathcal{P} := \mathcal{P}_{[Z]}$ as in Definition IV.6.5. Investigate the relation between the existence of a fundamental system of locally pseudoconvex neighborhoods of Z in X and the analytic and formal triviality of $\mathcal{P}$. In particular, under what additional hypotheses does the existence of such a system, together with $\widehat{\mathcal{P}} \cong \mathcal{O}_{\mathcal{X}^{(\infty)}}$, imply $\mathcal{P} \cong \mathcal{O}_{\mathcal{X}}$?* □

4 Finite Ueda type and pluripotential geometry

4.1 Mildest divergence and singular metrics

We first ask how slowly a plurisubharmonic function can tend to $+\infty$ along the center. Let $\mathcal{X} = [(X, \mathcal{I})]$ be an analytic neighborhood space with compact reduced center $Z := Z(\mathcal{X})$. We discuss divergence when every neighborhood of Z meets $X \setminus Z$; an empty punctured-neighborhood germ has no approach to the center from its complement.

DEFINITION V.4.1. A function $\psi \in \mathrm{Psh}(V \setminus Z)$ on a punctured neighborhood of Z *tends to $+\infty$ along Z* if, for every real number A, there is an open neighborhood $V_A \subset V$ of Z such that $\psi > A$ on $V_A \setminus Z$. After shrinking, take $\psi > 0$. For a real-valued function u on a punctured neighborhood, write $u = o(\psi)$ along Z if, for each $\varepsilon > 0$, one has $|u| \leq \varepsilon\psi$ on a sufficiently small common punctured neighborhood. We say that ψ has *the mildest divergence along Z* if there is no plurisubharmonic function u on a possibly smaller punctured neighborhood which both tends to $+\infty$ along Z and satisfies $u = o(\psi)$. □

These are uniform conditions along the entire compact center. They are unchanged by shrinking a model or by an isomorphism of analytic neighborhood spaces. Divergence concerns the end at Z; it does not require a proper exhaustion at the outer boundary of $V \setminus Z$. Mildest divergence excludes a slower *divergent* psh function, rather than requiring every psh function of little-o growth to be constant, or requiring a fixed pointwise lower bound for all divergent psh functions.

PROPOSITION V.4.2. *Let $\mathcal{X} = [(X, \mathcal{I})]$ have compact reduced center Z and nonempty punctured-neighborhood germ as above. Suppose that $h \colon V \setminus Z \to \mathbb{R}$ tends to $+\infty$ along Z and that both h and $-h$ belong to $\mathrm{Psh}(V \setminus Z)$. Then every plurisubharmonic $u = o(h)$ is bounded above on a smaller punctured neighborhood. In particular, h has the mildest divergence.*

PROOF. The two upper semi-continuity conditions make h continuous. Choose an open neighborhood W of Z whose closure is compact and contained in the common domain of h and u together with Z. We may take W to be a union of finitely many connected open neighborhoods meeting Z. Every component of W then meets Z, and every component of

$W \setminus Z$ has closure in W meeting Z: otherwise that component would be both open and closed in W. Here local connectedness follows from the local triangulations of Theorem III.2.55, applied in local embeddings of the complex space. Choose real constants M and $H \geq 0$ with $u \leq M$ and $|h| \leq H$ on the compact set ∂W, which is disjoint from Z.

For $\varepsilon > 0$, the function $v_\varepsilon := u - \varepsilon h$ is plurisubharmonic and tends uniformly to $-\infty$ at Z. Extend it by $-\infty$ on Z for the purpose of taking its maximum on $\overline{W}$. If its maximum exceeded $M + \varepsilon H$, it would be attained in $W \setminus Z$. The maximum principle on complex spaces (Lemma III.2.90) would make it constant on that component. This contradicts $v_\varepsilon \to -\infty$ at a point of Z in its closure. Consequently

$$u - \varepsilon h \leq M + \varepsilon H \quad \text{on } W \setminus Z.$$

Letting $\varepsilon \searrow 0$ at each point gives $u \leq M$. Since the punctured-neighborhood germ is nonempty, this bound is incompatible with divergence to $+\infty$ along Z. □

On a non-singular model a real-valued pluriharmonic h satisfies the two-sign hypothesis. On a singular model we use that hypothesis as stated, without resolving Question III.2.92.

Example V.4.3. On $\Delta^* := \{w \in \mathbb{C} \mid 0 < |w| < 1\}$, put $h(w) := -\log|w|$. Both h and $-h$ are harmonic, so Proposition V.4.2 shows that h has the mildest divergence at zero. A nonconstant subharmonic function can still have little-o growth relative to h; $|w|^2$ is one example. □

Divisor scales

Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center Z, and let D be an effective Cartier divisor on $\mathcal{X}$ whose support is Z. Choose an analytic neighborhood model $(X, \mathcal{I})$, a representative of D on X, and a C^∞ Hermitian metric h_0 on $[D]$ near Z. Put $r_D := |\sigma_D|_{h_0}$. The ideal specifying the jets of $\mathcal{X}$ need not be $\mathcal{O}_X(-D)$ (Compare Chapter IV, §8.5).

For a function u on a punctured neighborhood of Z and $a > 0$, we write $u = O(r_D^{-a})$ if $|u| r_D^a$ is bounded on a sufficiently small punctured neighborhood, and $u = o(r_D^{-a})$ if $|u| r_D^a \to 0$ uniformly as the point approaches Z. For positive functions u, v, we write $u \asymp v$ if $cv \leq u \leq Cv$ there for some constants $0 < c \leq C$. These conditions concern germs on shrinking punctured neighborhoods.

Changing h_0 multiplies r_D by a positive C^∞ function which, near the compact set Z, is bounded above and below by positive constants. The conditions just defined are therefore independent of h_0. If X is non-singular and $D = Z$ is a non-singular reduced hypersurface, r_Z is also comparable to the distance to Z for a fixed Riemannian metric. Replacing D by mD changes the scale from r_D to a function comparable to r_D^m.

From growth bounds to minimal singular metrics

For an effective divisor D on a complex manifold, let h be a singular Hermitian metric on $[D]$ with semi-positive curvature current. In holomorphic frames e_j on open sets U_j, write

$\sigma_D = f_j e_j$ and $|e_j|_h^2 = e^{-\varphi_j}$, with φ_j plurisubharmonic. On the complement of the support,

$$\psi_h = \varphi_j - \log|f_j|^2.$$

On overlaps, $e_k = g_{jk}e_j$ gives $f_j = g_{jk}f_k$ and $\varphi_k = \varphi_j - \log|g_{jk}|^2$, so the displayed expressions agree. Since $\log|f_j|^2$ is pluriharmonic where $f_j \neq 0$, they define one plurisubharmonic function there. The canonically attached singular Hermitian metric $h_{\mathrm{sing},D}$ has weight $\log|f_j|^2$ and hence $\psi_{h_{\mathrm{sing},D}} = 0$ (Example III.2.140). For a smooth metric, $\psi_h = -2\log r_D + O(1)$. The following criterion translates a growth restriction into the minimality of $h_{\mathrm{sing},D}$.

LEMMA V.4.4. *Let X be a complex manifold, let $D \neq 0$ be an effective divisor with compact support Z, and put $r_D = |\sigma_D|_{h_0}$ for a C^∞ Hermitian metric h_0 near Z. Suppose that there is $a > 0$ such that every nonnegative plurisubharmonic function $u = o(r_D^{-a})$ on a punctured neighborhood of Z is bounded above on a smaller punctured neighborhood. Then $h_{\mathrm{sing},D}$ is a minimal singular metric as a germ along Z: for every singular Hermitian metric h on $[D]$ near Z with semi-positive curvature current, its weights satisfy $\varphi_j \leq \log|f_j|^2 + C$ on smaller patches. In particular, $[D]$ is not semi-positive near Z.*

PROOF. Take any singular Hermitian metric h on $[D]$ near Z with semi-positive curvature current. Use the maximum of the weights of h and $h_{\mathrm{sing},D}$, as in [Koi3, Lemma 3.4 and the proof of Theorem 1.1]. The functions

$$u = \max\{\varphi_j - \log|f_j|^2, 0\}$$

define one nonnegative plurisubharmonic function off Z by the preceding transition calculation. Choose finitely many open sets W_j covering Z such that each $\overline{W_j}$ is compact and contained in a trivializing open set U_j on which h and h_0 are defined. Local upper bounds for φ_j and smoothness of h_0 give $\varphi_j + \log|e_j|_{h_0}^2 \leq C$ on every W_j for one constant $C \geq 0$. Since $r_D = |f_j||e_j|_{h_0}$, this gives $u \leq C - 2\log r_D$ on a smaller punctured neighborhood with $0 < r_D < 1$. The limit $t^a(1 + |\log t|) \to 0$ as $t \searrow 0$ therefore gives $u = o(r_D^{-a})$ uniformly along Z.

The hypothesis bounds u above on a smaller punctured neighborhood, giving $\varphi_j \leq \log|f_j|^2 + C$ there after increasing C. The set $Z \cap U_j = \{f_j = 0\}$ is closed and pluripolar in U_j, and the limsup formula in Lemma III.2.94 recovers the values of both plurisubharmonic weights on Z. Taking limsup thus extends the inequality across Z. It says that $h_{\mathrm{sing},D}$ is less singular than h in the sense of Definition V.1.6. Since h was arbitrary, this proves minimality as a metric germ along Z. A C^∞ Hermitian metric with semi-positive curvature would have bounded weights near Z, contradicting this inequality along any component of D. □

REMARK V.4.5. In Lemma V.4.4, the scale r_D^{-a} can be replaced by $(-\log r_D)^\lambda$ for any fixed $\lambda > 1$: if every nonnegative plurisubharmonic function $u = o((-\log r_D)^\lambda)$ on a punctured neighborhood of Z is bounded above on a smaller punctured neighborhood, the same conclusions hold. Indeed, the function used in the proof satisfies $0 \leq u \leq C - 2\log r_D$, and hence this little-o estimate uniformly along Z, after shrinking so that $0 < r_D < 1$. □

4.2 Ueda's finite-type theorems

The following is a principal function-theoretic feature of Ueda's class (α).

THEOREM V.4.6 (Ueda, [U1, Theorems 1 and 2]). *Let $\mathcal{X}$ be an analytic neighborhood space with a model $(X, \mathcal{I}_Z)$, where X is a non-singular complex surface and $Z \subset X$ is a compact non-singular connected curve with unitary flat normal bundle. Suppose that $\mathcal{X}$ has finite Ueda type μ.*

(i) For every $a > \mu$, there are an open neighborhood V of Z and a C^∞ strictly plurisubharmonic function Φ on $V \setminus Z$ such that $\Phi \asymp r_Z^{-a}$.

(ii) If $0 < b < \mu$ and u is plurisubharmonic on a punctured neighborhood of Z with $u = o(r_Z^{-b})$, then u is constant on a smaller punctured neighborhood.

The construction has four steps.

Step 1. A pluriharmonic primitive supplies the degree-zero coefficient on the compact Kähler center (Lemma V.4.7).

Step 2. A finite normal jet with pluriharmonic coefficients supplies corrections with controlled derivatives (Lemma V.4.8).

Step 3. The nonzero obstruction enters a two-dimensional part of the leading Levi form; a torus illustrates the case without zeros.

Step 4. For a curve, local corrections at isolated zeros give the functions used for the existence and rigidity assertions.

We carry out the preparations and the leading Levi calculation for a non-singular Kähler hypersurface, and then return explicitly to the surface case of Theorem V.4.6.

Step 1: Pluriharmonic primitives on a compact Kähler center

We first solve the additive cocycle equation for the degree-zero coefficient by pluriharmonic local functions on a compact Kähler manifold (See [U1, Proposition 2 and pp. 586–587], and [KUed1, Proposition 2.1 and its proof]).

LEMMA V.4.7. *Let Z be a compact connected Kähler manifold and let F be a unitary flat holomorphic line bundle on Z. Fix a unitary flat structure on F and a sufficiently fine finite covering $\mathcal{U} = \{U_j\}_{j\in J}$ by coordinate polydiscs, with holomorphic frames compatible with this structure. For every $a = \{(U_{jk}, a_{jk})\} \in \check{Z}^1(\mathcal{U}, \mathcal{O}_Z(F))$, there is a smooth 0-cochain $h = \{(U_j, h_j)\}$ with $\check{\delta} h = a$, whose components in these frames are pluriharmonic functions.*

PROOF. Since the sheaf of smooth sections is fine, choose a smooth zero-cochain $h = \{(U_j, h_j)\}_{j\in J}$ with $\check{\delta} h = a$ (Proposition II.3.63). The forms $\overline{\partial} h_j$ glue to a smooth F-valued $(0,1)$-form η on Z.

Fix a Kähler metric on Z and the flat metric on F. The Hodge decomposition for the Dolbeault complex (Proposition III.2.120) gives a harmonic representative η_0 of the Dolbeault class of η. There is a global smooth section v of F with $\eta - \eta_0 = \overline{\partial} v$. Replacing h_j by $h_j - v$ preserves its coboundary and gives $\overline{\partial} h_j = \eta_0$.

The Kähler identities with unitary flat coefficients (Proposition III.2.124) make η_0 closed for the flat connection. In these frames this gives $\partial\overline{\partial} h_j = 0$, including all mixed tangential derivatives. Thus h_j is pluriharmonic and, on a smaller coordinate polydisc, has an expression

$$h_j = p_j + \overline{g_j},$$

with p_j and g_j holomorphic (Proposition III.2.77). □

Step 2: Pluriharmonic coefficients in a finite normal jet

Let Z be a non-singular hypersurface in a non-singular complex manifold X of pure dimension $n+1$, with $n \geq 1$. In an adapted coordinate chart (z_j, w_j), write $z_j = (z_j^1, z_j^2, \ldots, z_j^n)$ and $Z = \{w_j = 0\}$. For $m \in \mathbb{Z}_{\geq 0}$ and a function R smooth across Z, the estimate $R = O(|w_j|^m)$, locally uniformly in z_j, is equivalent to the vanishing of all its real normal Taylor coefficients of total degree less than m. These coefficients vanish identically as functions of z_j, so Taylor's formula also gives, on compact subsets of a fixed base patch,

$$\left[\prod_{\ell=1}^{n} \left(\frac{\partial}{\partial z_j^\ell}\right)^{\alpha_\ell} \left(\frac{\partial}{\partial \overline{z_j^\ell}}\right)^{\beta_\ell}\right] \left(\frac{\partial}{\partial w_j}\right)^p \left(\frac{\partial}{\partial \overline{w_j}}\right)^q R = O(|w_j|^{m-p-q}) \qquad (p+q \leq 2), \tag{365}$$

where $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ and p, q are nonnegative integers, for every fixed finite number of tangential derivatives. In particular, the estimate controls each mixed derivative $\frac{\partial^2 R}{\partial z_j^a \, \partial \overline{z_j^b}}$.

The following finite induction is the construction of [U1, §3, Lemma 1], using the preceding pluriharmonic primitive on the Kähler center. We include the transition and remainder calculation in this dimension.

Lemma V.4.8. *Let Z be a compact connected non-singular Kähler hypersurface in a non-singular complex manifold X of pure dimension $n+1$, where $n \geq 1$, with unitary flat normal bundle. Fix a positive integer μ and a defining-function system $\{(V_j, w_j)\}$ of order μ with the prescribed unitary transitions t_{jk}. Choose adapted product coordinates (z_j, w_j) on these patches. Let $\widetilde{N}$ be the corresponding unitary flat extension of $N_{Z/X}$, and put $U_j = V_j \cap Z$. Let F be a unitary flat line bundle on a neighborhood of Z, and fix a unitary flat structure on F. Use holomorphic frames of F and $\widetilde{N}$ compatible with their fixed unitary flat structures. Let $0 \leq k \leq \mu$ be an integer. Let $a = \{(V_{jk}, a_{jk})\}$ be a holomorphic F-valued 1-cochain whose coboundary vanishes to normal order at least $k+1$. After shrinking the patches, there is a smooth F-valued 0-cochain $h := \{(V_j, h_j)\}_{j \in J}$ with components*

$$h_j(z_j, w_j) = \sum_{p+q \leq k} h_{j,p,q}(z_j) w_j^p \overline{w}_j^q, \tag{366}$$

whose coefficients are pluriharmonic in z_j, such that $a - \check{\delta} h = O(|w|^{k+1})$. A prescribed pluriharmonic primitive of $a|_Z$ may be used as $h_{j,0,0}$.

PROOF. For $k = 0$, take the primitive from Lemma V.4.7, or the prescribed one, and extend it by the fixed holomorphic projection onto the base coordinate. This extension is pluriharmonic on the product patch. The residual is smooth and zero on Z, as required.

For the induction step, subtract the coboundary of this degree-zero primitive. The residual cochain is locally pluriharmonic and zero on Z. After refining the patches, split each component into a holomorphic part and an anti-holomorphic part. On Z these restrictions sum to zero, so they are opposite constants on each connected overlap. Subtract these constants to make both parts zero on Z. This normalized decomposition is unique, since the intersection of the holomorphic and anti-holomorphic functions consists of locally constant functions. Consequently the decompositions agree under restriction and changes between these frames on overlaps meeting Z.

The triple-overlap defects of the two parts have respectively pure holomorphic and pure anti-holomorphic normal Taylor expansions. They vanish on Z, and their sum has zero normal jets through degree k. The monomials w^r and $\overline{w}^r$, $r \geq 1$, are linearly independent in these real Taylor jets, so the two defects separately have zero jets through degree k.

Write b_{jk} for the holomorphic part and set $b := \{(V_{jk}, b_{jk})\}_{j,k\in J}$. Let ρ_{jk} be the transitions of F for scalar components in the chosen frames. Set $d := \{(V_{jk}, d_{jk})\}_{j,k\in J}$, where $d_{jk} := b_{jk}/w_j$. This is a holomorphic $F \otimes \widetilde{N}^{-1}$-valued cochain, with transitions $\rho_{jk} t_{jk}^{-1}$. Its coboundary in the j-frame is

$$(\check{\delta} d)_{jk\ell} = \frac{(\check{\delta} b)_{jk\ell}}{w_j} + \rho_{jk} b_{k\ell} \left(\frac{1}{t_{jk} w_k} - \frac{1}{w_j} \right).$$

The first term has order at least k. Since $t_{jk} w_k = w_j + O(w_j^{\mu+1})$ and $b_{k\ell} = O(w_j)$, the second has order at least μ, hence at least k. Apply the induction hypothesis with $k - 1$ to d. For the anti-holomorphic part, first conjugate and use the dual flat bundle F^*, perform the same division and induction, and conjugate back.

Multiply the resulting polynomial zero-cochains by w_j and $\overline{w}_j$, respectively, and add the degree-zero primitive. Multiplication restores the coefficient bundle F. The difference between $w_j t_{jk}^{-1}$ and w_k has order $\mu+1$, so the additional transition error is $O(|w|^{\mu+1})$ and does not affect the required order $k + 1$. Every coefficient remains pluriharmonic in the chosen base coordinates and the total normal degree is at most k. This proves the statement by finite induction. □

Step 3: The leading Levi form in arbitrary dimension

Let $\mathcal{X} = [(X, \mathcal{I}_Z)]$ have compact connected non-singular Kähler hypersurface center Z of dimension $n \geq 1$, where X is non-singular of pure dimension $n + 1$. Suppose that $N_{Z/X}$ is unitary flat and that the specified comparison has finite Ueda type μ. Let $\widetilde{N}$ represent its specified unitary flat extension on a neighborhood V of Z, as in Chapter IV, §8.1. Choose a

finite covering $\mathcal{V} = \{V_j\}_{j\in J}$ of a neighborhood of Z by product coordinate patches (z_j, w_j), where $z_j = (z_j^1, z_j^2, \dots, z_j^n)$. Put $U_j := V_j \cap Z$ and $V_{jk} := V_j \cap V_k$. A system of order μ satisfies

$$t_{jk} w_k = w_j + f_{jk,\mu+1}(z_j) w_j^{\mu+1} + O(w_j^{\mu+2}), \qquad |t_{jk}| = 1. \tag{367}$$

Its degree-μ Ueda obstruction $u_\mu(\varphi_\mu)$ is nonzero; its representative is the coefficient of degree $\mu+1$ in (311). We now carry out the finite-jet and leading-form part of Ueda's construction in these coordinates, without assuming that $n = 1$.

Put $F := \widetilde{N}^{-\mu}$ and $\rho_{jk} := t_{jk}^{-\mu}$. We use ρ_{jk} as the transition rule for the scalar components of sections of F: a section has components $a_j = \rho_{jk} a_k$. The functions

$$c_{jk} := \rho_{jk} w_k^{-\mu} - w_j^{-\mu}$$

extend holomorphically across Z. Indeed, (367) gives

$$\rho_{jk} w_k^{-\mu} = w_j^{-\mu}\big(1 + f_{jk,\mu+1}(z_j) w_j^{\mu} + O(w_j^{\mu+1})\big)^{-\mu} = w_j^{-\mu} - \mu f_{jk,\mu+1}(z_j) + O(w_j).$$

Thus $c_{jk}|_Z = -\mu f_{jk,\mu+1}$, and $c := \{(V_{jk}, c_{jk})\}_{j,k\in J}$ belongs to $\check{Z}^1(\mathcal{V}, \mathcal{O}_V(F))$. It is the coboundary of the meromorphic cochain $\{(V_j, w_j^{-\mu})\}_{j\in J}$. Its restriction to Z represents a nonzero class in $H^1(Z, \mathcal{O}_Z(N_{Z/X}^{-\mu}))$.

The functions $w_j^{-\mu}$ themselves do not glue. We will correct their finite normal jets by functions pluriharmonic in z_j, and then glue the remaining high-order error smoothly. Pluriharmonicity is what will control all mixed tangential second derivatives of the eventual squared norm.

Apply Lemma V.4.7 to $c|_Z$. Choose the primitive $h = \{(U_j, h_j)\}_{j\in J}$ constructed in its proof, and write $h_j = p_j + \overline{g_j}$ with p_j, g_j holomorphic there. If the harmonic representative of $c|_Z$ were zero, the corrected h_j would be holomorphic and the class of $c|_Z$ would vanish. Therefore that harmonic representative is nonzero. Removing the holomorphic terms p_j by changing the defining functions according to

$$\widehat{w}_j^{-\mu} = w_j^{-\mu} - p_j(z_j), \qquad \widehat{w}_j = w_j\big(1 - p_j(z_j) w_j^{\mu}\big)^{-1/\mu}, \tag{368}$$

we obtain another system of order μ with the same first-order part. The branch in (368) is the one equal to 1 at $w_j = 0$. It is holomorphic after shrinking.

For this system, dropping the hats, the degree-zero discrepancy has the anti-holomorphic primitive $\overline{g_j}$:

$$c_{jk}|_Z = \rho_{jk}\overline{g_k} - \overline{g_j}. \tag{369}$$

The right-hand side is also holomorphic and is consequently locally constant. Its $\overline{\partial}$ derivative is zero. Thus the forms $\overline{\partial}\overline{g_j}$ glue and are not identically zero. Equivalently, their conjugates are locally holomorphic one-forms with unitary transition factors. Denote their common zero set by Σ. In higher dimension this analytic subset need not be a finite set. The form $\overline{\partial}\overline{g_j}$ is the local expression of the nonzero harmonic representative η_0.

Apply Lemma V.4.8 with $F = \widetilde{N}^{-\mu}$, $k = \mu$, $a = c$, and the primitive in (369) for the degree-zero term. We obtain a smooth F-valued cochain $h = \{(V_j, h_j)\}_{j\in J}$ with

$$h_j = \overline{g_j(z_j)} + \sum_{1\le p+q\le \mu} h_{j,p,q}(z_j) w_j^p \overline{w}_j^q,$$

with all $h_{j,p,q}$ pluriharmonic. Put $q_j = w_j^{-\mu} - h_j$. Its discrepancy $\rho_{jk}q_k - q_j$ is $O(|w|^{\mu+1})$ and extends smoothly across Z.

Use a smooth partition of unity $\{\chi_k\}$ on a smaller neighborhood subordinate to the coordinate patches. In the j-frame set

$$S_j := \sum_k \chi_k \rho_{jk} q_k.$$

The expressions define a global smooth section S of F on $V\setminus Z$. Since $S_j - q_j = \sum_k \chi_k(\rho_{jk}q_k - q_j)$ extends smoothly across Z, we have the expansion

$$S_j = w_j^{-\mu} - \overline{g_j(z_j)} - \sum_{1\le p+q\le \mu} h_{j,p,q}(z_j) w_j^p \overline{w}_j^q + R_j, \qquad R_j = O(|w_j|^{\mu+1}). \tag{370}$$

The flat metric on F gives a global function $|S|$, with $|S| = |S_j|$ in these unitary frames. After shrinking, S has no zero and $|S_j| \asymp |w_j|^{-\mu}$ uniformly on smaller base charts. Here S is a smooth section with a pole.

The order of these operations matters. If we had patched the functions $w_j^{-\mu}$ by an arbitrary partition of unity before making their coefficients pluriharmonic through degree μ, their tangential Levi derivatives could contain terms of size $|w_j|^{-\mu}$. Formula (370) is what prevents these terms from obscuring the required sign.

Fix one coordinate patch and suppress j. Put $s := |S|^2$ and $g_a := \frac{\partial g}{\partial z^a}$ for $1 \le a \le n$. Differentiate (370), using (365) for the remainder. Uniformly on a smaller base patch, for $1 \le a, b \le n$,

$$\begin{aligned}
s &= |w|^{-2\mu}\big(1 + O(|w|^{\mu})\big),\\
\frac{\partial s}{\partial w} &= -\mu w^{-\mu-1}\overline{w}^{-\mu} + O(|w|^{-\mu-1}),\\
\frac{\partial s}{\partial z^a} &= -w^{-\mu} g_a + O(|w|^{1-\mu}),\\
\frac{\partial^2 s}{\partial w\, \partial \overline{w}} &= \mu^2 |w|^{-2\mu-2} + o(|w|^{-2\mu-2}),\\
\frac{\partial^2 s}{\partial w\, \partial \overline{z^b}} &= O(|w|^{-\mu}),\\
\frac{\partial^2 s}{\partial z^a\, \partial \overline{z^b}} &= g_a \overline{g_b} + O(|w|).
\end{aligned} \tag{371}$$

For the last equality, all the mixed tangential derivatives of the terms linear in the pluriharmonic coefficients vanish. The product of the bounded correction with its conjugate contributes $g_a\overline{g_b} + O(|w|)$, and the product of $w^{-\mu}$ with the remainder contributes only $O(|w|)$.

Let $\tau > 0$. For a real-valued smooth function v, let $H(v)$ be its complex Hessian in the variable order $(w, z^1, z^2, \dots, z^n)$. Put $v_s := {}^t(\frac{\partial s}{\partial w}, \frac{\partial s}{\partial z^1}, \frac{\partial s}{\partial z^2}, \dots, \frac{\partial s}{\partial z^n})$. The chain rule gives

$$\frac{2}{\tau}|S|^{2-\tau}H(|S|^\tau) = H(s) + \left(\frac{\tau}{2} - 1\right) s^{-1} v_s v_s^\dagger. \tag{372}$$

The positive factor relating the complex Hessian to dd^c preserves its signature. Define the column $G := {}^t(g_1, g_2, \dots, g_n)$ and put

$$D_w := \begin{pmatrix} |w|^{\mu+1}/\mu & 0 \\ 0 & I_n \end{pmatrix}, \qquad \xi(w) := |w|^{\mu+1} w^{-1} \overline{w}^{-\mu},$$

where I_n is the identity matrix of order n; thus $|\xi(w)| = 1$. Equations (371) and (372) give

$$\begin{aligned} \mathcal{M}_\tau &:= D_w^\dagger \left(\frac{2}{\tau}|S|^{2-\tau}H(|S|^\tau)\right) D_w \\ &= \begin{pmatrix} \tau/2 & (\tau/2-1)\xi(w)G^\dagger \\ (\tau/2-1)\overline{\xi(w)}G & (\tau/2)GG^\dagger \end{pmatrix} + o(1). \end{aligned} \tag{373}$$

The remainder tends to zero in matrix norm uniformly on the smaller base patches, including at the zeros of G.

Where $G \neq 0$, let E_G be the matrix with the two orthonormal columns ${}^t(1, 0, \dots, 0)$ and ${}^t(0, g_1, g_2, \dots, g_n)/|G|$, and set $\mathcal{B}_\tau := E_G^\dagger \mathcal{M}_\tau E_G$. Here $|G|$ is the Euclidean norm of the column. Then

$$\mathcal{B}_\tau = \begin{pmatrix} \tau/2 & (\tau/2-1)\xi(w)|G| \\ (\tau/2-1)\overline{\xi(w)}|G| & (\tau/2)|G|^2 \end{pmatrix} + o(1),$$

and consequently

$$(\mathcal{B}_\tau)_{11} = \tau/2 + o(1), \qquad \det \mathcal{B}_\tau = (\tau - 1)|G|^2 + o(1). \tag{374}$$

With the convention ${}^t v H \overline{v}$ for Hermitian forms, this compression corresponds to the tangent plane spanned by $\partial/\partial w$ and $\sum_a \overline{g_a}\, \partial/\partial z^a$ after the normal scaling. Congruence and positive scalar multiplication preserve signature. Thus $H(|S|^\tau)$ has at least two positive eigenvalues near every compact subset of $Z \setminus \Sigma$ when $\tau > 1$, and at least one positive and one negative eigenvalue there when $0 < \tau < 1$. These assertions are uniform on such compact subsets. For $n \geq 2$ they do not determine the signs in the remaining $n - 1$ directions. Nor do they give a conclusion at Σ, which need not consist of isolated points.

Illustration of Step 3: Compact complex tori

For a torus the harmonic representative has no zero, so the preceding calculation is uniform over the entire center.

Proposition V.4.9. *Let $\mathcal{X} = [(X, \mathcal{I}_Z)]$ have center Z, a compact complex torus of dimension $n \geq 1$ embedded as a hypersurface in a non-singular complex manifold X of pure*

dimension $n+1$. Suppose that $N_{Z/X}$ is unitary flat and that the specified comparison has finite Ueda type μ. Then $N_{Z/X}^{\mu} \cong \mathbb{I}_Z$. For every $a > \mu$, there are an open neighborhood V of Z and a positive smooth function Φ_a on $V \setminus Z$ such that $\Phi_a \asymp r_Z^{-a}$ and the Levi form of Φ_a has at least two positive eigenvalues at every point of $V \setminus Z$.

PROOF. Choose a translation-invariant flat Kähler metric on Z and put $F_Z := N_{Z/X}^{-\mu}$, with its unitary flat metric. Let η_0 be the nonzero harmonic $(0,1)$-form representing the class of $c|_Z$ above. In flat local coordinates and unitary frames, the Laplacian of the flat connection acts coefficientwise as the scalar Euclidean Laplacian. The flat-coefficient Kähler identities and integration by parts on Z therefore give

$$\int_Z |\nabla \eta_0|^2 \, dV = 0,$$

where ∇ is the connection on F_Z-valued forms induced by these flat metrics and dV is the volume form of the chosen metric. It follows that η_0 is parallel.

Take a global parallel holomorphic coframe $\theta^1, \theta^2, \dots, \theta^n$ on the torus and write $\eta_0 = \sum_a s_a \overline{\theta^a}$. Each s_a is a parallel section of F_Z, and at least one is nonzero. Its norm is a positive constant, so it trivializes F_Z as a unitary flat bundle. In particular, $N_{Z/X}^{\mu} \cong \mathbb{I}_Z$. Also η_0 has positive constant norm, so its zero set Σ is empty. In a finite covering by smaller base charts the columns G are consequently bounded away from zero.

Apply the preceding construction with this harmonic representative and $\tau = a/\mu > 1$. Equation (374) gives a uniform positive margin for the two-dimensional compression. After shrinking the normal radius, the smooth function $\Phi_a := |S|^{a/\mu}$ has at least two positive Levi eigenvalues everywhere on the punctured neighborhood. Its growth follows from $|S| \asymp |w_j|^{-\mu}$ and $r_Z \asymp |w_j|$ on a finite covering. No correction supported near zeros is required. □

The torsion conclusion is necessary here; it does not assert that a torsion normal bundle forces finite Ueda type. When $n = 1$, two positive eigenvalues mean strict plurisubharmonicity. For $n \geq 2$, the proposition asserts neither plurisubharmonicity nor strict plurisubharmonicity of Φ_a.

Step 4: Corrections at the zeros and rigidity in the surface case

Proof of Theorem V.4.6. From this point on, X is a non-singular complex surface and Z is a compact non-singular connected curve, as in the theorem. Apply the preceding preparations, writing $g' = dg/dz$ in each base disc. The nonzero twisted holomorphic one-form has only finitely many zeros, so Σ is finite. Since $n = 1$, the compression $\mathcal{B}_\tau$ has the same determinant and first diagonal entry as $\mathcal{M}_\tau$ wherever $g' \neq 0$. Thus (373)–(374) give

$$(\mathcal{M}_\tau)_{11} = \tau/2 + o(1), \qquad \det \mathcal{M}_\tau = (\tau - 1)|g'|^2 + o(1),$$

with uniform remainders also at the zeros of g'. We finish Ueda's construction by the isolated-zero corrections and comparison argument of [U1, §3].

At the points of Σ the leading determinant vanishes. We now correct the function there.

For each $q \in \Sigma$, take a base coordinate disc centered at q, with these discs mutually disjoint on Z. Choose a nonnegative smooth function $\chi_q(z)$ supported in the disc, equal to $|z - z(q)|^2$ on a smaller disc. Its transition annulus contains no zero of g'. Thus $|g'|$ is bounded below there by a positive constant.

Put $k := (2 - \tau)\mu$ and define, near q,

$$\beta_q(z, w) := \chi_q(z)|w|^k. \tag{375}$$

Extend this function by zero outside the base patch. Formally, one may first insert a transverse cutoff equal to one near the relevant part of Z; after shrinking V, that cutoff is one everywhere involved in the following estimates. Thus β_q is a well-defined smooth function on the punctured neighborhood under consideration.

The derivatives needed for the Hessian are explicit:

$$\begin{aligned} \frac{\partial^2 \beta_q}{\partial z\, \partial \overline{z}} &= \frac{\partial^2 \chi_q}{\partial z\, \partial \overline{z}} |w|^k, \\ \frac{\partial^2 \beta_q}{\partial w\, \partial \overline{z}} &= \frac{k}{2} \frac{\partial \chi_q}{\partial \overline{z}} \frac{|w|^k}{w}, \\ \frac{\partial^2 \beta_q}{\partial w\, \partial \overline{w}} &= \frac{k^2}{4} \chi_q |w|^{k-2}. \end{aligned} \tag{376}$$

Since $|S|^{2-\tau} = |w|^{(\tau-2)\mu}(1 + o(1))$, the same normalization as in (373) gives

$$D_w^\dagger \left(\frac{2}{\tau} |S|^{2-\tau} H(\beta_q) \right) D_w = \begin{pmatrix} O(|w|^{2\mu}) & O(|w|^\mu) \\ O(|w|^\mu) & (2/\tau) \frac{\partial^2 \chi_q}{\partial z\, \partial \overline{z}} + o(1) \end{pmatrix}. \tag{377}$$

This explains the exponent in (375): the correction changes the tangential leading term, while its effect on the normalized normal and mixed entries tends to zero.

Suppose first that $\tau > 1$, and set

$$\Phi_\tau := |S|^\tau + \varepsilon \sum_{q \in \Sigma} \beta_q.$$

On the smaller disc about q, $\frac{\partial^2 \chi_q}{\partial z\, \partial \overline{z}} = 1$. The normalized determinant of its Hessian is therefore

$$(\tau - 1)|g'|^2 + \varepsilon + o(1),$$

while the first diagonal entry is $\tau/2 + o(1)$. On the transition annulus, the derivatives of χ_q are bounded and $|g'|$ is bounded below. Choose $\varepsilon > 0$ sufficiently small that the perturbation preserves the positive determinant on every such annulus. On the remaining compact part of $Z \setminus \Sigma$, (374) already gives a positive margin. Then shrink the normal radius so that the uniform error terms are smaller than all these margins. We obtain a strictly plurisubharmonic function on an entire punctured neighborhood of Z.

For $0 < \tau < 1$, use instead

$$\Psi_\tau := |S|^\tau - \varepsilon \sum_{q \in \Sigma} \beta_q.$$

The corresponding determinant on the smaller discs is $(\tau-1)|g'|^2-\varepsilon+o(1)$. Choose ε and then the normal radius in the same order. The normal diagonal remains positive and the determinant is negative everywhere. Thus $H(\Psi_\tau)$ has one positive and one negative eigenvalue at every point of the smaller punctured neighborhood.

The correction does not alter the order of growth. More precisely,

$$\frac{\beta_q}{|S|^\tau}=O(|w|^{2\mu}), \tag{378}$$

so both functions are positive after shrinking and are comparable to $|w|^{-\mu\tau}$. For $\tau>2$ the correction itself need not be bounded. What matters is the relative estimate (378).

For the exponent $a>\mu$ in part (i), take $\tau=a/\mu$. Since r_Z is comparable to $|w_j|$ on a finite covering of the non-singular curve, the function Φ_τ satisfies $\Phi_\tau\asymp r_Z^{-a}$. This proves part (i). It also supplies strongly pseudoconcave neighborhoods: the leading transverse derivative is nonzero for sufficiently small $|w_j|$, and the same is true after the correction. The large-level sets are smooth. The component of $Z\cup\{\Phi_\tau>c\}$ containing Z has defining function $c-\Phi_\tau$ off Z, whose Levi form is negative on its complex tangent space. Taking c large gives relatively compact such neighborhoods inside any prescribed sufficiently small neighborhood.

We next prove the preliminary bounded case of part (ii), corresponding to [U1, §3, Lemma 2]. Let u be plurisubharmonic on a punctured neighborhood and bounded above near Z. Lemma III.2.94 extends it plurisubharmonically across Z. Take a connected relatively compact strongly pseudoconcave neighborhood Ω as constructed above, with its closure in the domain of the extension.

The extension attains its maximum on $\overline{\Omega}$. If the maximum is attained in Ω, the maximum principle, Lemma III.2.90, makes it constant there. Suppose instead that a maximum point lies on the boundary. The boundary normal form in Lemma III.1.14 supplies a small holomorphic disc through that point, tangent to a negative Levi direction, whose punctured disc lies in Ω (Compare Observation III.2.148). The submean inequality on this disc moves the maximum to a point in Ω: the value at its center is the maximum and the values on its small circles are bounded by the same number, so equality occurs at an interior point of Ω. Apply the maximum principle again. Thus u is constant on a smaller punctured neighborhood. The case $u\equiv-\infty$ is immediate.

Let $0<b<\mu$ and suppose that $u=o(r_Z^{-b})$ as in part (ii). Use the function Ψ_τ constructed above with $\tau=b/\mu$. It is positive, comparable to r_Z^{-b}, and its Hessian has one negative eigenvalue everywhere on a sufficiently small punctured neighborhood.

Take a relatively compact strongly pseudoconcave neighborhood Ω with closure in that neighborhood. Subtract a constant from u so that $u\le 0$ on $\partial\Omega$. If u is bounded above near Z, the bounded case just proved applies. Otherwise it is positive at some point of $\Omega\setminus Z$. Since $u/\Psi_\tau\to 0$ uniformly near Z, the upper semi-continuous function u/Ψ_τ attains a positive maximum B at some point $p_0\in\Omega\setminus Z$. Indeed, the outer boundary has nonpositive values, and a sufficiently small neighborhood of Z has values less than any fixed positive value already attained.

Hence

$$u\le B\Psi_\tau \text{ on } \Omega\setminus Z,\qquad u(p_0)=B\Psi_\tau(p_0).$$

Choose a holomorphic disc f centered at p_0 with tangent vector in a negative Levi direction of Ψ_τ. For a sufficiently small positive radius r, the second-order Taylor expansion on the disc gives

$$\frac{1}{2\pi}\int_0^{2\pi} \Psi_\tau(f(re^{\sqrt{-1}\theta}))\,d\theta < \Psi_\tau(p_0).$$

The submean inequality for the plurisubharmonic function u then yields

$$\begin{aligned} u(p_0) &\le \frac{1}{2\pi}\int_0^{2\pi} u(f(re^{\sqrt{-1}\theta}))\,d\theta \\ &\le \frac{B}{2\pi}\int_0^{2\pi} \Psi_\tau(f(re^{\sqrt{-1}\theta}))\,d\theta < B\Psi_\tau(p_0) = u(p_0), \end{aligned}$$

a contradiction. No differentiability of u has been used. Therefore u is bounded above near Z, and the bounded case proves part (*ii*).

The two conclusions come from the opposite signs of $\tau - 1$ in (374), supplemented by the corrections at Σ. The argument gives no conclusion at $\tau = 1$, which corresponds to the exponent μ. The affine line bundle discussed next has additional structure and will supply a function at its critical exponent. □

4.3 Affine line bundles and the critical exponent one

Let M be a compact connected Kähler manifold, and let $\pi\colon A \to M$ be a holomorphic affine line bundle. Its associated line bundle F is specified by the linear parts of its transition maps. Assume that $c_1(F)$ is torsion and that the translation class of A in $H^1(M, \mathcal{O}_M(F))$ is nonzero. Equivalently, A has no holomorphic section. Let $\overline{A} \to M$ be the fiberwise compactification by projective lines, and let Z be its section at infinity. The analytic neighborhood space under consideration is $\mathcal{X}_\infty := [(\overline{A}, \mathcal{I}_Z)]$, whereas A itself is the complement of Z in $\overline{A}$.

THEOREM V.4.10 (Koike–Ueda, [KUed1, Theorems 1.1–1.3]). *In the setting above, the following holds.*

(i) The space A admits a continuous plurisubharmonic exhaustion Ψ which is C^∞ near infinity and satisfies $\Psi \asymp r_Z^{-1}$.

(ii) If u is plurisubharmonic on $V \setminus Z$ for an open neighborhood V of Z and $u = o(r_Z^{-1})$, then u is constant on a smaller punctured neighborhood.

(iii) If $\dim M = 1$, A admits a C^∞ strictly plurisubharmonic exhaustion of the same order r_Z^{-1} and is Stein. If $\dim M \ge 2$, there is no strictly plurisubharmonic function on all of A.

PROOF. We give the function in (i), the mechanism of (ii), and the dimension distinction in (iii); for the auxiliary leaf and perturbation estimates used below, see [KUed1, §§2–4]. The unitary flat structure on F and the Kähler Hodge theory with flat coefficients allow one to

choose affine coordinates w_j and an anti-holomorphic section with local expressions s_j such that

$$w_j = a_{jk} w_k + b_{jk}, \qquad s_j = a_{jk} s_k + b_{jk}, \qquad |a_{jk}| = 1,$$

where a_{jk} and b_{jk} are locally constant on the overlaps (See [KUed1, Proposition 2.1 and its preparation]). Set $q_j := w_j - s_j$ and define $\Psi := |q_j|$. Since $q_j = a_{jk} q_k$, the local expressions glue.

Write $s_j = \overline{f_j}$ with f_j holomorphic on the base chart. For a tangent vector represented by (ξ, b) in base and fiber directions, respectively, direct differentiation away from $q_j = 0$ gives

$$\mathcal{L}_\Psi(\xi, b) = \frac{|\overline{q_j} b + q_j \, df_j(\xi)|^2}{4|q_j|^3} \geq 0. \tag{379}$$

At $q_j = 0$, use the decreasing regularizations $(\varepsilon + |q_j|^2)^{1/2}$, whose Levi forms are also semi-positive. Thus Ψ is plurisubharmonic on A, although it need not be C^∞ on the anti-holomorphic section. On relatively compact base charts the functions s_j are bounded. With the local defining function $v_j := 1/w_j$ at infinity, one has

$$|w_j - s_j| = |v_j|^{-1} |1 - v_j s_j| \asymp |v_j|^{-1}.$$

A finite covering of M makes this comparison uniform. It proves the growth assertion and properness of the sublevel sets of Ψ.

The Levi form in (379) has rank one. Its kernel on $A \setminus \{\Psi = 0\}$ integrates to leaves which are holomorphically immersed and on which Ψ is pluriharmonic. Two additional facts used in the rigidity argument are that Ψ is unbounded above on each leaf and that a plurisubharmonic function bounded above near infinity is constant on a smaller punctured neighborhood ([KUed1, Lemmata 4.3 and 4.4]). These facts use the nonzero translation class.

To see how they yield (ii), work on an end $\{\Psi > R\}$ contained in $V \setminus Z$. Subtract a constant from u so that $u < 0$ on the inner boundary. If u is not bounded above on a smaller end, it is positive somewhere. Since $u/\Psi \to 0$, the positive supremum $B := \sup_{\Psi \geq R} u/\Psi$ is attained at an interior point. On the leaf through that point, $u - B\Psi$ is plurisubharmonic and has its maximum zero. The maximum principle makes it zero on the connected part of that leaf in the end. This part cannot exit through the inner boundary, where $u - B\Psi < 0$. The leaf property just recalled then contradicts $u/\Psi \to 0$. Hence u is bounded above on a smaller end, and the bounded case gives (ii).

For (iii), put $\Phi := (1 + \Psi^2)^{1/2}$. This is C^∞ on all of A, and

$$4\Phi^3 \mathcal{L}_\Phi(\xi, b) = 2\big(|b|^2 + |df_j(\xi)|^2\big) + |\overline{q_j} b + q_j \, df_j(\xi)|^2. \tag{380}$$

When M is a curve, the local differentials df_j have only finitely many zeros, because the translation class is nonzero. Away from these zeros the displayed form is positive definite. At each zero one adds a small base bump multiplied by Φ^{-3}. In a smaller base disc the bump has strictly positive Levi form; on its annular support the first term of (380) absorbs its derivatives when its coefficient is small. The weighted estimates, including derivatives in the unbounded fiber direction, are given in [KUed1, Lemma 3.1]. The resulting strictly

plurisubharmonic function differs from Φ by a bounded function, so it is an exhaustion of the same order. The Stein conclusion uses the characterization by a strictly plurisubharmonic exhaustion (Chapter III, §2.10).

For $\dim M \geq 2$, suppose that u were strictly plurisubharmonic on A. Its restriction to the compact level $\{\Psi = R\}$ has a maximum at some point (z_0, w_0). In a base chart, the analytic set

$$\{(z, w) \mid w = w_0,\ f_j(z) = f_j(z_0)\}$$

has a positive-dimensional germ at that point and lies in the same level. The maximum principle on an analytic curve in that germ contradicts strict plurisubharmonicity. This proves the assertion for functions on all of A. □

For a curve M, this is a type-one neighborhood. Indeed, $v_j = (a_{jk}/v_k + b_{jk})^{-1}$ has a quadratic term representing the nonzero translation class, with the normal-bundle convention of Chapter IV, §8.3. In this family the exponent one is realized, and little-o growth below the corresponding function is rigid (Compare [KUed1, p. 142]).

4.4 Singular curves and their growth scales

We next describe growth near rational curves with a node or a cusp. The normal bundle and the local defining equation determine the hypotheses and scales in these examples.

Let Z be a rational curve with one ordinary node in a non-singular complex surface X, with $(Z.Z) = 0$. Under the identification $\mathrm{Pic}^0(Z) \cong \mathbb{C}^*$, write $a \in \mathbb{C}^*$ for its normal bundle $[Z]|_Z$. Suppose that $|a| \neq 1$. Ueda's theorem [U2, Theorems 1 and 2] constructs strictly plurisubharmonic functions tending to $+\infty$ at Z, with upper bounds by powers $(\log(1/d))^{2\lambda}$ for every $\lambda > 1$, where d is the distance to Z in a fixed Riemannian metric. It also gives a constancy conclusion for plurisubharmonic functions with growth bounded by $(\log(1/d))^{2\lambda}$ with $0 < \lambda < 1$. The normal bundle here is $\mathbb{C}^*$-flat but is not unitary flat. Neither the unitary flat hypothesis of Theorem V.4.6 nor its power scale $r_Z^{-\mu}$ describes this case. Near a node with equation $xy = 0$, d is comparable to $\min\{|x|, |y|\}$, whereas the norm of the Cartier defining function is comparable to $|xy|$. Their logarithms are comparable after restricting to a small coordinate bidisc, although the functions themselves are not comparable by fixed positive constants.

For a rational curve Z with one ordinary cusp in a non-singular complex surface X, assume again that $(Z.Z) = 0$. The result [KUeh2, Theorem 1.1] has two alternatives. If the normal bundle is holomorphically trivial and the neighborhood is of infinite type, Z occurs as a fiber of a holomorphic fibration on a neighborhood. If the normal bundle is not holomorphically trivial, or the type is finite, there is a positive number $\ell \in \frac{1}{6}\mathbb{Z}_{>0}$ with the following properties. For each $\lambda > 1$ there is a C^∞ strictly plurisubharmonic function comparable to $r_Z^{-\lambda\ell}$ near Z. For $0 < \lambda < 1$, a plurisubharmonic function which is $o(r_Z^{-\lambda\ell})$ is constant as a punctured-neighborhood germ. Here r_Z uses the reduced Cartier divisor Z, including its singular defining equation; it has not been replaced by the distance to the cusp. The fractional scale is part of the theorem.

For reducible nodal curves, the normalization components, their dual graph, and the restrictions of the obstruction class enter the results of [Koi4], with the corrections in [KUeh2, Appendix A]. In particular, the type assumption in Propositions 7.8 and 7.9 of [Koi4] is imposed on the pair after the indicated contraction on the finite covering. The power growth in Proposition 7.9 is measured using the absolute value of a local defining function of the specified divisor, including its multiplicities, rather than the distance to its reduced support. These corrected hypotheses and scales are retained when applying those propositions. The local extension estimate for a node and the corresponding convergence issues were described in Chapter IV, §8.9.

4.5 Minimal singular metrics and obstructions to semi-positivity

Let $\mathcal{X}$ and D be as in § 4.1, with a non-singular ambient representative. If $[D]$ has a C^∞ Hermitian metric of semi-positive curvature near Z, the metric–function correspondence gives a plurisubharmonic function

$$\psi = -\log|\sigma_D|_h^2 = -2\log r_D + O(1)$$

on a punctured neighborhood. It tends to $+\infty$ at Z and is $o(r_D^{-b})$ for every $b > 0$.

For the non-singular finite-type curve of Theorem V.4.6, choose $0 < a < \mu$. Its rigidity assertion (ii) satisfies the hypothesis of Lemma V.4.4. Thus $h_{\mathrm{sing},Z}$ is a minimal singular metric as a germ along Z, and $[Z]$ is not semi-positive there, recovering [Koi3, Theorem 1.1]. For the infinity divisor in Theorem V.4.10, its little-o rigidity with exponent one gives the same two conclusions, in every base dimension allowed by that theorem. In the power-growth alternative for a cusp described in §4.4, choose $0 < \lambda < 1$. The function ψ above would be $o(r_Z^{-\lambda\ell})$ and hence constant as a punctured-neighborhood germ, contradicting its divergence. This again excludes semi-positivity. Determining a minimal singular metric leaves the finer growth question: which divergent plurisubharmonic functions have the mildest divergence?

There is also a sufficient condition in the opposite direction. For a compact non-singular hypersurface with unitary flat normal bundle, assume infinite type. Either of the additional arithmetic hypotheses in the Ueda-type linearization theorem (Theorem IV.8.13) gives defining functions with unitary constant transition factors. Thus the divisor bundle is unitary flat on a neighborhood. By Proposition V.2.1, it is semi-positive on any ambient manifold on which the effective divisor is defined.

Problem V.4.11. *Let $\mathcal{X}$ be an analytic neighborhood space with compact reduced center Z and nonempty punctured-neighborhood germ. Suppose that a plurisubharmonic function tending to $+\infty$ along Z exists. Does there exist such a function with the mildest divergence along Z?* □

Theorem V.4.6 gives, for each $a > \mu$, a strictly plurisubharmonic function $\Phi_a \asymp r_Z^{-a}$, but this function is not mildest: taking $\mu < b < a$ gives another divergent psh function $\Phi_b = o(\Phi_a)$. That theorem makes no assertion at the exponent μ. Even existence of a psh function comparable to $r_Z^{-\mu}$ would not by itself prove mildest divergence, since the stated rigidity

concerns exponents strictly less than μ. In contrast, the function $\Psi \asymp r_Z^{-1}$ in Theorem V.4.10 is mildest, by its sharp little-o rigidity. This holds also for a higher-dimensional Kähler base, without asserting the existence of a globally strictly plurisubharmonic function on the affine bundle.

5 Intersection geometry and FPLB on surfaces

We now compare two sources of geometric information. Intersection matrices produce metrics near compact curves, whereas Runge approximation and Serre duality remove additive formal obstructions. The latter argument is the surface case of [Koi13, Corollary 1.4 and §5.3]; it gives FPLB for all invertible sheaves.

Let $\mathcal{X}$ be an analytic neighborhood space admitting a model $(X, \mathcal{I}_Z)$ in which X is a non-singular complex surface and Z is a connected compact reduced curve. Write $Z_1, Z_2, \dots, Z_N$ for its irreducible components, and set

$$A_Z := ((Z_\nu . Z_\mu))_{1 \le \nu, \mu \le N}, \qquad \lambda := \max\{\text{eigenvalues of } A_Z\}.$$

Intersection numbers can be computed in any sufficiently small representative neighborhood, so the matrix belongs to the specified neighborhood geometry. It is symmetric, has integer entries, and its off-diagonal entries are non-negative. The graph in which ν and μ are adjacent when Z_ν meets Z_μ is connected. These are the hypotheses used in Chapter II, §2. By Proposition II.2.2, one can choose positive integers $m_1, m_2, \dots, m_N$ so that the divisor $D := \sum_\nu m_\nu Z_\nu$ satisfies, respectively,

$$(D.Z_\nu) < 0 \ (\lambda < 0), \qquad (D.Z_\nu) = 0 \ (\lambda = 0), \qquad (D.Z_\nu) > 0 \ (\lambda > 0) \tag{381}$$

for every ν. In the zero case the kernel is generated by a positive integer vector, uniquely determined by requiring that its coordinates have greatest common divisor 1. To record these multiplicities, we also use the analytic neighborhood model $(X, \mathcal{O}_X(-D))$, whose reduced center is Z.

We will use the following passage from degree to a metric.

Theorem V.5.1 ([KUed2, Theorem 1.3]). *Let X be a non-singular complex surface, let $Z \subset X$ be a connected compact reduced curve, and let L be a holomorphic line bundle on a neighborhood of Z. For each irreducible component Z_ν, let $\iota_\nu \colon \widetilde{Z}_\nu \to Z_\nu \subset X$ be its normalization. Then L has a C^∞ Hermitian metric of positive curvature on a neighborhood of Z if and only if $\deg \iota_\nu^* L > 0$ for every ν.*

Proof. The extension and gluing of positive local weights at the singular points is the content of the cited theorem; we refer to its proof for that step. On each compact Riemann surface $\widetilde{Z}_\nu$, positivity is equivalent to positive degree by Theorem III.2.177. The necessity follows also by pulling back a Hermitian metric of positive curvature: its curvature is non-negative on the normalization, and strictly positive away from the finitely many points lying over the singular locus. Its integral is therefore positive. The converse uses the compatible

ambient weights constructed in [KUed2, Lemma 2.4 and Proposition 4.1]. In the proof of that lemma, the sections obtained from the normalization are continuous on the curve and holomorphic on its regular part. Their local components need not be holomorphic at a singular point. This is sufficient for the metric construction: if $\sigma_j^{(\ell)}$ are these components, then $\log \sum_\ell |\sigma_j^{(\ell)}|^2$ tends to $-\infty$ at the singular point. The regularized maximum used there agrees with the auxiliary smooth strictly plurisubharmonic weight on a smaller neighborhood of that point, and with the logarithmic weight near the outer edge of the patch. It therefore extends smoothly and strictly plurisubharmonically across the point. On the intervening regular part the usual holomorphic-section calculation applies. This argument constructs a metric on the tensor power used in that proof; taking its positive root gives the metric on L. Proposition 4.1 of the same reference extends the compatible weights to the ambient neighborhood. □

5.1 Negative largest eigenvalue

Let $\mathcal{X} = [(X, \mathcal{I}_Z)]$ be as above and assume $\lambda < 0$. Choose D as in (381). Apply Theorem V.5.1 to $[-D]$, and let h be a Hermitian metric of positive curvature on $[-D]$ near Z. The dual metric h^{-1} on $[D]$ gives the function

$$u := \log |\sigma_D|^2_{h^{-1}} \quad \text{on } V \setminus Z, \qquad \sqrt{-1}\partial\overline{\partial} u = \sqrt{-1}\Theta_h > 0. \tag{382}$$

It tends to $-\infty$ along Z. Equivalently, $\rho := e^u = |\sigma_D|^2_{h^{-1}}$ is continuous on V, vanishes exactly on Z, and is strictly plurisubharmonic off Z. It is plurisubharmonic also across Z, by the removable singularity theorem for plurisubharmonic functions (Lemma III.2.94). Small sublevels $\{\rho < c\}$ are relatively compact in V and form a neighborhood basis of Z. The local gradient argument used in the proof of Theorem V.2.5 gives regular small levels; their Levi forms, oriented toward the sublevel side, are positive. Thus these are strongly pseudoconvex neighborhoods. Each is strongly 1-convex: for example $\rho/(c - \rho)$ is an exhaustion which is strictly plurisubharmonic off the compact curve.

This is the analytic direction of the Grauert–Mumford criterion: a connected compact curve in a smooth surface is exceptional precisely when its intersection matrix is negative definite (See [Gr3] and [Riem, Chapter 9, §9.8, Theorem 9.14]). Here negative definiteness is equivalent to $\lambda < 0$. The contraction and Remmert reduction are those of Theorem III.2.183 and Remark III.2.184. Indeed, the maximum principle for the strictly plurisubharmonic function off Z excludes other compact irreducible curves in a small neighborhood. We therefore obtain a contraction of Z to an isolated point of a normal complex surface. Theorem IV.6.15 now gives the FPLB on $\mathcal{X}$.

There is also a direct answer to the semi-positivity question for effective divisors supported on this center. If $E = \sum a_\nu Z_\nu \neq 0$ is effective and $[E]$ is semi-positive near Z, then $(E.Z_\nu) \geq 0$ for every ν, by integrating the pulled-back curvature on $\widetilde{Z}_\nu$. It follows that $(E.E) = \sum a_\nu (E.Z_\nu) \geq 0$, contrary to negative definiteness. Thus no such non-zero E is semi-positive.

5.2 Positive largest eigenvalue

We next treat the positive case; the zero case will be discussed in §5.3. The following formulation keeps track of logarithmic growth and is valid for reducible and singular curves.

THEOREM V.5.2 ([KUed2, Theorem 1.1]). *Let X be a non-singular complex surface and $Z \subset X$ a connected compact reduced curve. The largest eigenvalue of A_Z is positive if and only if there are a neighborhood V of Z and a C^∞ strictly plurisubharmonic function ψ on $V \setminus Z$ with the following property: near every point of Z one can write*

$$\psi = -\log|f|^2 + \varphi,$$

where f is holomorphic with zero set Z in that neighborhood and φ is C^∞ strictly plurisubharmonic across Z.

PROOF. The forward direction follows directly from the common tools above. Choose a positive integer vector with $A_Z m > 0$, put $D = \sum m_\nu Z_\nu$, and use Theorem V.5.1 to obtain a Hermitian metric h of positive curvature on $[D]$ near Z. Then $\psi = -\log|\sigma_D|_h^2$ has the stated local expressions. For the converse, take local expressions with equations f_j. On an overlap, $\log|f_j/f_k|^2 = \varphi_j - \varphi_k$ off Z, and the right side is bounded across Z. At a generic smooth point of each component, boundedness forces equal vanishing orders of f_j and f_k. The local factorial property on a smooth surface then implies that f_j/f_k extends as a holomorphic unit everywhere on the overlap. The local equations therefore define one effective divisor D with support Z, and the weights φ_j define a Hermitian metric of positive curvature on $[D]$. Hence $(D.Z_\nu) > 0$ for every ν. If m is its positive multiplicity vector, then ${}^t m A_Z m > 0$, so A_Z has a positive eigenvalue. This also explains the divisor and metric passage in the cited theorem (Its convention uses $-\log|f|$; multiplication of the function and its weight by 2 gives the normalization here). □

For this function, $\psi \to +\infty$ along Z. The neighborhoods $Z \cup \{\psi > c\}$ are therefore on the *superlevel* side. Their regular boundaries are strongly pseudoconcave with respect to these neighborhoods, since the Levi form of the defining function $c - \psi$ is negative on their complex tangent spaces.

Formal obstructions and compact-support cohomology

For the additive Cousin problem, a 1-cocycle with values in a coherent analytic sheaf $\mathcal{F}$ on $\mathcal{X}$ is solvable precisely when its class in $H^1(\mathcal{X}, \mathcal{F})$ is zero. The higher groups, their natural topologies, and estimates for primitives develop this gluing question further. For the multiplicative problem, the formal principle for line bundles asks whether an invertible sheaf which is formally trivial is already trivial on an analytic neighborhood. By Corollary IV.6.12, its obstruction is measured by $K^1_{\text{for}}(\mathcal{X}, \mathcal{O}_\mathcal{X})$. This is injectivity of the formal restriction on Picard groups.

To connect the neighborhood with a complement, let X be a connected compact complex manifold, let $D \neq 0$ be an effective divisor, and put

$$Z := \operatorname{Supp} D, \qquad M := X \setminus Z, \qquad \mathcal{X}_D := [(X, \mathcal{O}_X(-D))].$$

Here $\mathcal{X}_D$ is the analytic neighborhood space with model $(X, \mathcal{O}_X(-D))$. The set Z is given its reduced structure; D retains its multiplicities. The punctured neighborhoods in Notation IV.1.24 describe the end of M associated with this compactification. Indeed, complements of neighborhoods of Z are compact in M, and conversely every compact subset of M is disjoint from a sufficiently small neighborhood of Z. Proposition II.3.72 therefore gives the exact sequence

$$\begin{aligned} \cdots \longrightarrow H^q(M, \mathcal{O}_M) &\longrightarrow H^q(\mathcal{X}_D^*, \mathcal{O}_{\mathcal{X}_D}) \\ &\longrightarrow H_c^{q+1}(M, \mathcal{O}_M) \longrightarrow H^{q+1}(M, \mathcal{O}_M) \longrightarrow \cdots . \end{aligned} \tag{383}$$

The degree-zero restriction asks whether functions on the end extend to M. In degree one, the sequence relates additive gluing on the end to an obstruction with compact support. The compact ambient manifold is part of this construction; the abstract neighborhood space alone does not specify a whole complement.

Let X be a connected compact Kähler surface, let $D \neq 0$ be an effective divisor, and put $Z := \operatorname{Supp} D$ and $M := X \setminus Z$. Consider the analytic neighborhood space $\mathcal{X}_D := [(X, \mathcal{O}_X(-D))]$. Recall $K^1_{\mathrm{for}}(\mathcal{X}_D, \mathcal{O}_{\mathcal{X}_D})$ from Definition IV.2.29. For a class a in this kernel, choose a neighborhood V of Z and a $\overline{\partial}$-closed form $\beta \in \mathcal{A}^{0,1}(V)$ representing it under Dolbeault and the shrinking-neighborhood comparison. Choose a real smooth cutoff χ on X, equal to 1 near Z and with support contained in V. Extend $\chi\beta$ by zero to X. Then

$$\gamma := \overline{\partial}(\chi\beta) \in \mathcal{A}_c^{0,2}(M), \qquad v(a) := \{\gamma\} \in H_c^{0,2}(M). \tag{384}$$

This is the compact-support obstruction obtained by restricting a to the punctured neighborhood and using (383).

Lemma V.5.3. *In the preceding setting, $v(a)$ is independent of the form, cutoff, and representative neighborhood. If $v(a) = 0$, then $a = 0$.*

Proof. Suppose that β' represents the same germ class. On a smaller neighborhood of Z, Dolbeault comparison gives $\beta' - \beta = \overline{\partial} u$ for a smooth function u. Extend u smoothly to X after multiplying it by a cutoff which is 1 near Z. If χ', χ are the two cutoffs, the smooth form $\chi'\beta' - \chi\beta - \overline{\partial}\widetilde{u}$ vanishes near Z and is compactly supported in M. Its $\overline{\partial}$ is the difference of the two forms γ. This proves independence, including shrinking and changing only the cutoff.

If $v(a) = 0$, there is $u \in \mathcal{A}_c^{0,1}(M)$ with $\overline{\partial} u = \gamma$. Extend it by zero across Z. The form $\chi\beta - u$ is globally $\overline{\partial}$-closed on X, so it defines $b \in H^1(X, \mathcal{O}_X)$ whose image on $\mathcal{X}_D$ is a. Exponentiate b to obtain a holomorphic line bundle F on X. The exponential sequence gives $c_1(F) = 0$. By the compact Kähler $\partial\overline{\partial}$-lemma and Proposition III.2.131, F is unitary flat. Since a vanishes on the zeroth jet, $F|_Z$ is holomorphically trivial. Lemma III.2.130 then shows that its flat structure on the reduced compact space Z is trivial as well.

Apply the compact-neighborhood comparison of Theorem II.3.112 and Remark II.3.113 to the locally constant sheaf $\mathrm{U}(1)$. The flat class of F, which restricts to zero on Z, is zero on some smaller neighborhood. Thus the invertible sheaf on $\mathcal{X}_D$ obtained by exponentiating a is trivial. Injectivity in Corollary IV.6.12 gives $a = 0$. □

This is the compact Kähler step in the argument: an additive class extended to X exponentiates to a globally topologically trivial line bundle, hence to a unitary flat one. The conclusion uses its triviality on the compact center to trivialize it on a whole neighborhood.

Runge approximation and Serre duality

THEOREM V.5.4 ([Koi13, Corollary 1.4 (i)]). *Let X be a connected compact Kähler surface, let $D \neq 0$ be an effective divisor, and put $Z = \operatorname{Supp} D$, $M = X \setminus Z$. Suppose that $H_c^{0,2}(M)$ is Hausdorff and that the image of*

$$\varinjlim_{\mu \geq 0} \Gamma(X, \mathcal{O}_X(K_X \otimes [D]^{\mu})) \longrightarrow \Gamma(M, \mathcal{O}_M(K_M)), \qquad \tau \longmapsto \frac{\tau}{\sigma_D^{\mu}}\Big|_M \tag{385}$$

is dense for uniform convergence on compact subsets of M. Then the analytic neighborhood space $\mathcal{X}_D := [(X, \mathcal{O}_X(-D))]$ satisfies FPLB. The same conclusion holds for any specified coherent ideal with reduced support Z.

We prepare the finite-pole test before proving the theorem. For X, D, Z, M, and $\mathcal{X}_D$ as in the theorem, fix $a \in K^1_{\mathrm{for}}(\mathcal{X}_D, \mathcal{O}_{\mathcal{X}_D})$, and let $v(a)$ and γ be as in (384).

LEMMA V.5.5. *Let $X, D, Z, M, \mathcal{X}_D$ and $a \in K^1_{\mathrm{for}}(\mathcal{X}_D, \mathcal{O}_{\mathcal{X}_D})$ be as above. For every $\mu \geq 0$ and every $\tau \in \Gamma(X, \mathcal{O}_X(K_X \otimes [D]^{\mu}))$, the Serre pairing of $v(a)$ with the holomorphic 2-form $\eta := \tau/\sigma_D^{\mu}$ on M is zero:*

$$\int_M \gamma \wedge \eta = 0. \tag{386}$$

PROOF. It is enough to treat $\mu \geq 1$, since forms with no pole can also be represented after multiplication by σ_D. Write $\mathcal{J}_D := \mathcal{O}_X(-D)$. Vanishing of a on $\mathcal{X}_D^{(\mu-1)}$ and the cohomology sequence of

$$0 \longrightarrow \mathcal{J}_D^{\mu} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{J}_D^{\mu} \longrightarrow 0$$

on the neighborhood germ give a lift of a to $H^1(\mathcal{X}_D, \mathcal{J}_D^{\mu})$. Exactness follows either on germ sheaves or by taking the exact direct limit of the neighborhood sequences. On a representative neighborhood V_μ, Dolbeault comparison therefore gives a closed scalar form β_μ representing a which is locally divisible by the μ-th power of a defining function of D. More precisely, $\beta_\mu = \sigma_D^{\mu} \widetilde{\beta}_\mu$, where $\widetilde{\beta}_\mu$ is a $\overline{\partial}$-closed smooth $(0,1)$-form with values in $[-\mu D]$.

Choose a cutoff χ_μ supported in V_μ and equal to 1 near Z. The form $\gamma_\mu := \overline{\partial}(\chi_\mu \beta_\mu)$ represents $v(a)$ by Lemma V.5.3. Although η may have poles on D, the product $\chi_\mu \beta_\mu \wedge \eta$ extends to a smooth $(2,1)$-form on all of X: the powers of the local defining function cancel, including their prescribed multiplicities. Its $\overline{\partial}$ is $\gamma_\mu \wedge \eta$, a smooth compactly supported top-degree form on M extended by zero. Stokes' theorem on the compact surface gives $\int_M \gamma_\mu \wedge \eta = 0$. Replacing γ_μ by γ does not change the integral, since their difference is the $\overline{\partial}$ of a compactly supported form on M and $\overline{\partial}\eta = 0$ there. This proves (386). □

The representatives may depend on μ, whereas the compact-support class is fixed. Formal triviality therefore annihilates every finite-pole test form.

Proof of Theorem V.5.4. Take $a \in K^1_{\mathrm{for}}(\mathcal{X}_D, \mathcal{O}_{\mathcal{X}_D})$ and the compactly supported form γ in (384). Pairing with γ is a continuous linear functional on $\Gamma(M, \mathcal{O}_M(K_M))$: it is an integral over the fixed compact set $\operatorname{Supp}\gamma$. The preceding lemma makes this functional zero on the image of (385). Density therefore makes it zero on every holomorphic 2-form on M.

By Serre duality (Theorem III.2.132, with $p = 2$, $q = 0$ and trivial coefficients), the assumed Hausdorffness of $H^{0,2}_c(M)$ makes this pairing non-degenerate on its compact-support factor. Thus $v(a) = 0$. Lemma V.5.3 gives $a = 0$, and Corollary IV.6.12 gives FPLB. Finally use Remark IV.6.3 for a coherent ideal with the same reduced support. □

Positive curvature and the formal principle

Corollary V.5.6 ([Koi13, Corollary 1.4 (*ii*)]). *Let X be a connected compact Kähler surface and let $D \neq 0$ be an effective divisor. If $[D]$ has a C^∞ Hermitian metric whose curvature is positive definite at every point of $Z := \operatorname{Supp} D$ as a form on $T_X|_Z$, then the analytic neighborhood space $\mathcal{X}_D := [(X, \mathcal{O}_X(-D))]$ satisfies FPLB.*

Proof. Fix a C^∞ Hermitian metric h on $[D]$ satisfying the curvature condition in the statement. The function $\psi = -\log|\sigma_D|^2_h$ on $M = X \setminus Z$ is an exhaustion strictly plurisubharmonic outside a compact set. A regularized maximum with a sufficiently large constant makes it plurisubharmonic everywhere without changing it near the end. Hence M is strongly 1-convex. The Andreotti–Grauert finiteness and Hausdorff result recalled at the end of Chapter III, §2.11, applies to $\mathcal{O}_M(K_M)$ in degree one. By Dolbeault comparison, $H^{2,1}(M)$ is Hausdorff. The equivalent conditions in Serre duality (Theorem III.2.132), with $p = 2$ and $q = 0$, give Hausdorffness of $H^{0,2}_c(M)$.

The remaining density is exactly Ohsawa's approximation theorem (Theorem III.2.194). Its proof in Chapter III constructs small corrections by a weighted $\overline{\partial}$ estimate and bounds their pole order. Apply Theorem V.5.4. □

The positivity in this corollary concerns all ambient directions along Z, not merely the tangent directions of its regular part. In particular, it is stronger than semi-positivity alone. Combining it with the positive weighted divisor from §V.5 gives the following global consequence.

Corollary V.5.7. *Let X be a connected compact non-singular complex surface and $Z \subset X$ a connected compact reduced curve. Assume that the largest eigenvalue of A_Z is positive. Then X is projective, $X \setminus Z$ is a proper modification of an affine variety, and the FPLB holds on the analytic neighborhood space $\mathcal{X} := [(X, \mathcal{I}_Z)]$.*

Proof. Projectivity and the affine conclusion are [KUed2, Corollary 5.1]. Here is the role of the neighborhood construction. Use the weighted divisor D and function ψ from the proof of Theorem V.5.2. A regularized maximum of this ψ and a large constant gives an exhaustion of $X \setminus Z$ which is strictly plurisubharmonic outside a compact set. The complement is strongly 1-convex. Its Remmert reduction contracts a compact exceptional set disjoint from a smaller neighborhood of Z. The cited corollary glues this reduction to that neighborhood, and applies Grauert's positivity results to the descended weighted divisor

to obtain the projective compactification with affine complement. Since a smooth projective surface is Kähler and $[D]$ has a Hermitian metric of positive curvature near Z, Corollary V.5.6 gives FPLB for $[(X, \mathcal{O}_X(-D))]$. Remark IV.6.3 gives the same conclusion for $[(X, \mathcal{I}_Z)]$. □

5.3 Zero largest eigenvalue

Let again $\mathcal{X} = [(X, \mathcal{I}_Z)]$ have smooth surface ambient space and connected compact reduced curve center, and assume that the largest eigenvalue of A_Z is zero. Write $D_0 = \sum m_\nu Z_\nu$ for the primitive positive integral kernel divisor. If an effective divisor $E = \sum a_\nu Z_\nu$ is semi-positive near Z, then $A_Z a \geq 0$ componentwise. But

$$0 = {}^t(A_Z m)a = {}^t m A_Z a.$$

All coordinates of m are positive, so $A_Z a = 0$. The one-dimensional kernel therefore implies $E = qD_0$ for a non-negative rational number q. This restricts the numerical candidates to one ray. It does not decide whether any non-zero member of the ray is semi-positive or unitary flat near Z.

The following criterion relates the FPLB to the geometry of a complement in a family with zero self-intersection.

Proposition V.5.8 ([Koi13, Proposition 6.9]). *Let $Z_0 \subset \mathbb{P}^2$ be a non-singular cubic, let X be obtained by nine successive point blow-ups whose centers lie on the successive strict transforms of Z_0, and let $Z \subset X$ be its final strict transform. Put $M := X \setminus Z$ and $\mathcal{X} := [(X, \mathcal{I}_Z)]$. Suppose that $N_{Z/X}$ is non-torsion in $\mathrm{Pic}^0(Z)$. Then the following are equivalent.*

(*i*) *$\mathcal{X}$ satisfies the FPLB.*

(*ii*) *K_X^{-1} is semi-positive and $H_c^{0,2}(M)$ is Hausdorff.*

(*iii*) *K_X^{-1} is semi-positive and $H^{2,1}(M) = 0$.*

(*iv*) *K_X^{-1} is semi-positive and $H^{2,1}(M)$ is finite-dimensional.*

Proof. Since X is rational, $H^1(X, \mathcal{O}_X) = 0$. The blow-up formula gives $K_X \cong [-Z]$ and $(Z.Z) = 0$. Write $N := N_{Z/X}$. If K_X^{-1} is semi-positive, then $[Z]$ is nef of numerical dimension one. Theorem V.2.5 makes it unitary flat near Z. Since N is non-torsion, Lemma V.2.8 (*ii*) and the identity principle give $\Gamma(M, \mathcal{O}_M) = \mathbb{C}$. Choose a nowhere vanishing section τ of $K_X \otimes [Z]$. Then $\sigma := \tau/\sigma_Z$ trivializes K_M, and $\Gamma(M, \mathcal{O}_M(K_M)) = \mathbb{C}\sigma$ consists of forms with at most a simple pole along Z. Under (*ii*), Theorem V.5.4 therefore gives (*i*).

Conversely, assume (*i*). The group $H^1(Z, \mathcal{O}_Z(N^{-\mu}))$ vanishes for each integer $\mu \geq 1$, by (241). Lemma IV.6.8 thus lifts a frame of any invertible sheaf trivial on Z through all finite jets, and the compatible frames give a formal trivialization (Lemma IV.6.6). The FPLB makes every such sheaf trivial on $\mathcal{X}$. Apply this to the Ueda invertible sheaf $\mathcal{P}_{[Z]}$ (Definition IV.6.5). Then $[Z]$ is unitary flat near Z, and Proposition V.2.1 makes $[Z] \cong K_X^{-1}$ semi-positive on X.

The same injectivity on Picard groups, together with Proposition IV.6.11 for $\mu = 0$, makes $H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to H^1(Z, \mathcal{O}_Z)$ injective. Constants on Z lift to a neighborhood, so the exact sequence defined by $\mathcal{I}_Z \subset \mathcal{O}_X$ gives $H^1(\mathcal{X}, \mathcal{I}_Z) = 0$, where the coefficient sheaf denotes the germ of $\mathcal{I}_Z$ along Z. It follows that extension by zero induces an injection $H^{0,2}_c(M) \to H^2(X, \mathcal{I}_Z)$, using $\mathcal{I}_Z|_M = \mathcal{O}_M$ and smooth forms with values in $[-Z]$. Indeed, if a compactly supported representative becomes $\overline{\partial} v$ on X, then v is closed near Z. The vanishing just proved gives $v = \overline{\partial} a$ on a smaller neighborhood. For a smooth cutoff χ supported there and equal to one near Z, $v - \overline{\partial}(\chi a)$ is a compactly supported primitive on M. Since $\mathcal{I}_Z \cong \mathcal{O}_X(K_X)$, compact Serre duality gives $\dim H^2(X, \mathcal{I}_Z) = 1$. We already have $\Gamma(M, \mathcal{O}_M(K_M)) = \mathbb{C}\sigma$; hence Serre duality (Theorem III.2.132) gives a continuous surjection from $H^{0,2}_c(M)$ onto $\Gamma(M, \mathcal{O}_M(K_M))^\vee \cong \mathbb{C}$. The dimension bound makes this map injective, so $H^{0,2}_c(M)$ is Hausdorff. This proves (ii).

Finally, under semi-positivity, [Koi12, Theorem 1.1] says that $H^{0,1}(M)$ is either zero or infinite-dimensional and non-Hausdorff, since $H^1(X, \mathcal{O}_X) = 0$. Multiplication by σ identifies $H^{0,1}(M)$ with $H^{2,1}(M)$ as topological vector spaces. Serre duality with $p = 2$, $q = 0$ identifies Hausdorffness of $H^{2,1}(M)$ with that of $H^{0,2}_c(M)$. The cohomology topologies are the natural ones fixed in Chapter III. This proves the equivalence of (ii), (iii), and (iv). □

We now let the ninth point, and hence the normal bundle, vary.

Example V.5.9 ([Koi13, Theorem 1.5]). There exist a compact Riemann surface R, a proper holomorphic submersion $\pi\colon \mathfrak{M} \to R$ from a complex threefold, and a compact hypersurface $\mathfrak{Z} \subset \mathfrak{M}$ such that, for each $t \in R$, the surface $X_t := \pi^{-1}(t)$ is projective and $Z_t := X_t \cap \mathfrak{Z}$ is a smooth elliptic curve with $(Z_t.Z_t) = 0$. The FPLB holds on the analytic neighborhood space $\mathcal{X}_t := [(X_t, \mathcal{I}_{Z_t})]$ for almost every $t \in R$, with respect to Lebesgue measure in local coordinates, but fails for uncountably many t.

For the construction, start with a smooth plane cubic Z_0. Perform eight successive point blow-ups, each at a point of the current strict transform of the cubic. Denote the resulting surface by X_8 and curve by Z_8. Set $R := Z_8$, and blow up $X_8 \times Z_8$ along the section $\{(t, t) \mid t \in Z_8\}$. This is $\mathfrak{M}$; take $\mathfrak{Z}$ to be the strict transform of $Z_8 \times Z_8$. Projection onto the second factor gives π. Its fiber over t is the ninth blow-up, now at the moving point t of Z_8. The strict transform remains a smooth elliptic curve and has self-intersection $9 - 9 = 0$. This construction allows successive, possibly infinitely near, centers; it does not assert that all nine images in the original plane are distinct. Local coordinates along the section give the usual product family of point blow-ups, so π is a submersion.

The almost-everywhere assertion and the uncountable exceptional set follow from the cited theorem. Its proof combines Proposition V.5.8 with the small-denominator behavior of the varying normal bundle. □

Classification and deformations of analytic neighborhoods

The preceding results suggest studying the classification of analytic neighborhood spaces with compact centers, including singular and reducible centers. As a first concrete setting, consider a compact Riemann surface Z in a non-singular complex surface X, and fix the

complex structure of Z and the holomorphic isomorphism class of $N_{Z/X}$. Here the intersection matrix has just one entry, so its largest eigenvalue is $\lambda = (Z.Z) = \deg N_{Z/X}$. Write $g(Z)$ for the genus of Z.

For $\lambda < 0$, Grauert's tubular neighborhood theorem gives the linear model whenever $\lambda < \min\{0, 4-4g(Z)\}$ (Chapter IV, Corollary IV.5.8). For genus at least two, the range $4-4g(Z) \leq \lambda < 0$ is not covered by this numerical sufficient condition; individual cohomological vanishing conditions may still apply. The case $\lambda > 0$ leads toward Kodaira's displacement and embedded-deformation theory (See [Kod, KodS]). Of particular interest here is $\lambda = 0$. For an elliptic Z and non-torsion $N_{Z/X} \in \mathrm{Pic}^0(Z)$, the finite comparison groups in Chapter IV, §IV.5, vanish, so the formal neighborhood is isomorphic to the linear model. Analytic comparison requires more: compare Arnol'd's theorem (Theorem IV.9.1) and the nine-point blow-up examples in §V.6.

Problem V.5.10. *Fix a compact Riemann surface Z and a holomorphic line bundle $N \to Z$. Classify analytic neighborhood spaces $\mathcal{X} = [(X, \mathcal{I}_Z)]$, with X a non-singular complex surface, together with specified identifications of their centers with Z and of their normal bundles with N, up to isomorphisms respecting these identifications. In particular, when $\deg N = 0$, formulate a deformation theory with these fixed data and determine conditions under which it admits a finite-dimensional Kuranishi-type parameter space (a locally complete analytic family with minimal infinitesimal parameters).* □

The specified identifications give a *marked* problem; the unmarked problem additionally identifies markings under automorphisms of the pair (Z, N). For a representative $(X, \mathcal{I})$ with $\mathcal{I} = \mathcal{I}_Z$, the relevant infinitesimal automorphisms suggest the sheaf

$$\mathfrak{g}_{(Z,N)} := \{\theta \in \mathcal{O}_X(T_X) \mid \theta(\mathcal{O}_X) \subset \mathcal{I}, \quad \theta(\mathcal{I}) \subset \mathcal{I}^2\}.$$

In coordinates (z, w) with $Z = \{w = 0\}$, its sections have the form

$$\theta = wa(z,w)\frac{\partial}{\partial z} + w^2 b(z,w)\frac{\partial}{\partial w}.$$

Thus fixing Z and N does not remove the tangential coefficient of order one, which involves $T_Z \otimes N^{-1}$. Using the germ of this sheaf on $\mathcal{X}$, the group $H^1(\mathcal{X}, \mathfrak{g}_{(Z,N)})$ is a candidate for the infinitesimal deformation space; identifying it as such requires specifying the deformation problem and its gluing equivalence. Neither formal uniqueness nor uniqueness of analytic isomorphism classes alone proves that this group is zero. Its Hausdorffness also implies neither finite-dimensionality nor injectivity of comparison with formal cohomology. These questions, the obstruction spaces, and convergence of the nonlinear gluing equations remain separate parts of the proposed construction.

Nine-point blow-ups of the projective plane provide some of the most interesting examples from the viewpoint of these notes. We study these surfaces in more detail in the next section.

6 Nine-point blow-ups of the projective plane

The blow-up of the projective plane at nine points brings together numerical positivity, singular metrics, neighborhood comparison, and the function theory of a complement. We

first allow arbitrary positions of nine distinct points, and then examine the configurations lying on a smooth or nodal cubic in more detail.

6.1 The anticanonical bundle

Let $X \to \mathbb{P}^2$ be the blow-up at nine distinct points. The anticanonical bundle has a nonzero section, since nine point conditions on homogeneous cubic polynomials leave a nonzero polynomial, and $c_1(K_X^{-1})^{\wedge 2} = 0$. The classification in [Koi4, Theorem 7.1], with the corrections in [KUeh2, Appendix A], gives the following alternatives:

(*i*) K_X^{-1} is semi-ample.

(*ii*) K_X^{-1} is semi-positive but not semi-ample.

(*iii*) K_X^{-1} is nef, and an effective anticanonical divisor D has canonically attached singular Hermitian metric $h_{\mathrm{sing},D}$ which is a minimal singular metric. In this case K_X^{-1} is not semi-positive.

(*iv*) K_X^{-1} is nef and has a reduced anticanonical curve Z with at most nodes such that $[Z]|_Z$ is unitary flat, non-torsion, and outside the Diophantine class of the cited theorem.

(*v*) K_X^{-1} is not nef, and the nef part of its Zariski decomposition is semi-ample.

The fourth item isolates the flat normal bundles not decided by the earlier torsion and Diophantine criteria; the results discussed below give further information in that range. In the last item, the *Zariski decomposition* of an effective anticanonical divisor is $D = P + N$, where P is a nef rational divisor, N is effective with negative definite intersection matrix on its components, and P has zero intersection with those components. Semi-ampleness of P means that an integral positive multiple has globally generated associated bundle. A smooth Hermitian metric of semi-positive curvature on such a multiple, together with the canonical singular factors of N, determines a minimal singular metric on K_X^{-1} (See [Koi4, Remark 7.3]). This description distinguishes the non-nef case from failure of semi-positivity for a nef bundle.

6.2 Cubic curves and elliptic pencils

Let $Z_0 \subset \mathbb{P}^2$ be a smooth cubic or an irreducible cubic with one node. Perform nine successive point blow-ups, each centered at a non-singular point of the current strict transform of Z_0, and denote the resulting surface and curve by X and Z. Put $M := X \setminus Z$. The blow-up formula gives $K_X^{-1} \cong [Z]$ and $(Z.Z) = 0$. Since Z is irreducible, its intersection with every other irreducible curve is non-negative; hence $[Z]$ is nef of numerical dimension one. When the centers are nine distinct points $p_1, p_2, \ldots, p_9$ of Z_0, the identification $Z \cong Z_0$ gives $[Z]|_Z \cong [3H|_{Z_0} - p_1 - \cdots - p_9]$, where H is a line in $\mathbb{P}^2$ not containing Z_0.

Fixing the first eight blow-ups leaves a curve Z_8 with normal bundle L whose pullback to the normalization has degree one. Blowing up a non-singular point $q \in Z_8$ replaces this bundle

by $L \otimes [-q]$. As q varies, this realizes all of $\mathrm{Pic}^0(Z_8)$: this is the Abel–Jacobi description in the smooth case and the corresponding description of the generalized Jacobian in the nodal case (See [Koi12, Example 1.2] and [Koi10, §4.2]). Allowing infinitely near centers is important here; requiring nine distinct points in the original plane excludes finitely many values of the ninth point. When $[Z]|_Z$ has finite order m, the map defined by the global sections of $[Z]^m$ gives an elliptic fibration $X \to \mathbb{P}^1$ with mZ as a fiber, the classical Halphen situation (See [Br, p. 441] for the smooth case and [Koi10, pp. 233–234] for the nodal case).

6.3 Flatness and comparison of neighborhoods

Suppose first that Z_0 is smooth. Then $[Z]|_Z$ is unitary flat, but this alone does not provide a holomorphic tubular neighborhood. For non-torsion $[Z]|_Z$, the formal neighborhood is isomorphic to that of the zero section in $[Z]|_Z$; Ogus asked whether the corresponding analytic comparison always exists ([Ogu, Problem (4.16)]). Under a Diophantine condition, Arnol'd's theorem (Theorem IV.9.1) provides this comparison. Its proof here uses Ueda's linearization of the defining functions (Theorem IV.8.13), followed by a separate correction of the coordinate along the curve. Such tubular neighborhoods are used in Koike–Uehara's gluing construction of K3 surfaces, recalled in Chapter IV, Remark IV.9.4. The related construction in [KUeh1] also yields projective non-Kummer K3 surfaces containing compact Levi-flat hypersurfaces.

Brunella's theorem concerns a different comparison. Assuming that $X \setminus Z$ contains no compact complex curve, he proved that a C^∞ Kähler metric on X with semi-positive Ricci curvature exists if and only if Z has a *pseudoflat neighborhood system* ([Br, Theorem 1]). Such a system is described by adjoining Z to sufficiently large superlevel sets of a pluriharmonic function F on a punctured neighborhood, where $F + \log|w|^2$ extends pluriharmonically across Z for each local defining function w of Z. By the Aubin–Yau theorem, the metric condition is equivalent to semi-positivity of K_X^{-1}. The relation between semi-positivity of $[Z]$ and its unitary flatness near Z was subsequently established for smooth connected hypersurfaces with unitary flat normal bundle in compact Kähler manifolds in [Koi11, Corollary 1.5]; Theorem V.2.5 gives the criterion for effective nef divisors of numerical dimension one, without requiring smooth or reduced support. Under the same no-compact-curve assumption, Brunella also proved that no *real-analytic Kähler metric with semi-positive Ricci curvature* exists. Thus the regularity of the metric is a further issue, even when a C^∞ metric exists. This is a statement about Kähler metrics and their Ricci curvature, not a non-existence statement for arbitrary real-analytic Hermitian metrics on a line bundle.

For a nodal Z_0, instead, $\mathrm{Pic}^0(Z) \cong \mathbb{C}^*$. If $a \in \mathbb{C}^*$ corresponds to $[Z]|_Z$, then $[Z]|_Z$ is unitary flat precisely when $|a| = 1$. Thus degree zero does not even imply unitary flatness of the normal bundle in this case. For the nine-point examples with $|a| \neq 1$, Ueda constructed a $\mathbb{C}^*$-flat extension F of $[Z]|_Z$ near Z such that $[Z]$ and F are formally isomorphic along Z but are not holomorphically isomorphic on any sufficiently small neighborhood ([U2, §4, Example 2 and Remark 3]). Here F is not unitary flat, and the comparison concerns these two line bundles on a fixed neighborhood, rather than an isomorphism of neighborhoods themselves. The nef but non-semi-positive anticanonical bundles in [Koi4, Theorem 1.1(ii)

and Corollary 1.3] give a further manifestation of the same distinction. For the nodal anti-canonical curves under consideration, [Koi10, Theorem 1.2] characterizes semi-positivity by unitary flatness of $[Z]|_Z$ (The cited theorem also treats cycles of rational curves). In the smooth case, [Koi10, Theorem 1.3] gives semi-positivity when the image of the unitary monodromy representation of $[Z]|_Z$ has free abelian rank at most one, a condition which includes non-Diophantine bundles.

Two preprints announced further results in 2026. For every fixed smooth plane cubic, Filip–Tosatti [FT2, Theorem 1.3.2] assert that the space of ordered configurations of nine distinct points contains a dense G_δ subset for which $[Z]|_Z$ is non-torsion and the embedding $Z \subset X$ is not linearizable. The topology here is the usual analytic topology on the open configuration space in Z_0^9, not the topology of a family obtained by fixing eight points. This answers the analytic comparison question above by producing non-linearizable examples. Chen–Filip–Sun–Tosatti–Zhang [CFSTZ, Corollary 1.2] assert that K_X^{-1} is semi-positive for every nine-point blow-up along a smooth cubic, including infinitely near centers. These preprints have appeared only recently. While we mention their results as important recent developments, the author has not yet examined the details of their proofs. Taken together with Theorem V.2.5, the announced results give examples for which $[Z]$ is unitary flat on a neighborhood of Z, but the whole neighborhood is not linearizable.

6.4 Cohomology of the complement

Return to a smooth cubic Z_0 and its strict transform $Z \subset X$, and put $M := X \setminus Z$. For non-torsion $[Z]|_Z$, recall the number $\alpha([Z]|_Z)$ in (242). Thus $[Z]|_Z$ is asymptotically positive precisely when there are $c, b > 0$ such that $d(\mathbb{I}_Z, ([Z]|_Z)^n) \geq ce^{-bn}$ for every $n \geq 1$, where d is the invariant distance induced from a Euclidean norm on the universal cover of $\mathrm{Pic}^0(Z)$. Equivalently, there are $\varepsilon, \delta > 0$ with $d(\mathbb{I}_Z, ([Z]|_Z)^n) > \varepsilon\delta^n$ for every $n \geq 1$, which is the condition used in [Koi12, Theorem 1.1]. Asymptotically zero means $\alpha([Z]|_Z) = 0$, not that the sequence $d(\mathbb{I}_Z, ([Z]|_Z)^n)$ converges to zero.

Using the semi-positivity announced in [CFSTZ, Corollary 1.2], the results of [Koi12, Theorem 1.1 and Proposition 2.6] give the following description over the whole of $\mathrm{Pic}^0(Z)$:

- If $[Z]|_Z$ is torsion, then $H^1(M, \mathcal{O}_M)$ is infinite-dimensional and Hausdorff.
- If $[Z]|_Z$ is non-torsion and asymptotically positive, then $H^1(M, \mathcal{O}_M) = 0$.
- If $[Z]|_Z$ is non-torsion and asymptotically zero, then $H^1(M, \mathcal{O}_M)$ is infinite-dimensional and non-Hausdorff.

Here the topology is the Dolbeault quotient topology. To check the application, X is a smooth compact rational surface, Z is a smooth connected reduced anticanonical divisor, and $\dim H^1(X, \mathcal{O}_X) = 0$. For non-torsion $[Z]|_Z$, [Koi12, Theorem 1.1] says that the finite-dimensional alternative has dimension $\dim H^1(X, \mathcal{O}_X)$, and that the other alternative is infinite-dimensional and non-Hausdorff. For torsion $[Z]|_Z$ one uses [Koi12, Proposition 2.6] instead: it gives infinite dimension and bounds the dimension of the closure of zero by

$\dim H^1(X, \mathcal{O}_X)$, hence gives Hausdorffness here. In this torsion case the semi-positivity already follows from the elliptic fibration, independently of the 2026 announcement.

Theorem 1.1 of [Koi12] itself is not restricted to a one-parameter subfamily: its input is semi-positivity, along with the geometric hypotheses just checked. Before the new semi-positivity announcement, [Koi12, Corollary 1.3] already gave this trichotomy for the bundles whose unitary monodromy, on a fixed pair of generators of $\pi_1(Z)$, is 1 and $e^{2\pi\sqrt{-1}\theta}$, with $\theta \in \mathbb{R}$. That case uses [Koi10, Theorem 1.3]. The new input thus extends the application from this circle of bundles to all of $\mathrm{Pic}^0(Z)$; it does not change the cohomological theorem. For the same circle subfamily, the relation with the $\partial\overline{\partial}$-lemma is also described in [Koi12, Proposition 7.2].

The comparison with the toroidal surface in Theorem III.2.198 is useful: there the finite-dimensional alternative is one-dimensional, whereas it is zero here. Its ruled-surface compactification S_θ, constructed in the proof of that theorem, has $\dim H^1(S_\theta, \mathcal{O}_{S_\theta}) = 1$, rather than the value zero for our rational surface (See [Koi12, §7.1]).

Problem V.6.1. *Investigate higher-dimensional analogues of the phenomena and problems in this section for blow-ups of del Pezzo manifolds of degree one at a point.* □

For the construction of these blow-ups and their distinguished elliptic curves, see [Koi6, §6.3].

Bibliography

[AK] Y. Abe, K. Kopfermann, Toroidal Groups: Line Bundles, Cohomology and Quasi-Abelian Varieties, Lecture Notes in Mathematics **1759**, Springer-Verlag, Berlin (2001).

[Ab1] W. Abikoff, The Real Analytic Theory of Teichmüller Space, Lecture Notes in Mathematics **820**, Springer-Verlag, Berlin (1980).

[Ab2] W. Abikoff, The uniformization theorem, Amer. Math. Monthly **88** (1981), no. 8, 574–592.

[Ad] M. Adachi, *On the ampleness of positive CR line bundles over Levi-flat manifolds*, Doctoral thesis, Nagoya University, 2013.

[Ah] L. V. Ahlfors, Complex Analysis, McGraw-Hill (1966).

[AS] M. Andersson, H. Samuelsson Kalm, On proper intersections on a singular analytic space, Math. Z. **309** (2025), article 23.

[AG] A. Andreotti, H. Grauert, Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France, **90** (1962), 193–259.

[Ar] N. Aronszajn, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, J. Math. Pures Appl. **36** (1957), 235–249.

[AM] M. F. Atiyah, I. G. Macdonald, Introduction to commutative algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. (1969).

[Au] M. Audin, Fatou, Julia, Montel: The Great Prize of Mathematical Sciences of 1918, and Beyond, Lecture Notes in Mathematics **2014**, Springer, Berlin, Heidelberg (2011).

[ABC] A. Avila, X. Buff, A. Chéritat, Siegel disks with smooth boundaries, Acta Math. **193** (2004), 1–30.

[Bana] S. Banach, Théorie des opérations linéaires, Warszawa (1932).

[Bani] C. Bănică, Le complété formel d'un espace analytique le long d'un sous-espace: un théorème de comparaison, Manuscr. math. **6** (1972), 207–244.

[BS] C. Bănică, O. Stănăşilă, Algebraic Methods in the Global Theory of Complex Spaces, revised English edition, Editura Academiei, Bucharest; John Wiley & Sons, London (1976).

[Bar] D. E. Barrett, Complex analytic realization of Reeb's foliation of S^3, Math. Z. **203** (1990), 355–361.

[BHPV] W. Barth, K. Hulek, C. Peters, A. Van de Ven, Compact complex surfaces, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 4. Folge, **4**, Springer-Verlag, Berlin (2004).

[BM1] E. Bierstone, P. D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, arXiv:alg-geom/9508005v2, Chapter I. Published in Invent. Math. **128** (1997), 207–302; theorem numbers cited here refer to the arXiv version.

[BM2] E. Bierstone, P. D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. **128** (1997), 207–302.

[BM3] E. Bierstone, P. D. Milman, Resolution of singularities, in Several Complex Variables, Mathematical Sciences Research Institute Publications **37**, Cambridge University Press (1999), 43–78.

[Bi] J. Bingener, Über formale komplexe Räume, Manuscr. math. **24** (1978), 253–293.

[BT] R. Bott, L. W. Tu, Differential Forms in Algebraic Topology, Graduate Texts in Mathematics **82**, Springer, Berlin, Heidelberg, New-York (1982).

[Bouc] S. Boucksom, Divisorial Zariski decompositions on compact complex manifolds, Ann. Sci. École Norm. Sup. (4) **37**, 1 (2004), 45–76.

[Bour] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 4–6, Hermann, Paris (1968).

[Bro] K. S. Brown, Lectures on the Cohomology of Groups, lecture notes. https://pi.math.cornell.edu/~kbrown/papers/cohomology_hangzhou.pdf.

[Br] M. Brunella, On Kähler surfaces with semipositive Ricci curvature, Riv. Mat. Univ. Parma, **1** (2010), 441–450.

[BC] X. Buff, A. Chéritat, Upper bound for the size of quadratic Siegel disks, Invent. Math. **156** (2004), 1–24. The theorem and section numbers cited here refer to arXiv:math/0305080v1.

[CC] A. Candel, L. Conlon, Foliations I, Graduate Studies in Mathematics **23**, Amer. Math. Soc. Providence, R.I. (2000).

[CG] L. Carleson, T. W. Gamelin, Complex Dynamics, Universitext, Springer-Verlag, New York (1993).

[Cas] A. Cassa, Coomologia separata sulle varietà analitiche complesse, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **25** (1971), no. 2, 291–323.

[CFSTZ] Y. Chen, S. Filip, S. Sun, V. Tosatti, J. Zhang, Compact Kähler surfaces with semipositive anticanonical bundle, arXiv:2609.26716v1.

[CJLO] R. Cheng, L. Ji, M. Larson, N. Olander, Theorem of the Base, in *Stacks Project Expository Collection*, London Math. Soc. Lecture Note Ser. **480**, Cambridge University Press, Cambridge (2022), 163–193. Authors' version of July 1, 2021: https://chngr.github.io/assets/theorem-of-the-base.pdf.

[Ch] A. Chéritat, Nouvelle preuve d'un théorème de Yoccoz, C. R. Acad. Sci. Paris, Ser. I **338** (2004), no. 4, 301–304.

[CLPT] B. Claudon, F. Loray, J. V. Pereira, F. Touzet, Compact leaves of codimension one holomorphic foliations on projective manifolds, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 6, 1457–1506.

[CJ] M. Colţoiu, C. Joiţa, Finite coverings of complex spaces by connected Stein open sets, Math. Z. **287** (2017), 929–946.

[Cr] H. Cremer, Zum Zentrumproblem, Math. Ann. **98** (1928), 151–163.

[De1] J.-P. Demailly, Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 3, 457–511.

[De2] J.-P. Demailly, Cohomology of q-convex spaces in top degrees, Math. Z. **204** (1990), 283–295.

[De3] J.-P. Demailly, Analytic Methods in Algebraic Geometry, 2nd ed. Surv. Modern Math. 1. Somerville: International Press, 2012.

[De4] J.-P. Demailly, Complex Analytic and Differential Geometry, available online.

[DP] J.-P. Demailly, M. Păun, Numerical characterization of the Kähler cone of a compact Kähler manifold, Ann. of Math. (2) **159** (2004), 1247–1274. The cited version is arXiv:math/0105176v2.

[DPS1] J.-P. Demailly, Th. Peternell, M. Schneider, Compact complex manifolds with numerically effective tangent bundles, J. Algebraic Geom. **3**, 2 (1994), 295–345.

[DPS2] J.-P. Demailly, Th. Peternell, M. Schneider, Pseudo-effective line bundles on compact Kähler manifolds, Internat. J. Math. **12** (2001), 689–741. The cited version is arXiv:math/0006205v2.

[DO1] K. Diederich, T. Ohsawa, A Levi problem on two-dimensional complex manifolds, Math. Ann. **261** (1982), 255–261.

[DO2] K. Diederich and T. Ohsawa, *Harmonic mappings and disc bundles over compact Kähler manifolds*, Publ. Res. Inst. Math. Sci. **21** (1985), 819–833.

[DS] J. Dieudonné, L. Schwartz, La dualité dans les espaces $(\mathcal{F})$ et $(\mathcal{LF})$, Ann. Inst. Fourier **1** (1949), 61–101.

[Doc] M. P. Do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall (1976).

[Dol] A. Dold, Lectures on Algebraic Topology, Springer-Verlag (1980).

[Dy] S. Dyatlov, Lecture notes for 18.155: distributions, elliptic regularity, and applications to PDEs, 10 December 2022. https://math.mit.edu/~dyatlov/18.155/155-notes.pdf.

[E] C. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable, Séminaire Bourbaki, Exp. **24** (1952), 153–168.

[Fa] H. Fang, A geometric criterion for prescribing residues and some applications, Ann. Inst. Fourier **71** (2021), no. 5, 1963–2018.

[FT1] S. Filip, V. Tosatti, Smooth and Rough Positive Currents, Ann. Inst. Fourier (Grenoble) **68**, 7 (2018), 2981–2999.

[FT2] S. Filip, V. Tosatti, Nonlinearizable embeddings of elliptic curves in rational surfaces, arXiv:2605.04160v2.

[FN] J. E. Fornæss, R. Narasimhan, The Levi problem on complex spaces with singularities, Math. Ann. **248** (1980), 47–72.

[FS] J. E. Fornæss and B. Stensønes, *Lectures on Counterexamples in Several Complex Variables*, Mathematical Notes 33, Princeton University Press, 1987; reprinted by AMS Chelsea Publishing, 2007.

[Forste] O. Forster, Lectures on Riemann surfaces, Graduate Texts in Mathematics **81**, Springer-Verlag, New York-Berlin (1981).

[Forstn] F. Forstnerič, Holomorphic embeddings and immersions of Stein manifolds: a survey, in Geometric Complex Analysis, Springer Proceedings in Mathematics & Statistics **246** (2018), 145–169.

[Fri] J. Frisch, Points de platitude d'un morphisme d'espaces analytiques complexes, Invent. Math. **4** (1967), 118–138.

[Fuj] O. Fujino, On Kawamata's theorem, arXiv:0910.1156v1.

[Gam] T. W. Gamelin, Conjugation Theorems via Neumann Series, MRS Report 99-1, UCLA (1999); author-posted version dated 3 September 2003, https://www.math.ucla.edu/~twg/mrs1.pdf.

[GS1] X. Gong, L. Stolovitch, Equivalence of neighborhoods of embedded compact complex manifolds and higher codimension foliations, Arnold Math. J. **8** (2022) 61–145.

[GS2] X. Gong, L. Stolovitch, On neighborhoods of embedded complex tori, Math. Ann. **391** (2025), 2509–2540.

[GM] C. Grant Melles, P. Milman, Classical Poincaré metric pulled back off singularities using a Chow-type theorem and desingularization, Ann. Fac. Sci. Toulouse Math. (6) **15**, 4 (2006), 689–771. Erratum to Lemma 4.5: ibid. **16**, 2 (2007), 425–426,.

[Gr1] H. Grauert, On Levi's problem and the imbedding of real-analytic manifolds, Ann. of Math. (2) **68** (1958), 460–472.

[Gr2] H. Grauert, Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, Publ. Math. Inst. Hautes Études Sci. **5** (1960), 5–64. Correction: Publ. Math. Inst. Hautes Études Sci. **16** (1963), 35–36.

[Gr3] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann. **146** (1962), 331–368.

[Gr4] H. Grauert, *Bemerkenswerte pseudokonvexe Mannigfaltigkeiten*, Math. Z. **81** (1963), 377–391.

[GPR] H. Grauert, T. Peternell, R. Remmert, Several Complex Variables VII, Encyclopaedia of Mathematical Sciences **74**, Springer Berlin, Heidelberg (1994).

[GrR] H. Grauert, R. Remmert, Coherent Analytic Sheaves, Grundlehren der mathematischen Wissenschaften **265**, Springer-Verlag, Berlin, Heidelberg (1984).

[GW] R. E. Greene, H. Wu, Embedding of open riemannian manifolds by harmonic functions, Ann. Inst. Fourier **25**, 1 (1975), 215–235.

[Gri] P. Griffiths, The Extension Problem in Complex Analysis II; Embeddings with Positive Normal Bundle, Amer. J. Math. **88**, 2 (1966), 366–446.

[GH] P. Griffiths, J. Harris, Principles of algebraic geometry, Wiley Classics Library, John Wiley & Sons Inc. (1994).

[GuR] R. C. Gunning, H. Rossi, Analytic functions of several complex variables, Prentice-Hall, Englewood Cliffs (1965).

[HaK] Y. Hashimoto, T. Koike, Ueda's lemma via uniform Hörmander estimates for flat line bundles, Kyoto J. Math. **66**, 1 (2026), 129–176.

[HKM] Y. Hashimoto, T. Koike, S. Matsumura, Uniform L^2-estimates for flat non-trivial line bundles on compact complex manifolds, Complex Manifolds, **12**, 1 (2025), Art. 20240009.

[Ha1] A. Hatcher, Algebraic topology, Cambridge University Press, Cambridge (2002).

[Ha2] A. Hatcher, Vector Bundles and K-Theory, available online.

[He1] M. R. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, Publ. Math. Inst. Hautes Études Sci. **49** (1979), 5–233.

[He2] M. R. Herman, Exemples de fractions rationnelles ayant une orbite dense sur la sphère de Riemann, Bull. Soc. Math. France **112** (1984), 93–142.

[He3] M. R. Herman, Simple proofs of local conjugacy theorems for diffeomorphisms of the circle with almost every rotation number, Bol. Soc. Brasil. Mat. **16** (1985), 45–83.

[Hö] L. Hörmander, An Introduction to Complex Analysis in Several Variables, third revised edition, North-Holland Mathematical Library **7**, North-Holland, Amsterdam (1990).

[HJ] R. A. Horn, C. R. Johnson, Matrix Analysis, 2nd Edition, Cambridge University Press, 2012.

[HoK] G. Hosono, T. Koike, On metrics with minimal singularities of line bundles whose stable base loci admit holomorphic tubular neighborhoods, Ann. Fac. Sci. Toulouse Math, Serie 6 **29**, 1 (2020), 149–175.

[I] B. Iversen, Cohomology of Sheaves, Springer Berlin, Heidelberg (1986).

[J] J. Jost, Compact Riemann Surfaces. An Introduction to Contemporary Mathematics, Universitext, Springer, Berlin (2006).

[Ka] H. Kazama, $\overline{\partial}$ cohomology of (H, C)-groups, Publ. Res. Inst. Math. Sci. **20** (1984), 297–317.

[KaT1] H. Kazama, S. Takayama, $\partial\overline{\partial}$-problem on weakly 1-complete Kähler manifolds, Nagoya Math. J. **155** (1999), 81–94.

[KaT2] H. Kazama, S. Takayama, On the $\partial\overline{\partial}$-equation over pseudoconvex Kähler manifolds, Manuscripta Math. **102** (2000), 25–39.

[Ke] J. L. Kelley, General Topology, Springer, New York (1955).

[KV] R. Kiehl, J.-L. Verdier, Ein einfacher Beweis des Kohärenzsatzes von Grauert, Math. Ann. **195** (1971), 24–50.

[Kim] D. Kim, Extension of pluriadjoint sections from a log-canonical center, Ph.D. thesis, Princeton University (2007).

[Kod] K. Kodaira, Complex manifolds and deformation of complex structures, Springer-Verlag Berlin Heidelberg (2005).

[KodS] K. Kodaira, D. C. Spencer, A Theorem of Completeness of Characteristic Systems of Complete Continuous Systems, Amer. J. Math. **81** 2 (1959), 477–500.

[Kœ] G. Kœnigs, Recherches sur les intégrales de certaines équations fonctionnelles, Ann. Sci. École Norm. Sup. (3) **1** (1884), S3–S41.

[Koi1] T. Koike, Minimal singular metrics of a line bundle admitting no Zariski-decomposition, Tohoku Math. J. (2) **67**, 2 (2015), 297–321.

[Koi2] T. Koike, On minimal singular metrics of certain class of line bundles whose section ring is not finitely generated, Ann. Inst. Fourier (Grenoble) **65**, 5 (2015), 1953–1967.

[Koi3] T. Koike, On the minimality of canonically attached singular Hermitian metrics on certain nef line bundles, Kyoto J. Math. **55**, 3 (2015), 607–616.

[Koi4] T. Koike, Ueda theory for compact curves with nodes, Indiana U. Math. J, Volume **66**, Number 3 (2017), 845–876. Corrections: [KUeh2, Appendix A].

[Koi5] T. Koike, Plurisubharmonic functions on a neighborhood of a torus leaf of a certain class of foliations, Forum Math. **31**, 6 (2019), 1457–1466.

[Koi6] T. Koike, Higher codimensional Ueda theory for a compact submanifold with unitary flat normal bundle, Nagoya Math. J., Volume **238** (2020), 104–136.

[Koi7] T. Koike, Linearization of transition functions of a semi-positive line bundle along a certain submanifold, Ann. Inst. Fourier (Grenoble), **71**, 5 (2021), 2237–2271. arXiv:2002.07830v2.

[Koi8] T. Koike, On the complement of a hypersurface with flat normal bundle which corresponds to a semipositive line bundle, Math. Ann., **383** (2022), 291–313.

[Koi9] T. Koike, Arnol'd's type theorem on a neighborhood of a cycle of rational curves, JAMA, **147** (2022), 41–67.

[Koi10] T. Koike, Hermitian metrics on the anti-canonical bundle of the blow-up of the projective plane at nine points, Ann. Fac. Sci. Toulouse Math, Serie 6 **32**, 2 (2023), 231–285.

[Koi11] T. Koike, Holomorphic foliation associated with a semi-positive class of numerical dimension one, J. Differential Geom. **127**, 3 (2024), 1207–1243.

[Koi12] T. Koike, $\overline{\partial}$ cohomology of the complement of a semi-positive anticanonical divisor of a compact surface, Math. Z. **308**, 22 (2024).

[Koi13] T. Koike, Formal principle for line bundles on neighborhoods of an analytic subset of a compact Kähler manifold, arXiv:2409.20367v3.

[Koi14] T. Koike, Pluripotential geometry on semi-positive effective divisors of numerical dimension one, arXiv:2511.13903v2.

[KO1] T. Koike, N. Ogawa, Local criteria for non embeddability of Levi-flat manifolds, J. Geom. Anal **28**, 2 (2018), 1052–1077.

[KO2] T. Koike, N. Ogawa, On the neighborhood of a torus leaf and dynamics of holomorphic foliations, Tohoku Math. J. (2) **77**, 3 (2025), 421–443.

[KoiS] T. Koike, L. Stolovitch, On neighborhoods of projective space bundles over elliptic curves, arXiv:2501.11497.

[KTa] T. Koike, M. Takakura, Rigidity of Pseudoconvex Neighborhoods and Ohsawa–Takegoshi Extension, arXiv:2610.29082.

[KoT] T. Koike, J. Tanaka, Cohomology groups with compact support for flat line bundles on certain complex Lie groups, to appear in Hiroshima Math. J.

[KUed1] T. Koike, T. Ueda, Plurisubharmonic functions on affine line bundles over compact Kähler manifolds, Publ. Res. Inst. Math. Sci. **61**, 1 (2025), 139–152.

[KUed2] T. Koike, T. Ueda, Intersection matrix and neighborhood structure of complex curves, preprint.

[KUeh1] T. Koike, T. Uehara, A gluing construction of projective K3 surfaces, Epijournal de Geometrie Algebrique **6** (2022),.

[KUeh2] T. Koike, T. Uehara, Neighborhood of a rational curve with an ordinary cusp, arXiv:2407.01877v1, including Appendix A, "Erratum to 'Ueda theory for compact curves with nodes'."

[KUeh3] T. Koike, T. Uehara, A gluing construction of K3 surfaces, Michigan Math. J., Advance Publication (2025), 1–53.

[Kom] H. Komatsu, Projective and injective limits of weakly compact sequences of locally convex spaces, J. Math. Soc. Japan **19**, 3 (1967), 366–383.

[Lan1] S. Lang, Complex Analysis, Second Edition, Graduate Texts in Mathematics **103**, Springer-Verlag, 1985.

[Lan2] S. Lang, Algebra, Graduate Texts in Mathematics **211**, Springer-Verlag, 2002.

[Lau] H. B. Laufer, On Serre duality and envelopes of holomorphy, Trans. Amer. Math. Soc. **128** (1967), 414–436.

[LL] C. Laurent-Thiébaut, J. Leiterer, A separation theorem and Serre duality for the Dolbeault cohomology, Ark. Mat. **40** (2002), 301–321.

[Laz] R. Lazarsfeld, *Positivity in Algebraic Geometry I*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **48**, Springer, Berlin (2004).

[Lee-Je] Jeffrey M. Lee, Manifolds and Differential Geometry, Graduate Studies in Mathematics **107**, American. Mathematical Society (2009).

[Lee-Jo1] John M. Lee, Introduction to Topological Manifolds, 2nd ed., Graduate Texts in Mathematics **202**, Springer (2011).

[Lee-Jo2] John M. Lee, Introduction to Smooth Manifolds, 2nd ed., Graduate Texts in Mathematics **218**, Springer (2013).

[Lee-Jo3] John M. Lee, Introduction to Riemannian manifolds, Graduate Texts in Mathematics **176**, Springer, Cham (2018).

[Lei] T. Leinster, Basic Category Theory, arXiv:1612.09375v2.

[Leq] F. Lequen, Presque toute surface K3 contient une infinité d'hypersurfaces Levi-plates linéaires, J. Éc. polytech. Math. **10** (2023), 815–836.

[Ll] R. de la Llave, A simple proof of a particular case of C. Siegel's center theorem, J. Math. Phys. **24** (1983), 2118–2121.

[Ło1] S. Łojasiewicz, Triangulation of semi-analytic sets, Ann. Scuola Norm. Sup. di Pisa, ser. 3, **18**, 4 (1964), 449–474.

[Ło2] S. Łojasiewicz, Ensembles semi-analytiques, Institut des Hautes Études Scientifiques, Bures-sur-Yvette, lecture notes (July 1965). LaTeX transcription by M. Coste, version of 29 August 2006.

[Mac] S. Mac Lane, Categories for the Working Mathematician, 2nd ed., Graduate Texts in Mathematics **5**, Springer (1997).

[Mar] S. Marmi, An Introduction to Small Divisors Problems, arXiv:math/0009232v1.

[Mat] H. Matsumura, Commutative ring theory, 2nd ed., Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, 1989.

[Mc] J. McCleary, A User's Guide to Spectral Sequences, 2nd ed., Cambridge University Press (2010).

[Mi1] J. Milnor, Morse Theory (Based on lecture notes by M. Spivak and R. Wells), Ann. of Math. Stud. **51**, Princeton Univ. Press,. Princeton (1963).

[Mi2] J. Milnor, Topology from the differentiable viewpoint, Princeton Landmarks in Mathematics, Princeton University Press, Princeton (1997).

[Mi3] J. Milnor, Dynamics in One Complex Variable, third edition, Annals of Mathematics Studies **160**, Princeton University Press, Princeton (2006).

[MST] S. Mongodi, Z. Słodkowski, G. Tomassini, Weakly complete complex surfaces, Indiana Univ. Math. J. **67**, 2 (2018), 899–935.

[MT] S. Mongodi, G. Tomassini, Minimal kernels and compact analytic objects in complex surfaces, in *Advancements in Complex Analysis*, D. Breaz and M. T. Rassias (eds.), Springer, Cham (2020), 329–362.

[Mu1] J. R. Munkres, Elementary Differential Topology, Ann. of Math. Stud. **54**, Princeton Univ. Press,. Princeton (1966).

[Mu2] J. R. Munkres, Topology, 2nd edition, Prentice-Hall (2000).

[Nee] A. Neeman, Ueda theory: theorems and problems, Mem. Amer. Math. Soc. **81** (1989), no. 415.

[Nem] S. Yu. Nemirovski, *Stein domains with Levi-flat boundaries on compact complex surfaces*, Math. Notes **66** (1999), 522–525.

[No1] J. Noguchi, Analytic Function Theory of Several Variables: Elements of Oka's Coherence, Springer, Singapore (2016), corrected publication (2023).

[No2] J. Noguchi, Basic Oka Theory in Several Complex Variables, Universitext, Springer Nature Singapore (2024).

[Og] S. Ogawa, On holomorphic tubular neighborhoods of compact Riemann surfaces, J. Geom. Anal. **35** (2025), article 6. The theorem and section numbers cited here refer to arXiv:2402.07050v1.

[Ogu] A. Ogus, The formal Hodge filtration, Invent. Math. **31** (1976), 193–228.

[Oh1] T. Ohsawa, *Finiteness theorems on weakly* 1*-complete manifolds*, Publ. Res. Inst. Math. Sci. **15** (1979), 853–870.

[Oh2] T. Ohsawa, *Supplement to "On the Levi-flats in Complex Tori of Dimension Two"*, Publ. Res. Inst. Math. Sci. **42** (2006), 379–382.

[Oh3] T. Ohsawa, *On the complement of Levi-flats in Kähler manifolds of dimension* ≥ 3, Nagoya Math. J. **185** (2007), 161–169.

[Oh4] T. Ohsawa, *Hartogs type extension theorems on some domains in Kähler manifolds*, Ann. Polon. Math. **106** (2012), 243–254.

[Oh5] T. Ohsawa, L^2 Approaches in Several Complex Variables: Development of Oka–Cartan Theory by L^2 Estimates for the $\overline{\partial}$ Operator, Springer (2015).

[Oh6] T. Ohsawa, *A remark on Hörmander's isomorphism*, in *Complex Analysis and Geometry*, Springer Proc. Math. Stat. **144**, Springer, 2015, 273–280.

[OP] T. Ohsawa, T. Pawlaschyk, Analytic Continuation and q-Convexity, SpringerBriefs in Mathematics, Springer, Singapore (2022).

[Ok] K. Oka, Sur les fonctions analytiques de plusieurs variables. IX. Domaines finis sans point critique intérieur, Japanese J. Math. **23** (1953), 97–155.

[Os] M. S. Osborne, Basic Homological Algebra, Graduate Texts in Mathematics **196**, Springer (2000).

[P] R. Pérez-Marco, Fixed points and circle maps, Acta Math. **179** (1997), no. 2, 243–294.

[Pet] Th. Peternell, Vektorraumbündel in der Nähe von kompakten komplexen Unterräumen, Math. Ann. **257** (1981), 111–134.

[Pom] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Grundlehren der mathematischen Wissenschaften **299**, Springer-Verlag, Berlin (1992).

[Re] J. Reppekus, Small divisors in discrete local holomorphic dynamics, Boll. Unione Mat. Ital. **16** (2023), 173–202.

[Ri] R. Richberg, Stetige streng pseudokonvexe Funktionen, Math. Ann. **175** (1968), 257–286.

[Riem] O. Riemenschneider, Lecture notes, Chapter 9: Blowing down and the Grauert–Mumford criterion, §9.8, Theorem 9.14. https://www.math.uni-hamburg.de/home/riemenschneider/bookchap09.pdf.

[Roc] R. T. Rockafellar, Convex analysis, Princeton University Press, Princeton (1970).

[Ros] H. Rossi, Strongly pseudoconvex manifolds, Lectures in Modern Analysis and Applications I, Lecture Notes in Mathematics **103**, (1969), 10–29.

[Ru1] W. Rudin, Real and Complex Analysis, 3rd edition, McGraw-Hill (1987).

[Ru2] W. Rudin, Functional Analysis, 2nd edition, McGraw-Hill (1991).

[Sa] J. H. Sampson, Some properties and applications of harmonic mappings, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 2, 211–228.

[SW] H. H. Schaefer, M. P. Wolff, Topological Vector Spaces, Graduate Texts in Mathematics **3** (second edition), Springer-Verlag, New. York (1999).

[Se1] J.-P. Serre, Un théorème de dualité, Comment. Math. Helv. **29** (1955), 9–26.

[Se2] J.-P. Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier **6** (1956), 1–42.

[Sie] C. L. Siegel, Iteration of analytic functions, Ann. of Math., **43** (1942), 607–612.

[Siu1] Y.-T. Siu, Noetherianness of rings of holomorphic functions on Stein compact subsets, Proc. Amer. Math. Soc. **21** (1969), 483–489.

[Siu2] Y. T. Siu, Every Stein subvariety admits a Stein neighborhood, Invent. Math., **38** (1976), 89–100.

[Sp] M. Spivak, Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus, Addison-. Wesley Publishing Company (1965).

[St] The Stacks Project Authors, Stacks Project, https://stacks.math.columbia.edu/ (accessed July 13, 2026).

[Ta] A. Takeuchi, *Domaines pseudoconvexes infinis et la métrique riemannienne dans un espace projectif*, J. Math. Soc. Japan **16** (1964), 159–181.

[Tos] V. Tosatti, Semipositive line bundles and $(1,1)$-classes, arXiv:2309.00580.

[Tot] B. Totaro, The topology of smooth divisors and the arithmetic of abelian varieties, Michigan Math. J. **48** (2000), 611–624.

[Tr] F. Trèves, Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, (1967).

[Ts1] H. Tsuji, Complex structures on $S^3 \times S^3$, Tohoku Math. J. (2) **36** (1984), 351–376.

[Ts2] H. Tsuji, Analytic Zariski Decomposition, RIMS Kôkyûroku **819** (1993), 203–217.

[U1] T. Ueda, On the neighborhood of a compact complex curve with topologically trivial normal bundle, J. Math. Kyoto Univ. **22** (1983), 583–607.

[U2] T. Ueda, Neighborhood of a rational curve with a node, Publ. RIMS, Kyoto Univ. **27** (1991), 681–693.

[V] J. Voigt, A Course on Topological Vector Spaces, Compact Textbooks in Mathematics, Birkhäuser Cham (2020).

[Weib] C. Weibel, An introduction to homological algebra, Cambridge studies in advanced mathematics **38**, Cambridge Univ. Press (1994).

[Weil] A. Weil, Sur les théorèmes de de Rham, Comment. Math. Helv. **26** (1952), 119–145.

[Wh] R. Whitley, Projecting m onto c_0, Amer. Math. Monthly **73** (1966), 285–286.

[Wł] J. Włodarczyk, Resolution of singularities of analytic spaces, in Proceedings of Gökova Geometry-Topology Conference 2008, Gökova Geometry/Topology Conference (GGT), Gökova (2009), 31–63.

[Yau] S.-T. Yau, On the curvature of compact Hermitian manifolds, Invent. Math. **25** (1974), 213–239.

[Y1] J.-C. Yoccoz, Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne, Ann. Sci. Éc. Norm. Supér. (4) **17** (1984), no. 3, 333–359.

[Y2] J.-C. Yoccoz, Théorème de Siegel, nombres de Bruno et polynômes quadratiques, Astérisque **231** (1995), 3–88.

[Y3] J.-C. Yoccoz, Analytic linearization of circle diffeomorphisms, in *Dynamical Systems and Small Divisors*, Lecture Notes in Math. **1784**, Springer (2002), 125–173.

[Zha] Y. Zhang, On the Non-Semipositivity of a Nef and Big Line Bundle on Grauert's Example, arXiv:2512.23446v1.

Appendix

1 A class of double complexes with controlled higher exactness

1.1 Set-up and notation

Here we fix a ring R and a first quadrant double complex

$$\big(C^{p,q}, d, \delta\big)_{p,q\geq 0}$$

of R–modules. Thus we have R–linear maps

$$d\colon C^{p,q} \longrightarrow C^{p,q+1}, \qquad \delta\colon C^{p,q} \longrightarrow C^{p+1,q},$$

for all $p, q \geq 0$, satisfying

$$d^2 = 0, \qquad \delta^2 = 0, \qquad d \circ \delta = \delta \circ d.$$

We use commuting unsigned differentials; the sign needed for the total complex will be inserted explicitly below. The conversion from anticommuting differentials, and other sign conventions used in applications, are recorded in Remark A.1.4. We only consider the first quadrant, i.e., $C^{p,q} = 0$ if $p < 0$ or $q < 0$.

For each $p, q \geq 0$ we write $d^{p,q}\colon C^{p,q} \to C^{p,q+1}$ and $\delta^{p,q}\colon C^{p,q} \to C^{p+1,q}$ when we wish to stress the bidegree.

We define

$$E^p := \{x \in C^{p,0} \mid d^{p,0}(x) = 0\} = \mathrm{Ker}\big(d^{p,0}\colon C^{p,0} \to C^{p,1}\big),$$
$$F^q := \{y \in C^{0,q} \mid \delta^{0,q}(y) = 0\} = \mathrm{Ker}\big(\delta^{0,q}\colon C^{0,q} \to C^{1,q}\big).$$

Thus E^p consists of those elements in the bottom row that are closed for d, and F^q consists of those elements in the leftmost column that are closed for δ.

The relation $d \circ \delta = \delta \circ d$ implies

$$d^{p+1,0} \circ \delta^{p,0} = \delta^{p,1} \circ d^{p,0}, \quad \text{for all } p \geq 0,$$

so $\delta^{p,0}(E^p) \subset E^{p+1}$. We set

$$\delta_E^p := \delta^{p,0}\big|_{E^p}\colon E^p \longrightarrow E^{p+1},$$

and obtain a cochain complex

$$\cdots \longrightarrow E^{p-1} \xrightarrow{\delta_E^{p-1}} E^p \xrightarrow{\delta_E^p} E^{p+1} \longrightarrow \cdots .$$

Similarly,

$$\delta^{0,q+1} \circ d^{0,q} = d^{1,q} \circ \delta^{0,q}, \quad \text{for all } q \geq 0,$$

implies $d^{0,q}(F^q) \subset F^{q+1}$, and we set

$$d_F^q := d^{0,q}\big|_{F^q} \colon F^q \longrightarrow F^{q+1},$$

so that we have a cochain complex

$$\cdots \longrightarrow F^{q-1} \xrightarrow{d_F^{q-1}} F^q \xrightarrow{d_F^q} F^{q+1} \longrightarrow \cdots .$$

We write

$$H^m(E^\bullet) := \begin{cases} \operatorname{Ker}\bigl(\delta_E^0 \colon E^0 \to E^1\bigr) & \text{if } m = 0 \\ \operatorname{Ker}\bigl(\delta_E^m \colon E^m \to E^{m+1}\bigr) \Big/ \operatorname{Image}\bigl(\delta_E^{m-1} \colon E^{m-1} \to E^m\bigr) & \text{otherwise} \end{cases},$$

$$H^m(F^\bullet) := \begin{cases} \operatorname{Ker}\bigl(d_F^0 \colon F^0 \to F^1\bigr) & \text{if } m = 0 \\ \operatorname{Ker}\bigl(d_F^m \colon F^m \to F^{m+1}\bigr) \Big/ \operatorname{Image}\bigl(d_F^{m-1} \colon F^{m-1} \to F^m\bigr) & \text{otherwise} \end{cases}.$$

We equip the total complex

$$\operatorname{Tot}^m C := \bigoplus_{p+q=m} C^{p,q}, \qquad m \geq 0,$$

with the differential

$$\Delta\big|_{C^{p,q}} := \delta^{p,q} + (-1)^p d^{p,q} \colon C^{p,q} \longrightarrow C^{p+1,q} \oplus C^{p,q+1},$$

so that $(\operatorname{Tot}^\bullet C, \Delta)$ is a cochain complex (the condition $d \circ \delta = \delta \circ d$ ensures $\Delta^2 = 0$).

We denote

$$Z^m(\operatorname{Tot} C) := \operatorname{Ker}\bigl(\Delta \colon \operatorname{Tot}^m C \to \operatorname{Tot}^{m+1} C\bigr),$$
$$B^m(\operatorname{Tot} C) := \operatorname{Image}\bigl(\Delta \colon \operatorname{Tot}^{m-1} C \to \operatorname{Tot}^m C\bigr),$$

and

$$H^m(\operatorname{Tot} C) := \begin{cases} Z^0(\operatorname{Tot} C) & \text{if } m = 0 \\ Z^m(\operatorname{Tot} C) \big/ B^m(\operatorname{Tot} C) & \text{otherwise} \end{cases}.$$

For each $m \geq 0$ the inclusions

$$E^m \subset C^{m,0} \subset \operatorname{Tot}^m C, \qquad F^m \subset C^{0,m} \subset \operatorname{Tot}^m C$$

define cochain maps

$$E^\bullet \longrightarrow \operatorname{Tot}^\bullet C, \qquad F^\bullet \longrightarrow \operatorname{Tot}^\bullet C,$$

and hence homomorphisms in cohomology

$$\alpha_m \colon H^m(E^\bullet) \longrightarrow H^m(\operatorname{Tot} C), \qquad \beta_m \colon H^m(F^\bullet) \longrightarrow H^m(\operatorname{Tot} C).$$

For the reader's convenience, we display a portion of the double complex (up to bidegree $(4,4)$):

$$\begin{array}{ccccccccccc}
 & & E^0 & \xrightarrow{\delta_E^0} & E^1 & \xrightarrow{\delta_E^1} & E^2 & \xrightarrow{\delta_E^2} & E^3 & \xrightarrow{\delta_E^3} & E^4 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F^0 & \to & C^{0,0} & \xrightarrow{\delta^{0,0}} & C^{1,0} & \xrightarrow{\delta^{1,0}} & C^{2,0} & \xrightarrow{\delta^{2,0}} & C^{3,0} & \xrightarrow{\delta^{3,0}} & C^{4,0} \\
\downarrow d_F^0 & & \downarrow d^{0,0} & & \downarrow d^{1,0} & & \downarrow d^{2,0} & & \downarrow d^{3,0} & & \downarrow d^{4,0} \\
F^1 & \to & C^{0,1} & \xrightarrow{\delta^{0,1}} & C^{1,1} & \xrightarrow{\delta^{1,1}} & C^{2,1} & \xrightarrow{\delta^{2,1}} & C^{3,1} & \xrightarrow{\delta^{3,1}} & C^{4,1} \\
\downarrow d_F^1 & & \downarrow d^{0,1} & & \downarrow d^{1,1} & & \downarrow d^{2,1} & & \downarrow d^{3,1} & & \downarrow d^{4,1} \\
F^2 & \to & C^{0,2} & \xrightarrow{\delta^{0,2}} & C^{1,2} & \xrightarrow{\delta^{1,2}} & C^{2,2} & \xrightarrow{\delta^{2,2}} & C^{3,2} & \xrightarrow{\delta^{3,2}} & C^{4,2} \\
\downarrow d_F^2 & & \downarrow d^{0,2} & & \downarrow d^{1,2} & & \downarrow d^{2,2} & & \downarrow d^{3,2} & & \downarrow d^{4,2} \\
F^3 & \to & C^{0,3} & \xrightarrow{\delta^{0,3}} & C^{1,3} & \xrightarrow{\delta^{1,3}} & C^{2,3} & \xrightarrow{\delta^{2,3}} & C^{3,3} & \xrightarrow{\delta^{3,3}} & C^{4,3} \\
\downarrow d_F^3 & & \downarrow d^{0,3} & & \downarrow d^{1,3} & & \downarrow d^{2,3} & & \downarrow d^{3,3} & & \downarrow d^{4,3} \\
F^4 & \to & C^{0,4} & \xrightarrow{\delta^{0,4}} & C^{1,4} & \xrightarrow{\delta^{1,4}} & C^{2,4} & \xrightarrow{\delta^{2,4}} & C^{3,4} & \xrightarrow{\delta^{3,4}} & C^{4,4}
\end{array}$$

1.2 Exactness hypotheses and the statement

Fix an integer $m \geq 1$. We make the following hypotheses:

($\mathbf{V}_m$) For every pair (p,q) with $p,q \geq 0$, $q > 0$ and $p+q \in \{m-1, m\}$, the vertical sequence

$$C^{p,q-1} \xrightarrow{d^{p,q-1}} C^{p,q} \xrightarrow{d^{p,q}} C^{p,q+1}$$

is exact at $C^{p,q}$, i.e.

$$\operatorname{Ker} d^{p,q} = \operatorname{Image} d^{p,q-1}.$$

($\mathbf{H}_m$) For every pair (p,q) with $p,q \geq 0$, $p > 0$ and $p+q \in \{m-1, m\}$, the horizontal sequence

$$C^{p-1,q} \xrightarrow{\delta^{p-1,q}} C^{p,q} \xrightarrow{\delta^{p,q}} C^{p+1,q}$$

is exact at $C^{p,q}$, i.e.

$$\operatorname{Ker} \delta^{p,q} = \operatorname{Image} \delta^{p-1,q}.$$

Under (V_m) and (H_m) we will show that α_m and β_m are isomorphisms, and hence that we obtain a natural isomorphism $\Phi_m := \beta_m^{-1} \circ \alpha_m$:

THEOREM A.1.1. *Let $m \geq 1$. Assume (V_m) and (H_m). Then the natural maps*

$$\alpha_m \colon H^m(E^\bullet) \longrightarrow H^m(\operatorname{Tot} C), \qquad \beta_m \colon H^m(F^\bullet) \longrightarrow H^m(\operatorname{Tot} C)$$

are isomorphisms. In particular the composite

$$\Phi_m := \beta_m^{-1} \circ \alpha_m \colon H^m(E^\bullet) \xrightarrow{\sim} H^m(F^\bullet)$$

is a well-defined isomorphism.

This type of statement seems often formulated under the simultaneous assumption of (V_m) and (H_m) for all $m \geq 1$, and its proof is an immediate application of the general machinery of *spectral sequences*, see [Mc] for example. Here, however, we prefer to work in a slightly more restricted setting, mainly for two reasons: first, in order to make the argument essentially self-contained and accessible without any prior knowledge of spectral sequences; second, in order to see more precisely which exactness conditions along the diagonals $p + q = m$ and $p + q = m - 1$ are actually needed for the isomorphism in degree m. Although the proof below seems to be essentially the same as the standard spectral–sequence type argument, we present it in an elementary form that makes all steps entirely explicit.

Before giving the proof for general m, we spell out the cases $m = 0, 1, 2$ for the convenience of the reader. In these low degrees the proof is purely elementary and one can see concretely how (V_m) and (H_m) are used.

1.3 The case $m = 0$

For $m = 0$ we simply by definition have

$$H^0(E^\bullet) = H^0(F^\bullet) = H^0(\operatorname{Tot} C) = \operatorname{Ker} d^{0,0} \cap \operatorname{Ker} \delta^{0,0}$$

and the natural maps

$$\alpha_0 \colon H^0(E^\bullet) \longrightarrow H^0(\operatorname{Tot} C), \qquad \beta_0 \colon H^0(F^\bullet) \longrightarrow H^0(\operatorname{Tot} C)$$

are both the identity on $\operatorname{Ker} d^{0,0} \cap \operatorname{Ker} \delta^{0,0}$.

1.4 The case $m = 1$

The portion of the double complex that contributes to the following argument for the case $m = 1$ is as follows:

$$\begin{array}{ccccccc}
 & & E^0 & \xrightarrow{\delta_E^0} & E^1 & \xrightarrow{\delta_E^1} & E^2 \\
 & & \downarrow & & \downarrow & & \downarrow \\
F^0 & \to & C^{0,0} & \xrightarrow{\delta^{0,0}} & C^{1,0} & \xrightarrow{\delta^{1,0}} & C^{2,0} \\
\downarrow d_F^0 & & \downarrow d^{0,0} & & \downarrow d^{1,0} & & \\
F^1 & \to & C^{0,1} & \xrightarrow{\delta^{0,1}} & C^{1,1} & & \\
\downarrow d_F^1 & & \downarrow d^{0,1} & & & & \\
F^2 & \to & C^{0,2} & & & &
\end{array}$$

The total complexes in degree 0 and 1 are

$$\mathrm{Tot}^0 C = C^{0,0}, \qquad \mathrm{Tot}^1 C = C^{1,0} \oplus C^{0,1}.$$

The total differential

$$\Delta^0 = \delta^{0,0} + d^{0,0} \colon C^{0,0} \to C^{1,0} \oplus C^{0,1}$$

and

$$\Delta^1 \colon C^{1,0} \oplus C^{0,1} \to C^{2,0} \oplus C^{1,1} \oplus C^{0,2}$$

are given explicitly by

$$\Delta^1(a, b) = \Big(\delta^{1,0}(a),\ -d^{1,0}(a) + \delta^{0,1}(b),\ d^{0,1}(b)\Big),$$

for $a \in C^{1,0}$, $b \in C^{0,1}$. Assume hypotheses (V_1) and (H_1).

Surjectivity of α_1 and β_1

Let $[(u, v)] \in H^1(\mathrm{Tot}\, C)$ be a cohomology class, represented by a 1–cocycle $(u, v) \in C^{1,0} \oplus C^{0,1}$:

$$\Delta^1(u, v) = 0.$$

This means

$$\delta^{1,0}(u) = 0, \qquad -d^{1,0}(u) + \delta^{0,1}(v) = 0, \qquad d^{0,1}(v) = 0.$$

By (V_1) we have $\mathrm{Ker}\, d^{0,1} = \mathrm{Image}\, d^{0,0}$, so from $d^{0,1}(v) = 0$ we find $c \in C^{0,0}$ with $v = d^{0,0}(c)$. Consider the 0–cochain c and its image

$$\Delta^0(c) = \big(\delta^{0,0}(c),\, d^{0,0}(c)\big) \in C^{1,0} \oplus C^{0,1}.$$

Replacing (u, v) by the cohomologous cocycle

$$(u', v') := (u, v) - \Delta^0(c) = \big(u - \delta^{0,0}(c),\, 0\big),$$

we obtain a representative of $[(u,v)]$ with vanishing $C^{0,1}$-component. The cocycle conditions for $(u',0)$ imply

$$d^{1,0}(u') = 0, \qquad \delta^{1,0}(u') = 0,$$

so $u' \in E^1$ and $\delta_E^1(u') = 0$. Thus u' defines a cohomology class $[u'] \in H^1(E^\bullet)$, and by construction

$$\alpha_1([u']) = [(u',0)] = [(u,v)] \in H^1(\operatorname{Tot} C).$$

Hence $\alpha_1 \colon H^1(E^\bullet) \to H^1(\operatorname{Tot} C)$ is surjective.

Similarly, starting from a cocycle (u,v) we can first use (H_1) to simplify the $C^{1,0}$–component. Concretely, the condition $\delta^{1,0}(u) = 0$ and the exactness

$$\operatorname{Ker} \delta^{1,0} = \operatorname{Image} \delta^{0,0}$$

at $C^{1,0}$ show that there exists $c' \in C^{0,0}$ such that $u = \delta^{0,0}(c')$. Then

$$\Delta^0(c') = (u, d^{0,0}(c')) \in C^{1,0} \oplus C^{0,1},$$

and replacing (u,v) by

$$(u'', v'') := (u,v) - \Delta^0(c') = \big(0,\, v - d^{0,0}(c')\big),$$

we obtain a representative with vanishing $C^{1,0}$–component. The equations $\Delta^1(u'', v'') = 0$ reduce to

$$\delta^{0,1}(v'') = 0, \qquad d^{0,1}(v'') = 0,$$

so $v'' \in F^1$ and $d_F^1(v'') = 0$. Thus v'' defines a class $[v''] \in H^1(F^\bullet)$ and

$$\beta_1([v'']) = [(0,v'')] = [(u,v)].$$

Hence $\beta_1 \colon H^1(F^\bullet) \to H^1(\operatorname{Tot} C)$ is surjective.

Injectivity of α_1 and β_1

Suppose $[u] \in H^1(E^\bullet)$ satisfies $\alpha_1([u]) = 0 \in H^1(\operatorname{Tot} C)$. We may choose a representative $u \in E^1 \subset C^{1,0}$ with $\delta_E^1(u) = 0$, i.e., $d^{1,0}(u) = 0$ and $\delta^{1,0}(u) = 0$. The condition $\alpha_1([u]) = 0$ means that the cocycle $(u,0)$ is a boundary in the total complex, i.e., there exists $c \in C^{0,0}$ with

$$(u,0) = \Delta^0(c) = \big(\delta^{0,0}(c),\, d^{0,0}(c)\big).$$

In particular $u = \delta^{0,0}(c)$ and $d^{0,0}(c) = 0$, so $c \in E^0$ and

$$u = \delta_E^0(c).$$

Thus $[u] = 0$ in $H^1(E^\bullet)$, and α_1 is injective. The argument for β_1 is completely the same.

Combining the above, we see that under (V_1) and (H_1) we have isomorphisms

$$\alpha_1 \colon H^1(E^\bullet) \xrightarrow{\sim} H^1(\operatorname{Tot} C), \qquad \beta_1 \colon H^1(F^\bullet) \xrightarrow{\sim} H^1(\operatorname{Tot} C),$$

and therefore a natural isomorphism

$$\Phi_1 := \beta_1^{-1} \circ \alpha_1 \colon H^1(E^\bullet) \xrightarrow{\sim} H^1(F^\bullet).$$

REMARK A.1.2. From the argument above, one can describe concretely how the isomorphism

$$\Phi_1 \colon H^1(E^\bullet) \xrightarrow{\sim} H^1(F^\bullet)$$

maps the classes for $m = 1$. Let $[u] \in H^1(E^\bullet)$ be represented by $u \in E^1 \subset C^{1,0}$ with

$$d^{1,0}(u) = 0, \qquad \delta^{1,0}(u) = 0.$$

The condition $\delta^{1,0}(u) = 0$ together with (H_1) implies that there exists $c \in C^{0,0}$ with

$$u = \delta^{0,0}(c).$$

For such c we have $d^{0,0}(c) \in F^1$ because $\delta^{0,1}(d^{0,0}(c)) = 0$ (this follows from $d \circ \delta = \delta \circ d$). The corresponding class in $H^1(F^\bullet)$ is represented by $-\,d^{0,0}(c)$, and this is precisely the value of Φ_1:

$$\Phi_1([u]) = \big[\, -d^{0,0}(c) \,\big] \in H^1(F^\bullet).$$

Thus the map Φ_1 is obtained, at the level of representatives, by solving $\delta(c) = u$ along the bottom row and then taking $-d(c)$ along the left column. □

1.5 The case $m = 2$

The portion of the double complex that contributes to the following argument for the case $m = 2$ is as follows:

$$\begin{array}{ccccccccc}
 & & & & E^1 & \xrightarrow{\delta_E^1} & E^2 & \xrightarrow{\delta_E^2} & E^3 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & C^{1,0} & \xrightarrow{\delta^{1,0}} & C^{2,0} & \xrightarrow{\delta^{2,0}} & C^{3,0} \\
 & & & & \downarrow d^{1,0} & & \downarrow d^{2,0} & & \\
F^1 & \to & C^{0,1} & \xrightarrow{\delta^{0,1}} & C^{1,1} & \xrightarrow{\delta^{1,1}} & C^{2,1} & & \\
\downarrow d_F^1 & & \downarrow d^{0,1} & & \downarrow d^{1,1} & & & & \\
F^2 & \to & C^{0,2} & \xrightarrow{\delta^{0,2}} & C^{1,2} & & & & \\
\downarrow d_F^2 & & \downarrow d^{0,2} & & & & & & \\
F^3 & \to & C^{0,3} & & & & & &
\end{array}$$

We have

$$\mathrm{Tot}^1 C = C^{1,0} \oplus C^{0,1}, \qquad \mathrm{Tot}^2 C = C^{2,0} \oplus C^{1,1} \oplus C^{0,2}.$$

The differential Δ^1 was described above; the differential

$$\Delta^2 \colon \mathrm{Tot}^2 C \to \mathrm{Tot}^3 C$$

is given by

$$\Delta^2(x, y, z) = \Big(\delta^{2,0}(x),\ d^{2,0}(x) + \delta^{1,1}(y),\ -d^{1,1}(y) + \delta^{0,2}(z),\ d^{0,2}(z)\Big),$$

for $x \in C^{2,0}$, $y \in C^{1,1}$, $z \in C^{0,2}$. Assume (V_2) and (H_2).

Surjectivity of α_2

Let $[(x,y,z)] \in H^2(\operatorname{Tot} C)$ be a cohomology class represented by a 2–cocycle $(x,y,z) \in C^{2,0} \oplus C^{1,1} \oplus C^{0,2}$:

$$\Delta^2(x,y,z) = 0.$$

Explicitly,

$$\delta^{2,0}(x) = 0, \quad d^{2,0}(x) + \delta^{1,1}(y) = 0, \quad -d^{1,1}(y) + \delta^{0,2}(z) = 0, \quad d^{0,2}(z) = 0.$$

Step 1: kill the $C^{0,2}$–component. By (V_2) we have $\operatorname{Ker} d^{0,2} = \operatorname{Image} d^{0,1}$ at $C^{0,2}$, so from $d^{0,2}(z) = 0$ we find $u \in C^{0,1}$ with $z = d^{0,1}(u)$. Consider the 1–cochain $(0,u) \in \operatorname{Tot}^1 C$, whose image is

$$\Delta^1(0,u) = \big(0, \delta^{0,1}(u), d^{0,1}(u)\big) \in C^{2,0} \oplus C^{1,1} \oplus C^{0,2}.$$

Replacing (x,y,z) by the cohomologous cocycle

$$(x_1,y_1,z_1) := (x,y,z) - \Delta^1(0,u) = \big(x,\ y - \delta^{0,1}(u),\ z - d^{0,1}(u)\big),$$

we obtain a representative with $z_1 = 0$.

Step 2: kill the $C^{1,1}$–component. The cocycle condition for $(x_1,y_1,0)$ implies $d^{1,1}(y_1) = 0$. By (V_2) at $(1,1)$ we have $\operatorname{Ker} d^{1,1} = \operatorname{Image} d^{1,0}$, so there exists $v \in C^{1,0}$ with $y_1 = -d^{1,0}(v)$. Consider $(v,0) \in \operatorname{Tot}^1 C$, whose image is

$$\Delta^1(v,0) = \big(\delta^{1,0}(v), -d^{1,0}(v), 0\big).$$

Replacing $(x_1,y_1,0)$ by

$$(x_2,y_2,z_2) := (x_1,y_1,0) - \Delta^1(v,0) = \big(x_1 - \delta^{1,0}(v),\ 0,\ 0\big),$$

we obtain a representative $(x_2,0,0)$ concentrated in $C^{2,0}$.

The remaining cocycle conditions for $(x_2,0,0)$ force $d^{2,0}(x_2) = 0$ and $\delta^{2,0}(x_2) = 0$, so $x_2 \in E^2$ and $\delta_E^2(x_2) = 0$. Thus x_2 defines a class $[x_2] \in H^2(E^\bullet)$, and by construction

$$\alpha_2([x_2]) = [(x_2,0,0)] = [(x,y,z)].$$

Hence $\alpha_2 \colon H^2(E^\bullet) \to H^2(\operatorname{Tot} C)$ is surjective.

Surjectivity of β_2

Starting again from a 2–cocycle (x,y,z), we can use the horizontal hypotheses (H_2) in the same fashion to kill the $C^{2,0}$-component and then the $C^{1,1}$–component.

Concretely, from $\delta^{2,0}(x) = 0$ and exactness at $(2,0)$ we find $w \in C^{1,0}$ with $x = \delta^{1,0}(w)$. Subtracting $\Delta^1(w,0)$ as before kills the $C^{2,0}$–component. The cocycle condition for the new representative implies $\delta^{1,1}(y') = 0$, and (H_2) at $(1,1)$ then allows us to write $y' = \delta^{0,1}(u')$ and to kill the $C^{1,1}$–component by subtracting $\Delta^1(0,u')$. We arrive at a representative $(0,0,z'')$ with $\delta^{0,2}(z'') = 0$ and $d^{0,2}(z'') = 0$, i.e., $z'' \in F^2$ and $d_F^2(z'') = 0$. Thus z'' defines a class $[z''] \in H^2(F^\bullet)$ and

$$\beta_2([z'']) = [(0,0,z'')] = [(x,y,z)].$$

Hence β_2 is surjective.

Injectivity of α_2

Assume $[x] \in H^2(E^\bullet)$ satisfies $\alpha_2([x]) = 0 \in H^2(\operatorname{Tot} C)$. We may choose a representative $x \in E^2 \subset C^{2,0}$ with $\delta_E^2(x) = 0$, i.e., $d^{2,0}(x) = 0$ and $\delta^{2,0}(x) = 0$. The condition $\alpha_2([x]) = 0$ means that there exists $(a, b) \in C^{1,0} \oplus C^{0,1}$ such that

$$(x, 0, 0) = \Delta^1(a, b) = \big(\delta^{1,0}(a),\ -d^{1,0}(a) + \delta^{0,1}(b),\ d^{0,1}(b)\big).$$

In particular

$$d^{0,1}(b) = 0.$$

By (V_2) at $(0, 1)$ we can write $b = d^{0,0}(c)$ for some $c \in C^{0,0}$. Then

$$\Delta^0(c) = \big(\delta^{0,0}(c),\, d^{0,0}(c)\big) = \big(\delta^{0,0}(c), b\big).$$

Subtracting $\Delta^0(c)$ from $(a, b) \in \operatorname{Tot}^1 C$ we obtain another pair

$$(a', b') := (a, b) - \Delta^0(c) = \big(a - \delta^{0,0}(c),\, 0\big)$$

such that

$$\Delta^1(a', b') = \Delta^1(a, b) = (x, 0, 0).$$

Therefore

$$(x, 0, 0) = \Delta^1(a', 0) = \big(\delta^{1,0}(a'), -d^{1,0}(a'), 0\big),$$

so $x = \delta^{1,0}(a')$. The identity $d^{1,0}(a') = 0$ shows that $a' \in E^1$, and hence

$$x = \delta_E^1(a').$$

Thus $[x] = 0$ in $H^2(E^\bullet)$, and α_2 is injective.

Injectivity of β_2

The argument for β_2 is again analogous. If $[z] \in H^2(F^\bullet)$ satisfies $\beta_2([z]) = 0$, we may choose a representative $z \in F^2 \subset C^{0,2}$ with $d_F^2(z) = 0$, and the condition $\beta_2([z]) = 0$ means that

$$(0, 0, z) = \Delta^1(a, b)$$

for some $(a, b) \in \operatorname{Tot}^1 C$. Writing out the components of this equation gives $\delta a = 0$, $-da + \delta b = 0$, and $db = z$. By (H_2) at $(1, 0)$, write $a = \delta c$. Subtracting $\Delta^0(c)$ replaces (a, b) by $(0, b - dc)$. Since the differentials commute, $\delta(b - dc) = \delta b - da = 0$; also $d(b - dc) = z$. Thus $b - dc \in F^1$ and z is a coboundary in $F^\bullet$. Hence β_2 is injective.

Combining the above, we obtain:

Lemma A.1.3. *Assume* (V_2) *and* (H_2). *Then*

$$\alpha_2 \colon H^2(E^\bullet) \xrightarrow{\sim} H^2(\operatorname{Tot} C), \qquad \beta_2 \colon H^2(F^\bullet) \xrightarrow{\sim} H^2(\operatorname{Tot} C)$$

are isomorphisms, and hence

$$\Phi_2 := \beta_2^{-1} \circ \alpha_2 \colon H^2(E^\bullet) \xrightarrow{\sim} H^2(F^\bullet)$$

is an isomorphism.

1.6 Proof of Theorem A.1.1 (for general m)

We first show that α_m is an isomorphism under the hypothesis (V_m). By symmetry (interchanging p and q, d and δ, $E^\bullet$ and $F^\bullet$) the same argument then applies to β_m under (H_m). We therefore only spell out the proof for α_m.

Surjectivity of α_m. Let $[c] \in H^m(\operatorname{Tot} C)$ be a cohomology class, represented by a cocycle

$$c = (c_{p,q})_{p+q=m} \in \operatorname{Tot}^m C, \qquad \Delta(c) = 0.$$

We will show that $[c]$ lies in the image of α_m by a zig–zag procedure along the diagonal $p + q = m$.

We proceed by downward induction on q to kill all components $c_{p,q}$ with $q > 0$, using only the vertical exactness (V_m).

Fix q with $1 \le q \le m$, and set $p = m - q$. Assume that the components with second index greater than q have already been killed. The component of Δc in $C^{p,q+1}$ is then $(-1)^p d^{p,q}(c_{p,q})$, so $d^{p,q}(c_{p,q}) = 0$. By (V_m) at (p, q), choose $u \in C^{p,q-1}$ with $d^{p,q-1}u = c_{p,q}$. Let b have the single component $b_{p,q-1} = (-1)^p u$. Then the $C^{p,q}$-component of Δb is exactly $c_{p,q}$, and its only other possible component lies in $C^{p+1,q-1}$. Replacing c by $c - \Delta b$ kills $c_{p,q}$ without restoring any component with larger second index.

Iterating this procedure for $q = m, m-1, \dots, 1$, we obtain a cocycle c' cohomologous to c and with $c'_{p,q} = 0$ for all $q > 0$. Thus c' is supported entirely in bidegree $(m, 0)$, say

$$c'_{m,0} = x \in C^{m,0}, \qquad c'_{p,q} = 0 \text{ for all } q > 0.$$

The cocycle condition $\Delta(c') = 0$ now reduces to

$$d^{m,0}(x) = 0, \qquad \delta^{m,0}(x) = 0,$$

so $x \in E^m$ and $\delta_E^m(x) = 0$. Hence x defines a class

$$[x] \in H^m(E^\bullet),$$

and by construction we have

$$\alpha_m([x]) = [c'] = [c] \in H^m(\operatorname{Tot} C).$$

This proves that α_m is surjective.

Injectivity of α_m. Suppose $[x] \in H^m(E^\bullet)$ satisfies $\alpha_m([x]) = 0 \in H^m(\operatorname{Tot} C)$. We choose a representative

$$x \in E^m \subset C^{m,0}$$

with $\delta_E^m(x) = 0$, i.e., $d^{m,0}(x) = 0$ and $\delta^{m,0}(x) = 0$. The condition $\alpha_m([x]) = 0$ means that the cocycle

$$c = (c_{p,q})_{p+q=m}, \qquad c_{m,0} = x,\ c_{p,q} = 0 \text{ for } q > 0,$$

is a coboundary in the total complex: there exists a cochain

$$b = (b_{p,q})_{p+q=m-1} \in \mathrm{Tot}^{m-1} C$$

such that

$$\Delta(b) = c.$$

We now use a descending induction on q along the diagonal $p+q = m-1$ to modify b by adding Δ of $(m-2)$–cochains so as to kill all components $b_{p,q}$ with $q > 0$. At each step we use only the vertical exactness (V_m).

More precisely, fix q with $1 \le q \le m-1$, and put $p = m-1-q$. Consider the component $b_{p,q} \in C^{p,q}$. The equality $\Delta(b) = c$ implies that the component of $\Delta(b)$ in $C^{p,q+1}$ is $(-1)^p d^{p,q}(b_{p,q})$ plus terms depending on $b_{p',q'}$ with $q' > q$. By the induction hypothesis we may assume that all $b_{p',q'}$ with $q' > q$ have already been killed, so that the component of $\Delta(b)$ in $C^{p,q+1}$ is simply $(-1)^p d^{p,q}(b_{p,q})$. On the other hand, since c has no component in $C^{p,q+1}$ for $q+1 > 0$, we must have

$$d^{p,q}(b_{p,q}) = 0.$$

By (V_m) at (p,q) (note that $q > 0$ and $p+q = m-1$), we have

$$\mathrm{Ker}\, d^{p,q} = \mathrm{Image}\, d^{p,q-1}.$$

Hence there exists $t_{p,q-1} \in C^{p,q-1}$ with

$$b_{p,q} = d^{p,q-1}(t_{p,q-1}).$$

Let s be the $(m-2)$–cochain supported in bidegree $(p, q-1)$ with $s_{p,q-1} := (-1)^p t_{p,q-1}$. Then $\Delta(s)$ has a component in $C^{p,q}$ equal to $d^{p,q-1}(t_{p,q-1}) = b_{p,q}$, and possibly components in neighbouring bidegrees on the diagonal $p+q = m-1$. After replacing b by $b - \Delta(s)$ we have a new cochain b' with $b'_{p,q} = 0$, without changing $\Delta(b)$.

Iterating this procedure for $q = m-1, m-2, \ldots, 1$, we obtain a cochain b' satisfying $\Delta(b') = \Delta(b)$ and supported entirely in $C^{m-1,0}$:

$$b'_{m-1,0} = y \in C^{m-1,0}, \qquad b'_{p,q} = 0 \text{ for } q > 0.$$

The equality $\Delta(b') = c$ now reduces to

$$\delta^{m-1,0}(y) = x, \qquad d^{m-1,0}(y) = 0,$$

so $y \in E^{m-1}$ and

$$x = \delta_E^{m-1}(y).$$

Thus $[x] = 0$ in $H^m(E^\bullet)$, and α_m is injective.

We have shown that under (V_m) the map $\alpha_m \colon H^m(E^\bullet) \to H^m(\mathrm{Tot}\, C)$ is an isomorphism. To treat β_m, one may either repeat the same argument with the roles of p and q (and hence of d and δ, $E^\bullet$ and $F^\bullet$) interchanged, using the horizontal hypothesis (H_m) in place of (V_m), or equivalently apply the above reasoning to the "transposed" double complex $\widetilde{C}^{p,q} := C^{q,p}$.

The map multiplying $C^{p,q}$ by $(-1)^{pq}$ identifies the original total complex with the transposed total complex: it intertwines $\delta + (-1)^p d$ and $d + (-1)^q \delta$. It is the identity on both axes, so it preserves the edge maps being compared. This shows that under (H_m) the map

$$\beta_m \colon H^m(F^\bullet) \longrightarrow H^m(\operatorname{Tot} C)$$

is also an isomorphism. Under the joint hypotheses (V_m) and (H_m) both α_m and β_m are therefore isomorphisms, and the composite

$$\Phi_m := \beta_m^{-1} \circ \alpha_m \colon H^m(E^\bullet) \xrightarrow{\sim} H^m(F^\bullet)$$

is a well-defined isomorphism, as claimed. □

Remark A.1.4 (Changing the sign convention). There are several useful ways of writing the same double-complex calculation (compare [St, Section 12.18]). With the commuting d, δ used here, set $\widehat{d}|_{C^{p,q}} = (-1)^p d$ and $\widehat{\delta} = \delta$. Then $\widehat{d}\widehat{\delta} + \widehat{\delta}\widehat{d} = 0$ and $\widehat{\delta} + \widehat{d} = \Delta$. Thus the anticommuting convention uses the *unsigned sum* of the already signed differentials. Inserting the total-complex sign a second time would in general destroy the relation $\Delta^2 = 0$. Conversely, the same twist turns an anticommuting pair into a commuting pair; it does not change exactness.

Another total differential for the commuting pair is $\Delta' = d + (-1)^q \delta$. The map $S|_{C^{p,q}} = (-1)^{pq}\mathrm{id}$ satisfies $S\Delta = \Delta' S$ and is the identity on both axes. Consequently it does not change Φ_m when the two edge complexes retain their names. Interchanging the names of the axes as well interchanges $E^\bullet$ and $F^\bullet$, so the comparison in the opposite direction is Φ_m^{-1}.

In contrast, actually replacing δ by $-\delta$, while keeping d, changes the bottom-edge differential to $-\delta_E$. Under the identity identifications of the resulting kernels and images with the old ones, the comparison becomes $\Phi_m^{\mathrm{new}} = (-1)^m \Phi_m$. Indeed, multiplication by $(-1)^p$ on $C^{p,q}$ is a chain isomorphism between the two total complexes, acting by $(-1)^m$ on E^m and by 1 on F^m. This change must be distinguished from merely moving the sign in the total differential. □

Remark A.1.5 (Signs of comparison representatives). Assume the hypotheses of Theorem A.1.1, and let $u \in E^m$ be a cocycle, with $m \geq 1$. Successive uses of (H_m) give elements $a_r \in C^{r,m-1-r}$ for $0 \leq r \leq m-1$ such that $\delta a_{m-1} = u$ and $\delta a_{r-1} = d a_r$ for $1 \leq r \leq m-1$. Then $d a_0$ is a cocycle in F^m, and our comparison is

$$\Phi_m([u]) = (-1)^{m(m+1)/2}[d a_0].$$

To check the sign, put $\varepsilon_{m-1} = 1$ and $\varepsilon_{r-1} = (-1)^{r+1}\varepsilon_r$. The interior terms cancel in the total differential, giving

$$\Delta\left(\sum_{r=0}^{m-1} \varepsilon_r a_r\right) = u - (-1)^{m(m+1)/2} d a_0.$$

In degree one this reads $\Phi_1([\delta a_0]) = [-d a_0]$. In degree two, if $n = \delta f$, $\delta b = df$ and $db = -F$, it reads $\Delta(b + f) = n - F$, hence $\Phi_2([n]) = [F]$. These are the formulas used in Example II.3.84 and Proposition III.2.22, respectively.

One can also choose a degree-dependent normalization of the comparison itself. In the Čech–de Rham setting, the explicit representative in [De4, Chapter IV, (6.11)–(6.12)] uses $(-1)^{m(m-1)/2}[da_0]$, and therefore gives $(-1)^m\Phi_m$ in our notation. It agrees with our comparison in degree two, but has the opposite sign in degree one. Thus an odd-degree formula cannot be imported from that normalization without its sign adjustment. The transition convention and the sign of the first Chern class of the dual tautological line bundle remain those fixed in Remark III.2.25. □

Remark A.1.6. Assume now that each $C^{p,q}$ is a TVS and that all differentials $d^{p,q}$ and $\delta^{p,q}$ are continuous $\mathbb{C}$-linear maps. Give the edge terms E^p, F^q their subspace topologies and each $\mathrm{Tot}^m C$ its finite direct sum topology. Equip

$$Z^m(E^\bullet) = \operatorname{Ker}\delta_E^m, \qquad B^m(E^\bullet) = \operatorname{Image}\delta_E^{m-1}$$

and similarly $Z^m(F^\bullet)$, $B^m(F^\bullet)$, and $Z^m(\mathrm{Tot}\,C)$, $B^m(\mathrm{Tot}\,C)$ with the subspace topology, and endow all cohomology groups $H^m(E^\bullet)$, $H^m(F^\bullet)$ and $H^m(\mathrm{Tot}\,C)$ with the quotient topology.

Because the inclusions

$$E^\bullet \hookrightarrow \mathrm{Tot}^\bullet C, \qquad F^\bullet \hookrightarrow \mathrm{Tot}^\bullet C$$

are continuous cochain maps, the induced maps

$$\alpha_m\colon H^m(E^\bullet) \longrightarrow H^m(\mathrm{Tot}\,C), \qquad \beta_m\colon H^m(F^\bullet) \longrightarrow H^m(\mathrm{Tot}\,C)$$

are automatically continuous.

Recall that TVSs need not be Hausdorff in our convention, whereas Fréchet spaces are Hausdorff by definition. If, moreover, each $C^{p,q}$ is Fréchet, the comparison is a TVS isomorphism in every degree covered by Theorem A.1.1, without any Hausdorffness assumption on the cohomology groups. The relevant open mapping argument takes place before passing to cohomology, as follows.

Let $P^\bullet, Q^\bullet$ be complexes of Fréchet spaces with continuous differentials $d_P^j\colon P^j \to P^{j+1}$ and $d_Q^j\colon Q^j \to Q^{j+1}$, respectively. Let $f\colon P^\bullet \to Q^\bullet$ be a continuous cochain map, with degree-j component $f^j\colon P^j \to Q^j$. Fix $m \in \mathbb{Z}_{\geq 0}$, with negative-degree terms interpreted as zero, and give cohomology its kernel-quotient topology. If $H^m(f)$ is surjective, it is open. Indeed, put $Z_P^m = \operatorname{Ker} d_P^m$ and $Z_Q^m = \operatorname{Ker} d_Q^m$. These are closed subspaces of Fréchet spaces, hence Fréchet, and surjectivity on cohomology makes

$$T\colon Z_P^m \oplus Q^{m-1} \longrightarrow Z_Q^m, \qquad T(z,b) = f^m(z) + d_Q^{m-1}b$$

a continuous surjection. Thus T is open by Theorem I.6.46. Let q_P, q_Q be the quotient maps from cocycles to cohomology. For any open set $W \subset H^m(P^\bullet)$, the Cartesian product $q_P^{-1}(W) \times Q^{m-1}$ is open in $Z_P^m \oplus Q^{m-1}$. Hence

$$q_Q^{-1}\big(H^m(f)(W)\big) = T\big(q_P^{-1}(W) \times Q^{m-1}\big)$$

is open in Z_Q^m. The quotient topology proves the assertion. The map $H^m(f)$ is also continuous, since f is continuous. Consequently, if $H^m(f)$ is an isomorphism of vector spaces, it is a TVS isomorphism, even when these quotient spaces are not Hausdorff.

In our double complex, each edge term is a closed kernel and each total degree is a finite direct sum of Fréchet spaces. Apply this argument to the two edge inclusions above, whose maps α_m, β_m are algebraic isomorphisms under the hypotheses of Theorem A.1.1. It follows that

$$\Phi_m = \beta_m^{-1} \circ \alpha_m \colon H^m(E^\bullet) \xrightarrow{\sim} H^m(F^\bullet)$$

is a TVS isomorphism. The boundary images have not been replaced by their closures. The argument uses Fréchet spaces at the cochain and cocycle levels; it does not automatically apply to complexes of LF-spaces or of strong duals. □

2 Proof of Lemma II.1.4

Here we prove Lemma II.1.4 in accordance with the argument in [Sie]. We denote the left-hand side of (62) by L.

First let us note that the assertion can be easily shown when $t = 2$. Indeed, in this case, we have $r = 0$, $s = 2$, $L = y_1^2 y_2^2$ with $y_1 + y_2 = k$ and $y_q \leq k/2$ for $q = 1, 2$. Hence we have

$$\begin{aligned} L = y_1^2 \cdot y_2^2 &= \frac{1}{4}\left((y_1 + y_2)^2 - \left(y_1^2 + y_2^2\right)\right)^2 \\ &\geq \frac{1}{4}\left(k^2 - 2 \cdot \left(\frac{k}{2}\right)^2\right)^2 = \frac{k^4}{16} = \frac{k}{2}\frac{k^3}{8} \geq k^3 8^{1-2}, \end{aligned} \tag{387}$$

which proves the assertion when $t = 2$.

2.1 Some fundamental lemmata

As a preparation, we first show the following lemmata.

Lemma A.2.1. *For any integer $t \in \mathbb{Z}_{\geq 2}$, one has*

$$t - 1 \leq 2^{t-2}.$$

Proof. For $t = 2$, we have $t - 1 = 1$ and $2^{t-2} = 2^0 = 1$, hence the claim holds with equality. To show the assertion for $t \geq 3$, consider the real function

$$f(x) := 2^{x-2} - (x - 1) \qquad (x \in \mathbb{R}).$$

Then

$$f'(x) = 2^{x-2}\log 2 - 1, \qquad f''(x) = 2^{x-2}(\log 2)^2 > 0.$$

Hence f' is strictly increasing on $\mathbb{R}$, and in particular

$$f'(x) \geq f'(3) = 2\log 2 - 1 > 0 \qquad (x \geq 3),$$

since $\log 2 > 1/2$. Therefore f is strictly increasing on $[3, \infty)$. As $f(3) = 2^1 - 2 = 0$, we obtain $f(t) \geq 0$ for every integer $t \geq 3$. This is exactly $t - 1 \leq 2^{t-2}$. □

Lemma A.2.2. *For any integer $t \in \mathbb{Z}_{\geq 3}$, one has*

$$2^{3t-4} \geq (2t - 1)^2.$$

Proof. Consider the real function

$$F(x) := 2^{3x-4} - (2x - 1)^2 \qquad (x \in \mathbb{R}).$$

Then

$$F'(x) = 3(\log 2)\, 2^{3x-4} - 4(2x - 1), \qquad F''(x) = 9(\log 2)^2\, 2^{3x-4} - 8.$$

For $x \geq 3$, we have $2^{3x-4} \geq 2^5 = 32$, and hence

$$F''(x) \geq 9\Big(\frac{1}{2}\Big)^2 \cdot 32 - 8 = 72 - 8 = 64 > 0.$$

Therefore F' is strictly increasing on $[3, \infty)$. Moreover,

$$F'(3) = 3(\log 2)\, 2^5 - 4 \cdot 5 = 96 \log 2 - 20 \geq 96 \cdot \frac{1}{2} - 20 = 28 > 0.$$

Since F' is increasing on $[3, \infty)$, it follows that $F'(x) \geq F'(3) > 0$ for all $x \geq 3$. As

$$F(3) = 2^5 - 5^2 = 32 - 25 = 7 > 0,$$

for every integer $t \geq 3$ we obtain $F(t) \geq 0$, which proves the assertion. □

LEMMA A.2.3. *Let $a \leq b$ be real numbers and define*

$$\varphi(x) := (b + 1 - x)\,(x - (a - 1))^2 \qquad (x \in [a, b]).$$

Then, for every $a_0, b_0 \in [a, b]$ with $a_0 \leq b_0$, the minimum of φ on the subinterval $[a_0, b_0]$ is attained at an endpoint.

PROOF. Set $d := b - a \geq 0$ and put $u := x - (a - 1)$. Then $x \in [a, b]$ corresponds to $u \in [1, d + 1]$, and

$$\varphi(x) = (d + 2 - u)u^2 =: f(u).$$

We compute

$$f'(u) = 2(d + 2)u - 3u^2 = u\big(2(d + 2) - 3u\big), \qquad f''(u) = 2(d + 2) - 6u.$$

The only possible critical point of f in $(1, d + 1)$ is $u_* := 2(d + 2)/3$, and it satisfies

$$f''(u_*) = -2(d + 2) < 0.$$

Hence u_* is a strict local maximum. Therefore f has no local minimum in the interior of any subinterval of $[1, d + 1]$, and so the minimum of f on $[u(a_0), u(b_0)]$ is attained at an endpoint. □

2.2 Proof of Lemma II.1.4

We treat separately two ranges of k:

Case 1: The case where $k < 2t - 2$;

Case 2: The case where $k \geq 2t - 2$.

The proof of Lemma II.1.4 in Case 1 (When $k < 2t-2$)

In this case, as $L \geq 1$, we have $k^{-3}L \geq k^{-3} > (2t-2)^{-3}$. By Lemma A.2.1, we have

$$(2t-2)^{-3} = 2^{-3}(t-1)^{-3} \geq 2^{-3} \cdot 2^{-3(t-2)} = 2^{3(1-t)} = 8^{1-t}.$$

Therefore $k^{-3}L > (2t-2)^{-3} \geq 8^{1-t}$, which gives the desired estimate. □

The proof of Lemma II.1.4 in Case 2 (When $k \geq 2t-2$)

To show the assertion in this case, set $g := \lfloor k/2 \rfloor$ (i.e., g is the largest element of $\{\ell \in \mathbb{Z} \mid \ell \leq k/2\}$) and $\eta := r + \sum_{q=1}^{s} y_q$. Then

$$t = r + s \leq \eta, \qquad \eta \leq \sum_{p=1}^{r} x_p + \sum_{q=1}^{s} y_q = k, \qquad \sum_{p=1}^{r} x_p = k - \eta + r.$$

LEMMA A.2.4. *In Case 2 (i.e., when $k \geq 2t-2$), the following holds.*
(*i*) $t \leq g+1 \leq g+1+r \leq g-1+t \leq k$.
(*ii*) $\prod_{p=1}^{r} x_p \geq k - \eta + 1$.
(*iii*)

$$\prod_{q=1}^{s} y_q \geq \begin{cases} \eta - t + 1, & \text{if } \eta \leq g-1+t, \\ (\eta - g - t + 2)\, g, & \text{if } \eta \geq g-1+t \end{cases}.$$

(*iv*) $(k-g-t+2)g^2 \geq (k-g)(g-t+2)^2$.

PROOF. (*i*) As $k \geq 2t-2$, we have $\frac{k}{2} \geq t-1$, from which $g+1 \geq t$ follows. The inequality $g+1 \leq g+1+r$ is clear from the assumption $r \geq 0$. Since $s \geq 2$ and $r+s=t$, we have $r = t-s \leq t-2$. Hence

$$g+1+r \leq g+1+(t-2) = g+t-1 = g-1+t.$$

Since $g = \lfloor k/2 \rfloor$, we have $k-g \geq k/2$. By the assumption $k \geq 2t-2$, this implies $k-g \geq t-1$, which is equivalent to $g-1+t \leq k$.

(*ii*) When $r \geq 1$, all $x_p - 1$ are non-negative, so

$$\prod_{p=1}^{r} x_p = \prod_{p=1}^{r} \big(1 + (x_p - 1)\big) \geq 1 + \sum_{p=1}^{r} (x_p - 1) = k - \eta + 1.$$

When $r = 0$, then $\eta = \sum_{q=1}^{s} y_q = k$, hence $k - \eta + 1 = 1$. Since the empty product equals 1, the inequality holds.

(*iii*) Set

$$S := \sum_{q=1}^{s} y_q = \eta - r.$$

Recall that y_q are positive integers and $y_q \leq g$ for all q.

As in (ii), expanding the product gives

$$\prod_{q=1}^{s} y_q \geq 1 + \sum_{q=1}^{s} (y_q - 1) = S - s + 1 = \eta - t + 1.$$

In what follows, let us assume that $\eta \geq g-1+t$, so $S \geq g+s-1$. If $g = 1$, all y_q are 1 and the assertion is immediate. Suppose that $g \geq 2$. Among the s-tuples of integers $(z_1, z_2, \ldots, z_s)$ satisfying $1 \leq z_q \leq g$ and $\sum_{q=1}^{s} z_q = S$, choose one for which the product is minimal (Such a tuple exists, since the set of these tuples is finite and contains $(y_1, y_2, \ldots, y_s)$). If every z_q were smaller than g, then $S \geq g + s - 1$ would force at least two of them to be larger than 1. For two such entries $1 < a \leq b < g$, replacing (a, b) by $(a - 1, b + 1)$ preserves the sum and the bounds, but

$$(a-1)(b+1) - ab = a - b - 1 < 0,$$

contrary to minimality. Thus we may reorder this minimizing tuple so that $z_s = g$. Applying the same product estimate to the remaining $s - 1$ entries gives

$$\prod_{q=1}^{s} y_q \geq \prod_{q=1}^{s} z_q = g \prod_{q=1}^{s-1} z_q \geq g\big(1 + (S-g) - (s-1)\big) = (\eta - g - t + 2)\, g,$$

which proves (iii).

(iv) If $t = 2$, the assertion is clear, since both sides coincide. In what follows, we assume $t \geq 3$. Consider the difference

$$D := (k - g - t + 2)g^2 - (k - g)(g - t + 2)^2.$$

A direct computation gives

$$\begin{aligned} D &= (k-g)\big(g^2 - (g-t+2)^2\big) - (t-2)g^2 \\ &= (k-g)(t-2)(2g-t+2) - (t-2)g^2 \\ &= (t-2)\Big((k-g)(2g-t+2) - g^2\Big). \end{aligned}$$

Since $g = \lfloor k/2 \rfloor$, we have $k - g \geq g$. Hence

$$(k-g)(2g-t+2) - g^2 \geq g(2g-t+2) - g^2 = g(g-t+2).$$

By (i) we have $g \geq t - 1$, and thus $g - t + 2 \geq 1$. Therefore $g(g - t + 2) \geq 0$, so $D \geq 0$, proving (iv). □

By Lemma A.2.4 (ii) and (iii), we have

$$L = \left(\prod_{p=1}^{r} x_p\right)\left(\prod_{q=1}^{s} y_q^2\right) \geq (k - \eta + 1) \cdot \begin{cases} (\eta - t + 1)^2, & \eta \leq g - 1 + t, \\ (\eta - g - t + 2)^2 g^2, & \eta \geq g - 1 + t. \end{cases}$$

We regard the right-hand side as a function $\Phi(\eta)$ on the integer interval $g+1 \le \eta \le k$: $\Phi\colon [g+1,k]\cap\mathbb{Z} \to \mathbb{Z}$. Here we recall

$$t \;\le\; g+1 \;\le\; g+1+r \;\le\; g-1+t \;\le\; k$$

(see Lemma A.2.4 (i)) and $\eta \in [g+1,k]$ holds (since $\sum_{q=1}^{s} y_q > k/2$ implies $\sum_{q=1}^{s} y_q \ge g+1$).

We consider separately the two subintervals $[g+1, g-1+t]$ and $[g-1+t, k]$. When $\eta \in [g+1, g-1+t]$, we have

$$\Phi(\eta) = (k-\eta+1)(\eta-t+1)^2 = ((k+1)-\eta)\,(\eta-(t-1))^2.$$

Applying Lemma A.2.3 with $a := t$, $b := k$, $a_0 := g+1$ and $b_0 := g-1+t$, we see that the minimum of the right-hand side on $[g+1, g-1+t]$ is attained at one of the endpoints. Hence

$$\Phi(\eta) \ge \min\bigl\{(k-g)(g-t+2)^2,\ (k-g-t+2)g^2\bigr\}.$$

When $\eta \in [g-1+t, k]$, we have

$$\Phi(\eta) = ((k+1)-\eta)(\eta-((g-1+t)-1))^2 g^2.$$

Applying Lemma A.2.3 with $a := g-1+t$, $b := k$, $a_0 := g-1+t$ and $b_0 := k$, we see that the minimum of the right-hand side on $[g-1+t, k]$ is attained at one of the endpoints. Hence

$$\Phi(\eta) \ge \min\bigl\{(k-g-t+2)^2 g^2,\ (k-g-t+2)g^2\bigr\} = (k-g-t+2)g^2,$$

since $k-g-t+2 = (k-(g-1+t))+1 \ge 1$.

Combining the two cases, we conclude that

$$L \ge \min\bigl\{(k-g)(g-t+2)^2,\ (k-g-t+2)g^2\bigr\}.$$

By Lemma A.2.4 (iv), from this it follows that

$$L \ge (k-g)(g-t+2)^2$$

holds. Consequently,

$$k^{-3}L \ge \frac{k-g}{k}\left(\frac{g-t+2}{k}\right)^2 \ge \frac{1}{2}\left(\frac{g-t+2}{k}\right)^2 \tag{388}$$

holds, since

$$\frac{k-g}{k} = \frac{k-\lfloor k/2\rfloor}{k} \ge \frac{1}{2}.$$

As $k \ge 2t-2$, one has $(g-t+1)/k \ge 0$, since

$$g = \left\lfloor \frac{k}{2} \right\rfloor \ge \left\lfloor \frac{2t-2}{2} \right\rfloor \ge t-1.$$

If one keeps t and replaces k with $k+2$, the value of $(g-t+2)/k$ does not decrease, since

$$\frac{\lfloor (k+2)/2 \rfloor - t + 2}{k+2} - \frac{\lfloor k/2 \rfloor - t + 2}{k} = \frac{k\lfloor (k+2)/2 \rfloor - (k+2)\lfloor k/2 \rfloor + 2(t-2)}{k(k+2)}$$
$$= \frac{k - 2\lfloor k/2 \rfloor + 2(t-2)}{k(k+2)} \geq 0.$$

Therefore, again by $k \geq 2t-2$, it follows that

$$\left(\frac{g-t+2}{k}\right)^2 \geq \min\left\{ \left(\frac{\lfloor (2t-2)/2 \rfloor - t + 2}{2t-2}\right)^2, \left(\frac{\lfloor (2t-1)/2 \rfloor - t + 2}{2t-1}\right)^2 \right\}$$
$$= \min\left\{ (2t-2)^{-2}, (2t-1)^{-2} \right\} = (2t-1)^{-2} \tag{389}$$

Combining (388) and (389), we obtain

$$k^{-3}L \geq \frac{1}{2}(2t-1)^{-2}.$$

When $t \geq 3$, it follows from Lemma A.2.2 that $2^{3t-4} \geq (2t-1)^2$ holds, and hence

$$\frac{1}{2}(2t-1)^{-2} \geq 2^{3-3t} = 8^{1-t},$$

which proves (62) in Case 2 for $t \geq 3$. As we have shown the assertion when $t = 2$ in (387), this completes the proof. □

3 Hints and solutions to the exercises

3.1 Hints and solutions to the exercises in Chapter I

Mini-Exercise I.1.7

Let

$$\mathcal{A} := \Big\{ (a_n)_{n\geq 1} \in \{0,1\}^{\mathbb{Z}_{>0}} \;\Big|\; \{n \geq 1 \mid a_n = 1\} \text{ is infinite} \Big\}.$$

Then $\mathcal{A}$ is uncountable. Indeed, for each subset $S \subset \mathbb{Z}_{>0}$, define $a^{(S)} = (a_n^{(S)})$ by $a_n^{(S)} = 1$ if $n \in S$ and $a_n^{(S)} = 0$ otherwise. Restricting to infinite subsets S gives an injection

$$\{S \subset \mathbb{Z}_{>0} \mid S \text{ infinite}\} \hookrightarrow \mathcal{A}.$$

Since the set of infinite subsets of $\mathbb{Z}_{>0}$ is uncountable (Indeed, the set of all finite subsets of $\mathbb{Z}_{>0}$ is countable, while the set of all the subsets of $\mathbb{Z}_{>0}$ is uncountable by Cantor's theorem), so is $\mathcal{A}$.

For each $a \in \mathcal{A}$, consider the real number

$$\alpha(a) := \sum_{n=1}^{\infty} \frac{a_n}{2^{n!}}.$$

Then the map $\mathcal{A} \to \mathbb{R}$, $a \mapsto \alpha(a)$, is injective. Indeed, if $a = (a_n)_n, b = (b_n)_n \in \mathcal{A}$ satisfies $a \neq b$, let N be the smallest positive integer such that $a_N \neq b_N$. Then

$$|\alpha(a) - \alpha(b)| \geq \frac{1}{2^{N!}} - \sum_{n>N} \frac{1}{2^{n!}} > \frac{1}{2^{N!}} - \sum_{k=1}^{\infty} \frac{1}{2^{N!+k}} = \frac{1}{2^{N!}} - \frac{1}{2^{N!}} \cdot \frac{1}{2-1} = 0.$$

Therefore, for each element $a \in \mathcal{A}$, it is sufficient to show that $\alpha(a)$ is not Diophantine. To show this, let us estimate the approximation quality of the rationals $p_k(a)/q_k$, where

$$p_k(a) := \sum_{n=1}^{k} a_n \, 2^{k!-n!} \in \mathbb{Z}, \qquad q_k := 2^{k!} \in \mathbb{Z}_{>0}.$$

For every $k \geq 1$,

$$\left|\alpha(a) - \frac{p_k(a)}{q_k}\right| = \sum_{n \geq k+1} \frac{a_n}{2^{n!}} \leq \sum_{n \geq k+1} \frac{1}{2^{n!}} < \frac{2}{2^{(k+1)!}}$$

(the last inequality follows since $n! \geq (k+1)! + (n - (k+1))$ for $n \geq k+1$). Therefore

$$\min_{m \in \mathbb{Z}} |q_k \alpha(a) - m| \leq |q_k \alpha(a) - p_k(a)| = q_k \left|\alpha(a) - \frac{p_k(a)}{q_k}\right| < 2 \cdot \frac{2^{k!}}{2^{(k+1)!}} = 2 \cdot 2^{-k \cdot k!}.$$

Since $q_k = 2^{k!}$, this can be rewritten as

$$\min_{m \in \mathbb{Z}} |q_k \alpha(a) - m| < 2\, q_k^{-k}.$$

Now fix arbitrary constants $C > 0$ and $\tau > 1$. Choose an integer $k > \tau$ so large that $2\, q_k^{-(k-\tau)} < C$ (possible since $q_k \to \infty$). Then

$$\min_{m\in\mathbb{Z}} |q_k\alpha(a) - m| < 2\, q_k^{-k} < C\, q_k^{-\tau},$$

so the Diophantine inequality fails for $q = q_k$. Hence $\alpha(a)$ is not Diophantine.

Mini-Exercise I.2.1

It suffices to show that $a_{jk} = 0$ whenever $j < k$ and $\ell < k$, since A is Hermitian and thus $a_{jk} = \overline{a_{kj}}$. Fix such j and k. Consider the 2×2 principal submatrix

$$A_{jk} := \begin{pmatrix} a_{jj} & a_{jk} \\ a_{kj} & a_{kk} \end{pmatrix}.$$

Since A is positive semi-definite, A_{jk} is also positive semi-definite. Hence the determinant $\det A_{jk}$ of A_{jk} is non-negative. Since $k > \ell$, we have $a_{kk} = 0$ by assumption, and therefore

$$0 \le \det A_{jk} = a_{jj}a_{kk} - |a_{jk}|^2 = -|a_{jk}|^2,$$

which implies $a_{jk} = 0$.

Mini-Exercise I.2.2

For the existence, one first treats the case where H is a diagonal matrix. The general case then follows by considering $U\sqrt{\Lambda}U^\dagger$ for U and Λ as in (2).

For the uniqueness, let A be a positive definite Hermitian matrix such that $A^2 = H$. Note first that

$$AH = A^3 = HA.$$

It follows from [HJ, Theorem 2.5.5 and Theorem 4.1.6] that A and H can be simultaneously diagonalized by the same unitary matrix. The uniqueness of $\sqrt{H}$ follows immediately.

Mini-Exercise I.3.1

Let $A \subset X$ be a closed subset. Since X is compact, the subset A is compact as well. The image $f(A)$ is therefore compact in Y, because f is continuous. Since Y is Hausdorff, every compact subset of Y is closed. Hence $f(A)$ is closed in Y, and thus f is a closed map.

Mini-Exercise I.3.2

Let $F \subset X \times K$ be a closed subset, and let $x_0 \in X \setminus \pi_X(F)$. Then, by definition of the projection, we have

$$\{x_0\} \times K \subset (X \times K) \setminus F.$$

Since $(X \times K) \setminus F$ is open in the product topology, for each $y \in K$ there exist open neighborhoods $U_y \subset X$ of x_0 and $V_y \subset K$ of y such that

$$(x_0, y) \in U_y \times V_y \subset (X \times K) \setminus F.$$

The family $\{V_y\}_{y \in K}$ is an open cover of K. As K is compact, there exist finitely many points $y_1, \ldots, y_n \in K$ such that

$$K \subset V_{y_1} \cup \cdots \cup V_{y_n}.$$

Define

$$U := U_{y_1} \cap \cdots \cap U_{y_n}.$$

Then U is an open neighborhood of x_0 in X.

We claim that

$$U \times K \subset (X \times K) \setminus F.$$

Indeed, let $(x, y) \in U \times K$ be arbitrary. Since $y \in K$, there exists an index j with $y \in V_{y_j}$. On the other hand, $x \in U \subset U_{y_j}$ by definition of U. Hence

$$(x, y) \in U_{y_j} \times V_{y_j} \subset (X \times K) \setminus F.$$

This proves the claimed inclusion.

As a consequence, we have $U \cap \pi_X(F) = \emptyset$. Since x_0 was an arbitrary point in $X \setminus \pi_X(F)$, this shows that $\pi_X(F)$ is closed in X.

Mini-ExerciseI.3.3

(i) Let

$$S := \left\{ v = (v_1, \ldots, v_n) \in \mathbb{C}^n \;\middle|\; |v_1|^2 + \cdots + |v_n|^2 = 1 \right\}.$$

Define $F \colon S \times \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$F(v, x, c) := v\big(A(x) - cB(x)\big)v^\dagger.$$

Then F is continuous, hence

$$W := \{(v, x, c) \in S \times \Omega \times \mathbb{R} \mid F(v, x, c) > 0\}$$

is an open subset of $S \times \Omega \times \mathbb{R}$.

Since $A(x_0) > 0$, we have $F(v, x_0, 0) = vA(x_0)v^\dagger > 0$ for all $v \in S$. In particular, $S \times \{(x_0, 0)\} \subset W$. For each $v \in S$, because $(v, x_0, 0) \in W$ and W is open, there exist open neighborhoods $V_v \subset S$ of v, $U_v \subset \Omega$ of x_0, and an open interval $I_v \subset \mathbb{R}$ containing 0 such that $V_v \times U_v \times I_v \subset W$. Then $\{V_v\}_{v \in S}$ is an open covering of S. Since S is compact, there exist finitely many points $v_1, \ldots, v_k \in S$ such that

$$S = V_{v_1} \cup \cdots \cup V_{v_k}.$$

Set $U := U_{v_1} \cap \cdots \cap U_{v_k}$ and $I := I_{v_1} \cap \cdots \cap I_{v_k}$. Then U is an open neighborhood of x_0 in Ω, and I is an open interval containing 0, and we have $S \times U \times I \subset W$. Hence $F(v, x, c) > 0$ for all $(v, x, c) \in S \times U \times I$, which means that $A(x) - cB(x)$ is positive definite for all $(x, c) \in U \times I$. In particular, shrinking I if necessary we may assume that $I = (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, and then $A(x) > cB(x)$ holds for all $x \in U$ and all $c \in (-\varepsilon, \varepsilon)$.

(*ii*) Apply (*i*) to $\widetilde{A}(x) := A(x) - B(x)$ and $\widetilde{B}(x) := 0$.

Mini-Exercise I.3.5

(*i*) Recall that a subset A of a metric space (X, d) is said to be *bounded* if there exist $x_0 \in X$ and $R > 0$ such that $d(x, x_0) \leq R$ for all $x \in A$. It is said to be *totally bounded* if for every $\varepsilon > 0$ there exist finitely many points $x_1, \ldots, x_N \in X$ such that

$$A \subset \bigcup_{j=1}^{N} B(x_j, \varepsilon),$$

where $B(x_j, \varepsilon) := \{x \in X \mid d(x, x_j) < \varepsilon\}$.

Assume that A is totally bounded. Taking $\varepsilon = 1$, we see that A is covered by finitely many balls of radius 1, and hence A is bounded. The converse does not hold in general. For example, let X be an infinite set equipped with the discrete metric

$$d(x, y) = \begin{cases} 0 & (x = y), \\ 1 & (x \neq y). \end{cases}$$

Then X is bounded, since $d(x, y) \leq 1$ for all $x, y \in X$, but X is not totally bounded, because for $\varepsilon < 1$ each ball $B(x, \varepsilon)$ consists of a single point, and hence no finite family of such balls can cover X.

As another important example, the closed unit ball of the Hilbert space ℓ^2 (see the following explanation in (*ii*) for the details) is bounded but not totally bounded. We shall return to this example in (iii).

(*ii*) Let $H = \ell^2$ be the Hilbert space of square-summable real sequences, equipped with its usual norm $\| \cdot \|$. Then H is a complete metric space. Consider the closed unit ball

$$B := \{x \in H \mid \|x\| \leq 1\}.$$

The set B is closed and bounded in H. However, B is not compact.

Indeed, let $\{e_n\}_{n \geq 1}$ be the standard orthonormal basis of ℓ^2. Then $\|e_n\| = 1$ for all n, so $\{e_n\} \subset B$, but for $n \neq m$ we have

$$\|e_n - e_m\| = \sqrt{2}.$$

Hence the sequence $\{e_n\}$ has no convergent subsequence in H, and therefore B is not compact.

(*iii*) The Heine–Borel theorem states that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Comparing this with the example in (*ii*), we see that the key difference lies in the notion of total boundedness.

In $\mathbb{R}^n$, every bounded subset is totally bounded: for any $\varepsilon > 0$, a bounded set can be covered by finitely many cubes (or balls) of diameter less than ε. As a consequence, in $\mathbb{R}^n$ the conditions "bounded" and "totally bounded" are equivalent.

In contrast, in infinite-dimensional complete metric spaces such as ℓ^2, bounded sets need not be totally bounded. The closed unit ball in ℓ^2 is bounded but not totally bounded, which explains why the conclusion of the Heine–Borel theorem fails in this setting.

Mini-Exercise I.3.6

(*i*) Let $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence in X. It is enough to show that it has an accumulation point in X. We may assume that the set $A := \{x_n \mid n \in \mathbb{Z}_{\geq 0}\}$ is infinite, since otherwise the statement is clear. By taking subsequence if necessary, we may assume that $x_n \neq x_m$ holds for any $n \neq m \in \mathbb{Z}_{\geq 0}$. Suppose that A has no accumulation point in X. Then for each $n \in \mathbb{Z}_{\geq 0}$, there exists an open neighborhood $U_n \subset X$ of x_n such that $U_n \cap A = \{x_n\}$. Then the family $\{U_n\}_{n\in\mathbb{Z}_{\geq 0}}$ forms an open cover of $A = \overline{A}$ with no finite subcover, contradicting the compactness of $\overline{A}$.

(*ii*) We prove the contrapositive. Assume that X is not totally bounded. Then there exists $\varepsilon > 0$ such that for every finite subset $F \subset X$,

$$X \not\subset \bigcup_{x\in F} B(x,\varepsilon),$$

where $B(x,\varepsilon) := \{y \in X \mid d(x,y) < \varepsilon\}$. Choose $x_1 \in X$ arbitrarily. Having chosen $x_1, \ldots, x_n$, the above property implies that

$$X \setminus \bigcup_{j=1}^{n} B(x_j,\varepsilon) \neq \emptyset,$$

so we can pick

$$x_{n+1} \in X \setminus \bigcup_{j=1}^{n} B(x_j,\varepsilon).$$

By construction, $d(x_m, x_n) \geq \varepsilon$ holds whenever $m \neq n$, from which it follows that X is not sequential compact.

(*iii*) Assume first that X is compact. Then X is sequentially compact by (*i*). Conversely, assume that X is sequentially compact. By (*ii*), X is totally bounded. Thus, by the fact recalled in §3.1, it suffices to show that X is complete. Let $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be a Cauchy sequence in X. By sequential compactness, it has a convergent subsequence. Denote by x its limit. Since $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is Cauchy, it follows that $x_n \to x$ as $n \to \infty$.

Mini-Exercise I.3.7

(*i*) We first show that

$$d_H(A,B) \leq \max\Big\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\Big\}.$$

Let

$$M := \max\Big\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\Big\}.$$

For every $\varepsilon > M$ and every $a \in A$, one has $d(a,B) \leq M < \varepsilon$, hence $a \in U(B,\varepsilon)$. Thus $A \subset U(B,\varepsilon)$, and similarly $B \subset U(A,\varepsilon)$. By definition of d_H, one has $d_H(A,B) \leq \varepsilon$ for every $\varepsilon > M$. Taking the infimum over such ε gives $d_H(A,B) \leq M$. Conversely, let $\varepsilon > 0$ be such that $A \subset U(B,\varepsilon)$ and $B \subset U(A,\varepsilon)$. Then for every $a \in A$ we have $d(a,B) \leq \varepsilon$, and for every $b \in B$ we have $d(b,A) \leq \varepsilon$. Taking suprema yields

$$\max\Big\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\Big\} \leq \varepsilon.$$

Taking the infimum over all such ε gives the desired equality.

(ii) Symmetry and non-negativity of d_H are immediate from the definition. If $d_H(A,B) = 0$, then by *(i)* we have $\sup_{a\in A} d(a,B) = 0$ and $\sup_{b\in B} d(b,A) = 0$, which implies $A \subset B$ and $B \subset A$ (refer to Mini-Exercise I.3.11 *(ii)*), hence $A = B$. To prove the triangle inequality, let $A, B, C \in \mathcal{H}_X$. For any $a \in A$ we have

$$d(a,C) \leq d(a,B) + \sup_{b\in B} d(b,C) \leq \sup_{a\in A} d(a,B) + \sup_{b\in B} d(b,C).$$

Taking the supremum over $a \in A$ yields

$$\sup_{a\in A} d(a,C) \leq \sup_{a\in A} d(a,B) + \sup_{b\in B} d(b,C).$$

A similar estimate holds with A and C exchanged. Applying *(i)*, we obtain

$$d_H(A,C) \leq d_H(A,B) + d_H(B,C).$$

(iii) Assume that A is not connected. Then there exist non-empty disjoint compact subsets $K_1, K_2 \subset A$ such that $A = K_1 \cup K_2$. Since K_1 and K_2 are compact and disjoint, there exists $\delta > 0$ such that $d(K_1,K_2) \geq \delta$.

Let ν be sufficiently large so that $d_H(A_\nu, A) < \delta/3$. Then we have

$$A_\nu \subset U(K_1,\delta/3) \cup U(K_2,\delta/3).$$

We claim that both intersections $A_\nu \cap U(K_1,\delta/3)$ and $A_\nu \cap U(K_2,\delta/3)$ are non-empty. To show this, by symmetry, it is sufficient to lead a contradiction by assuming $A_\nu \cap U(K_1,\delta/3) = \emptyset$. In this case, one has $A_\nu \subset U(K_2,\delta/3)$. Since $d_H(A_\nu, A) < \delta/3$, we have $A \subset U(A_\nu,\delta/3)$. Let $x \in K_1$ be an arbitrary point. Then there exists $y \in A_\nu$ such that $d(x,y) < \delta/3$. By the above inclusion, we have $y \in U(K_2,\delta/3)$, hence there exists $z \in K_2$ such that $d(y,z) < \delta/3$. By the triangle inequality, we obtain

$$d(x,z) \leq d(x,y) + d(y,z) < \delta/3 + \delta/3 = 2\delta/3,$$

which contradicts the assumption that $d(K_1,K_2) \geq \delta$. Therefore, $A_\nu \cap U(K_1,\delta/3) \neq \emptyset$.

As $U(K_1, \delta/3) \cap U(K_2, \delta/3) = \emptyset$, from this it follows that A_ν is disconnected.

(*iv*) Let $\{A_n\}_{n \in \mathbb{Z}_{\geq 0}}$ be a Cauchy sequence in $(\mathcal{H}_X, d_H)$. Choose a subsequence $\{n_k\}_{k \in \mathbb{Z}_{\geq 0}}$ such that

$$d_H(A_{n_k}, A_{n_{k+1}}) < 2^{-k} \qquad (k \in \mathbb{Z}_{\geq 0}).$$

Define a subset $A \subset X$ by

$$A := \Big\{ x \in X \;\Big|\; \text{there exists } \{x_k \in A_{n_k}\}_{k \in \mathbb{Z}_{\geq 0}} \text{ such that } x_k \to x \Big\}.$$

We first show that A is non-empty. Fix $x_0 \in A_{n_0}$. Assume $x_k \in A_{n_k}$ has been chosen. Since $A_{n_k} \subset U(A_{n_{k+1}}, 2^{-k})$, there exists $x_{k+1} \in A_{n_{k+1}}$ such that $d(x_k, x_{k+1}) < 2^{-k}$. Then for $m > \ell$ we have

$$d(x_\ell, x_m) \leq \sum_{j=\ell}^{m-1} d(x_j, x_{j+1}) < \sum_{j=\ell}^{m-1} 2^{-j} \leq \sum_{j=\ell}^{\infty} 2^{-j} = 2^{-\ell+1}.$$

Hence $\{x_k\}$ is a Cauchy sequence in X. Since (X, d) is complete, x_k converges to some $x \in X$, and by definition we have $x \in A$. Thus $A \neq \emptyset$.

Next we show that A is totally bounded. Let $\varepsilon > 0$. Choose $k \in \mathbb{Z}_{\geq 0}$ such that $2^{-k+1} < \varepsilon/3$. Since A_{n_k} is compact, it is totally bounded. Hence there exists a finite subset $F \subset A_{n_k}$ such that

$$A_{n_k} \subset \bigcup_{p \in F} U(\{p\}, \varepsilon/3).$$

We claim that

$$A \subset \bigcup_{p \in F} U(\{p\}, \varepsilon).$$

Let $x \in A$. By definition of A, there exists a sequence $\{x_j \in A_{n_j}\}_{j \in \mathbb{Z}_{\geq 0}}$ such that $x_j \to x$. Choose $j > k$ such that $d(x, x_j) < \varepsilon/3$. Since

$$d_H(A_{n_k}, A_{n_j}) \leq \sum_{r=k}^{j-1} d_H(A_{n_r}, A_{n_{r+1}}) < \sum_{r=k}^{j-1} 2^{-r} \leq \sum_{r=k}^{\infty} 2^{-r} = 2^{-k+1} < \varepsilon/3,$$

we have $A_{n_j} \subset U(A_{n_k}, \varepsilon/3)$. Hence there exists $a \in A_{n_k}$ such that $d(x_j, a) < \varepsilon/3$. Choose $p \in F$ such that $d(a, p) < \varepsilon/3$. Then, by the triangle inequality,

$$d(x, p) \leq d(x, x_j) + d(x_j, a) + d(a, p) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

This proves the claim, and therefore A is totally bounded.

Let us show that A is closed in X. Let $\{x^{(\mu)}\}_{\mu \in \mathbb{Z}_{\geq 0}} \subset A$ be a sequence converging to a point $x \in X$. For each μ, choose a sequence $\{x_k^{(\mu)} \in A_{n_k}\}_{k \in \mathbb{Z}_{\geq 0}}$ such that $x_k^{(\mu)} \to x^{(\mu)}$ as $k \to \infty$. Choose integers $k(\mu) \in \mathbb{Z}_{\geq 0}$ such that $k(\mu)$ is strictly increasing and

$$d\big(x^{(\mu)}, x_{k(\mu)}^{(\mu)}\big) < 2^{-\mu} \qquad (\mu \in \mathbb{Z}_{\geq 0}).$$

Set

$$y_\mu := x^{(\mu)}_{k(\mu)} \in A_{n_{k(\mu)}}.$$

Since $x^{(\mu)} \to x$ and $d(x^{(\mu)}, y_\mu) \le 2^{-\mu}$, we have $y_\mu \to x$.

We now construct a sequence $\{z_k \in A_{n_k}\}_{k\in\mathbb{Z}_{\ge 0}}$ converging to x. Choose the finitely many terms with $k < k(0)$ arbitrarily. For every μ, set $z_{k(\mu)} := y_\mu$. Within each block $k(\mu) < k < k(\mu+1)$, choose $z_k \in A_{n_k}$ inductively so that

$$d(z_{k-1}, z_k) < 2^{-(k-1)}.$$

This is possible because $A_{n_{k-1}} \subset U(A_{n_k}, 2^{-(k-1)})$. No estimate across the last step into the next prescribed term $z_{k(\mu+1)}$ is needed. Then for any $k(\mu) \le k < k(\mu+1)$ we have

$$d\big(z_k, z_{k(\mu)}\big) \le \sum_{j=k(\mu)}^{k-1} 2^{-j} \le 2^{-k(\mu)+1}.$$

Hence

$$d(z_k, x) \le d\big(z_k, z_{k(\mu)}\big) + d\big(z_{k(\mu)}, x\big) \le 2^{-k(\mu)+1} + d(y_\mu, x).$$

Letting $\mu \to \infty$, the right-hand side tends to 0, so $z_k \to x$. By definition of A, this shows $x \in A$. Therefore A is closed.

Since A is closed and totally bounded, A is compact. Thus $A \in \mathcal{H}_X$.

We now show that $d_H(A_{n_k}, A) \to 0$. First, we prove that

$$A_{n_k} \subset U(A, 2^{-k+1}).$$

Let $x \in A_{n_k}$ and set $x_k := x$. Choose inductively $x_{k+1} \in A_{n_{k+1}}, x_{k+2} \in A_{n_{k+2}}, \dots$ such that $d(x_j, x_{j+1}) < 2^{-j}$ for all $j \ge k$. Then $\{x_j\}_{j\ge k}$ is Cauchy, hence converges to some $x_\infty \in X$. After choosing the finitely many terms with index less than k arbitrarily, this is a sequence of the kind used to define A. Thus $x_\infty \in A$. Moreover,

$$d(x, x_\infty) \le d(x, x_{k+1}) + \sum_{j=k+1}^{\infty} 2^{-j} < 2^{-k+1},$$

which proves the inclusion.

Next, we prove that

$$A \subset U(A_{n_k}, 3 \cdot 2^{-k}).$$

Let $x \in A$ and choose $\{x_j \in A_{n_j}\}_{j\in\mathbb{Z}_{\ge 0}}$ with $x_j \to x$. For $j > k$, we use the distance to the set A_{n_k} rather than an estimate for the arbitrarily chosen point x_k:

$$\begin{aligned} d(x, A_{n_k}) &\le d(x, x_j) + d(x_j, A_{n_k}) \\ &\le d(x, x_j) + d_H(A_{n_j}, A_{n_k}) \\ &\le d(x, x_j) + \sum_{m=k}^{j-1} 2^{-m}. \end{aligned}$$

Letting $j \to \infty$ gives $d(x, A_{n_k}) \le 2^{-k+1} < 3 \cdot 2^{-k}$, hence the inclusion.

These two inclusions imply $d_H(A_{n_k}, A) \le 3 \cdot 2^{-k}$, and therefore $d_H(A_{n_k}, A) \to 0$.

Finally, we show that $A_n \to A$. Let $\varepsilon > 0$. Since $\{A_n\}$ is Cauchy, choose N such that $d_H(A_n, A_m) < \varepsilon/2$ for all $n, m \ge N$. Then choose k with $n_k \ge N$ and $d_H(A_{n_k}, A) < \varepsilon/2$. Then for $n \ge N$ we have

$$d_H(A_n, A) \le d_H(A_n, A_{n_k}) + d_H(A_{n_k}, A) < \varepsilon.$$

Hence $A_n \to A$ in $(\mathcal{H}_X, d_H)$. This proves that $(\mathcal{H}_X, d_H)$ is complete.

(v) Assume that (X, d) is totally bounded. Let $\varepsilon > 0$. Choose a finite set $S = \{s_1, s_2, \ldots, s_N\} \subset X$ such that

$$X = \bigcup_{j=1}^{N} U(\{s_j\}, \varepsilon/2).$$

For $A \in \mathcal{H}_X$, define a subset $T_A \subset S$ by

$$T_A := \{s \in S \mid d(s, A) < \varepsilon/2\}.$$

Since A is non-empty, T_A is also non-empty.

We claim that $d_H(A, T_A) < \varepsilon$. First we show $A \subset U(T_A, \varepsilon)$. Let $a \in A$. Choose $s \in S$ such that $d(a, s) < \varepsilon/2$. Then $d(s, A) \le d(s, a) < \varepsilon/2$, hence $s \in T_A$. Therefore $d(a, T_A) \le d(a, s) < \varepsilon/2 < \varepsilon$, and thus $a \in U(T_A, \varepsilon)$. Next we show $T_A \subset U(A, \varepsilon)$. If $s \in T_A$, then by definition $d(s, A) < \varepsilon/2 < \varepsilon$, so $s \in U(A, \varepsilon)$. Thus $A \subset U(T_A, \varepsilon)$ and $T_A \subset U(A, \varepsilon)$, which implies $d_H(A, T_A) < \varepsilon$.

Since S is finite, there are only finitely many non-empty subsets of S. Hence the finite set

$$\{T \subset S \mid T \neq \emptyset\} \subset \mathcal{H}_X$$

forms an ε-net of $\mathcal{H}_X$. Therefore $(\mathcal{H}_X, d_H)$ is totally bounded.

(vi) If (X, d) is compact, then it is complete and totally bounded. By (iv) and (v), $(\mathcal{H}_X, d_H)$ is complete and totally bounded. Therefore $(\mathcal{H}_X, d_H)$ is compact.

Mini-Exercise I.3.9

We first prove uniqueness. Suppose that $\xi, \eta \in X$ are fixed points of T. By Lemma A.3.1 below, we have

$$d(\xi, \eta) \le \frac{d(\xi, T(\xi)) + d(\eta, T(\eta))}{1 - r}.$$

Since $T(\xi) = \xi$ and $T(\eta) = \eta$, the right-hand side is 0, hence $d(\xi, \eta) = 0$ and thus $\xi = \eta$.

Next we prove existence. Fix $x_0 \in X$ and define $x_{n+1} := T(x_n)$ for $n \ge 0$. It suffices to show that $\{x_n\}$ is a Cauchy sequence, because then completeness yields $x_n \to x^* \in X$, and continuity of T (indeed, T is r-Lipschitz) gives

$$T(x^*) = T\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x^*,$$

so x^* is a fixed point.

We claim that

$$d(x_n, x_{n+1}) \le r^n\, d(x_0, x_1) \qquad (n \ge 0).$$

Indeed, this follows by induction from $d(x_{n+1}, x_{n+2}) = d(T(x_n), T(x_{n+1})) \le r\, d(x_n, x_{n+1})$.

Now let $\varepsilon > 0$ and choose N such that

$$\frac{2r^N}{1-r}\, d(x_0, x_1) < \varepsilon.$$

For $n, m \ge N$, applying Lemma A.3.1 again to $\zeta = x_n$ and $\eta = x_m$, we obtain

$$\begin{aligned} d(x_n, x_m) &\le \frac{d(x_n, T(x_n)) + d(x_m, T(x_m))}{1-r} \\ &= \frac{d(x_n, x_{n+1}) + d(x_m, x_{m+1})}{1-r} \\ &\le \frac{r^n + r^m}{1-r}\, d(x_0, x_1) \le \frac{2r^N}{1-r}\, d(x_0, x_1) < \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is Cauchy and therefore converges to a point $x^* \in X$, which is the unique fixed point of T. This also shows $x_n \to x^*$ for every $x_0 \in X$.

LEMMA A.3.1. *Let (X, d) be a metric space and $T\colon X \to X$ a contraction with constant $0 < r < 1$. Then for all $\zeta, \eta \in X$ one has*

$$d(\zeta, \eta) \le \frac{d(\zeta, T(\zeta)) + d(\eta, T(\eta))}{1-r}.$$

PROOF. By the triangle inequality and the contraction property,

$$\begin{aligned} d(\zeta, \eta) &\le d(\zeta, T(\zeta)) + d(T(\zeta), T(\eta)) + d(T(\eta), \eta) \\ &\le d(\zeta, T(\zeta)) + r\, d(\zeta, \eta) + d(\eta, T(\eta)). \end{aligned}$$

Rearranging gives $(1-r)d(\zeta, \eta) \le d(\zeta, T(\zeta)) + d(\eta, T(\eta))$. □

Mini-Exercise I.3.10

(i) Since X satisfies T_3, the point x and the closed subset $X \setminus U$ admit disjoint open neighborhoods. Thus there exist open subsets $V, W \subset X$ such that

$$x \in V, \qquad X \setminus U \subset W, \qquad V \cap W = \emptyset.$$

Since W is open and disjoint from V, one has $\overline{V} \cap W = \emptyset$. Therefore $\overline{V} \subset X \setminus W \subset U$.

(ii) Since

$$Z \subset \bigcup_{j \in J} U_j$$

and $W_j \cap Z = U_j \cap Z$, the family $\{W_j\}_{j \in J}$ also covers Z.

For every $z \in Z$, choose an index $j(z) \in J$ such that $z \in W_{j(z)}$. By (i), there exists an open neighborhood O_z of z such that $\overline{O_z} \subset W_{j(z)}$. The family $\{O_z\}_{z \in Z}$ is an open covering of the compact set Z. Hence there exist points $z_1, z_2, \ldots, z_m \in Z$ such that $Z \subset O_{z_1} \cup O_{z_2} \cup \cdots \cup O_{z_m}$.

For every $j \in J$, set $V_j := \bigcup_{\substack{1 \leq \ell \leq m \\ j(z_\ell)=j}} O_{z_\ell}$. If no such ℓ exists, we set $V_j := \emptyset$. Then the open subsets V_j cover Z. Moreover, since the union defining V_j is finite,

$$\overline{V_j} \subset \bigcup_{\substack{1 \leq \ell \leq m \\ j(z_\ell)=j}} \overline{O_{z_\ell}} \subset W_j.$$

Thus

$$V_j \subset \overline{V_j} \subset W_j \subset U_j.$$

(iii) Apply (ii) to the open subsets $W_{jj} \subset U_j$. We obtain open subsets V_j which cover Z and satisfy $\overline{V_j} \subset W_{jj} \subset U_j$.

Fix a pair $(j, k) \in J^2$ with $j \neq k$, and set

$$T_{jk} := \overline{V_j \cap V_k \setminus W_{jk}}.$$

Since W_{jk} is open,

$$T_{jk} \subset \overline{V_j} \cap \overline{V_k} \setminus W_{jk}.$$

Suppose that $z \in T_{jk} \cap Z$. Then $z \in \overline{V_j} \cap Z \subset U_j$ and $z \in \overline{V_k} \cap Z \subset U_k$. Hence $z \in U_j \cap U_k \cap Z = W_{jk} \cap Z$. On the other hand, $T_{jk} \subset X \setminus W_{jk}$, which is a contradiction. Therefore $T_{jk} \cap Z = \emptyset$.

Since T_{jk} is closed, $V_j \setminus T_{jk}$ is open. The equality $T_{jk} \cap Z = \emptyset$ shows that this replacement removes no point of Z. Furthermore, if $x \in (V_j \setminus T_{jk}) \cap V_k$ and $x \notin W_{jk}$, then

$$x \in V_j \cap V_k \setminus W_{jk} \subset T_{jk},$$

which contradicts $x \notin T_{jk}$. Thus $(V_j \setminus T_{jk}) \cap V_k \subset W_{jk}$.

Replacing V_j with $V_j \setminus T_{jk}$ only shrinks one member of the family. It therefore preserves the covering of Z, the inclusions $\overline{V_j} \subset U_j$, and all intersection conditions which have already been obtained.

(iv) As there are only finitely many pairs in J^2 with distinct entries, enumerate them as $(j_1, k_1), (j_2, k_2), \ldots, (j_N, k_N)$. Set

$$V_j^{(0)} := V_j \qquad (j \in J).$$

Suppose that the open subsets

$$V_j^{(r-1)} \qquad (j \in J)$$

have been defined for some integer r with $1 \leq r \leq N$. Set

$$T_r := \overline{V_{j_r}^{(r-1)} \cap V_{k_r}^{(r-1)} \setminus W_{j_r k_r}}.$$

By the argument as above, one has $T_r \cap Z = \emptyset$. Therefore, by letting

$$V_j^{(r)} := \begin{cases} V_{j_r}^{(r-1)} \setminus T_r & \text{if } j = j_r, \\ V_j^{(r-1)} & \text{if } j \neq j_r \end{cases},$$

the family $\{V_j^{(r)}\}_{j\in J}$ still covers Z, and

$$V_{j_r}^{(r)} \cap V_{k_r}^{(r)} \subset W_{j_r k_r}.$$

Moreover, since the open subsets are only being shrunk, all the intersection conditions obtained at the preceding steps remain valid.

After completing these finitely many steps, set

$$U_j' := V_j^{(N)} \qquad (j \in J).$$

Then the open subsets U_j' cover Z and satisfy the desired properties.

Mini-Exercise I.3.11

(i) Assume that f is upper semi-continuous. For $y \in Y$, since Z is non-empty and compact and $f(y,\cdot)$ is upper semi-continuous on Z, the maximum

$$g(y) := \max_{z\in Z} f(y,z)$$

exists, so g is well-defined. Let $\pi_Y : Y\times Z \to Y$ be the first projection. By Mini-Exercise I.3.2, π_Y is a closed map. For any $a \in \mathbb{R}$, one has

$$\{y \in Y \mid g(y) \geq a\} = \pi_Y\big(\{(y,z) \in Y \times Z \mid f(y,z) \geq a\}\big).$$

The subset of $Y \times Z$ on the right-hand side is closed by the upper semi-continuity of f. Since π_Y is closed, the set on the left-hand side is closed in Y. Thus g is upper semi-continuous on Y.

When f is lower semi-continuous, then $-f$ is upper semi-continuous, and

$$\min_{z\in Z} f(y,z) = -\max_{z\in Z}(-f)(y,z)$$

holds. Therefore the assertion in this case follows from the upper semi-continuous case.

(ii) Let $A \subset X$ be non-empty and define $d(x,A) := \inf_{a\in A} d(x,a)$. We show that $x \mapsto d(x,A)$ is 1-Lipschitz. Fix $x, x' \in X$. For any $a \in A$, the triangle inequality gives

$$d(x,a) \leq d(x,x') + d(x',a),$$

hence

$$d(x,A) = \inf_{a\in A} d(x,a) \leq d(x,x') + \inf_{a\in A} d(x',a) = d(x,x') + d(x',A).$$

By symmetry (interchanging x and x'), we also have

$$d(x',A) \leq d(x,x') + d(x,A).$$

Therefore

$$|d(x,A) - d(x',A)| \leq d(x,x'),$$

so $d(\,\cdot\,, A)$ is continuous.

Assume moreover that A is closed. If $x \in A$, then $d(x, A) = 0$ is clear. Conversely, if $d(x, A) = 0$, then there exists a sequence $\{a_n\}_{n\in\mathbb{Z}_{\geq 1}}$ in A such that $d(x, a_n) \to 0$. Then $a_n \to x$ in X. Since A is closed, this implies $x \in A$.

(iii) Let $A, B \subset X$ be non-empty disjoint closed subsets. By (ii), the functions $x \mapsto d(x, A)$ and $x \mapsto d(x, B)$ are continuous on X. We claim that

$$d(x, A) + d(x, B) > 0 \qquad (x \in X).$$

Indeed, if $d(x, A) + d(x, B) = 0$, then $d(x, A) = d(x, B) = 0$, hence $x \in A \cap B$ by (ii), a contradiction. Therefore this function is well-defined on X. Since it is a quotient of continuous functions with non-vanishing denominator, it is continuous.

Mini-Exercise I.3.14

(i) By Lemma I.3.12, the space X satisfies the separation axiom T_4. Therefore the assertion follows from the standard Urysohn-type construction for T_4 spaces (see [Ke, Chapter 4 4] for example). If X is also second countable, it is metrizable by Urysohn's metrization theorem, and the assertion can alternatively be verified directly by considering the function (3) when both A and B are non-empty. If $A = \emptyset$, use the constant function 1; if $B = \emptyset$, use the constant function 0.

(ii) By Lemma I.3.12, X satisfies T_3. For each $x \in X$, choose an index $\iota(x) \in I$ with $x \in U_{\iota(x)}$. By Mini-Exercise I.3.10 (i), there is an open neighborhood B_x of x such that

$$\overline{B_x} \subset U_{\iota(x)}.$$

Then $\{B_x\}_{x\in X}$ is an open covering of X. By paracompactness, it admits a locally finite open refinement $\{W_\alpha\}_{\alpha\in A}$. For each α, fix $x(\alpha) \in X$ such that

$$W_\alpha \subset B_{x(\alpha)},$$

and set $i(\alpha) := \iota(x(\alpha))$. Then

$$\overline{W_\alpha} \subset \overline{B_{x(\alpha)}} \subset U_{i(\alpha)}.$$

For each $i \in I$, define

$$V_i := \bigcup \{W_\alpha \mid \alpha \in A,\ i(\alpha) = i\}.$$

Then $\{V_i\}_{i\in I}$ is a locally finite open covering of X and $V_i \subset U_i$ for all i. Moreover, since $\{W_\alpha\}_{\alpha\in A}$ is locally finite, by Lemma I.3.13 we obtain

$$\overline{V_i} = \bigcup \{\overline{W_\alpha} \mid \alpha \in A,\ i(\alpha) = i\} \subset U_i.$$

Consider $A_i := \overline{V_i}$ and $B_i := X \setminus U_i$. Then A_i and B_i are disjoint closed subsets of X. By (i), there exists a continuous function $\chi_i \colon X \to [0, 1]$ such that $\chi_i|_{A_i} = 1$ and $\chi_i|_{B_i} = 0$. Set $\psi_i := \max\{2\chi_i - 1, 0\}$. Then $0 \leq \psi_i \leq 1$ and $\psi_i|_{A_i} = 1$, and

$$\operatorname{Supp} \psi_i \subset \{x \in X \mid \chi_i(x) \geq 1/2\} \subset U_i.$$

Since $\{U_i\}_{i\in I}$ is locally finite and $\operatorname{Supp}\psi_i \subset U_i$, the family $\{\psi_i\}_{i\in I}$ is locally finite. Hence the sum

$$\psi := \sum_{i\in I} \psi_i$$

is a well-defined continuous function on X.

We claim that $\psi(x) > 0$ for all $x \in X$. Indeed, for $x \in X$, there exists i with $x \in V_i$, since $\{V_i\}_{i\in I}$ covers X. Then $x \in A_i = \overline{V_i}$, hence $\psi_i(x) = 1$, and thus $\psi(x) \geq \psi_i(x) = 1$.

Therefore we have the assertion by considering $\varphi_i := \psi_i/\psi$.

Mini-Exercise I.3.16

(i) Let $F \subset X$ be closed. We show that $f(F)$ is closed in Y. Let $y \in Y$ and choose an open neighborhood V of y such that $\overline{V}$ is compact; this is possible since Y is locally compact Hausdorff. By properness of f, the inverse image $f^{-1}(\overline{V})$ is compact. Hence $F \cap f^{-1}(\overline{V})$ is compact. Therefore the restricted map

$$f|_{F\cap f^{-1}(\overline{V})} \colon F \cap f^{-1}(\overline{V}) \to Y$$

is a continuous map from a compact space to a Hausdorff space. By the previous Mini-Exercise I.3.1, its image

$$f(F \cap f^{-1}(\overline{V})) = f(F) \cap \overline{V}$$

is closed in Y. Since y was arbitrary, this shows that $f(F)$ is closed in a neighborhood of every point of Y, and hence $f(F)$ is closed in Y (Indeed, if $f(F)$ were not closed, then there would exist a point of its closure having no neighborhood on which $f(F)$ is closed).

(ii) If f is bijective, then by (i) it is a closed bijection. Hence the inverse map f^{-1} is continuous.

Mini-Exercise I.4.1

Fix $y \in \mathbb{R}$. If $y \neq 0$, then

$$h(x, y) = \frac{xy}{x^2 + y^2}$$

is a quotient of C^∞ functions of x whose denominator does not vanish. Hence it is of class C^∞ as a function of x. If $y = 0$, then $h(x, 0) = 0$ for every $x \in \mathbb{R}$, and hence this function is also of class C^∞. By the same argument, for every fixed $x \in \mathbb{R}$, the function $y \mapsto h(x, y)$ is of class C^∞.

On the other hand, for every non-zero real number t, we have

$$h(t, t) = \frac{t^2}{2t^2} = \frac{1}{2}.$$

Therefore,

$$\lim_{t\to 0} h(t, t) = \frac{1}{2} \neq 0 = h(0, 0).$$

Thus h is not continuous at $(0, 0)$.

Mini-Exercise I.4.3

First assume $a \neq -1$. As

$$\int t^a \, dt = \frac{t^{a+1}}{a+1} + C,$$

where C is a constant, for $0 < b \leq \varepsilon$ we have

$$\int_b^\varepsilon t^a \, dt = \left[\frac{t^{a+1}}{a+1} \right]_{t=b}^{t=\varepsilon} = \frac{\varepsilon^{a+1} - b^{a+1}}{a+1}.$$

Thus the integral equals

$$\int_0^\varepsilon t^a \, dt := \lim_{b \to 0+} \int_b^\varepsilon t^a \, dt = \lim_{b \to 0+} \frac{\varepsilon^{a+1} - b^{a+1}}{a+1}.$$

We now consider the behavior of b^{a+1} as $b \to 0+$.

- If $a + 1 > 0$, then $b^{a+1} \to 0$, and hence

$$\int_0^\varepsilon t^a \, dt = \frac{\varepsilon^{a+1}}{a+1},$$

  which is finite.

- If $a + 1 < 0$, then $b^{a+1} = 1/b^{-(a+1)} \to +\infty$, and therefore

$$\left| \int_b^\varepsilon t^a \, dt \right| = \left| \frac{\varepsilon^{a+1} - b^{a+1}}{a+1} \right| \to \infty \qquad (b \to 0+),$$

  so the improper integral diverges.

Finally, if $a = -1$, then

$$\int_b^\varepsilon t^{-1} \, dt = \int_b^\varepsilon \frac{1}{t} \, dt = [\log t]_{t=b}^{t=\varepsilon} = \log \varepsilon - \log b = \log \frac{\varepsilon}{b},$$

and since $\log(\varepsilon/b) \to +\infty$ as $b \to 0+$, the integral diverges in this case as well.

Mini-Exercise I.4.4

This fact is nothing but the content of [Koi7, Lemma 2.1]. For the reader's convenience, we nevertheless include here a proof, with slightly more details. Set

$$\Psi(w) := \sum_{p=0}^{n} c_p \, w^p \overline{w}^{n-p}.$$

Since Φ is real-valued, the leading homogeneous term in its expansion is also real-valued.

We first show that $\Psi(w) \geq 0$ holds for all $w \in \mathbb{C}$. Fix $w \in \mathbb{C}$. Since Ω is a neighborhood of the origin, there exists $\varepsilon > 0$ such that $rw \in \Omega$ for all $0 < r < \varepsilon$. By the assumed expansion of Φ, as $r \to 0$ we have

$$\frac{\Phi(rw)}{r^n} = \Psi(w) + O(r).$$

Since $\Phi(rw) \geq 0$ for all sufficiently small $r > 0$, letting $r \to 0$ yields $\Psi(w) \geq 0$.

Next, since Ω is a neighborhood of 0, there exists $\varepsilon > 0$ such that the open disc

$$\Delta_\varepsilon := \{w \in \mathbb{C} \mid |w| < \varepsilon\}$$

is contained in Ω. Using polar coordinates $w = \rho e^{\sqrt{-1}\theta}$ and the Lebesgue measure $d\lambda = \rho\, d\rho\, d\theta$, we compute

$$\int_{\Delta_\varepsilon} w^p \overline{w}^{n-p}\, d\lambda = \int_0^\varepsilon \int_0^{2\pi} \rho^n e^{\sqrt{-1}(2p-n)\theta}\, \rho\, d\theta\, d\rho.$$

When $2p \neq n$, the integral vanishes. If n is even and $p = n/2$, then the integrand reduces to ρ^n, and we obtain

$$\int_{\Delta_\varepsilon} |w|^n\, d\lambda = \int_0^\varepsilon \rho^{n+1}\, d\rho \cdot \int_0^{2\pi} 1\, d\theta = \frac{2\pi}{n+2}\, \varepsilon^{n+2}.$$

Consequently,

$$\int_{\Delta_\varepsilon} \Psi(w)\, d\lambda = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \dfrac{2\pi}{n+2}\, \varepsilon^{n+2}\, c_{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Assume first that n is odd. Then $\int_{\Delta_\varepsilon} \Psi\, d\lambda = 0$. Since Ψ is continuous and non-negative, this implies $\Psi \equiv 0$ on Δ_ε, from which $c_p = 0$ follows for all p. Assume next that n is even. Since $\Psi \geq 0$, we have $\int_{\Delta_\varepsilon} \Psi\, d\lambda \geq 0$, which implies $c_{n/2} \geq 0$. Moreover, Ψ is real-valued, hence in particular $c_{n/2} \in \mathbb{R}$. If $c_{n/2} = 0$, then again $\int_{\Delta_\varepsilon} \Psi\, d\lambda = 0$, so $\Psi \equiv 0$ and therefore $c_p = 0$ for all p.

Mini-Exercise I.4.5

Write points of $\mathbb{R}^2$ as (x, y) and set

$$A := \frac{\partial F}{\partial y}(a, b) \in \mathbb{R},$$

which is nonzero by assumption. By continuity of $\partial F/\partial y$, there exist $\delta > 0$ and $\rho > 0$ such that the rectangle

$$R := [a - \delta, a + \delta] \times [b - \rho, b + \rho] \subset U$$

satisfies

$$\left| 1 - \frac{1}{A} \frac{\partial F}{\partial y}(x, y) \right| \leq r \qquad \text{for all } (x, y) \in R$$

for the constant $r := 1/2$. Note that

$$\left|\frac{\partial F}{\partial y}(x,y)\right| \geq |A| \cdot \left(1 - \left|1 - \frac{1}{A}\frac{\partial F}{\partial y}(x,y)\right|\right) \geq |A|(1-r) > 0 \tag{390}$$

holds. Moreover, since $F(a,b) = 0$ and F is continuous, shrinking $\delta > 0$ if necessary we may assume

$$\frac{1}{|A|} \sup_{x \in [a-\delta, a+\delta]} |F(x,b)| \leq (1-r)\rho.$$

Set

$$I := [a-\delta, a+\delta] \subset \mathbb{R}, \qquad J := [b-\rho, b+\rho] \subset \mathbb{R},$$

and

$$X := \{f \colon I \to J \mid f : \text{ continuous}\}.$$

Let d be the distance of X defined by

$$d(f,g) := \max_{x \in I} |f(x) - g(x)|.$$

Then it is fundamental that (X,d) is a complete metric space.

For any $f \in X$, define by letting

$$(T(f))(x) := f(x) - \frac{1}{A}F\big(x, f(x)\big)$$

for each $x \in I$.

First we show that $T(f) \in X$ for any $f \in X$. Let $f \in X$. Then $T(f)$ is continuous, since it is obtained from f by composition with the continuous map $(x,y) \mapsto y - \frac{1}{A}F(x,y)$. It remains to prove that $(T(f))(x) \in J$ for every $x \in I$. Fix $x \in I$ and set $y := f(x) \in J$. Applying the mean value theorem to the function $t \mapsto F(x,t)$ on the interval joining b and y, we obtain some $\xi \in J$ such that

$$F(x,y) - F(x,b) = \frac{\partial F}{\partial y}(x,\xi)\,(y-b).$$

Hence

$$\begin{aligned}(T(f))(x) - b &= y - b - \frac{1}{A}F(x,y) \\ &= \left(1 - \frac{1}{A}\frac{\partial F}{\partial y}(x,\xi)\right)(y-b) - \frac{1}{A}F(x,b).\end{aligned}$$

Taking absolute values and using $(x,\xi) \in I \times J \subset R$, we get

$$|(T(f))(x) - b| \leq \left|1 - \frac{1}{A}\frac{\partial F}{\partial y}(x,\xi)\right| |y-b| + \frac{1}{|A|}|F(x,b)| \leq r\,|y-b| + (1-r)\rho \leq \rho,$$

which shows $T(f) \in X$.

Next we show that T is a contraction on (X, d). Let $f, g \in X$ and fix $x \in I$. Set $y_1 := f(x)$ and $y_2 := g(x)$, both in J. Applying the mean value theorem to the function $t \mapsto F(x, t)$ on the interval joining y_1 and y_2, we obtain some $\eta \in J$ such that

$$F(x, y_1) - F(x, y_2) = \frac{\partial F}{\partial y}(x, \eta)\,(y_1 - y_2).$$

Hence

$$(T(f))(x) - (T(g))(x) = (y_1 - y_2) - \frac{1}{A}\big(F(x, y_1) - F(x, y_2)\big) = \left(1 - \frac{1}{A}\frac{\partial F}{\partial y}(x, \eta)\right)(y_1 - y_2).$$

Taking absolute values and using $(x, \eta) \in R$, we get

$$|(T(f))(x) - (T(g))(x)| \le r\,|y_1 - y_2| = r\,|f(x) - g(x)|.$$

Taking the supremum over $x \in I$, we conclude that

$$d(T(f), T(g)) \le r\,d(f, g).$$

Therefore, by Banach fixed point theorem (Theorem I.3.8), T has a unique fixed point $f \in X$. For each $x \in I$, the identity $T(f) = f$ is equivalent to $F\big(x, f(x)\big) = 0$. Conversely, for any $(x, y) \in I \times J$, $F(x, y) = 0$ implies $y = f(x)$ in virtue of the inequality (390).

Mini-Exercise I.4.10

By Proposition I.4.7, all we have to do is to check the positive (semi-)definiteness of the Hessians of f and g.

Put

$$S(x) := \sum_{k=1}^{n} e^{x_k}.$$

For $1 \le i \le n$, define

$$p_i(x) := \frac{e^{x_i}}{S(x)} \qquad \text{and} \qquad q_i(x) := \frac{e^{x_i}}{1 + S(x)}.$$

We also recall that the *Kronecker delta* δ_{ij} is defined by

$$\delta_{ij} := \begin{cases} 1 & (i = j), \\ 0 & (i \ne j). \end{cases}$$

The Hessian of f. Let $f(x) = \log S(x)$. Then

$$\frac{\partial f}{\partial x_i}(x) = \frac{1}{S(x)}\frac{\partial S}{\partial x_i}(x) = \frac{e^{x_i}}{S(x)} = p_i(x).$$

To compute the second derivatives, we differentiate $p_i(x) = e^{x_i} S(x)^{-1}$:

$$\frac{\partial p_i}{\partial x_j}(x) = \frac{\partial}{\partial x_j}\big(e^{x_i}\big) \cdot \frac{1}{S(x)} + e^{x_i} \cdot \frac{\partial}{\partial x_j}\Big(\frac{1}{S(x)}\Big).$$

Since $\frac{\partial}{\partial x_j}(e^{x_i}) = \delta_{ij} e^{x_i}$ and

$$\frac{\partial}{\partial x_j}\Big(\frac{1}{S(x)}\Big) = -\frac{1}{S(x)^2}\frac{\partial S}{\partial x_j}(x) = -\frac{e^{x_j}}{S(x)^2},$$

we obtain

$$\frac{\partial p_i}{\partial x_j}(x) = \delta_{ij}\frac{e^{x_i}}{S(x)} - \frac{e^{x_i}e^{x_j}}{S(x)^2} = \delta_{ij} p_i(x) - p_i(x)p_j(x).$$

Therefore,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial p_i}{\partial x_j}(x) = p_i(x)\big(\delta_{ij} - p_j(x)\big).$$

Hence the Hessian matrix $\mathrm{Hess}_x(f) = \big(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\big)_{i,j}$ is given by

$$\mathrm{Hess}_x(f) = \begin{pmatrix} p_1(x)(1-p_1(x)) & -p_1(x)p_2(x) & \cdots & -p_1(x)p_n(x) \\ -p_2(x)p_1(x) & p_2(x)(1-p_2(x)) & \cdots & -p_2(x)p_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ -p_n(x)p_1(x) & -p_n(x)p_2(x) & \cdots & p_n(x)(1-p_n(x)) \end{pmatrix}.$$

For $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, we compute

$$\begin{aligned} {}^t\xi\, \mathrm{Hess}_x(f)\, \xi &= \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\, \xi_i \xi_j \\ &= \sum_{i,j=1}^{n} \big(\delta_{ij} p_i(x) - p_i(x)p_j(x)\big)\xi_i\xi_j \\ &= \sum_{i=1}^{n} p_i(x)\xi_i^2 - \Big(\sum_{i=1}^{n} p_i(x)\xi_i\Big)^2. \end{aligned}$$

By the Cauchy–Schwarz inequality applied to the weights $p_i(x) \geq 0$ with $\sum_i p_i(x) = 1$, we have

$$\Big(\sum_{i=1}^{n} p_i(x)\xi_i\Big)^2 \leq \Big(\sum_{i=1}^{n} p_i(x)\Big)\Big(\sum_{i=1}^{n} p_i(x)\xi_i^2\Big) = \sum_{i=1}^{n} p_i(x)\xi_i^2,$$

and thus ${}^t\xi\, \mathrm{Hess}_x(f)\, \xi \geq 0$. Hence $\mathrm{Hess}_x(f)$ is positive semi-definite for every x.

The Hessian of g. Let $g(x) = \log(1 + S(x))$. Then

$$\frac{\partial g}{\partial x_i}(x) = \frac{1}{1+S(x)}\frac{\partial S}{\partial x_i}(x) = \frac{e^{x_i}}{1+S(x)} =: q_i(x).$$

As in the same manner as above, differentiating $q_i(x) = e^{x_i}(1 + S(x))^{-1}$ gives

$$\frac{\partial q_i}{\partial x_j}(x) = \delta_{ij} q_i(x) - q_i(x) q_j(x),$$

hence

$$\frac{\partial^2 g}{\partial x_i \partial x_j}(x) = q_i(x)\big(\delta_{ij} - q_j(x)\big).$$

Therefore

$$\mathrm{Hess}_x(g) = \begin{pmatrix} q_1(x)(1-q_1(x)) & -q_1(x)q_2(x) & \cdots & -q_1(x)q_n(x) \\ -q_2(x)q_1(x) & q_2(x)(1-q_2(x)) & \cdots & -q_2(x)q_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ -q_n(x)q_1(x) & -q_n(x)q_2(x) & \cdots & q_n(x)(1-q_n(x)) \end{pmatrix}.$$

For $\xi \in \mathbb{R}^n$,

$${}^t\xi\, \mathrm{Hess}_x(g)\, \xi = \sum_{i=1}^n q_i(x)\xi_i^2 - \Big(\sum_{i=1}^n q_i(x)\xi_i\Big)^2.$$

Using Cauchy–Schwarz again,

$$\Big(\sum_{i=1}^n q_i(x)\xi_i\Big)^2 \le \Big(\sum_{i=1}^n q_i(x)\Big)\Big(\sum_{i=1}^n q_i(x)\xi_i^2\Big).$$

Since $\sum_{i=1}^n q_i(x) = \frac{S(x)}{1+S(x)} \in (0,1)$, we obtain

$${}^t\xi\, \mathrm{Hess}_x(g)\, \xi \ge \Big(1 - \sum_{i=1}^n q_i(x)\Big) \sum_{i=1}^n q_i(x)\xi_i^2.$$

The right-hand side is strictly positive for $\xi \neq 0$ because $q_i(x) > 0$ for all i. Thus $\mathrm{Hess}_x(g)$ is positive definite for every x.

Mini-Exercise I.4.11

(i) Choose a non-negative function $\rho\colon \mathbb{R} \to \mathbb{R}$ of class C^∞ such that

$$\mathrm{Supp}\,\rho \subset \Big(-\frac{1}{4}, \frac{1}{4}\Big), \qquad \int_{\mathbb{R}} \rho(t)\, dt = 1, \qquad \int_{\mathbb{R}} t\, \rho(t)\, dt < 0.$$

Set

$$c := -\int_{\mathbb{R}} t\, \rho(t)\, dt \;>\; 0.$$

Let

$$A_n := \frac{n^2}{2} + n \qquad (n \in \mathbb{Z}_{>0}).$$

We now construct a sequence of positive numbers $\{b_n\}_{n\in\mathbb{Z}_{>0}}$ so that the following requirement is satisfied for each $n \in \mathbb{Z}_{>0}$: **($\mathbf{R}_n$)** $A_n + \sum_{k=1}^{n} b_k\, c > y_n$.
First, since $c > 0$ and $A_1 = \frac{3}{2}$, we can choose $b_1 > 0$ such that

$$A_1 + b_1\, c > y_1,$$

i.e., (R_1) holds. Assume that $b_1, \dots, b_n$ have been chosen so that (R_m) holds for every $m \in \mathbb{Z}_{>0}$ with $1 \le m \le n$. Since $c > 0$, we may choose $b_{n+1} > 0$ so large that

$$A_{n+1} + \sum_{k=1}^{n+1} b_k\, c > y_{n+1},$$

so (R_{n+1}) holds. Thus, by induction, we obtain $\{b_n\}_{n\ge1}$ satisfying (R_n) for all n.

Now define

$$\xi(x) := 1 + \sum_{n=1}^{\infty} b_n\, \rho(x-n) \qquad (x > 0),$$

where the sum is locally finite because $\operatorname{Supp}\rho \subset (-1/4, 1/4)$. In particular, ξ is a C^∞ function on $\mathbb{R}_{>0}$ and $\xi(x) > 0$ for all $x > 0$. Define η and θ by

$$\eta(x) := 1 + \int_0^x \xi(t)\, dt, \qquad \theta(x) := \int_0^x \eta(s)\, ds \qquad (x > 0).$$

Then θ is also a C^∞ function on $\mathbb{R}_{>0}$, and

$$\theta'(x) = \eta(x) > 0, \qquad \theta''(x) = \xi(x) > 0 \quad (x > 0).$$

Fix an integer $n \ge 1$. Since

$$\theta(n) = \int_0^n \eta(t)\, dt = \int_0^n \Big(1 + t + \sum_{k=1}^{\infty} b_k \int_0^t \rho(s-k)\, ds\Big) dt,$$

we obtain

$$\theta(n) = n + \frac{n^2}{2} + \sum_{k=1}^{\infty} b_k \int_0^n \int_0^t \rho(s-k)\, ds\, dt.$$

The integrand is bounded on the set $\{(t,s) \in \mathbb{R}^2 \mid 0 \le s \le t \le n\}$, which has finite measure. Hence, by Fubini's theorem, we may change the order of integration to get

$$\theta(n) = \frac{n^2}{2} + n + \sum_{k=1}^{\infty} b_k \int_0^n (n-s)\rho(s-k)\, ds.$$

For $k \ge 1$, set

$$I_{k,n} := \int_0^n (n-s)\rho(s-k)\, ds.$$

If $k > n$, then for every $s \in [0, n]$ we have

$$s - k \le n - k \le -1 < -\tfrac{1}{4},$$

so $s - k \notin \operatorname{Supp}\rho$ and thus $\rho(s-k) = 0$ on $[0, n]$. Hence $I_{k,n} = 0$ for $k > n$.

Assume $1 \le k \le n$. Since $\operatorname{Supp}\rho \subset (-1/4, 1/4)$, we have $\rho(s-k) \neq 0$ only if $s \in (k - 1/4, k + 1/4)$. Extending the integral beyond n adds only non-positive terms, and therefore

$$I_{k,n} \ge \int_{k-\frac{1}{4}}^{k+\frac{1}{4}} (n - s)\rho(s-k)\, ds.$$

With the change of variables $u := s - k$, we obtain

$$\begin{aligned} I_{k,n} &\ge \int_{-\frac{1}{4}}^{\frac{1}{4}} (n - k - u)\rho(u)\, du \\ &= (n-k)\int_{\mathbb{R}} \rho(u)\, du - \int_{\mathbb{R}} u\rho(u)\, du \\ &= (n-k) + c. \end{aligned}$$

Combining these observations, we conclude that

$$\theta(n) \ge \frac{n^2}{2} + n + \sum_{k=1}^{n} b_k\big((n-k) + c\big) \;\ge\; \frac{n^2}{2} + n + \sum_{k=1}^{n} b_k\, c = A_n + \sum_{k=1}^{n} b_k\, c > y_n,$$

where the last inequality follows from (R_n). This proves (i).

(ii) Let $f\colon \mathbb{R}_{\ge 0} \to \mathbb{R}$ be continuous and define

$$y_n := \max_{x \in [0,n]} f(x) \qquad (n \in \mathbb{Z}_{>0}),$$

which is finite by compactness of $[0, n]$. By (i), we obtain a C^∞ function $\theta\colon \mathbb{R}_{>0} \to \mathbb{R}$ with $y_n < \theta(n)$ for all n and $\theta'(x) > 0$, $\theta''(x) > 0$ for all $x > 0$. Since $\theta'(x) > 0$, the function θ is strictly increasing. For any $x > 0$, let $n := \min\{m \in \mathbb{Z} \mid x \le m\}$ so that $x \le n \le x + 1$. Then

$$f(x) \le y_n < \theta(n) \le \theta(x+1).$$

Hence $\chi(x) := \theta(x+1)$ satisfies $f(x) < \chi(x)$ and

$$\chi'(x) = \theta'(x+1) > 0, \qquad \chi''(x) = \theta''(x+1) > 0 \quad (x > 0),$$

which proves (ii).

Mini-Exercise I.4.13

We first verify that d defines a metric on X. For $\varphi, \psi \in X$, we clearly have $d(\varphi, \psi) \geq 0$, and $d(\varphi, \psi) = 0$ if and only if $\varphi(x) = \psi(x)$ for all $x \in I$, i.e. $\varphi = \psi$. Symmetry is obvious from the definition. Finally, for $\varphi, \psi, \chi \in X$ and every $x \in I$, the triangle inequality for the norm $\|\cdot\|$ yields

$$\|\varphi(x) - \chi(x)\| \leq \|\varphi(x) - \psi(x)\| + \|\psi(x) - \chi(x)\|,$$

and taking the maximum over $x \in I$ we obtain

$$d(\varphi, \chi) \leq d(\varphi, \psi) + d(\psi, \chi).$$

Thus (X, d) is a metric space.

We now show that (X, d) is complete. Let $\{\varphi_\nu\}_{\nu=0}^{\infty} \subset X$ be a Cauchy sequence, i.e., for every $\varepsilon > 0$ there exists $N \in \mathbb{Z}_{\geq 0}$ such that $d(\varphi_\nu, \varphi_\mu) < \varepsilon$ for all $\nu, \mu \geq N$.

Writing

$$\varphi_\nu = (\varphi_{\nu 1}, \varphi_{\nu 2}, \ldots, \varphi_{\nu n}),$$

we observe that for each $j \in \{1, 2, \ldots, n\}$:

- for every $x \in I$,

$$|\varphi_{\nu j}(x)| \leq |\varphi_{\nu j}(x) - \beta_j| + |\beta_j| \leq \|\varphi_\nu - \beta\| + \|\beta\| \leq b + \|\beta\|,$$

  so that the sequence $\{\varphi_{\nu j}\}_{\nu=0}^{\infty}$ is uniformly bounded;

- for every $\varepsilon > 0$, setting $\delta := \varepsilon / M$, we have

$$|x_0 - x_1| < \delta \implies |\varphi_{\nu j}(x_0) - \varphi_{\nu j}(x_1)| \leq M|x_0 - x_1| < \varepsilon,$$

  which shows that the family $\{\varphi_\nu\}_{\nu=0}^{\infty}$ is equicontinuous.

By applying the Ascoli–Arzelà theorem successively for each j, we obtain a subsequence $\{\varphi_{\nu_k}\}_{k=0}^{\infty}$ converging uniformly on I to an M-Lipschitz continuous function $\varphi_{\infty j} \colon I \to \mathbb{R}$ for each j.

A standard argument shows that the sequence $\{\varphi_{\nu j}\}_{\nu=0}^{\infty}$ converges uniformly to $\varphi_{\infty j}$. Defining

$$\varphi_\infty(x) := (\varphi_{\infty 1}(x), \ldots, \varphi_{\infty n}(x)),$$

we obtain

$$\lim_{\nu \to \infty} \sup_{x \in I} \|\varphi_\nu(x) - \varphi_\infty(x)\| = 0.$$

In particular,

$$\max_{x \in I} \|\varphi_\infty(x) - \beta\| \leq b,$$

and φ_∞ is M-Lipschitz continuous. Hence $\varphi_\infty \in X$, which proves completeness.

Mini-Exercise I.4.14

For $\varphi \in X$:

- As observed above, for every $t \in I$ and every $\varphi \in X$ we have $(t, \varphi(t)) \in D$, so that $f(t, \varphi(t))$ is well-defined. Moreover, for each $j = 1, \dots, n$, the function $t \mapsto f_j(t, \varphi(t))$ is continuous on the compact interval I. Hence each integral
$$\int_\alpha^x f_j(t, \varphi(t))\, dt$$
is well-defined as a finite real number, and thus $T(\varphi)(x)$ is well-defined for all $x \in I$.

- For arbitrary $x_0, x_1 \in I$, we compute
$$\begin{aligned} \|T(\varphi)(x_0) - T(\varphi)(x_1)\| &= \left\| \int_{x_1}^{x_0} f(t, \varphi(t))\, dt \right\| \\ &\le \left| \int_{x_1}^{x_0} \|f(t, \varphi(t))\|\, dt \right| \le \left| \int_{x_1}^{x_0} M\, dt \right| = M|x_0 - x_1|. \end{aligned}$$
Thus $T(\varphi)$ is M-Lipschitz continuous on I.

- Finally, for any $x \in I$, using the M-Lipschitz continuity of $T(\varphi)$ and the fact that $T(\varphi)(\alpha) = \beta$, we obtain
$$\|T(\varphi)(x) - \beta\| = \|T(\varphi)(x) - T(\varphi)(\alpha)\| \le M|x - \alpha| \le b,$$
where the last inequality follows from the definition of the interval I.

Thus $T(\varphi) \in X$, and we obtain a map $T\colon X \to X$.

For $\varphi, \psi \in X$, consider the continuous function
$$x \longmapsto \|T(\varphi)(x) - T(\psi)(x)\|$$
on the compact interval I, and let $x_0 \in I$ be a point at which this function attains its maximum. Then we have
$$\begin{aligned} d(T(\varphi), T(\psi)) &= \|T(\varphi)(x_0) - T(\psi)(x_0)\| \\ &= \left\| \int_\alpha^{x_0} \big(f(t, \varphi(t)) - f(t, \psi(t))\big)\, dt \right\| \\ &\le \left| \int_\alpha^{x_0} \|f(t, \varphi(t)) - f(t, \psi(t))\|\, dt \right| \\ &\le \left| \int_\alpha^{x_0} L\, \|\varphi(t) - \psi(t)\|\, dt \right| \\ &\le \left| \int_\alpha^{x_0} L\, d(\varphi, \psi)\, dt \right| \\ &= |x_0 - \alpha|\, L\, d(\varphi, \psi). \end{aligned}$$

Since $x_0 \in I$, we have $|x_0 - \alpha| \le 1/(2L)$ by the definition of I, and hence

$$d(T(\varphi), T(\psi)) \le \frac{1}{2}\, d(\varphi, \psi).$$

Hence T is a contraction.

Mini-Exercise I.4.15

Set

$$J(x) := \int_0^x t^{a-1} F(t)\, dt \qquad (0 \le x < r).$$

Then, it easily follows from Lemma I.4.2 that J is continuous on $[0, r)$: Even more precisely, one has

$$|J(x)| = \left| \int_0^x t^{a-1} F(t)\, dt \right| \le C \int_0^x t^{a-1}\, dt = O(x^a) \qquad (x \to 0)$$

for some constant $C > 0$, since F is bounded near 0. By construction, J is of class C^1 on $(0, r)$ and $J'(x) = x^{a-1} F(x)$ holds on $(0, r)$.

Define ρ on $(0, r)$ by

$$\rho(x) := \frac{e^{-J(x)}}{x}.$$

By construction, ρ is of class C^1 on $(0, r)$. Moreover,

$$\rho'(x) = -\frac{1}{x^2} e^{-J(x)} - \frac{1}{x} J'(x) e^{-J(x)} \qquad (x \in (0, r)).$$

Using $J'(x) = x^{a-1} F(x)$, we obtain

$$\rho'(x) + \Big(\frac{1}{x} + x^{a-1} F(x) \Big) \rho(x) = 0 \qquad (x \in (0, r)).$$

Define H on $(0, r)$ by

$$H(x) := \int_0^x \left(\rho(t) - \frac{1}{t} \right) dt.$$

Since $J(x) = O(x^a)$ as $x \to 0$, we have

$$\rho(t) - \frac{1}{t} = \frac{e^{-J(t)} - 1}{t} = O(t^{a-1}),$$

which is integrable near 0 again by Lemma I.4.2. Hence H extends continuously to $[0, r)$ with $H(0) = 0$. Moreover, H is of class C^2 on $(0, r)$ and $\chi(x) := \log x + H(x)$ satisfies

$$\chi''(x) + G(x) \cdot \chi'(x) = \rho'(x) + G(x) \cdot \rho(x) = 0$$

for all $x \in (0, r)$, by construction.

For uniqueness, let χ_1, χ_2 be two such functions and set $u := \chi_1 - \chi_2$. Then $u'' + Gu' = 0$ on $(0, r)$, hence

$$u'(x) = C \cdot \rho(x)$$

for some constant C, by virtue of Theorem I.4.12 (Strictly speaking, the theorem yields the existence of such a constant C locally on a neighborhood of each point of $(0, r)$; however, since $(0, r)$ is connected, these local constants must coincide, and therefore a single constant C exists on the whole interval $(0, r)$). If $C \neq 0$, then u is unbounded as $x \to 0$, contradicting the continuity of $\chi_1 - \chi_2$ at 0. Thus $C = 0$ and u is constant. Since $(\chi_1 - \chi_2)(0) = 0$, we obtain $u \equiv 0$, proving uniqueness.

Mini-Exercise I.4.18

By translating the coordinates in $\mathbb{R}^n$, we may assume without loss of generality that $x_0 = 0$. Write the given vector field v on $U \subset \mathbb{R}^n$ in the standard coordinates $x = (x^1, \dots, x^n)$ as

$$v = \sum_{j=1}^{n} f_j(x) \, \frac{\partial}{\partial x^j},$$

where each $f_j \colon U \to \mathbb{R}$ is a C^1 function. Set

$$f(x) := (f_1(x), \dots, f_n(x)) \in \mathbb{R}^n \qquad (x \in U),$$

so that $v_x = \sum_{j=1}^{n} f_j(x) \, (\partial/\partial x^j)|_x$.

Let $\varepsilon > 0$ and $V \subset U$ be an open set, and let $\varphi \colon I_\varepsilon \times V \to U$ be a differentiable map. By definition,

$$(\dot\varphi)_{(t,x)} = \sum_{j=1}^{n} \frac{\partial \varphi_j}{\partial t}(t, x) \; \left. \frac{\partial}{\partial x^j} \right|_{\varphi(t,x)},$$

where $\varphi(t, x) = (\varphi_1(t, x), \dots, \varphi_n(t, x))$. Hence the condition that φ is a flow of v, namely

$$(\dot\varphi)_{(t,x)} = v_{\varphi(t,x)} \qquad (\forall (t, x) \in I_\varepsilon \times V),$$

is equivalent to the system of ordinary differential equations

$$\frac{\partial \varphi_j}{\partial t}(t, x) = f_j(\varphi(t, x)) \quad (j = 1, \dots, n),$$

together with the initial condition $\varphi(0, x) = x$.

We now construct such a φ by solving, for each initial point $x \in V$, the initial value problem

$$y'(t) = f(y(t)), \qquad y(0) = x.$$

Since f is of class C^1, it is locally Lipschitz. Choose $r > 0$ such that the closed cube

$$\overline{B} := \{y \in \mathbb{R}^n \mid \|y\| \leq r\}$$

is contained in U, and set

$$V := \{x \in \mathbb{R}^n \mid \|x\| < r/2\}.$$

Then $V \subset U$ and, moreover, if $x \in V$ and $\|y - x\| \leq r/2$, then $\|y\| \leq r$, so $y \in \overline{B}$.

Since $\overline{B}$ is compact and f is continuous, there exists $M > 0$ such that $\|f(y)\| \leq M$ for all $y \in \overline{B}$. Also, since f is of class C^1, its derivative is bounded on $\overline{B}$, and therefore there exists $L > 0$ such that

$$\|f(y) - f(z)\| \leq L\,\|y - z\| \qquad (\forall y, z \in \overline{B}).$$

Fix $a := 1$ and $b := r/2$. For each $x \in V$, consider the set

$$D_x := \{(t, y) \in \mathbb{R} \times \mathbb{R}^n \mid |t| \leq a,\ \|y - x\| \leq b\}.$$

As explained above, $D_x \subset [-a, a] \times \overline{B}$, hence on D_x the map $(t, y) \mapsto f(y)$ is bounded by M and Lipschitz in the second variable with constant L.

Apply Theorem I.4.12 with $\alpha = 0$, $\beta = x$, and with the function $\widetilde{f}(t, y) := f(y)$ on D_x. We obtain a unique differentiable solution $y_x \colon I_\varepsilon \to \mathbb{R}^n$, where

$$\varepsilon := \min\left\{a,\ \frac{b}{M},\ \frac{1}{2L}\right\} > 0.$$

Note that ε depends only on a, b, M, L, hence is independent of the choice of $x \in V$.

Define

$$\varphi(t, x) := y_x(t) \qquad ((t, x) \in I_\varepsilon \times V).$$

Then $\varphi(0, x) = y_x(0) = x$ for all $x \in V$. Moreover, for each fixed $x \in V$, the curve $t \mapsto \varphi(t, x)$ satisfies $y_x'(t) = f(y_x(t))$, i.e. $\partial\varphi_j/\partial t(t, x) = f_j(\varphi(t, x))$ for $j = 1, \ldots, n$. Therefore

$$(\dot{\varphi})_{(t,x)} = v_{\varphi(t,x)} \qquad (\forall (t, x) \in I_\varepsilon \times V).$$

The regularity result recalled after Mini-Exercise I.4.18 (see [Lee-Jo2, Theorem D.5]) shows that φ is of class C^1 as a map of (t, x). Thus φ is a (local) flow of v.

Finally, let $\psi \colon I_{\varepsilon'} \times V' \to U$ be another (local) flow of v defined near $(0, 0)$. Fix $x \in V \cap V'$. Then both curves $t \mapsto \varphi(t, x)$ and $t \mapsto \psi(t, x)$ solve the same initial value problem $y'(t) = f(y(t))$ with $y(0) = x$. By the local uniqueness statement in Theorem I.4.12, they coincide on the whole common interval $I_{\min\{\varepsilon, \varepsilon'\}}$ (Their coincidence set contains 0, is closed by continuity, and is open by local uniqueness). Hence $\varphi = \psi$ on $I_{\min\{\varepsilon, \varepsilon'\}} \times (V \cap V')$, which proves the claimed uniqueness.

Mini-Exercise I.4.20

Refer to [Lee-Jo2, Example 9.1 (b)].

Mini-Exercise I.5.1

Refer to [AM, Lan2, Mat] for example.

Mini-Exercise I.5.2

Refer to [AM, Lan2, Mat] for example.

Mini-Exercise I.5.3

Refer to [AM, Lan2, Mat] for example. The fraction fields $\mathrm{Frac}(R)$ and $\mathrm{Frac}(S^{-1}R)$ are canonically isomorphic. This fact can be shown as follows: By definition, the fraction field $\mathrm{Frac}(R)$ consists of equivalence classes of fractions a/b with $a, b \in R$ and $b \neq 0$. On the other hand, since $S \subset R \setminus \{0\}$, the localization $S^{-1}R$ is an integral domain, and $\mathrm{Frac}(S^{-1}R)$ consists of fractions of elements of $S^{-1}R$, that is, of expressions of the form

$$\frac{a/s}{b/t} \qquad (a, b \in R,\ s, t \in S).$$

Such an element can be rewritten as

$$\frac{at}{bs},$$

which is a fraction of elements of R with nonzero denominator. Conversely, for any fraction $a/b \in \mathrm{Frac}(R)$, we may write

$$\frac{a}{b} = \frac{a/1}{b/1},$$

which is a well-defined element of $\mathrm{Frac}(S^{-1}R)$ since $1 \in S$. Hence $\mathrm{Frac}(R)$ and $\mathrm{Frac}(S^{-1}R)$ coincide.

Note that, from the universal point of view, the answer can be described more simply by applying [AM, Proposition 3.1].

Mini-Exercise I.5.4

(i) Assume that R is Noetherian. Since M is a finitely generated R-module, there exists a surjective R-module homomorphism

$$\varphi\colon R^{\oplus n} \longrightarrow M$$

for some integer $n \geq 1$. Set $N' := \varphi^{-1}(N) \subset R^{\oplus n}$. Then N' is an R-submodule of $R^{\oplus n}$. Since $R^{\oplus n}$ is a finitely generated R-module and R is Noetherian, the module $R^{\oplus n}$ satisfies the ascending chain condition on R-submodules. Hence every R-submodule of $R^{\oplus n}$, in particular N', is finitely generated. Since $\varphi|_{N'}\colon N' \twoheadrightarrow N$ is surjective, it follows that N is finitely generated as an R-module.

(ii) Let $R := \mathbb{C}[x_1, x_2, x_3, \dots]$ be the polynomial ring in countably many indeterminates over $\mathbb{C}$. Let $M := R$ and let $N \subset M$ be the ideal

$$N := (x_1, x_2, x_3, \dots).$$

Clearly, M is finitely generated as an R-module. However, the ideal N is not finitely generated, since any finite set of generators involves only finitely many variables. Thus N is an R-submodule of a finitely generated R-module which is not finitely generated.

Mini-Exercise I.5.5

Since r is not a zero-divisor, multiplication by r on R is injective. Consider the short exact sequence

$$0 \longrightarrow R \xrightarrow{\times r} R \longrightarrow R/rR \longrightarrow 0.$$

Applying the long exact Tor-sequence, we obtain an exact sequence

$$\mathrm{Tor}_1^R(R, A) \longrightarrow \mathrm{Tor}_1^R(R/rR, A) \longrightarrow R \otimes_R A \xrightarrow{\times r} R \otimes_R A.$$

Since R is free (as an R-module), $\mathrm{Tor}_1^R(R, A) = 0$, and using the natural identification $R \otimes_R A \cong A$, the map $R \otimes_R A \to R \otimes_R A$ is identified with $A \xrightarrow{\times r} A$. Hence

$$\mathrm{Tor}_1^R(R/rR, A) \cong \mathrm{Ker}(A \xrightarrow{\times r} A) = \{a \in A \mid ra = 0\}.$$

Mini-Exercise I.5.6

By the structure theorem for finitely generated abelian groups, there exist an integer $r \geq 0$ and integers $n_1, \ldots, n_k \geq 2$ such that

$$A \;\cong\; \mathbb{Z}^{\oplus r} \;\oplus\; \bigoplus_{i=1}^{k} \mathbb{Z}/n_i\mathbb{Z}.$$

In particular,

$$A_{\mathrm{tor}} \;\cong\; \bigoplus_{i=1}^{k} \mathbb{Z}/n_i\mathbb{Z}.$$

Let $e_1, \ldots, e_{r+k}$ be the standard basis of $\mathbb{Z}^{\oplus(r+k)}$. Define a surjective homomorphism

$$\pi : \mathbb{Z}^{\oplus(r+k)} \longrightarrow A$$

as follows: for $1 \leq j \leq r$, let $\pi(e_j)$ be the j-th standard basis vector in the free summand $\mathbb{Z}^{\oplus r} \subset A$, and for $1 \leq i \leq k$, let $\pi(e_{r+i})$ be the class of 1 in $\mathbb{Z}/n_i\mathbb{Z} \subset A$. Then π is surjective and its kernel is given by

$$\mathrm{Ker}\,\pi \;=\; \bigoplus_{i=1}^{k} n_i\mathbb{Z}\, e_{r+i} \;\cong\; \mathbb{Z}^{\oplus k}.$$

Let

$$\mu : \mathbb{Z}^{\oplus k} \longrightarrow \mathbb{Z}^{\oplus(r+k)}$$

be the homomorphism defined by $\mu(\varepsilon_i) = n_i e_{r+i}$ for the standard basis $\varepsilon_1, \ldots, \varepsilon_k$ of $\mathbb{Z}^{\oplus k}$. Then we obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z}^{\oplus k} \xrightarrow{\mu} \mathbb{Z}^{\oplus(r+k)} \xrightarrow{\pi} A \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ and using $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^{\oplus m}, \mathbb{Z}) = 0$, we obtain an exact sequence

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\oplus(r+k)}, \mathbb{Z}) \xrightarrow{\mu^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\oplus k}, \mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \longrightarrow 0.$$

Identifying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\oplus(r+k)}, \mathbb{Z}) \cong \mathbb{Z}^{\oplus(r+k)}$ via the dual basis, the map μ^* is described by

$$\mu^* : \mathbb{Z}^{\oplus(r+k)} \longrightarrow \mathbb{Z}^{\oplus k}, \qquad (a_1, a_2, \ldots, a_{r+k}) \longmapsto (n_1 a_{r+1}, n_2 a_{r+2}, \ldots, n_k a_{r+k}).$$

Hence

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \;\cong\; \mathrm{Coker}\, \mu^* \;\cong\; \bigoplus_{i=1}^{k} \mathbb{Z}/n_i\mathbb{Z} \;\cong\; A_{\mathrm{tor}},$$

which proves the claim.

Mini-Exercise I.5.11

For $R = \mathbb{C}[z]$ and $I = (z)$, each class in R/I^n has a unique representative of degree less than n. Compatibility under reduction therefore identifies a family in $\widehat{R}$ with a unique sequence $a_0, a_1, a_2, \ldots$ of complex numbers. Conversely, these coefficients give the compatible truncations, so

$$\mathbb{C}[[z]] \longrightarrow \widehat{R}, \qquad \sum_{k\geq 0} a_k z^k \longmapsto \left(\sum_{k=0}^{n-1} a_k z^k + I^n \right)_{n\geq 1}$$

is a bijection. It preserves addition and multiplication modulo I^n for every $n \geq 1$, hence is a ring isomorphism.

Mini-Exercise I.6.1

(i) The map T_b is continuous, since it is the composition of the addition map $V \times V \to V$ with the map $x \mapsto (x, b)$. Its inverse is T_{-b}, which is also continuous. Hence T_b is a homeomorphism.

(ii) Let U be a neighborhood of 0 and $x \in V$ be a point. Consider the map $f_x : \mathbb{C} \to V$ defined by $f_x(r) := rx$. Since f_x is continuous and U is a neighborhood of 0, the preimage $f_x^{-1}(U)$ is a neighborhood of 0 in $\mathbb{C}$. Hence there exists $\varepsilon > 0$ such that $rx \in U$ for all $r \in \mathbb{C}$ with $|r| < \varepsilon$. It follows that $x \in tU$ for $t = 2\varepsilon^{-1}$. Thus U is absorbing.

(iii) Consider the addition map $h : V \times V \to V$, $h(x, y) = x + y$. By continuity of h at $(0, 0)$, there exist neighborhoods W_1 and W_2 of 0 such that $W_1 + W_2 \subset U$. Setting $W := W_1 \cap W_2$, we obtain $W + W \subset U$.

(iv) By (i), for each $b \in B$ the translation T_b is a homeomorphism, hence $A + b$ is open if A is open. Therefore

$$A + B = \bigcup_{b\in B} (A + b)$$

is open.

(v) Let U be a neighborhood of 0. By continuity of scalar multiplication $\mathbb{C} \times V \to V$ at $(0, 0)$, there exist a neighborhood U_1 of 0 in V and a number $\delta > 0$ such that

$$\lambda U_1 \subset U \quad \text{for all } \lambda \in \mathbb{C} \text{ with } |\lambda| < \delta.$$

Set

$$W := \bigcup_{|\lambda|\leq 1} \lambda((\delta/2)U_1).$$

Then W is a neighborhood of 0 and satisfies $W \subset U$. Moreover, W is balanced by construction. Finally, W is absorbing by (ii).

(vi) Consider the map

$$\Phi : V \times B \longrightarrow V \times B, \qquad \Phi(x, y) = (x + y, y).$$

This map is a homeomorphism, with inverse given by

$$\Phi^{-1}(u, v) = (u - v, v).$$

Since A is closed in V, the subset $A \times B$ is closed in the subspace $V \times B$. Hence $\Phi(A \times B)$ is closed in $V \times B$.

Let $\pi : V \times B \to V$ be the projection onto the first factor. By Mini-Exercise I.3.2, the map π is closed because B is compact. Therefore

$$A + B = \pi\big(\Phi(A \times B)\big)$$

is closed in V.

(vii) First, consider A with the subspace topology. Let $+_A : A \times A \to A$ and $\cdot_A : \mathbb{C} \times A \to A$ be the addition and scalar multiplication on A. Let $\iota : A \hookrightarrow V$ be the inclusion map. Then we have

$$\iota \circ +_A \;=\; + \circ (\iota \times \iota), \qquad \iota \circ \cdot_A \;=\; \cdot \circ (\mathrm{id}_{\mathbb{C}} \times \iota),$$

where $+ : V \times V \to V$ and $\cdot : \mathbb{C} \times V \to V$ denote the corresponding operations on V. Since $\iota \times \iota : A \times A \to V \times V$ and $\mathrm{id}_{\mathbb{C}} \times \iota : \mathbb{C} \times A \to \mathbb{C} \times V$ are continuous (by definition of product and subspace topologies), and since $+$ and $\cdot$ are continuous on V, it follows that $\iota \circ +_A$ and $\iota \circ \cdot_A$ are continuous. By the definition of the subspace topology on A, this implies that $+_A$ and $\cdot_A$ are continuous. Hence A is a TVS.

Next, consider V/A with the quotient topology. The quotient map $q\colon V \to V/A$ is open, since $q^{-1}(q(O)) = O + A$ is open for every open subset $O \subset V$ by (iv). Hence $q \times q$ and $\mathrm{id}_{\mathbb{C}} \times q$ are open surjections, and therefore quotient maps. Define addition and scalar multiplication on V/A by

$$q(x) + q(y) := q(x + y), \qquad \lambda\, q(x) := q(\lambda x) \qquad (x, y \in V,\ \lambda \in \mathbb{C}).$$

These are well-defined since A is a vector subspace. Let $+_{V/A} : (V/A) \times (V/A) \to V/A$ and $\cdot_{V/A} : \mathbb{C} \times (V/A) \to V/A$ denote these operations. We claim that both maps are continuous.

For addition, consider the map

$$\alpha : V \times V \longrightarrow V/A, \qquad \alpha(x, y) := q(x + y).$$

It is continuous because $+$ on V and q are continuous. Let $q \times q : V \times V \to (V/A) \times (V/A)$ be the product map. By the universal property of the quotient topology, a map $f : (V/A) \times (V/A) \to V/A$ is continuous if and only if $f \circ (q \times q)$ is continuous. Since

$$+_{V/A} \circ (q \times q) = \alpha,$$

it follows that $+_{V/A}$ is continuous.

Similarly for scalar multiplication, consider the map

$$\beta : \mathbb{C} \times V \longrightarrow V/A, \qquad \beta(\lambda, x) := q(\lambda x).$$

It is continuous because scalar multiplication on V and q are continuous. Again, by the universal property of the quotient topology and since

$$\cdot_{V/A} \circ (\mathrm{id}_{\mathbb{C}} \times q) = \beta,$$

we conclude that $\cdot_{V/A}$ is continuous. Hence V/A is a TVS.

Mini-Exercise I.6.3

Assume that V is complete in the sense of Cauchy filter bases. Let $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be a Cauchy sequence in the metric space (V, d). For each integer $N \geq 0$, set

$$A_N := \{x_n \mid n \geq N\} \subset V.$$

Let $\mathcal{F} := \{A_N\}_{N\geq 0}$. Then $\mathcal{F}$ is a filter base. We claim that $\mathcal{F}$ is Cauchy. Let U be an open neighborhood of 0 in V. Since the topology comes from d, there exists $\varepsilon > 0$ such that

$$B_d(0, \varepsilon) := \{v \in V \mid d(v, 0) < \varepsilon\} \subset U.$$

Since $\{x_n\}$ is Cauchy, there exists N such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. If $a, b \in A_N$, then $a = x_n$, $b = x_m$ for some $n, m \geq N$, hence

$$d(a - b, 0) = d\big((a - b) + b,\ 0 + b\big) = d(a, b) < \varepsilon.$$

Thus $a - b \in B_d(0, \varepsilon) \subset U$, so $A_N - A_N \subset U$, which proves that $\mathcal{F}$ is Cauchy.

By completeness of V, there exists $x \in V$ such that $\mathcal{F} \to x$. We show that $x_n \to x$ in the metric space (V, d). Let $\varepsilon > 0$ be arbitrary and set $U := B_d(0, \varepsilon)$. Then U is an open neighborhood of 0 in V, so $B_d(x, \varepsilon) := x + U$ is an open neighborhood of x. Since $\mathcal{F} \to x$, there exists N such that $A_N \subset x + U$. This means $x_n \in B_d(x, \varepsilon)$ for all $n \geq N$, i.e. $d(x_n, x) < \varepsilon$. Hence $x_n \to x$. Thus (V, d) is complete, since any Cauchy sequence converges.

Conversely, assume that the metric space (V, d) is complete. Let $\mathcal{F}$ be a Cauchy filter base in V. We will construct a Cauchy sequence $\{x_n\}_{n\in\mathbb{Z}_{\geq 1}}$ in (V, d) whose limit will be a point x, and show that $\mathcal{F} \to x$.

For each integer $n \geq 1$, choose a balanced open neighborhood U_n of 0 in V such that

$$U_n \subset B_d(0, 2^{-n}) \qquad \text{and} \qquad U_{n+1} \subset U_n$$

(Such a sequence exists since the d-balls form a neighborhood basis at 0: Refer to Mini-Exercise I.6.1 (v)). Since $\mathcal{F}$ is Cauchy, for each n we can choose $A_n \in \mathcal{F}$ with $A_n - A_n \subset U_n$. Using that $\mathcal{F}$ is a filter base, we may moreover choose A_n so that

$$A_{n+1} \subset A_n \qquad (n \geq 1).$$

Pick a point $x_n \in A_n$ for each n. Then for $m \geq n$ we have $x_m \in A_m \subset A_n$, hence $x_m - x_n \in A_n - A_n \subset U_n$, and therefore

$$d(x_m, x_n) = d(x_m - x_n, 0) < 2^{-n}.$$

Thus $\{x_n\}$ is a Cauchy sequence in (V, d). By completeness of (V, d), there exists $x \in V$ such that $x_n \to x$.

We now prove that $\mathcal{F} \to x$. Let U be an arbitrary neighborhood of x in V. Choose $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$. Since $x_n \to x$, there exists N such that $x_n \in B_d(x, \varepsilon/2)$ for every $n \geq N$. Choose $n \geq N$ such that $2^{-n} < \varepsilon/2$. For any $y \in A_n$, we have $y - x_n \in A_n - A_n \subset U_n \subset B_d(0, 2^{-n})$, hence

$$d(y, x) \leq d(y, x_n) + d(x_n, x) < 2^{-n} + \varepsilon/2 < \varepsilon.$$

Thus $y \in B_d(x, \varepsilon) \subset U$, and we obtain $A_n \subset U$. Since $A_n \in \mathcal{F}$, this proves $\mathcal{F} \to x$.

Mini-Exercise I.6.6

(i) Let $A \subset V$ be a finite-dimensional vector subspace. Take $x \in \overline{A}$ and assume that $x \notin A$. Set $B := A + \mathbb{C}x$. Then B is a finite-dimensional vector subspace of V. Endow B with the subspace topology induced from V. Since V is Hausdorff, so is B.

Choose a basis $(e_1, \dots, e_n)$ of A, where $n := \dim A$. Then $(e_1, \dots, e_n, x)$ is a basis of B. By Theorem I.6.4 (i) applied to B with this basis, the linear map

$$f : \mathbb{C}^{n+1} \longrightarrow B, \qquad f(z_1, \dots, z_{n+1}) = \sum_{j=1}^{n} z_j e_j + z_{n+1} x$$

is a TVS isomorphism. In particular, $A = f(\mathbb{C}^n \times \{0\})$. Since $x \in \overline{A}$ in V and B carries the subspace topology, we also have $x \in \overline{A}$ in B. Applying f^{-1}, we obtain

$$(0, \dots, 0, 1) = f^{-1}(x) \in \overline{\mathbb{C}^n \times \{0\}} \quad \text{in } \mathbb{C}^{n+1},$$

which is a contradiction. Therefore we have $A = \overline{A}$.

(ii) Assume that V is compact and Hausdorff. If $V \neq \{0\}$, choose $x \in V$ with $x \neq 0$ and consider the one-dimensional subspace $\mathbb{C}x \subset V$ with the subspace topology. Then $\mathbb{C}x$ is a one-dimensional Hausdorff TVS. By Theorem I.6.4 (i), $\mathbb{C}x$ is linearly homeomorphic to $\mathbb{C}$, hence it is not compact. On the other hand, by (i) of this Mini-Exercise applied to the finite-dimensional subspace $\mathbb{C}x$, the subspace $\mathbb{C}x$ is closed in V. Thus $\mathbb{C}x$ is a closed subset of the compact space V, hence compact, a contradiction.

(iii) Let $A \subset V$ be a vector subspace of codimension one. Assume that A is not closed. Then, by Lemma I.6.2, it follows that $\overline{\{0\}} = V/A$, since $\dim V/A = 1$. Therefore we have that $\overline{A} = V$ in this case.

(iv) Let V be the vector space consisting of all complex sequences

$$x = (x_0, x_1, x_2, \dots)$$

such that

$$\sum_{j=0}^{\infty} |x_j|^2 < \infty,$$

endowed with the norm

$$\|x\| := \Big(\sum_{j=0}^{\infty} |x_j|^2\Big)^{1/2}.$$

Then V is a Hausdorff TVS.

For each $k \in \mathbb{Z}_{\geq 0}$, let $e_k \in V$ be the sequence defined by

$$(e_k)_j := \begin{cases} 1 & (j = k), \\ 0 & (j \neq k). \end{cases}$$

Then $\|e_k\| = 1$ for all k.

Let W be the vector subspace of V consisting of all sequences with finite support, i.e.,

$$W := \operatorname{span}\{e_k \mid k \in \mathbb{Z}_{\geq 0}\}.$$

Define a linear functional $f_0 \colon W \to \mathbb{C}$ by

$$f_0\Big(\sum_{k=0}^{N} a_k e_k\Big) := \sum_{k=0}^{N} k\, a_k.$$

Then f_0 is well-defined and linear. Moreover, f_0 is not continuous on W. Indeed, $\|e_k\| = 1$ for all k, while

$$|f_0(e_k)| = k \longrightarrow \infty \qquad (k \to \infty),$$

so f_0 is unbounded on the unit sphere of W.

Now consider the family of pairs (U, f_U), where U is a vector subspace of V containing W and $f_U \colon U \to \mathbb{C}$ is a linear extension of f_0. This family is partially ordered by extension of the functionals. Every totally ordered subfamily has an upper bound given by the union of the domains together with the induced linear functional. By Zorn's lemma, there exists a maximal element $(U_{\max}, f_{\max})$.

We claim that $U_{\max} = V$. Otherwise, take $x \in V \setminus U_{\max}$. Then $f_{\max}$ extends linearly to $U_{\max} + \mathbb{C}x$ by assigning an arbitrary value to x, contradicting maximality. Thus $f_{\max} \colon V \to \mathbb{C}$ is a linear functional extending f_0.

Since $f_{\max}|_W = f_0$ and f_0 is unbounded, $f_{\max}$ is not continuous.

(v) Let V and $f = f_{\max} \colon V \to \mathbb{C}$ be as in (iv) and set $A := \operatorname{Ker} f$. Then A is a vector subspace of V of codimension one. If A were closed, then V/A would be Hausdorff by Lemma I.6.2 (iii). Since V/A is one-dimensional over $\mathbb{C}$, by Theorem I.6.4 (i) it would be homeomorphic to $\mathbb{C}$, and hence the induced linear map $V \to V/A \cong \mathbb{C}$ would be continuous. But this induced map coincides with f up to a nonzero scalar, so f would be continuous, a contradiction. Therefore A is not closed.

Mini-Exercise I.6.8

Let $n := \dim A$ and choose a basis $(a_1, \dots, a_n)$ of A. Let $(\alpha_1, \dots, \alpha_n) \subset A^\vee$ be the dual basis. By surjectivity of the restriction $r\colon V^\vee \to A^\vee$, for each i there exists $\widetilde{\alpha}_i \in V^\vee$ such that $\widetilde{\alpha}_i|_A = \alpha_i$. Define a linear map

$$P\colon V \longrightarrow A, \qquad P(v) := \sum_{i=1}^{n} \widetilde{\alpha}_i(v)\, a_i \qquad (v \in V).$$

Then P is continuous, since it is a finite linear combination of continuous linear functionals. Moreover, for $a = \sum_{j=1}^{n} c_j a_j \in A$ we have

$$P(a) = \sum_{i=1}^{n} \widetilde{\alpha}_i(a)\, a_i = \sum_{i=1}^{n} \alpha_i(a)\, a_i = \sum_{i=1}^{n} c_i a_i = a,$$

hence $P|_A = \mathrm{id}_A$. In particular, P is a continuous projection onto A.

Set $B := \operatorname{Ker} P$. Then B is a closed vector subspace of V. We claim that $V = A \oplus B$ as TVSs. Indeed, for any $v \in V$ we have

$$v = P(v) + \big(v - P(v)\big) \in A + B,$$

and $v - P(v) \in B$ since $P(v - P(v)) = P(v) - P(v) = 0$. If $a \in A \cap B$, then $a = P(a) = 0$, so $A \cap B = \{0\}$. Thus $V = A \oplus B$ algebraically. Finally, the canonical projection $\mathrm{Pr}_A\colon V \to A$ associated with the algebraic decomposition coincides with P, hence is continuous, and $\mathrm{Pr}_B = \mathrm{id}_V - P$ is also continuous. Therefore $V = A \oplus B$ holds as TVSs.

Mini-Exercise I.6.10

Here we denote by $\| \bullet \|_X$ and $\| \bullet \|_Y$ the norms of X and Y, respectively.

(*i*) Let $\varphi \in Y^\vee$. Then for any $x \in X$ with $\|x\|_X \leq 1$ we have

$$|(T^\vee\varphi)(x)| = |\varphi(T(x))| \leq \|\varphi\|_{\mathrm{op}}\, \|T(x)\|_Y \leq \|\varphi\|_{\mathrm{op}}\, \|T\|_{\mathrm{op}}\, \|x\|_X \leq \|\varphi\|_{\mathrm{op}}\, \|T\|_{\mathrm{op}}.$$

Taking the supremum over $\{x \in X \mid \|x\|_X \leq 1\}$, we obtain

$$\|T^\vee\varphi\|_{\mathrm{op}} \leq \|T\|_{\mathrm{op}}\, \|\varphi\|_{\mathrm{op}}.$$

(*ii*) Let $x \in X$. For any $\varphi \in X^\vee$, we have

$$|(J_X(x))(\varphi)| = |\varphi(x)| \leq \|\varphi\|_{\mathrm{op}}\, \|x\|_X.$$

Hence $J_X(x)$ is a bounded linear functional on $X^\vee$, that is, $J_X(x) \in X^{\vee\vee}$, and moreover

$$\|J_X(x)\|_{\mathrm{op}} \leq \|x\|_X$$

(In particular, this inequality implies the continuity of $J_X(x)\colon X^\vee \to \mathbb{C}$, from which it follows that $J_X(x) \in X^{\vee\vee}$).

Since injectivity of J_X follows from the isometric property, it suffices to prove the reverse inequality

$$\|x\|_X \le \|J_X(x)\|_{\mathrm{op}}$$

for each $x \in X$. The case $x = 0$ being trivial, we assume $x \neq 0$. Consider the one-dimensional subspace $\mathbb{C}x \subset X$ (with the induced norm). Define $f\colon \mathbb{C}x \to \mathbb{C}$ by

$$f(\lambda x) := \lambda \qquad (\lambda \in \mathbb{C}).$$

Then f is linear and

$$\|f\|_{\mathrm{op}} = \sup_{\lambda \neq 0} \frac{|f(\lambda x)|}{\|\lambda x\|_X} = \sup_{\lambda \neq 0} \frac{|\lambda|}{|\lambda|\,\|x\|_X} = \frac{1}{\|x\|_X}.$$

By the Hahn–Banach theorem, f extends to some $\varphi \in X^\vee$ with $\|\varphi\|_{\mathrm{op}} = \|f\|_{\mathrm{op}} = 1/\|x\|_X$. Set $\psi := \|x\|_X\,\varphi \in X^\vee$. Then $\|\psi\|_{\mathrm{op}} = 1$ and $\psi(x) = \|x\|_X$. Therefore

$$\|J_X(x)\|_{\mathrm{op}} = \sup\bigl\{|\eta(x)| \mid \eta \in X^\vee,\ \|\eta\|_{\mathrm{op}} \le 1\bigr\} \ge |\psi(x)| = \|x\|_X.$$

(*iii*) Set

$$A := \bigl\{\, y \in Y \;\bigm|\; \forall\, \varphi \in \operatorname{Ker} T^\vee,\ \varphi(y) = 0\bigr\}.$$

First let us show $\overline{\operatorname{Image} T} \subset A$. Take $y \in \overline{\operatorname{Image} T}$ and $\varphi \in \operatorname{Ker} T^\vee$. Then $\varphi(T(x)) = 0$ for all $x \in X$. Therefore $\varphi|_{\operatorname{Image} T} = 0$. By continuity of φ, it follows that $\varphi(y) = 0$.

Conversely, assume $y \notin \overline{\operatorname{Image} T}$. Since $\overline{\operatorname{Image} T}$ is a closed subspace of Y, by the Hahn–Banach theorem there exists $\varphi \in Y^\vee$ such that

$$\varphi|_{\overline{\operatorname{Image} T}} \equiv 0 \qquad \text{and} \qquad \varphi(y) \neq 0.$$

Then $\varphi(T(x)) = 0$ for all $x \in X$, hence $\varphi \in \operatorname{Ker} T^\vee$, while $\varphi(y) \neq 0$. Thus $y \notin A$.

(*iv*) The inclusion $\operatorname{Ker} T^\vee \subset \{\varphi \in Y^\vee \mid \varphi|_{\operatorname{Image} T} \equiv 0\}$ can be checked easily because, for any $\varphi \in \operatorname{Ker} T^\vee$, $T^\vee(\varphi) = \varphi \circ T = 0$ holds as an element of $X^\vee$. Conversely, let $\varphi \in Y^\vee$ satisfy $\varphi|_{\operatorname{Image} T} \equiv 0$. Then for every $x \in X$ we have $\varphi(T(x)) = 0$, that is, $(T^\vee(\varphi))(x) = 0$ for all $x \in X$. Hence $T^\vee(\varphi) = 0$ in $X^\vee$.

(*v*) Let $\{\varphi_n\}_{n \in \mathbb{Z}_{\ge 0}}$ be a Cauchy sequence in $X^\vee$ with respect to the operator norm. Then for each $x \in X$, the sequence $\{\varphi_n(x)\}_n$ is Cauchy in $\mathbb{C}$, hence converges. Define $\varphi\colon X \to \mathbb{C}$ by

$$\varphi(x) := \lim_{n \to \infty} \varphi_n(x).$$

Then φ is linear. Moreover, since $\{\varphi_n\}$ is Cauchy in the operator norm, there exists $M > 0$ such that $\|\varphi_n\|_{\mathrm{op}} \le M$ for all n. Hence for every $x \in X$ we have

$$|\varphi(x)| \le \sup_n |\varphi_n(x)| \le M\,\|x\|_X,$$

so φ is continuous and $\|\varphi\|_{\mathrm{op}} \le M$. Finally, for any $x \in X$ with $\|x\|_X \le 1$ we have

$$|\varphi_n(x) - \varphi(x)| = \lim_{m \to \infty} |\varphi_n(x) - \varphi_m(x)| \le \limsup_{m \to \infty} \|\varphi_n - \varphi_m\|_{\mathrm{op}},$$

which tends to 0 as $n \to \infty$ since $\{\varphi_n\}$ is Cauchy. Taking the supremum over $\{x \in X \mid \|x\|_X \le 1\}$, we obtain $\|\varphi_n - \varphi\|_{\mathrm{op}} \to 0$. Thus $X^\vee$ is complete.

Mini-Exercise I.6.16

(i) Assume first that T is closable, and let $\{x_n\} \subset \operatorname{Dom} T$ satisfy $x_n \to 0$ in X and $T(x_n) \to y$ in Y. Then $(x_n, T(x_n)) \to (0, y)$ in $X \oplus Y$, hence $(0, y) \in \overline{\operatorname{Graph}(T)}$. Since T is closable, $\overline{\operatorname{Graph}(T)}$ is the graph of an operator; in particular, over $0 \in X$ there can be only one point. As $(0, 0) \in \operatorname{Graph}(T) \subset \overline{\operatorname{Graph}(T)}$, we must have $y = 0$.

Conversely, assume that the implication in (i) holds. We show that $\overline{\operatorname{Graph}(T)}$ is the graph of an operator. Let (x, y) and (x, y') be points in $\overline{\operatorname{Graph}(T)}$. Choose sequences $\{u_n\}$ and $\{v_n\}$ in $\operatorname{Dom} T$ such that

$$(u_n, T(u_n)) \to (x, y), \qquad (v_n, T(v_n)) \to (x, y') \qquad (n \to \infty)$$

in $X \oplus Y$. Then $u_n - v_n \to 0$ in X and

$$T(u_n - v_n) = T(u_n) - T(v_n) \to y - y' \qquad (n \to \infty)$$

in Y. By the assumed implication, this forces $y - y' = 0$, hence $y = y'$. Therefore, above each $x \in X$ there is at most one $y \in Y$ with $(x, y) \in \overline{\operatorname{Graph}(T)}$. Define

$$\operatorname{Dom} \widehat{T} := \{x \in X \mid \exists\, y \in Y \text{ with } (x, y) \in \overline{\operatorname{Graph}(T)}\}, \qquad \widehat{T}(x) := y.$$

Then $\overline{\operatorname{Graph}(T)} = \operatorname{Graph}(\widehat{T})$, so T is closable.

(ii) Assume that T is densely defined and bounded. By the boundedness assumption, there exists $C \geq 0$ such that $\|T(x)\|_Y \leq C\|x\|_X$ for all $x \in \operatorname{Dom} T$. Let $\{x_n\} \subset \operatorname{Dom} T$ satisfy $x_n \to 0$ in X and $T(x_n) \to y$ in Y. Then $\|T(x_n)\|_Y \leq C\|x_n\|_X \to 0$, hence $T(x_n) \to 0$ in Y. Thus $y = 0$, and by (i) the operator T is closable. Let $\widehat{T}$ be its closure.

On the other hand, since T is bounded on $\operatorname{Dom} T$, it extends uniquely to a continuous linear map (still denoted by $\widehat{T}$)

$$\widehat{T} \colon \overline{\operatorname{Dom} T} \longrightarrow Y$$

by continuity. We claim that $\operatorname{Graph}(\widehat{T}) = \overline{\operatorname{Graph}(T)}$ in $X \oplus Y$. Indeed, let $(x, y) \in \operatorname{Graph}(\widehat{T})$ with $x \in \overline{\operatorname{Dom} T}$ and $y = \widehat{T}(x)$. Choose $\{x_n\} \subset \operatorname{Dom} T$ with $x_n \to x$ in X. Then by continuity we have $T(x_n) = \widehat{T}(x_n) \to \widehat{T}(x) = y$ in Y, so $(x_n, T(x_n)) \to (x, y)$ in $X \oplus Y$. Hence $(x, y) \in \overline{\operatorname{Graph}(T)}$, and thus $\operatorname{Graph}(\widehat{T}) \subset \overline{\operatorname{Graph}(T)}$. The reverse inclusion follows because $\operatorname{Graph}(\widehat{T})$ is closed and contains $\operatorname{Graph}(T)$. Therefore $\operatorname{Graph}(\widehat{T}) = \overline{\operatorname{Graph}(T)}$ and $\widehat{T}$ is the closure of T.

(iii) Let $X = Y = L^2(\mathbb{R})$ with respect to the Lebesgue measure, and define $T \colon X \xrightarrow{\text{op}} Y$ by

$$(T(f))(x) := f'(x) \quad \left(:= \frac{df}{dx}(x)\right) \qquad (x \in \mathbb{R})$$

with domain $\operatorname{Dom} T := C_c^\infty(\mathbb{R}) := \{f \colon \mathbb{R} \to \mathbb{C} : C^\infty \mid \text{The support of } f \text{ is compact}\}$. Then $\operatorname{Dom} T$ is dense in $L^2(\mathbb{R})$, so T is densely defined.

We show that T is closable. Let $\{f_n\} \subset C_c^\infty(\mathbb{R})$ satisfy $f_n \to 0$ in $L^2(\mathbb{R})$ and $f_n' \to g$ in $L^2(\mathbb{R})$. For any $\varphi \in C_c^\infty(\mathbb{R})$, integration by parts gives

$$\int_{\mathbb{R}} g\,\overline{\varphi}\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} f_n'\,\overline{\varphi}\,dx = -\lim_{n\to\infty} \int_{\mathbb{R}} f_n\,\overline{\varphi'}\,dx = 0,$$

since $f_n \to 0$ in $L^2(\mathbb{R})$ and $\varphi' \in L^2(\mathbb{R})$. Hence $g = 0$ as an element of $L^2(\mathbb{R})$. By (i), the operator T is closable.

We finally show that T cannot be extended to a continuous operator defined on the whole space $L^2(\mathbb{R})$. Assume to the contrary that there exists a continuous operator $\widetilde{T}\colon L^2(\mathbb{R}) \xrightarrow{\text{op}} L^2(\mathbb{R})$ with $\operatorname{Dom}\widetilde{T} = L^2(\mathbb{R})$ and $\widetilde{T}|_{C_c^\infty(\mathbb{R})} = T$. Since $\widetilde{T}$ is continuous, there exists a constant $C \geq 0$ such that

$$\|\widetilde{T}(f)\|_{L^2(\mathbb{R})} \leq C\,\|f\|_{L^2(\mathbb{R})} \qquad (f \in L^2(\mathbb{R})).$$

Restricting this estimate to $f \in C_c^\infty(\mathbb{R})$, we obtain

$$\|f'\|_{L^2(\mathbb{R})} \leq C\,\|f\|_{L^2(\mathbb{R})} \qquad (f \in C_c^\infty(\mathbb{R})).$$

This is impossible. Indeed, choose a nonzero function $\phi \in C_c^\infty(\mathbb{R})$, and set

$$f_n(x) := \phi(nx) \qquad (n \in \mathbb{Z}_{>0}).$$

Then $f_n \in C_c^\infty(\mathbb{R})$ and

$$\|f_n\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |\phi(nx)|^2\,dx = \frac{1}{n}\int_{\mathbb{R}} |\phi(u)|^2\,du,$$

hence $\|f_n\|_{L^2(\mathbb{R})} = n^{-1/2}\|\phi\|_{L^2(\mathbb{R})}$. On the other hand,

$$f_n'(x) = n\,\phi'(nx),$$

so

$$\|f_n'\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |n\phi'(nx)|^2\,dx = n\int_{\mathbb{R}} |\phi'(u)|^2\,du,$$

hence $\|f_n'\|_{L^2(\mathbb{R})} = n^{1/2}\|\phi'\|_{L^2(\mathbb{R})}$. Therefore

$$\frac{\|f_n'\|_{L^2(\mathbb{R})}}{\|f_n\|_{L^2(\mathbb{R})}} = n\,\frac{\|\phi'\|_{L^2(\mathbb{R})}}{\|\phi\|_{L^2(\mathbb{R})}} \longrightarrow \infty \qquad (n \to \infty),$$

contradicting the inequality $\|f'\|_{L^2} \leq C\|f\|_{L^2}$.

Mini-Exercise I.6.20

(i) By the von Neumann theorem for closed densely defined operators (Theorem I.6.18),

$$(\operatorname{Ran} T)^\perp = \operatorname{Ker} T^\dagger, \qquad (\operatorname{Ran} S^\dagger)^\perp = \operatorname{Ker} S.$$

Hence

$$\mathcal{H} = \operatorname{Ker} S \cap \operatorname{Ker} T^\dagger = (\operatorname{Ran} S^\dagger)^\perp \cap (\operatorname{Ran} T)^\perp = (\operatorname{Ran} T + \operatorname{Ran} S^\dagger)^\perp.$$

(ii) Taking orthogonal complements, by (i) we obtain

$$\overline{\operatorname{Ran} T + \operatorname{Ran} S^\dagger} = \mathcal{H}^\perp.$$

Since $\operatorname{Ran} T \perp \operatorname{Ran} S^\dagger$ (because $S \circ T = 0$), this yields the orthogonal decomposition

$$H_2 = \overline{\operatorname{Ran} T} \oplus \overline{\operatorname{Ran} S^\dagger} \oplus \mathcal{H}.$$

(*iii*) If $T^\dagger(u_1) = T^\dagger(u_2)$, then the condition implies

$$|\langle u_1 - u_2, v\rangle_{H_2}| \leq C_v \, \|T^\dagger(u_1 - u_2)\|_{H_1} = 0,$$

from which one has $\ell(T^\dagger(u_1)) = \ell(T^\dagger(u_2))$. Therefore this map is well-defined. As

$$\big|\ell(T^\dagger(u))\big| = |\langle u, v\rangle_{H_2}| \leq C_v \, \|T^\dagger(u)\|_{H_1}$$

holds for $u \in \operatorname{Dom} T^\dagger$, one has

$$\|\ell\|_{\mathrm{op}} \leq C_v.$$

Therefore ℓ is bounded, hence it is continuous.

(*iv*) Assume first that $v \in \operatorname{Ran} T$. Then there exists $x \in \operatorname{Dom} T$ such that $v = T(x)$. For any $u \in \operatorname{Dom} T^\dagger$, we have

$$|\langle u, v\rangle_{H_2}| = |\langle u, T(x)\rangle_{H_2}| = |\langle T^\dagger(u), x\rangle_{H_1}| \leq \|T^\dagger(u)\|_{H_1} \, \|x\|_{H_1}.$$

Conversely, assume that there exists $C_v \geq 0$ such that the condition in (10) holds. Then the linear map $\ell\colon \operatorname{Ran} T^\dagger \to \mathbb{C}$ as in (*iii*) is well-defined and continuous. Hence, by the Hahn–Banach Theorem, it extends to a continuous linear functional $\widetilde{\ell}\colon H_1 \to \mathbb{C}$. By the Riesz representation theorem, there exists $x \in H_1$ such that

$$\langle u, v\rangle_{H_2} = \langle T^\dagger(u), x\rangle_{H_1} \qquad (\forall\, u \in \operatorname{Dom} T^\dagger).$$

By definition of the von Neumann adjoint, this implies that $x \in \operatorname{Dom} (T^\dagger)^\dagger$ and $(T^\dagger)^\dagger(x) = v$. As $(T^\dagger)^\dagger = T$ by Theorem I.6.18, $v \in \operatorname{Ran} T$, from which (10) follows.

Mini-Exercise I.6.22

Let (X, Σ, μ) satisfy the assumptions of Lemma I.6.21. By σ-finiteness, choose an increasing sequence $\{X_n\}_{n\in\mathbb{Z}_{\geq 0}}$ in Σ with $X = \bigcup_{n=0}^{\infty} X_n$ and $\mu(X_n) < \infty$ for all n. Let $\mathcal{G}$ be a countable family generating Σ modulo μ-null sets, and enlarge it by the sets X_n. The algebra $\mathcal{A}$ obtained from $\mathcal{G}$ by finite unions, finite intersections, and complements is countable. For each n, the family $\mathcal{A}_n := \{A \cap X_n \mid A \in \mathcal{A}\}$ is a countable algebra of subsets of X_n.

Every measurable subset E of X_n can be approximated in measure by members of $\mathcal{A}_n$: for every $\varepsilon > 0$, there is $B \in \mathcal{A}_n$ with $\mu((E \setminus B) \cup (B \setminus E)) < \varepsilon$. Indeed, the family of measurable sets with this approximation property is closed under relative complements and finite unions. It is also closed under countable unions: since $\mu(X_n) < \infty$, a countable union can first be approximated in measure by a finite union, and then by a member of $\mathcal{A}_n$. This family is therefore a σ-algebra containing $\mathcal{A}_n$, and the approximation property is unchanged by a μ-null set. The hypothesis on $\mathcal{G}$ gives the assertion for every E.

For a measurable set E, write $\mathbf{1}_E$ for the function equal to 1 on E and to 0 elsewhere. For $E, B \subset X_n$ we have

$$\|\mathbf{1}_E - \mathbf{1}_B\|^2_{L^2(X_n,\mu)} = \mu\big((E \setminus B) \cup (B \setminus E)\big).$$

Thus indicators of members of $\mathcal{A}_n$ approximate all measurable indicators on X_n in L^2. Let $\mathcal{S}_n$ consist of the functions

$$\sum_{j=1}^{k} q_j \mathbf{1}_{A_j}, \qquad k \in \mathbb{Z}_{>0}, \quad q_j \in \mathbb{Q} + \sqrt{-1}\,\mathbb{Q}, \quad A_j \in \mathcal{A}_n.$$

The set $\mathcal{S}_n$ is countable. Truncation of an L^2 function, followed by approximation of its bounded real and imaginary parts by simple functions, reduces density to the indicators just considered. The coefficients of each simple function can be approximated by members of $\mathbb{Q} + \sqrt{-1}\,\mathbb{Q}$, since $\mu(X_n) < \infty$. Hence $\mathcal{S}_n$ is dense in $L^2(X_n, \mu)$. Extend its elements by zero outside X_n, and set $\mathcal{S} := \bigcup_{n=0}^{\infty} \mathcal{S}_n \subset L^2(X, \mu)$. This set is countable.

For $f \in L^2(X, \mu)$ and $\varepsilon > 0$, dominated convergence and $X_n \nearrow X$ give an n such that

$$\|f - \mathbf{1}_{X_n} f\|_{L^2(X,\mu)} < \tfrac{\varepsilon}{2}.$$

Choose $g \in \mathcal{S}_n$ with $\|\mathbf{1}_{X_n} f - g\|_{L^2(X,\mu)} < \varepsilon/2$. Then $\|f - g\|_{L^2(X,\mu)} < \varepsilon$. Thus $\mathcal{S}$ is dense, and $L^2(X, \mu)$ is separable.

Mini-Exercise I.6.24

- *convex*: if $x, y \in B_r(p)$ and $0 \le t \le 1$, then

$$p(tx + (1-t)y) \le t\,p(x) + (1-t)\,p(y) < r;$$

- *balanced*: if $x \in B_r(p)$ and $|\lambda| \le 1$, then

$$p(\lambda x) = |\lambda|\,p(x) \le p(x) < r;$$

- *absorbing*: for every $x \in V$ there exists $t > 0$ with $x \in tB_r(p)$, since

$$p\big(\tfrac{x}{t}\big) = \tfrac{1}{t} p(x) < r$$

whenever $t > p(x)/r$.

Mini-Exercise I.6.25

$(i) \Rightarrow (iii)$ Assume that f is continuous. Then the set

$$U := \{v \in V \mid |f(v)| < 1\}$$

is an open neighborhood of 0 in V. Since the topology of V is generated by the family $\{p_i\}_{i\in I}$, there exist indices $i_1,\dots,i_k \in I$ and $\varepsilon > 0$ such that

$$\{v \in V \mid p_{i_j}(v) < \varepsilon \text{ for all } j = 1,\dots,k\} \subset U.$$

Hence, for any $v \in V$ satisfying $\max_{1\le j\le k} p_{i_j}(v) < \varepsilon$, we have $|f(v)| < 1$. For $v \in V$ with $\max_{1\le j\le k} p_{i_j}(v) > 0$, homogeneity of the seminorms gives

$$|f(v)| = \frac{2}{\varepsilon} \max_{1\le j\le k} p_{i_j}(v) \cdot \left| f\left(\frac{\varepsilon}{2 \cdot \max_{1\le j\le k} p_{i_j}(v)} \cdot v \right) \right| < \frac{2}{\varepsilon} \max_{1\le j\le k} p_{i_j}(v).$$

If $\max_{1\le j\le k} p_{i_j}(v) = 0$, then $\lambda v \in U$ for every $\lambda > 0$, so $\lambda|f(v)| < 1$ for every $\lambda > 0$ and hence $f(v) = 0$. Thus (iii) holds with $C = 2/\varepsilon$.

$(iii) \Rightarrow (ii)$ Set

$$p(v) := C \max_{1\le j\le k} p_{i_j}(v).$$

Then p is a continuous seminorm on V, and $|f(v)| \le p(v)$ for all $v \in V$.

$(ii) \Rightarrow (i)$ Assume that there exists a continuous seminorm p such that $|f(v)| \le p(v)$ for all $v \in V$. Then

$$f^{-1}\big(\{z \in \mathbb{C} \mid |z| < 1\}\big) \supset \{v \in V \mid p(v) < 1\},$$

and the latter set is an open neighborhood of 0. Hence f is continuous.

Mini-Exercise I.6.27

(i) First, $d_p(f,g) \ge 0$ is clear, and $d_p(f,g) = d_p(g,f)$ holds because $|f-g| = |g-f|$. Next, $d_p(f,g) = 0$ implies $f = g$ almost everywhere, since it means $\int_0^1 |f-g|^p = 0$, hence $|f-g|^p = 0$ almost everywhere. In what follows we verify the triangle inequality. Let $f,g,h \in L^p([0,1])$. If $p \ge 1$, then Minkowski's inequality gives

$$\|f-h\|_p \le \|f-g\|_p + \|g-h\|_p,$$

hence $d_p(f,h) \le d_p(f,g) + d_p(g,h)$. If $0 < p < 1$, then for any $a,b \in \mathbb{R}_{\ge 0}$ one has

$$(a+b)^p \le a^p + b^p$$

(since $t \mapsto t^p$ is concave on $\mathbb{R}_{\ge 0}$). Applying this pointwise with $a = |f-g|$ and $b = |g-h|$, we obtain

$$|f-h|^p \le (|f-g| + |g-h|)^p \le |f-g|^p + |g-h|^p,$$

hence by integration,

$$d_p(f,h) = \int_0^1 |f-h|^p \le \int_0^1 |f-g|^p + \int_0^1 |g-h|^p = d_p(f,g) + d_p(g,h).$$

(*ii*) Nonnegativity is clear. If $\|f\|_p = 0$, then $\int_0^1 |f|^p = 0$, hence $f = 0$ almost everywhere, i.e. $f = 0$ in $L^p([0,1])$. For $\lambda \in \mathbb{C}$, we have

$$\|\lambda f\|_p = \left(\int_0^1 |\lambda|^p |f|^p \right)^{1/p} = |\lambda| \, \|f\|_p.$$

Finally, Minkowski's inequality (valid for $p \geq 1$) yields

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Thus $\| \bullet \|_p$ is a norm.

(*iii*) Assume $0 < p < 1$. Then $\| \bullet \|_p$ is not a norm, because the triangle inequality fails. Let

$$f := \mathbf{1}_{[0,1/2]}, \qquad g := \mathbf{1}_{(1/2,1]} \in L^p([0,1]).$$

Then $f + g = \mathbf{1}_{[0,1]}$, hence

$$\|f + g\|_p = \left(\int_0^1 1^p \right)^{1/p} = 1.$$

On the other hand,

$$\|f\|_p = \left(\int_0^{1/2} 1^p \right)^{1/p} = (1/2)^{1/p}, \qquad \|g\|_p = (1/2)^{1/p},$$

so

$$\|f\|_p + \|g\|_p = 2\,(1/2)^{1/p} = 2^{\,1-1/p}.$$

Since $0 < p < 1$ implies $1 - 1/p < 0$, we have $2^{1-1/p} < 1$, hence

$$\|f + g\|_p > \|f\|_p + \|g\|_p.$$

(*iv*) Assume $0 < p < 1$, and let $\varphi \in (L^p([0,1]))^\vee$. We show that $\varphi = 0$ by contradiction. Assume to the contrary that $\varphi \neq 0$. Then there exists $f \in L^p([0,1])$ with $\varphi(f) \neq 0$. Replacing f and φ by suitable non-zero scalar multiples, we may assume

$$\|f\|_p^p = 1 \qquad \text{and} \qquad |\varphi(f)| \geq 1.$$

Set $f_0 := f$.

We shall construct inductively a sequence $\{f_n\}_{n \in \mathbb{Z}_{\geq 0}} \subset L^p([0,1])$ such that

$$|\varphi(f_n)| \geq 1 \qquad \text{and} \qquad \|f_n\|_p^p = (2^{p-1})^n$$

for all n.

Assume f_n has been constructed. Define

$$F_n(s) := \int_0^s |f_n(x)|^p \, dx \qquad (s \in [0,1]).$$

Then F_n is continuous with $F_n(0) = 0$ and $F_n(1) = \|f_n\|_p^p$, so there exists $s \in (0,1)$ such that

$$F_n(s) = \frac{1}{2}\|f_n\|_p^p.$$

Set

$$g_1 := f_n \cdot \mathbf{1}_{[0,s]}, \qquad g_2 := f_n \cdot \mathbf{1}_{(s,1]}.$$

Then $f_n = g_1 + g_2$ and

$$\|g_1\|_p^p = \|g_2\|_p^p = \frac{1}{2}\|f_n\|_p^p.$$

Since $\varphi(f_n) = \varphi(g_1) + \varphi(g_2)$ and $|\varphi(f_n)| \geq 1$, we have $|\varphi(g_j)| \geq \frac{1}{2}$ for $j = 1$ or 2. Fix such j and define

$$f_{n+1} := 2\, g_j.$$

Then

$$|\varphi(f_{n+1})| \geq 1, \qquad \|f_{n+1}\|_p^p = 2^p \|g_j\|_p^p = 2^{p-1}\|f_n\|_p^p,$$

which completes the induction.

Since $2^{p-1} < 1$, we have $\|f_n\|_p^p \to 0$, hence $f_n \to 0$ with respect to the metric d_p. By continuity of φ, this implies $\varphi(f_n) \to 0$, contradicting $|\varphi(f_n)| \geq 1$.

Mini-Exercise I.6.30

It suffices to verify the following two statements:
(i) If $x \in V$ satisfies $\varphi(x) = 0$ for all $\varphi \in V^\vee$, then $x = 0$.
(ii) If $\varphi \in V^\vee$ satisfies $\varphi(x) = 0$ for all $x \in V$, then $\varphi = 0$.

Assertion (ii) is obvious. To prove (i), assume that $x \neq 0$. Set $W := \mathbb{C}x \subset V$. Since W is one-dimensional and Hausdorff, clearly there exists a linear functional $\ell \in W^\vee$ such that $\ell(x) \neq 0$. By the Hahn–Banach theorem for locally convex spaces (Theorem I.6.26), ℓ extends to a continuous linear functional $\varphi \in V^\vee$. Then $\varphi(x) = \ell(x) \neq 0$, which proves the claim.

Mini-Exercise I.6.31

This follows immediately from the definition of the weak-$*$ topology.

Mini-Exercise I.6.32

Let $\langle E, F\rangle$ be a separating dual pair. To show that $(E, \sigma(E,F))$ is Hausdorff, it suffices to show that

$$\bigcap_{y \in F} \operatorname{Ker} p_y = \{0\}$$

(refer to Lemma I.6.2). Now, by definition,

$$\bigcap_{y \in F} \operatorname{Ker} p_y = \{x \in E \mid \langle x, y\rangle = 0 \text{ for all } y \in F\}.$$

Since the dual pair $\langle E, F\rangle$ is separating, the condition $\langle x, y\rangle = 0$ for all $y \in F$ implies $x = 0$, from which we have the assertion.

Mini-Exercise I.6.36

If $B = \emptyset$ or $I = \emptyset$, both conditions hold. Assume henceforth that B and I are non-empty. Assume first that B is TVS bounded. Fix $i \in I$ and $\varepsilon > 0$, and consider the neighborhood

$$U := \{x \in V \mid p_i(x) < \varepsilon\}$$

of 0. By TVS boundedness, there exists $t > 0$ such that $B \subset tU$, hence $p_i(x) < t\varepsilon$ for all $x \in B$. This shows $\sup_{x\in B} p_i(x) < \infty$.

Conversely, assume that $\sup_{x\in B} p_i(x) < \infty$ holds for every $i \in I$. Let U be an arbitrary neighborhood of 0. By definition of the locally convex topology, there exist $i_1, \ldots, i_k \in I$ and $\varepsilon > 0$ such that

$$\Big\{x \in V \mid \max_{1\le j\le k} p_{i_j}(x) < \varepsilon\Big\} \subset U.$$

Put

$$M := \max_{1\le j\le k} \sup_{x\in B} p_{i_j}(x).$$

Then $B \subset \frac{M+1}{\varepsilon} U$, showing that B is TVS bounded.

Mini-Exercise I.6.38

By Mini-Exercise I.6.36, a subset $B \subset V$ is TVS bounded if and only if it is norm-bounded, i.e., if $\sup_{x\in B} \|x\|_V < \infty$, where $\| \bullet \|_V$ denotes the norm of V.

Let $A \subset V$ be a non-empty TVS bounded subset. Then there exists $M > 0$ such that $A \subset MB_V$, where $B_V := \{x \in V \mid \|x\|_V \le 1\}$ is the closed unit ball. For $\varphi \in V^\vee$, we have

$$p_A(\varphi) = \sup_{x\in A} |\varphi(x)| \le M \sup_{\|x\|_V\le 1} |\varphi(x)| = M\|\varphi\|_{\mathrm{op}}.$$

Thus every seminorm defining $\beta(V^\vee, V)$ is continuous with respect to the operator norm topology, from which it follows that $\beta(V^\vee, V)$ is weaker than the operator norm topology on $V^\vee$.

Conversely, taking $A = B_V$, which is TVS bounded in V, we obtain

$$p_{B_V}(\varphi) = \|\varphi\|_{\mathrm{op}}.$$

Hence the operator norm itself is one of the seminorms defining $\beta(V^\vee, V)$, from which it follows that $\beta(V^\vee, V)$ is stronger than the operator norm topology on $V^\vee$.

Mini-Exercise I.6.39

(i) Take a neighborhood U of $0 \in W$. As f is continuous, $f^{-1}(U)$ is a neighborhood of $0 \in V$. As B is TVS bounded, there exists a real number $t > 0$ such that $B \subset tf^{-1}(U)$. Thus one has $f(B) \subset tU$.

(ii) Since $u \circ f \in V^\vee$ for every $u \in W^\vee$, the map $f\colon (V, \sigma(V, V^\vee)) \to (W, \sigma(W, W^\vee))$ is continuous. Applying (i) to these weak topologies, we see that $f(B)$ is weakly TVS bounded whenever B is. Therefore the topology on $W^\vee$ defined by the system of seminorms

$$\left\{ u \mapsto \sup_{y \in f(B)} |u(y)| \,\middle|\, B \subset V : \text{ non-empty and weakly TVS bounded} \right\}$$

is weaker than the strong topology $\beta(W^\vee, W)$. From this we have that $f^\vee$ is continuous, since

$$\sup_{x \in B} |(f^\vee(u))(x)| = \sup_{x \in B} |u(f(x))| = \sup_{y \in f(B)} |u(y)|$$

for any $u \in W^\vee$ and for any non-empty TVS bounded set $B \subset (V, \sigma(V, V^\vee))$, and hence

$$(f^\vee)^{-1}\left(\left\{ \ell \in V^\vee \,\middle|\, \sup_{x \in B} |\ell(x)| < \varepsilon \right\} \right) = \left\{ u \in W^\vee \,\middle|\, \sup_{y \in f(B)} |u(y)| < \varepsilon \right\}$$

for any $\varepsilon > 0$ and any non-empty TVS bounded set $B \subset (V, \sigma(V, V^\vee))$.

Mini-Exercise I.6.42

(i) First we show that every neighborhood of 0 in the sense of LCTVS topology contains a d-ball. Fix finitely many indices $j_1, \ldots, j_m$ and positive numbers $\varepsilon_1, \ldots, \varepsilon_m$, and set

$$U := \bigcap_{k=1}^{m} \{x \in F \mid p_{j_k}(x) < \varepsilon_k\}.$$

Let

$$\delta := \min_{1 \le k \le m} \frac{1}{2^{j_k}} \cdot \frac{\varepsilon_k}{1 + \varepsilon_k} \; > \; 0.$$

If $x \in F$ satisfies $d(0, x) < \delta$, then for each k we have

$$\frac{1}{2^{j_k}} \frac{p_{j_k}(x)}{1 + p_{j_k}(x)} \le d(0, x) < \delta \le \frac{1}{2^{j_k}} \frac{\varepsilon_k}{1 + \varepsilon_k}.$$

Since $t \mapsto \frac{t}{1+t}$ is strictly increasing on $\mathbb{R}_{\ge 0}$, this implies $p_{j_k}(x) < \varepsilon_k$ for every k, hence $x \in U$. Thus $B_d(0, \delta) \subset U$, where $B_d(0, \delta) := \{x \in F \mid d(0, x) < \delta\}$.

Next, let us show that every d-ball contains a neighborhood of 0 in the sense of LCTVS topology. Fix $\varepsilon > 0$. Choose $N \in \mathbb{Z}_{\ge 0}$ such that

$$\sum_{j > N} \frac{1}{2^j} < \frac{\varepsilon}{2}.$$

Next choose $\eta > 0$ such that

$$\sum_{j=0}^{N} \frac{1}{2^j} \frac{\eta}{1+\eta} < \frac{\varepsilon}{2}.$$

Set

$$V := \bigcap_{j=0}^{N} \{x \in F \mid p_j(x) < \eta\}.$$

If $x \in V$, then

$$d(x,0) = \sum_{j=0}^{N} \frac{1}{2^j} \frac{p_j(x)}{1+p_j(x)} + \sum_{j>N} \frac{1}{2^j} \frac{p_j(x)}{1+p_j(x)} \leq \sum_{j=0}^{N} \frac{1}{2^j} \frac{\eta}{1+\eta} + \sum_{j>N} \frac{1}{2^j} < \varepsilon.$$

Hence $V \subset B_d(0,\varepsilon)$.

Therefore, the neighborhood bases at 0 for the original locally convex topology and for the metric topology coincide (Note that, by translation invariance of d, the same holds at every point).

(ii) Set $A := F$. As

$$d(0,x) \leq \sum_{j=0}^{\infty} \frac{1}{2^j} = 2,$$

A is bounded in (F,d). On the other hand, since $F \neq \{0\}$ and F is Hausdorff, one of its defining seminorms is non-zero and hence is unbounded on F. Thus A is not TVS bounded.

(iii) The family of seminorms $\{p_j|_A\}_{j\in\mathbb{Z}_{\geq 0}}$ defines the (induced) topology of A. As A is closed in the metric space (F,d) by (i), it is complete with respect to the restricted metric $d|_{A\times A}$. Hence A is a Fréchet space.

Let $q\colon F \to F/A$ be the quotient map, and endow F/A with the quotient topology. Put $\widetilde{p}_j := \max_{0\leq k\leq j} p_k$ for $j \in \mathbb{Z}_{\geq 0}$. These seminorms generate the same topology on F. For each j, define a seminorm $\overline{p}_j$ on F/A by

$$\overline{p}_j(q(x)) := \inf_{a\in A} \widetilde{p}_j(x+a) \qquad (x \in F).$$

This is well-defined and satisfies the seminorm axioms. Moreover, the family $\{\overline{p}_j\}_{j\in\mathbb{Z}_{\geq 0}}$ generates the quotient topology: since the sequence $\{\widetilde{p}_j\}_j$ is increasing, the sets $U = \{x \in F \mid \widetilde{p}_j(x) < \varepsilon\}$, for $j \in \mathbb{Z}_{\geq 0}$ and $\varepsilon > 0$, form a neighborhood base at 0 in F, and

$$q(U) = \{\xi \in F/A \mid \overline{p}_j(\xi) < \varepsilon\}.$$

Since q is open, these images form a neighborhood base at 0 in the quotient topology. Hence F/A is a Hausdorff LCTVS with a countable defining family of seminorms (because A is closed, F/A is Hausdorff by Lemma I.6.2 (iii)).

To prove completeness, set

$$\overline{d}(q(x),q(y)) := \inf_{a\in A} d(x-y+a,0) \qquad (x,y \in F).$$

Since A is closed, this is a well-defined translation-invariant distance function. For every $\varepsilon > 0$, one has

$$q(B_d(0,\varepsilon)) = B_{\overline{d}}(0,\varepsilon),$$

so $\overline{d}$ induces the quotient topology. Take a Cauchy sequence $\{\xi_n\}_{n\in\mathbb{Z}_{\geq 0}}$ in the metric space $(F/A, \overline{d})$. For each n, choose an element $x_n \in F$ such that $q(x_n) = \xi_n$.

Since $\{\xi_n\}$ is Cauchy in $(F/A, \overline{d})$, for every $\varepsilon > 0$ there exists N such that for all $m, n \geq N$ one has

$$\overline{d}(\xi_n, \xi_m) < \varepsilon.$$

By definition of $\overline{d}$, this means that for each such pair (n, m) there exists an element $a_{n,m} \in A$ with

$$d(x_n - x_m + a_{n,m}, 0) < \varepsilon.$$

Using this fact, we now modify the representatives $\{x_n\}$ inductively. Set $\varepsilon_k := 2^{-k}$ for $k \in \mathbb{Z}_{\geq 0}$. Since $\{\xi_n\}$ is Cauchy, we may choose an increasing sequence of indices $n_1 < n_2 < \cdots$ such that

$$\overline{d}(\xi_{n_{k+1}}, \xi_{n_k}) < \varepsilon_k \qquad (k \geq 1).$$

For each k, choose $a_k \in A$ satisfying

$$d(x_{n_{k+1}} - x_{n_k} + a_k, 0) < \varepsilon_k.$$

Define inductively

$$b_1 := 0, \qquad b_{k+1} := b_k + a_k,$$

and set

$$y_k := x_{n_k} + b_k \in F.$$

Then $q(y_k) = \xi_{n_k}$ for all k.

By construction, one has

$$d(y_{k+1} - y_k, 0) < \varepsilon_k,$$

and therefore for $r > s$,

$$d(y_r - y_s, 0) \leq \sum_{k=s}^{r-1} d(y_{k+1} - y_k, 0) \leq \sum_{k=s}^{\infty} \varepsilon_k.$$

Since the series on the right converges, the sequence $\{y_k\}$ is Cauchy in (F, d). As F is complete, there exists $y \in F$ such that $y_k \to y$ in F.

Finally, since the quotient map $q\colon F \to F/A$ is continuous, we obtain

$$\xi_{n_k} = q(y_k) \longrightarrow q(y) \quad \text{in } F/A.$$

Because $\{\xi_n\}$ is Cauchy and has a convergent subsequence, it follows that $\xi_n \to q(y)$ in $(F/A, \overline{d})$. Thus $(F/A, \overline{d})$ is complete, and hence F/A is a Fréchet space.

(*iv*) Let $\{F^\nu\}_{\nu\in\mathbb{Z}_{\geq 0}}$ be Fréchet spaces. For each ν, choose a countable defining family of continuous seminorms $\{p_{\nu,j}\}_{j\in\mathbb{Z}_{\geq 0}}$ for F^ν. Consider the product $F := \prod_{\nu=0}^{\infty} F^\nu$ with the product topology. For each pair (ν, j) define a seminorm $P_{\nu,j}$ on F by

$$P_{\nu,j}\big((x_\mu)_\mu\big) := p_{\nu,j}(x_\nu).$$

Then the family $\{P_{\nu,j}\}_{(\nu,j)\in\mathbb{Z}^2_{\geq 0}}$ is countable and generates the product topology (since it is exactly the family obtained by pulling back seminorms via the coordinate projections). Thus F is a Hausdorff LCTVS with a countable defining family of seminorms.

It remains to show completeness. Let $\{x^{(k)}\}_{k\in\mathbb{Z}_{\geq 0}}$ be a Cauchy sequence in F with respect to the translation-invariant metric associated with $\{P_{\nu,j}\}$. Then for each fixed ν, the projected sequence

$$\mathrm{pr}_\nu\big(x^{(k)}\big) \in F^\nu$$

is Cauchy in F^ν (because each $P_{\nu,j}$ depends only on the ν-th coordinate). Since F^ν is complete, $\mathrm{pr}_\nu(x^{(k)})$ converges to some $x_\nu \in F^\nu$. Let $x := (x_\nu)_\nu \in \prod_\nu F^\nu$. By the definition of the product topology (or directly by the seminorm description), the convergence of every coordinate implies $x^{(k)} \to x$ in F. Hence F is complete, and therefore Fréchet.

Mini-Exercise I.6.43

Put

$$P := \prod_{\nu=0}^{\infty} F^\nu,$$

endowed with the product topology, and consider the subset

$$L := \varprojlim_{\nu} F^\nu = \Big\{ (x_\nu)_\nu \in P \;\Big|\; \pi^\nu(x_{\nu+1}) = x_\nu \text{ for all } \nu \in \mathbb{Z}_{\geq 0} \Big\}.$$

Let $i\colon L \hookrightarrow P$ denote the inclusion map, and for each $\nu \in \mathbb{Z}_{\geq 0}$ let

$$\mathrm{pr}_\nu\colon P \longrightarrow F^\nu$$

be the canonical projection. Then the canonical maps

$$f_\nu\colon L \longrightarrow F^\nu$$

appearing in the statement are given by

$$f_\nu = \mathrm{pr}_\nu \circ i.$$

Hence, by definition of the subspace topology on $L \subset P$, each f_ν is continuous.

We now show the claimed universal property. Let $\mathcal{T}$ be the topology on L induced from the product topology of P (equivalently, the topology defined above), and let $\mathcal{T}'$ be an

arbitrary topology on the same underlying set L such that all maps $f_\nu\colon (L,\mathcal{T}') \to F^\nu$ are continuous. Consider the map

$$\Phi\colon (L,\mathcal{T}') \longrightarrow P, \qquad \Phi(x) := (f_\nu(x))_\nu.$$

Since the product topology on P is the weakest topology for which all projections pr_ν are continuous, and since $\mathrm{pr}_\nu \circ \Phi = f_\nu$ is continuous for every ν, it follows that Φ is continuous.

On the other hand, for $x = (x_\nu)_\nu \in L$ one has $f_\nu(x) = x_\nu$ by definition of f_ν, hence $\Phi(x) = (x_\nu)_\nu = i(x)$. Therefore $\Phi = i$ as maps of sets, and thus $i\colon (L,\mathcal{T}') \to P$ is continuous. By definition of the subspace topology, this implies that $\mathcal{T}'$ is finer than $\mathcal{T}$. In other words, $\mathcal{T}$ is the weakest topology on L for which all f_ν are continuous, as required.

Mini-Exercise I.6.45

Assume that $C_c^\infty(\Omega)$ is a Fréchet space. As by construction one has

$$C_c^\infty(\Omega) = \bigcup_{\nu=0}^{\infty} C^\infty_{K_\nu},$$

from the Baire category theorem it follows that there exists $\nu \in \mathbb{Z}_{\geq 0}$ such that $C^\infty_{K_\nu}$ has non-empty interior. By considering a transformation, it follows that $C^\infty_{K_\nu}$ includes a neighborhood of 0, which is absorbing by Mini-Exercise I.6.1 (ii). Therefore one has $C^\infty_{K_\nu} = C_c^\infty(\Omega)$, which is a contradiction.

Mini-Exercise I.6.51

(i) First, the closedness of each $A_\nu^\pm = C^\infty_{[-\nu-1,\nu+1]}$ is clear, since convergence in $C^\infty(\mathbb{R})$ implies uniform convergence on compact sets.

For $\bullet \in \{+,-\}$, let us show the map

$$I^\bullet\colon C^\infty(\mathbb{R}) \longrightarrow (C_c^\infty(\mathbb{R}))^\vee, \qquad I^\bullet(f)(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)\,dx$$

is injective. Take $f \in C^\infty(\mathbb{R})$. Assume $f \neq 0$. Then one can take a point $x_0 \in \mathbb{R}$ with $f(x_0) \neq 0$. By continuity of f, there exists an open neighborhood U of x_0 such that $\mathrm{Re}(f/f(x_0)) > 1/2$ on U. Taking $\varphi \in C_c^\infty(U)$ with $\varphi \geq 0$ and $\varphi \not\equiv 0$, one gets $\mathrm{Re}(I^\bullet(f)(\varphi)/f(x_0)) > 0$, and hence $I^\bullet(f) \neq 0$.

Next, let us check that the restriction

$$\langle -,-\rangle\colon C^\infty(\mathbb{R}) \times C^\infty_{[-\nu,\nu]} \longrightarrow \mathbb{C}$$

is continuous for each $\nu \in \mathbb{Z}_{\geq 1}$. Let $\{f_j\}_j \subset C^\infty(\mathbb{R})$ and $\{\varphi_j\}_j \subset C^\infty_{[-\nu,\nu]}$ be sequences such that $f_j \to 0$ and $\varphi_j \to 0$ as $j \to \infty$. Then

$$\left|\langle f_j,\varphi_j\rangle\right| = \left|\int_{-\nu}^{\nu} f_j(x)\varphi_j(x)\,dx\right| \leq 2\nu \sup_{x\in[-\nu,\nu]} |f_j(x)| \sup_{x\in[-\nu,\nu]} |\varphi_j(x)| \longrightarrow 0.$$

Hence the restriction of the pairing is continuous.

The continuity of the operator

$$T\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad T(f) = f'$$

is clear, since the defining seminorms on $C^\infty(\mathbb{R})$ are given by suprema of derivatives on compact sets.

Finally let us check the existence and continuity of the formal adjoint. Define

$$T^*_{\text{formal}}\colon C^\infty_c(\mathbb{R}) \longrightarrow C^\infty_c(\mathbb{R}), \qquad T^*_{\text{formal}}(\varphi) := -\varphi'.$$

Then for $f \in C^\infty(\mathbb{R})$ and $\varphi \in C^\infty_c(\mathbb{R})$, integration by parts yields

$$\int_{\mathbb{R}} f'(x)\varphi(x)\,dx = -\int_{\mathbb{R}} f(x)\varphi'(x)\,dx = \int_{\mathbb{R}} f(x)\,T^*_{\text{formal}}(\varphi)(x)\,dx,$$

where the boundary term vanishes because φ has compact support. Thus T^*_{formal} is a formal adjoint of T with respect to the pairing. Moreover, T^*_{formal} is continuous on each $C^\infty_{[-\nu,\nu]}$, since its topology is defined by seminorms given by suprema of derivatives on compact sets. As $C^\infty_c(\mathbb{R})$ is endowed with the locally convex final topology, we have that T^*_{formal} is continuous on $C^\infty_c(\mathbb{R})$.

(*ii*) Define the *Dirac delta distribution at* 0 by

$$\delta_0(\varphi) := \varphi(0) \qquad (\varphi \in C^\infty_c(\mathbb{R})).$$

Let $H \in (C^\infty_c(\mathbb{R}))^\vee$ be the element that corresponds to h via the "abuse of notation" mentioned in Example I.6.50: i.e., H is the distribution defined by

$$H(\varphi) := \int_{\mathbb{R}} h(x)\varphi(x)\,dx = \int_0^\infty \varphi(x)\,dx.$$

Then one computes:

$$\begin{aligned}(\widetilde{T}(H))(\varphi) &= H(T^*_{\text{formal}}(\varphi)) = H(-\varphi') \\ &= -\int_{\mathbb{R}} h(x)\,\varphi'(x)\,dx = -\int_0^\infty \varphi'(x)\,dx = -\big[\varphi(x)\big]_{x=0}^\infty = \varphi(0),\end{aligned}$$

because $\varphi(x) = 0$ for $x \gg 1$. Therefore the derivative of h in the sense of distributions is δ_0.

Mini-Exercise I.6.57

(*i*) Take $z \in \operatorname{Ker} g_-$. As the map $\operatorname{Ker} g_+/\operatorname{Image} f_+ \to \operatorname{Ker} g_-/\operatorname{Image} f_-$ is surjective, one can take $x \in \operatorname{Ker} g_+$ such that $r^\natural(x) - z \in \operatorname{Image} f_-$. Therefore one has $r^\natural(x) - z = f_-(y)$ for some $y \in C^\flat_-$. The equation $T(-x,-y) = r^\natural(x) - f_-(y) = z$ shows the assertion.

(*ii*) Let $K\colon \operatorname{Ker} g_+ \oplus C^\flat_- \to \operatorname{Ker} g_-$ be the map defined by

$$K(x,y) := r^\natural(x).$$

By the assumption, K is compact. Therefore, from a theorem of L. Schwartz (Theorem I.6.56), it follows that

$$\frac{\operatorname{Ker} g_-}{\operatorname{Image}(T+K)} = \frac{\operatorname{Ker} g_-}{\operatorname{Image} f_-}$$

is a finite-dimensional Hausdorff TVS.

Mini-Exercise I.7.1

(*i*) If $X = \emptyset$, then X is not path-connected by our convention. Assume now that X is non-empty and not connected. Then there exist disjoint non-empty open subsets $U, V \subset X$ with $X = U \sqcup V$. Take $x \in U$ and $y \in V$. If there were a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, then $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ would be disjoint non-empty open subsets of $[0,1]$ whose union is $[0,1]$. This contradicts the connectedness of $[0,1]$. Hence no such path exists, and therefore X is not path-connected.

(*ii*) Assume that X is locally path-connected and connected (hence non-empty). Fix $x \in X$, and let

$$A := \{y \in X \mid \exists\, \gamma\colon [0,1] \to X\colon \text{ continuous such that } \gamma(0) = x,\ \gamma(1) = y\}.$$

Then A is non-empty (since $x \in A$). By local path connectedness, one can easily show that A is open and closed. Since X is connected, we get $A = X$. Hence X is path-connected.

(*iii*) Assume that X is path-connected. Let $x, y \in X$ and choose a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Define a map

$$\Phi_\gamma\colon \pi_1(X,x) \longrightarrow \pi_1(X,y)$$

by

$$\Phi_\gamma([\alpha]) := \big[\operatorname{Inv}_{\text{path}}(\gamma) *_{\text{path}} \alpha *_{\text{path}} \gamma\big],$$

where α is a loop based at x and $[\alpha]$ denotes its class. One can easily show that this is a group isomorphism.

(*iv*) Take the one-point union of two circles. By the Seifert–Van Kampen theorem, its fundamental group is the free group on two generators, which is non-commutative (see [Ha1, § 1.2]).

Mini-Exercise I.7.5

First let us check that $\operatorname{wind}\colon \pi_1(\mathrm{U}(1),1) \to \mathbb{Z}$ is a group homomorphism. Let $[\gamma_1], [\gamma_2] \in \pi_1(\mathrm{U}(1),1)$. Choose lifts $\widetilde{\gamma}_1, \widetilde{\gamma}_2\colon [0,1] \to \mathbb{R}$ with $\widetilde{\gamma}_1(0) = \widetilde{\gamma}_2(0) = 0$. Then the lift of the concatenation $\gamma_1 *_{\text{path}} \gamma_2$ starting at 0 is obtained by following $\widetilde{\gamma}_1$ on $[0,1]$ and then $\widetilde{\gamma}_2$ shifted by $\widetilde{\gamma}_1(1)$. In particular,

$$\operatorname{wind}(\gamma_1 *_{\text{path}} \gamma_2) = \widetilde{\gamma}_1(1) + \widetilde{\gamma}_2(1) = \operatorname{wind}(\gamma_1) + \operatorname{wind}(\gamma_2).$$

Hence wind is a group homomorphism.

For each $n \in \mathbb{Z}$, consider the loop

$$\gamma_n \colon [0,1] \longrightarrow \mathrm{U}(1), \qquad \gamma_n(t) := e^{2\pi\sqrt{-1}\,nt}.$$

Then the lift $\widetilde{\gamma}_n(t) = nt$ satisfies $\widetilde{\gamma}_n(0) = 0$ and $\widetilde{\gamma}_n(1) = n$. Thus $\mathrm{wind}([\gamma_n]) = n$, which proves that wind is surjective.

To show the injectivity, let $[\gamma] \in \pi_1(\mathrm{U}(1), 1)$ and assume $\mathrm{wind}([\gamma]) = 0$. Let $\widetilde{\gamma} \colon [0,1] \to \mathbb{R}$ be the lift of γ with $\widetilde{\gamma}(0) = 0$. The assumption $\mathrm{wind}([\gamma]) = 0$ means $\widetilde{\gamma}(1) = 0$, so $\widetilde{\gamma}$ is a loop in $\mathbb{R}$ based at 0. Since $\mathbb{R}$ is contractible, $\widetilde{\gamma}$ is homotopic (relative to endpoints) to the constant loop at 0. Composing this homotopy with $p \colon \mathbb{R} \to \mathrm{U}(1)$, we obtain an endpoint-fixed homotopy between $\gamma = p \circ \widetilde{\gamma}$ and the constant loop at 1. Therefore $[\gamma] = 1$ in $\pi_1(\mathrm{U}(1), 1)$, and wind is injective.

Mini-Exercise I.7.13

Since X is contractible, there exists a point $x_0 \in X$ such that id_X is homotopic to the constant map $c \colon X \to X$ with $c(x) \equiv x_0$. Thus it suffices to compute $H_q(P)$ for $P := \{x_0\}$. For every $q \in \mathbb{Z}_{\geq 0}$, any continuous map $\sigma \colon (\mathrm{Simplex})_q \to P$ is necessarily the constant map. Hence $C_q^{\mathrm{sing}}(P)$ is a free abelian group of rank one:

$$C_q^{\mathrm{sing}}(P) = \mathbb{Z} \cdot \sigma_q,$$

where $\sigma_q \colon (\mathrm{Simplex})_q \to P$ denotes the unique singular q-simplex. For $q \geq 1$, by definition,

$$\partial_q(\sigma_q) = \sum_{j=0}^{q} (-1)^j \sigma_{q-1}.$$

Therefore we obtain

$$\partial_q = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \mathrm{id}_{\mathbb{Z}} & \text{if } q \text{ is even,} \end{cases} \qquad \text{(under the identifications } C_q^{\mathrm{sing}}(P) \cong \mathbb{Z}\text{).}$$

Now we compute $H_q(P) = \mathrm{Ker}\,\partial_q / \mathrm{Image}\,\partial_{q+1}$. If $q = 0$, then $C_0^{\mathrm{sing}}(P) \cong \mathbb{Z}$ and $\mathrm{Image}\,\partial_1 = 0$, hence $H_0(P) \cong \mathbb{Z}$. If $q \geq 1$ is odd, then $\partial_q = 0$ and $\partial_{q+1} = \mathrm{id}_{\mathbb{Z}}$, so $\mathrm{Ker}\,\partial_q = C_q^{\mathrm{sing}}(P) \cong \mathbb{Z}$ and $\mathrm{Image}\,\partial_{q+1} = C_q^{\mathrm{sing}}(P) \cong \mathbb{Z}$. Thus $H_q(P) = 0$. If $q \geq 2$ is even, then $\partial_q = \mathrm{id}_{\mathbb{Z}}$, so $\mathrm{Ker}\,\partial_q = 0$ and hence $H_q(P) = 0$. Consequently,

$$H_q(X) \cong H_q(P) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{if } q \geq 1 \end{cases}.$$

Mini-Exercise I.7.16

Refer to [Ha1, Example 0.6]. Also one can refer to [Mi1, p. 26–27] after reviewing Morse theory.

Mini-Exercise I.7.17

(*i*) We first note that S^1 is path-connected and non-empty, hence Example I.7.11 gives $H_0(S^1) \cong \mathbb{Z}$. Next, by the discussion preceding Mini-Exercise I.7.5, we have $\pi_1(S^1, 1) \cong \mathbb{Z}$ via the winding number. Since $\mathbb{Z}$ is abelian, its abelianization is itself, and thus Example I.7.12 yields $H_1(S^1) \cong \pi_1(S^1, 1)^{\mathrm{ab}} \cong \mathbb{Z}$. Finally, we realize S^1 as a finite CW complex with one 0-cell and one 1-cell: identify S^1 with the quotient space $[0, 1]/(0 \sim 1)$. Then the image of 0 is a 0-cell, and the image of $(0, 1)$ is a single 1-cell. In particular, this CW structure has no q-cell for $q \geq 2$, hence we have $H_q(S^1) = 0$ for all $q \geq 2$ by the fact stated in this subsection.

(*ii*) The set U_n is the complement in S^n of a neighborhood of the south pole, and V_n is the complement of a neighborhood of the north pole. Hence each of U_n and V_n is homeomorphic to $\mathbb{R}^n$ by the stereographic projection, and in particular they are contractible.

(*iii*) Regard S^{n-1} as a subset of S^n defined by $x_n = 0$. Define a homotopy

$$H \colon (U_n \cap V_n) \times [0, 1] \longrightarrow U_n \cap V_n$$

by

$$H(x, t) := \frac{(x_0, \ldots, x_{n-1}, (1-t)x_n)}{\sqrt{x_0^2 + \cdots + x_{n-1}^2 + (1-t)^2 x_n^2}} \qquad \big(x = (x_0, \ldots, x_n) \in U_n \cap V_n\big).$$

Then $H(-, 0) = \mathrm{id}_{U_n \cap V_n}$, $H(-, 1)$ takes values in S^{n-1}, and $H(y, t) = y$ for all $y \in S^{n-1}$ and $t \in [0, 1]$. Thus H is a deformation retraction of $U_n \cap V_n$ onto S^{n-1}.

(*iv*) Assume (19) holds for S^n. Consider the cover $S^{n+1} = U_{n+1} \cup V_{n+1}$ and the Mayer–Vietoris long exact sequence in singular homology. Using (ii), we have $H_q(U_{n+1}) = H_q(V_{n+1}) = 0$ for $q > 0$ and $H_0(U_{n+1}) \cong H_0(V_{n+1}) \cong \mathbb{Z}$. Using (*iii*), we have $U_{n+1} \cap V_{n+1} \simeq S^n$, hence $H_q(U_{n+1} \cap V_{n+1}) \cong H_q(S^n)$ for all q. From the exact sequence, one deduces

$$H_q(S^{n+1}) \cong H_{q-1}(S^n) \quad (q \geq 2), \qquad H_1(S^{n+1}) = 0,$$

and together with $H_0(S^{n+1}) \cong \mathbb{Z}$ this implies (19) for S^{n+1}. For details, see (the proof of) [Lee-Jo1, Theorem 13.23].

Mini-Exercise I.7.18

Assume that there exists a homeomorphism $h \colon \mathbb{R}^n \to \mathbb{R}^m$. Replacing h by the homeomorphism $x \longmapsto h(x) - h(0)$, we may assume that $h(0) = 0$. Then the restriction

$$h|_{\mathbb{R}^n \setminus \{0\}} \colon \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^m \setminus \{0\}$$

is also a homeomorphism.

We claim that $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}. Let $S^{n-1} := \{u \in \mathbb{R}^n \mid \|u\| = 1\}$, where $\| \bullet \|$ is the usual (L^2-)norm. Define continuous maps

$$r_n \colon \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}, \qquad r_n(x) := \frac{x}{\|x\|},$$

and

$$j_n\colon S^{n-1} \longrightarrow \mathbb{R}^n \setminus \{0\}, \qquad j_n(u) := u.$$

Then $r_n \circ j_n = \mathrm{id}_{S^{n-1}}$. Moreover, $j_n \circ r_n$ is homotopic to $\mathrm{id}_{\mathbb{R}^n\setminus\{0\}}$ via the explicit homotopy

$$H_n\colon (\mathbb{R}^n \setminus \{0\}) \times [0,1] \longrightarrow \mathbb{R}^n \setminus \{0\}, \qquad H_n(x,t) := \left((1-t) + \frac{t}{\|x\|}\right) x.$$

Indeed, $H_n(x,0) = x$ and $H_n(x,1) = x/\|x\|$, and the scalar factor $(1-t) + t/\|x\|$ is strictly positive for all $t \in [0,1]$. Hence r_n is a homotopy equivalence, so $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ (Equivalently, one may first use the homeomorphism $\mathbb{R}^n \setminus \{0\} \cong (0,\infty) \times S^{n-1}$ given by polar coordinates).

Therefore, by homotopy invariance of singular homology, we have

$$H_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong H_{n-1}(S^{n-1}).$$

Similarly,

$$H_{m-1}(\mathbb{R}^m \setminus \{0\}) \cong H_{m-1}(S^{m-1}).$$

Therefore the assertion follows from (19).

Mini-Exercise I.7.19

(*i*) By construction, each 1-cell e_{ij} is the image of the relative interior of $(\mathrm{Simplex})_1$ under the affine map Φ_{ij}, and each 0-cell v_i is the image of $(\mathrm{Simplex})_0$ under the constant map Φ_i. Moreover, the restriction of Φ_{ij} to each endpoint coincides with the characteristic map of the corresponding 0-cell. Hence all characteristic maps satisfy the compatibility conditions required in the definition of a simplicial structure. Therefore $(X, \mathfrak{E})$ admits a simplicial structure.

(*ii*) With respect to the ordered bases

$$C_1^{\mathrm{simp}}(X,\mathfrak{E}) = \mathbb{Z}\cdot e_{12} \oplus \mathbb{Z}\cdot e_{23} \oplus \mathbb{Z}\cdot e_{31}, \qquad C_0^{\mathrm{simp}}(X,\mathfrak{E}) = \mathbb{Z}\cdot v_1 \oplus \mathbb{Z}\cdot v_2 \oplus \mathbb{Z}\cdot v_3,$$

the boundary homomorphism ∂_1 is given by

$$\partial_1(e_{12}) = -v_1 + v_2, \qquad \partial_1(e_{23}) = -v_2 + v_3, \qquad \partial_1(e_{31}) = -v_3 + v_1.$$

Hence ∂_1 is represented by the integer matrix

$$\partial_1 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

(*iii*) Let $(a,b,c) \in \mathbb{Z}^3$ correspond to the element $ae_{12} + be_{23} + ce_{31} \in C_1^{\mathrm{simp}}(X,\mathfrak{E})$. Then

$$\partial_1(ae_{12} + be_{23} + ce_{31}) = (-a+c)v_1 + (a-b)v_2 + (b-c)v_3.$$

Thus $(a, b, c) \in \operatorname{Ker} \partial_1$ if and only if

$$-a + c = 0, \qquad a - b = 0, \qquad b - c = 0,$$

which is equivalent to $a = b = c$. Therefore

$$\operatorname{Ker} \partial_1 = \mathbb{Z} \cdot (e_{12} + e_{23} + e_{31}) \cong \mathbb{Z}.$$

On the other hand, $\operatorname{Image} \partial_1$ is generated by

$$-v_1 + v_2, \qquad -v_2 + v_3,$$

and hence

$$\operatorname{Image} \partial_1 = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1 + x_2 + x_3 = 0\} \cong \mathbb{Z}^2.$$

Since $C_2^{\mathrm{simp}}(X, \mathfrak{E}) = 0$, we have $\operatorname{Image} \partial_2 = 0$. Therefore the simplicial homology groups are given by

$$H_1\big(C_\bullet^{\mathrm{simp}}(X, \mathfrak{E})\big) = \operatorname{Ker} \partial_1 \cong \mathbb{Z},$$

$$H_0\big(C_\bullet^{\mathrm{simp}}(X, \mathfrak{E})\big) = C_0^{\mathrm{simp}}(X, \mathfrak{E}) / \operatorname{Image} \partial_1 \cong \mathbb{Z},$$

and

$$H_q\big(C_\bullet^{\mathrm{simp}}(X, \mathfrak{E})\big) = 0 \qquad (q \geq 2).$$

Mini-Exercise I.7.22

(i) Since G is torsion-free, we have $\operatorname{Hom}_{\mathbb{Z}}(H_q(X)_{\mathrm{tor}}, G) = 0$. Since $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ kills torsion, we also have $\operatorname{Hom}_{\mathbb{Z}}(H_q(X), \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_q(X)_{\mathrm{free}}, \mathbb{Z})$. Thus (22) reduces to the case where $H_q(X)$ is free. If F is a finitely generated free $\mathbb{Z}$-module, then the map $\operatorname{Hom}_{\mathbb{Z}}(F, \mathbb{Z}) \otimes_{\mathbb{Z}} G \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(F, G)$ is an isomorphism (e.g., by identifying F with $\mathbb{Z}^n$ and checking that it is independent of the choice of a basis), which proves the assertion.

(ii) Combining (i) with (21), we obtain a natural isomorphism

$$H^q(X, G) \;\cong\; \operatorname{Hom}_{\mathbb{Z}}(H_q(X), \mathbb{Z}) \otimes_{\mathbb{Z}} G \qquad (G = \mathbb{R}, \mathbb{C}). \tag{391}$$

On the other hand, the universal coefficient theorem with $G = \mathbb{Z}$ yields a short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_{q-1}(X), \mathbb{Z}) \longrightarrow H^q(X, \mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_q(X), \mathbb{Z}) \longrightarrow 0.$$

Tensoring it with $G = \mathbb{R}$ or $G = \mathbb{C}$, we obtain

$$H^q(X, \mathbb{Z}) \otimes_{\mathbb{Z}} G \;\cong\; \operatorname{Hom}_{\mathbb{Z}}(H_q(X), \mathbb{Z}) \otimes_{\mathbb{Z}} G,$$

because $\operatorname{Ext}^1_{\mathbb{Z}}(H_{q-1}(X), \mathbb{Z})$ is a torsion group and hence its tensor product with G vanishes (Refer to Mini-Exercise I.5.6). Therefore, the assertion follows from (391).

Mini-Exercise I.7.23

By letting $G = \mathbb{Z}$ and $q = 2$, from (20) we have the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X), \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_2(X), \mathbb{Z}) \longrightarrow 0.$$

Since $\mathrm{Hom}_{\mathbb{Z}}(H_2(X), \mathbb{Z})$ is torsion–free, it follows that the torsion subgroup of $H^2(X, \mathbb{Z})$ coincides with the image of the torsion subgroup of left-hand term. Therefore, from the isomorphism mentioned in Mini-Exercise I.5.6, we obtain an isomorphism $H^2(X, \mathbb{Z})_{\mathrm{tor}} \cong H_1(X)_{\mathrm{tor}}$.

Mini-Exercise I.7.26

From fiber bundles to Cousin data. Let (E, π) and (E', π') be isomorphic as fiber bundles with structure group G, and let

$$\Psi \colon E \xrightarrow{\sim} E'$$

be a fiber bundle isomorphism over X. By passing to a common refinement of open coverings if necessary, we may assume that there exists an open covering $\{U_j\}_{j \in J}$ of X such that both bundles admit local trivializations

$$\Phi_{U_j} \colon \pi^{-1}(U_j) \to U_j \times F, \qquad \Phi'_{U_j} \colon (\pi')^{-1}(U_j) \to U_j \times F$$

with transition functions taking values in G. For each $j \in J$, the map

$$\Phi'_{U_j} \circ \Psi \circ \Phi_{U_j}^{-1} \colon U_j \times F \longrightarrow U_j \times F$$

is a fiber-preserving homeomorphism over U_j. Since Ψ preserves the chosen G-structures, it is, by definition, of the form

$$(x, \xi) \longmapsto (x, \psi_j(x)(\xi))$$

for a continuous map $\psi_j \colon U_j \to G$. A direct computation then shows that the corresponding transition functions $\{f_{jk}\}$ and $\{g_{jk}\}$ satisfy

$$\psi_j(x) \cdot f_{jk}(x) \cdot \psi_k(x)^{-1} = g_{jk}(x) \qquad (x \in U_j \cap U_k),$$

so that the associated G-valued continuous Cousin data are equivalent in the sense of the statement.

From Cousin data to fiber bundles. Let

$$\mathfrak{f} = \{f_{jk} \colon U_j \cap U_k \to G\}_{j,k \in J}, \qquad \mathfrak{g} = \{g_{jk} \colon U_j \cap U_k \to G\}_{j,k \in J}$$

be two G-valued continuous Cousin data with respect to a common open covering $\{U_j\}_{j \in J}$ of X, and assume that they are equivalent. Thus there exist continuous maps

$$\psi_j \colon U_j \to G \qquad (j \in J)$$

such that

$$\psi_j(x) \cdot f_{jk}(x) \cdot \psi_k(x)^{-1} = g_{jk}(x) \qquad (x \in U_j \cap U_k).$$

Let $E_{\mathfrak{f}}$ and $E_{\mathfrak{g}}$ be the fiber bundles over X constructed from $\mathfrak{f}$ and $\mathfrak{g}$, respectively, by the gluing construction described above. Define a map

$$\Psi\colon E_{\mathfrak{f}} \longrightarrow E_{\mathfrak{g}}$$

as follows: on each $U_j \times F \subset \coprod_j (U_j \times F)$, set

$$(x, \xi) \longmapsto (x, \psi_j(x)(\xi)).$$

The compatibility condition above ensures that this map is well-defined with respect to the equivalence relations defining $E_{\mathfrak{f}}$ and $E_{\mathfrak{g}}$. Moreover, Ψ is a homeomorphism, fiber-preserving, and compatible with the G-structures. Hence Ψ induces an isomorphism $E_{\mathfrak{f}} \cong_{G\text{-str}} E_{\mathfrak{g}}$.

Bijectivity. As is clear by construction, the two constructions are inverse to each other, so the correspondence described above induces a canonical bijection between the two quotient sets.

Mini-Exercise I.7.30

(i) The family $\{\gamma^{-1}(U_j)\}_{j\in J}$ is an open covering of the compact space $[0,1]$. Hence it admits a Lebesgue number: there exists $\delta > 0$ such that any subinterval $I \subset [0,1]$ of length $< \delta$ is contained in $\gamma^{-1}(U_j)$ for some $j \in J$. Choose a subdivision $0 = t_0 < t_1 < \cdots < t_{N+1} = 1$ such that $t_{\nu+1} - t_\nu < \delta/2$ for all ν. Since $\gamma(0) = \gamma(1) = x_0 \in U_{j_0}$, choose the first two subintervals and the last subinterval sufficiently short that $\gamma([0, t_2]) \subset U_{j_0}$ and $\gamma([t_N, 1]) \subset U_{j_0}$. Then for each $\nu \in \{0, 1, \ldots, N-1\}$ we have $t_{\nu+2} - t_\nu < \delta$, hence the interval $[t_\nu, t_{\nu+2}]$ is contained in $\gamma^{-1}(U_{j_\nu})$ for some index $j_\nu \in J$. Equivalently,

$$\gamma([t_\nu, t_{\nu+2}]) \subset U_{j_\nu}.$$

For $\nu = 0$ we use the prescribed index j_0, and we set $j_N = j_0$. The last-subinterval condition in (i) then holds as well.

(ii) Fix $\xi \in \pi^{-1}_{\text{discrete}}(x_0)$. Let $\widetilde{\gamma}$ be the lift of γ in the connected component of E_{discrete} containing ξ, with $\widetilde{\gamma}(0) = \xi$. We compute $\widetilde{\gamma}(1)$ in the local trivializations.

For each $\nu \in \{0, 1, \ldots, N-1\}$, we have

$$\gamma([t_{\nu+1}, t_{\nu+2}]) \subset U_{j_\nu} \cap U_{j_{\nu+1}},$$

by the condition in (i). Therefore, the transition function $\varphi_{U_{j_{\nu+1}} U_{j_\nu}}$ is defined along this subpath. Since $\varphi_{U_{j_{\nu+1}} U_{j_\nu}}$ is locally constant and $\gamma([t_{\nu+1}, t_{\nu+2}])$ is connected, it is constant on $\gamma([t_{\nu+1}, t_{\nu+2}])$. Hence, for any $s \in [t_{\nu+1}, t_{\nu+2}]$,

$$\varphi_{U_{j_{\nu+1}} U_{j_\nu}}(\gamma(s)) = \varphi_{U_{j_{\nu+1}} U_{j_\nu}}(\gamma(t_{\nu+1})).$$

Now follow the standard path-lifting construction: on each piece $[t_\nu, t_{\nu+1}]$ the lift is described in the chart U_{j_ν}, and when we move from the chart U_{j_ν} to $U_{j_{\nu+1}}$ at the time $t_{\nu+1}$, the fiber coordinate is changed by applying $\varphi_{U_{j_{\nu+1}} U_{j_\nu}}(\gamma(t_{\nu+1})) \in G$. Iterating this along $\nu =$

$0, 1, \ldots, N-1$, we obtain that the terminal point $\widetilde{\gamma}(1)$, viewed in the fiber over x_0 and identified with F via $\Phi_{U_{j_0}}$, is given by the successive action

$$\varphi_{U_{j_N}U_{j_{N-1}}}(\gamma(t_N)) \circ \varphi_{U_{j_{N-1}}U_{j_{N-2}}}(\gamma(t_{N-1})) \circ \cdots \circ \varphi_{U_{j_1}U_{j_0}}(\gamma(t_1))$$

applied to ξ. By definition of $\rho_{E,j_0,\xi}$, this precisely means the desired identity in (ii).

(iii) By (ii), the element $\rho_{E,j_0,\xi}([\gamma]) \in G$ is computed solely from the transition functions and the chosen index j_0 (i.e., the chosen identification of $\pi^{-1}(x_0)$ with F). The point $\xi \in \pi^{-1}(x_0)$ does not enter the formula. Hence for any $\xi_0, \xi_1 \in \pi^{-1}_{\text{discrete}}(x_0)$ we have

$$\rho_{E,j_0,\xi_1}([\gamma]) = \rho_{E,j_0,\xi_0}([\gamma]) \quad \text{for all } [\gamma] \in \pi_1(X, x_0),$$

which proves $\rho_{E,j_0,\xi_1} = \rho_{E,j_0,\xi_0}$.

(iv) Let $k \in J$ with $x_0 \in U_k$. Consider the transition function

$$\varphi_{U_{j_0}U_k} \colon U_{j_0} \cap U_k \to G.$$

Since it is locally constant and $x_0 \in U_{j_0} \cap U_k$, after shrinking to a connected neighborhood of x_0 if necessary we may regard its value at x_0 as a single element

$$g := \varphi_{U_{j_0}U_k}(x_0) \in G.$$

This g is exactly the change-of-trivialization element between the identification of $\pi^{-1}(x_0)$ with F given by Φ_{U_k} and that given by $\Phi_{U_{j_0}}$. Let $[\gamma] \in \pi_1(X, x_0)$. Compute $\rho_{E,j_0,\xi_0}([\gamma])$ and $\rho_{E,k,\xi_0}([\gamma])$ by the formula in (ii), using a common subdivision and indices along the loop γ but starting/ending in U_{j_0} or in U_k. The only difference is the initial identification of the fiber at x_0 with F, and the change of identification is given by g. Therefore the two resulting group elements are conjugate by g, namely

$$\rho_{E,j_0,\xi_0}([\gamma]) = g \circ \rho_{E,k,\xi_0}([\gamma]) \circ g^{-1},$$

which proves (iv).

Mini-Exercise I.8.1

(i) The assertion is local, and hence it follows from the inverse function theorem (or equivalently, the implicit function theorem) in Euclidean space. Refer to the proof of [Lee-Jo2, Theorem 4.12] for example.

(ii) Let $q \in Y$ be a regular value of f. Then for every point $p \in f^{-1}(q)$, the point p is a regular point of f. By (i), there exist local coordinates $(x_1, \ldots, x_m)$ near p in X and $(y_1, \ldots, y_n)$ near q in Y such that f is locally written as

$$f(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_n).$$

In these coordinates, the fiber $f^{-1}(q)$ is locally given by the equations

$$x_1 = x_2 = \cdots = x_n = 0,$$

which define a C^∞ submanifold of dimension $m - n$. Since this description is valid in a neighborhood of every point of $f^{-1}(q)$ and is compatible under changes of local coordinates, it follows that $f^{-1}(q)$ is a C^∞ submanifold of X.

Mini-Exercise I.8.3

It is sufficient to show the assertion when X is connected. Let $n := \dim_{\mathbb{R}} X$. Fix a C^∞ Riemannian metric g on X, and let ∇ be its Levi–Civita connection.

We first recall the standard local description of ∇. Let $(x^1, x^2, \dots, x^n)$ be a local coordinate system on an open set $U \subset X$. Write

$$\partial_i := \frac{\partial}{\partial x^i} \qquad (i = 1, 2, \dots, n).$$

Since ∇ is a connection, for each $i, j \in \{1, 2, \dots, n\}$ the covariant derivative $\nabla_{\partial_i} \partial_j$ is a vector field on U, hence can be uniquely written as

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \, \partial_k \qquad \text{on } U$$

for certain C^∞ functions $\Gamma_{ij}^k \colon U \to \mathbb{R}$ (These functions Γ_{ij}^k are called the *Christoffel symbols* (of the connection ∇) with respect to the coordinate system $(x^1, x^2, \dots, x^n)$). It is well-known that a C^∞ curve $\gamma \colon (-\varepsilon, \varepsilon) \to U$ written as

$$\gamma(t) = \big(x^1(t), x^2(t), \dots, x^n(t)\big)$$

is a geodesic with respect to ∇ (equivalently, with respect to g) if and only if it satisfies

$$\frac{d^2 x^k}{dt^2}(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \Gamma_{ij}^k\big(x^1(t), x^2(t), \dots, x^n(t)\big) \frac{dx^i}{dt}(t) \frac{dx^j}{dt}(t) = 0 \qquad (k = 1, 2, \dots, n),$$

see [Lee-Jo3, Theorem 4.27 and the proof of Proposition 5.19].

Next we define the vector field G on TX locally. Over U, the tangent bundle TU is identified with $U \times \mathbb{R}^n$ by the coordinate frame $(\partial_1, \partial_2, \dots, \partial_n)$: any tangent vector $\xi \in T_x X$ $(x \in U)$ is uniquely written as

$$\xi = \sum_{i=1}^{n} v^i \, \partial_i\big|_x,$$

and we use $(x^1, \dots, x^n; v^1, \dots, v^n)$ as local coordinates on $TU \subset TX$. On this coordinate chart of TX, define a C^∞ vector field G by

$$G := \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} - \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \Gamma_{ij}^k(x) \, v^i v^j \right) \frac{\partial}{\partial v^k}. \tag{392}$$

We claim that these local expressions glue to a globally well-defined C^∞ vector field on TX. Indeed, the geodesic equation is intrinsic (it is the coordinate-free equation $\nabla_{\dot\gamma} \dot\gamma = 0$), so when one changes local coordinates on X, the Christoffel symbols Γ_{ij}^k transform in such a way that the system of ODEs written below is preserved under coordinate changes. Equivalently, the

right-hand side of (392) transforms as a vector field on TX. Hence the locally defined vector fields agree on overlaps, and therefore glue to a global C^∞ vector field G on TX.

Now let $\varepsilon \in \mathbb{R}_{>0}$ and let

$$(-\varepsilon, \varepsilon) \ni t \longmapsto (\gamma(t), \delta(t)) \in TX$$

be a C^∞ curve. Assume that $\gamma(t) \in U$ for all t (after shrinking ε if necessary). Write

$$\gamma(t) = \big(x^1(t), x^2(t), \ldots, x^n(t)\big), \qquad \delta(t) = \sum_{i=1}^n v^i(t)\, \partial_i\big|_{\gamma(t)}.$$

Then the condition that $t \mapsto (\gamma(t), \delta(t))$ is an integral curve of G is, by definition of G in (392), equivalent to the following system of ODEs:

$$\frac{dx^i}{dt}(t) = v^i(t) \qquad (i = 1, 2, \ldots, n),$$

and

$$\frac{dv^k}{dt}(t) = -\sum_{i=1}^n \sum_{j=1}^n \Gamma^k_{ij}\big(x^1(t), x^2(t), \ldots, x^n(t)\big)\, v^i(t) v^j(t) \qquad (k = 1, \ldots, n),$$

from which (and the local expression of the geodesic equation above), the equivalence stated in Mini-Exercise I.8.3 follows.

Mini-Exercise I.8.10

Write $x = x(u)$ for the C^∞ coordinate change from the u-coordinates to the x-coordinates. Let

$$J := \left(\frac{\partial x^j}{\partial u^a}(0)\right)_{\substack{1 \le j \le n \\ 1 \le a \le n}}$$

be the Jacobian matrix of this coordinate change at the point p.

Since p is a critical point of f, we have

$$\frac{\partial f}{\partial x^j}(0) = 0 \qquad \text{for all } j \in \{1, 2, \ldots, n\}.$$

By the chain rule, for any $a, b \in \{1, 2, \ldots, n\}$ we compute

$$\begin{aligned}\frac{\partial^2 (f \circ x)}{\partial u^a \partial u^b}(0) = &\sum_{j,k=1}^n \frac{\partial^2 f}{\partial x^j \partial x^k}(0)\, \frac{\partial x^j}{\partial u^a}(0)\, \frac{\partial x^k}{\partial u^b}(0) \\ &+ \sum_{j=1}^n \frac{\partial f}{\partial x^j}(0)\, \frac{\partial^2 x^j}{\partial u^a \partial u^b}(0).\end{aligned}$$

The second sum vanishes because p is a critical point of f. Therefore we obtain

$$\left(\frac{\partial^2 f}{\partial u^a \partial u^b}(0)\right)_{1 \le a, b \le n} = {}^t J \left(\frac{\partial^2 f}{\partial x^j \partial x^k}(0)\right)_{1 \le j, k \le n} J.$$

Mini-Exercise I.8.11

In the parameters (u, v), the function $f\colon X \to \mathbb{R}$ corresponds to

$$F(u, v) := (2 + \cos v) \cos u.$$

We compute

$$F_u(u, v) = -(2 + \cos v) \sin u, \qquad F_v(u, v) = -\sin v \cos u.$$

Since $2 + \cos v \in [1, 3]$, one has $2 + \cos v \neq 0$ for all v. Therefore $F_u = 0$ is equivalent to $\sin u = 0$, i.e., $u \equiv 0, \pi \pmod{2\pi}$. Substituting this into $F_v = 0$, we obtain $\sin v = 0$, i.e., $v \equiv 0, \pi \pmod{2\pi}$. Hence f has exactly four critical points, represented by

$$(u, v) = (0, 0),\ (0, \pi),\ (\pi, 0),\ (\pi, \pi).$$

In $\mathbb{R}^3$-coordinates they are

$$\begin{aligned} (0, 0) &\mapsto (3, 0, 0), & (0, \pi) &\mapsto (1, 0, 0), \\ (\pi, 0) &\mapsto (-3, 0, 0), & (\pi, \pi) &\mapsto (-1, 0, 0), \end{aligned}$$

and the corresponding critical values are $3, 1, -3, -1$, respectively.

Next we determine the indices. A direct computation gives

$$F_{uu}(u, v) = -(2 + \cos v) \cos u, \qquad F_{uv}(u, v) = \sin v \, \sin u, \qquad F_{vv}(u, v) = -\cos v \, \cos u.$$

At the critical points one has $\sin u = \sin v = 0$, hence $F_{uv} = 0$ there and the Hessian is diagonal:

$$\operatorname{Hess}_{(u,v)}(F) = \begin{pmatrix} F_{uu}(u, v) & 0 \\ 0 & F_{vv}(u, v) \end{pmatrix} \qquad \text{at critical points.}$$

Evaluating at each critical point, we obtain:

- $(u, v) = (0, 0)$: $\cos u = 1,\ \cos v = 1$, hence

  $$F_{uu} = -3, \qquad F_{vv} = -1,$$

  so the index is 2.

- $(u, v) = (0, \pi)$: $\cos u = 1,\ \cos v = -1$, hence

  $$F_{uu} = -1, \qquad F_{vv} = +1,$$

  so the index is 1.

- $(u, v) = (\pi, 0)$: $\cos u = -1,\ \cos v = 1$, hence

  $$F_{uu} = +3, \qquad F_{vv} = +1,$$

  so the index is 0.

- $(u, v) = (\pi, \pi)$: $\cos u = -1$, $\cos v = -1$, hence

$$F_{uu} = +1, \qquad F_{vv} = -1,$$

  so the index is 1.

In particular, all critical points of f are non-degenerate.

Finally, we describe the handle decomposition / CW structure. Let $X_c := \{x \in X \mid f(x) < c\}$. As c increases from $-\infty$ to $+\infty$, the topology of X_c changes only when c passes a critical value of f. For regular c, we describe handles on the compact sublevel set $\overline{X_c} = \{x \in X \mid f(x) \le c\}$; its interior X_c has the same homotopy type. More precisely, denoting by D^ℓ the closed real ℓ-dimensional ball, we have the following (To gain a visual intuition for the following handle decomposition, we recommend to look at the figures on pages 1–2 of [Mi1]):

1. If $c < -3$, then $\overline{X_c} = \emptyset$.

2. If $-3 < c < -1$, we pass the unique critical point of index 0. Thus $\overline{X_c}$ is obtained by attaching a 0-handle $D^0 \times D^2 \approx D^2$. In particular, $\overline{X_c}$ is diffeomorphic to a closed 2-disk, hence has the homotopy type of a point (a 0-cell).

3. If $-1 < c < 1$, we pass one critical point of index 1. Thus a 1-handle $D^1 \times D^1$ is attached. Topologically this produces an annulus $\mathrm{U}(1) \times [0, 1]$.

4. If $1 < c < 3$, we pass the second critical point of index 1. Attaching another 1-handle produces a surface diffeomorphic to a torus with an open disk removed.

5. If $c > 3$, we pass the unique critical point of index 2. Attaching a 2-handle $D^2 \times D^0 \cong D^2$ caps off the remaining boundary component, and we recover the compact torus X.

From the homotopy viewpoint, this describes a CW structure on X with one 0-cell, two 1-cells, and one 2-cell.

Mini-Exercise I.8.38

(i) For each ν, we have

$$\widetilde{f}_\nu(x) - \widetilde{f}(x) = \int_{\mathbb{C}^N} \big(\widetilde{f}(x - y) - \widetilde{f}(x)\big)\, \theta_\nu(y)\, dV_{\mathrm{Euc}}(y).$$

Let Q be the half-open fundamental domain $\{z \in \mathbb{C}^N \mid -1/2 \le \operatorname{Re} z^j, \operatorname{Im} z^j < 1/2,\ j = 1, 2, \dots, N\}$. For periodic functions, the $L^2(Q)$ norm is the $L^2(X)$ norm. Taking $L^2(Q)$ norms and using the triangle inequality, we obtain

$$\|\widetilde{f}_\nu - \widetilde{f}\|_{L^2(Q)} \le \int_{\mathbb{C}^N} \|\widetilde{f}(\,\cdot - y) - \widetilde{f}(\,\cdot)\|_{L^2(Q)}\, \theta_\nu(y)\, dV_{\mathrm{Euc}}(y).$$

Since $\widetilde{f} \in L^2(Q)$, the continuity of translations in $L^2(X)$ implies

$$\|\widetilde{f}(\,\cdot - y) - \widetilde{f}(\,\cdot)\|_{L^2(Q)} \longrightarrow 0 \qquad (y \to 0).$$

Because $\operatorname{Supp}\theta_\nu$ is contained in the ball centered at the origin with radius $1/\nu$ and $\int_{\mathbb{C}^N} \theta_\nu \, dV_{\mathrm{Euc}} = 1$, the right-hand side converges to 0 as $\nu \to \infty$. Hence,

$$\widetilde{f}_\nu \longrightarrow \widetilde{f} \quad \text{in } L^2(Q),$$

and therefore $F_\nu \to F$ in $L^2(X)$. Any compact subset of $\mathbb{C}^N$ is covered by finitely many lattice translates of Q. Periodicity therefore also gives $\widetilde{f}_\nu \to \widetilde{f}$ in $L^2_{\mathrm{loc}}(\mathbb{C}^N)$.

(ii) For each $j \in \{1, 2, \ldots, N\}$, the definition of $\widetilde{f}_\nu$ implies

$$\frac{\partial \widetilde{f}_\nu}{\partial \overline{z^j}}(x) = \int_{\mathbb{C}^N} \frac{\partial \widetilde{f}}{\partial \overline{z^j}}(x - y)\, \theta_\nu(y)\, dV_{\mathrm{Euc}}(y),$$

where we regard $\partial \widetilde{f}/\partial \overline{z^j}$ as a distribution: The cut-off computation leading to (37) shows that this distribution is represented by a periodic L^2_{loc} function. Therefore, the convolution $\big(\partial \widetilde{f}/\partial \overline{z^j}\big) * \theta_\nu$ is well-defined as a periodic L^2_{loc} function, and coincides with $\partial \widetilde{f}_\nu/\partial \overline{z^j}$.

Consequently,

$$\frac{\partial \widetilde{f}_\nu}{\partial \overline{z^j}}(x) - \frac{\partial \widetilde{f}}{\partial \overline{z^j}}(x) = \int_{\mathbb{C}^N} \Big(\frac{\partial \widetilde{f}}{\partial \overline{z^j}}(x - y) - \frac{\partial \widetilde{f}}{\partial \overline{z^j}}(x) \Big)\, \theta_\nu(y)\, dV_{\mathrm{Euc}}(y).$$

Taking $L^2(Q)$ norms and applying the triangle inequality again, we obtain

$$\Big\| \frac{\partial \widetilde{f}_\nu}{\partial \overline{z^j}} - \frac{\partial \widetilde{f}}{\partial \overline{z^j}} \Big\|_{L^2(Q)} \le \int_{\mathbb{C}^N} \Big\| \frac{\partial \widetilde{f}}{\partial \overline{z^j}}(\,\cdot - y) - \frac{\partial \widetilde{f}}{\partial \overline{z^j}}(\,\cdot) \Big\|_{L^2(Q)}\, \theta_\nu(y)\, dV_{\mathrm{Euc}}(y).$$

Since $\partial \widetilde{f}/\partial \overline{z^j} \in L^2(Q)$, the continuity of translations in $L^2(X)$ implies that the right-hand side converges to 0 as $\nu \to \infty$. Thus,

$$\frac{\partial \widetilde{f}_\nu}{\partial \overline{z^j}} \longrightarrow \frac{\partial \widetilde{f}}{\partial \overline{z^j}} \quad \text{in } L^2(Q),$$

and hence $\partial F_\nu/\partial \overline{z^j} \to \partial F/\partial \overline{z^j}$ in $L^2(X)$. The finite-translate argument from (i) also gives the asserted $L^2_{\mathrm{loc}}(\mathbb{C}^N)$ convergence of the derivatives.

(iii) For each $j \in \{1, 2, \ldots, N\}$, set

$$\omega_j := \frac{\sqrt{-1}}{2}\, dz^j \wedge d\overline{z^j} = dx^j \wedge dy^j,$$

and define the $(2N-1)$-form

$$a_j := \frac{\sqrt{-1}}{2}\, \omega_1 \wedge \cdots \wedge \omega_{j-1} \wedge d\overline{z^j} \wedge \omega_{j+1} \wedge \cdots \wedge \omega_N.$$

By Stokes' theorem, we have

$$\int_X d\left(\sum_{j=1}^N \frac{\partial u}{\partial \overline{z^j}}\,\overline{u}\,a_j\right) = 0.$$

On the other hand, a direct computation using the definition of a_j shows that

$$d\left(\sum_{j=1}^N \frac{\partial u}{\partial \overline{z^j}}\,\overline{u}\,a_j\right) = \left(\sum_{j=1}^N \left(\frac{\partial^2 u}{\partial z^j \partial \overline{z^j}}\,\overline{u} + \left|\frac{\partial u}{\partial \overline{z^j}}\right|^2\right)\right) dV_{\mathrm{Euc}}.$$

Using the identity (36), we conclude that

$$(\Delta u,\, u)_{L^2(X,E;g,h)} = 4\sum_{j=1}^N \int_X \left|\frac{\partial u}{\partial \overline{z^j}}\right|^2 dV_{\mathrm{Euc}} = 4\sum_{j=1}^N \left\|\frac{\partial u}{\partial \overline{z^j}}\right\|^2_{L^2(X,E;g,h)}.$$

This proves the desired identity.

Mini-Exercise I.8.48

(*i*) Put $r := \sqrt{x^2+y^2}$ and $\eta := -y\,dx + x\,dy$, so that $\omega = r^{-2}\eta$. Since $d(r^{-2}) = -2r^{-4}(x\,dx + y\,dy)$ and

$$d\eta = d(-y\,dx) + d(x\,dy) = -dy\wedge dx + dx\wedge dy = 2\,dx\wedge dy,$$

we obtain

$$\begin{aligned} d\omega &= d(r^{-2})\wedge\eta + r^{-2}\,d\eta \\ &= -2r^{-4}(x\,dx + y\,dy)\wedge(-y\,dx + x\,dy) + 2r^{-2}\,dx\wedge dy. \end{aligned}$$

Expanding the wedge product, we have

$$\begin{aligned} (x\,dx + y\,dy)\wedge(-y\,dx + x\,dy) &= x\,dx\wedge(x\,dy) + y\,dy\wedge(-y\,dx) \\ &= x^2\,dx\wedge dy + y^2\,dx\wedge dy \\ &= r^2\,dx\wedge dy. \end{aligned}$$

Hence

$$d\omega = -2r^{-4}\cdot r^2\,dx\wedge dy + 2r^{-2}\,dx\wedge dy = 0.$$

Thus ω is d-closed.

(*ii*) Consider the C^∞ map

$$\gamma\colon [0,1]\to X, \qquad \gamma(t) := \big(\cos(2\pi t),\, \sin(2\pi t)\big),$$

which parametrizes $\mathrm{U}(1)\subset\mathbb{C}$ via $t\mapsto e^{2\pi\sqrt{-1}\,t}$ under the identification $\mathbb{R}^2\cong\mathbb{C}$, $(x,y)\leftrightarrow x+\sqrt{-1}y$. Along γ we have $x^2+y^2=1$ and

$$dx = -2\pi\sin(2\pi t)\,dt, \qquad dy = 2\pi\cos(2\pi t)\,dt.$$

Therefore

$$\begin{aligned}\gamma^*\omega &= -\sin(2\pi t)\cdot\big(-2\pi\sin(2\pi t)\,dt\big) + \cos(2\pi t)\cdot\big(2\pi\cos(2\pi t)\,dt\big)\\ &= 2\pi\big(\sin^2(2\pi t)+\cos^2(2\pi t)\big)\,dt\\ &= 2\pi\,dt.\end{aligned}$$

Hence

$$\int_{\mathrm{U}(1)}\omega = \int_0^1 \gamma^*\omega = \int_0^1 2\pi\,dt = 2\pi \neq 0.$$

If ω were d-exact on X, say $\omega = d\varphi$ for some $\varphi \in \mathcal{A}^0(X)$, then by Stokes' theorem (or equivalently, by the fundamental theorem of calculus applied to $\varphi\circ\gamma$), we would have

$$\int_{\mathrm{U}(1)}\omega = \int_0^1 \gamma^* d\varphi = \int_0^1 d(\varphi\circ\gamma) = (\varphi\circ\gamma)(1) - (\varphi\circ\gamma)(0) = 0,$$

a contradiction. Thus ω is not d-exact.

(*iii*) Let $r\colon X \to \mathrm{U}(1)$ be the radial projection

$$r(x,y) := \frac{x+\sqrt{-1}y}{\sqrt{x^2+y^2}},$$

and let $\iota\colon \mathrm{U}(1) \hookrightarrow X$ be the inclusion. Then $\iota\circ r$ is homotopic to the identity map id_X on X (shrink the radius to 1), and $r\circ\iota = \mathrm{id}_{\mathrm{U}(1)}$. Hence r is a homotopy equivalence, so Lemma I.8.42 implies

$$r^*\colon H^1_{\mathrm{DR}}(\mathrm{U}(1)) \xrightarrow{\cong} H^1_{\mathrm{DR}}(X).$$

We compute $H^1_{\mathrm{DR}}(\mathrm{U}(1))$ by the parametrization $\gamma(t) = e^{2\pi\sqrt{-1}t}$. Any $\eta \in \mathcal{A}^1(\mathrm{U}(1))$ can be written uniquely as $\eta = f(t)\,dt$ on $[0,1]$ with f 1-periodic. Define

$$\mathcal{I}\colon H^1_{\mathrm{DR}}(\mathrm{U}(1)) \to \mathbb{C}, \qquad [\eta] \longmapsto \int_{\mathrm{U}(1)}\eta := \int_0^1 f(t)\,dt.$$

This is well-defined because if $\eta = d\psi$, then $f = \psi'$ and $\int_0^1 f(t)\,dt = \psi(1)-\psi(0) = 0$.

Moreover, if $\int_0^1 f(t)\,dt = 0$, then the function

$$F(t) := \int_0^t f(\tau)\,d\tau$$

satisfies $F(1) = F(0)$, hence defines a smooth function on $\mathrm{U}(1)$, and $dF = \eta$. Therefore $\operatorname{Ker}\mathcal{I} = 0$, so $\mathcal{I}$ is injective. It is also surjective, since the class of $2\pi\,dt$ maps to $2\pi \neq 0$. Hence $\mathcal{I}$ is an isomorphism and $H^1_{\mathrm{DR}}(\mathrm{U}(1)) \cong \mathbb{C}$ is one-dimensional ($H^1_{\mathrm{DR}}(\mathrm{U}(1)) \cong \mathbb{C}$ can also be shown by using the Mayer–Vietoris sequence, Poincaré duality Proposition I.8.49, or the singular–de Rham correspondence Theorem I.8.54). Thus, by (*ii*), we have that the class of ω forms a basis of $H^1_{\mathrm{DR}}(X)$.

Mini-Exercise I.8.59

Fix $f \in C^\infty(\mathbb{R}^n)$. Since d increases degree by 1, we have $d^\dagger f = 0$ on 0-forms. Hence

$$\Delta f = dd^\dagger f + d^\dagger\, df = d^\dagger(df).$$

Write

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}\, dx^j.$$

On $(\mathbb{R}^n, g_{\mathrm{Euc}})$, the Hodge star satisfies

$$*\, dx^j = (-1)^{n-j}\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n,$$

so

$$*df = \sum_{j=1}^{n} (-1)^{n-j}\, \frac{\partial f}{\partial x^j}\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n.$$

Applying d gives

$$d * df = \sum_{j=1}^{n} (-1)^{n-j} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x^k\, \partial x^j}\, dx^k \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n.$$

All terms with $k \neq j$ vanish, because dx^k already appears in $dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$. Thus

$$\begin{aligned} d * df &= \sum_{j=1}^{n} (-1)^{n-j}\, \frac{\partial^2 f}{\partial (x^j)^2}\, dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n \\ &= (-1)^{n-1} \left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial (x^j)^2} \right) dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

By the definition of $d^\dagger$ via the Hodge star on 1-forms, we have $d^\dagger df = (-1)^n * d * (df)$ in our sign convention, hence

$$d^\dagger df = (-1)^n * \, d * df = -\sum_{j=1}^{n} \frac{\partial^2 f}{\partial (x^j)^2}.$$

Therefore $\Delta f = d^\dagger df = -\sum_{j=1}^{n} (\partial/\partial x^j)^2 f$, as desired.

Mini-Exercise I.9.2

(i) The sum and product of representatives define a well-defined ring structure on $\mathcal{O}_{\mathbb{C},z_0}$. Let $w := z - z_0$. For a germ $[(V, f)] \in \mathcal{O}_{\mathbb{C},z_0}$, the Taylor expansion of f at z_0 gives a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n\, w^n \qquad \text{on a neighborhood of } z_0,$$

which depends only on the germ of f at z_0. This yields a ring homomorphism

$$\mathcal{O}_{\mathbb{C},z_0} \longrightarrow \mathbb{C}\{w\}, \qquad [f] \longmapsto \sum_{n=0}^{\infty} a_n w^n.$$

Conversely, any convergent power series $\sum_{n\geq 0} a_n w^n$ defines a holomorphic function on a small disc around z_0 by $w = z - z_0$, hence defines an element of $\mathcal{O}_{\mathbb{C},z_0}$. These two constructions are inverse to each other and give the desired ring isomorphism.

(*ii*) For a non-zero germ $[f] \in \mathcal{O}_{\mathbb{C},z_0}$, set $m := \operatorname{ord}_{z_0}(f)$. Then one can factor

$$f(z) = (z - z_0)^m \, g(z)$$

on a neighborhood of z_0, where g is holomorphic near z_0 and satisfies $g(z_0) \neq 0$. Indeed, taking the Taylor expansion $f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n$ with $a_m \neq 0$, we put $g(z) := \sum_{n=0}^{\infty} a_{n+m}(z - z_0)^n$. This series has the same radius of convergence as the original one, hence defines a holomorphic function near z_0, and $g(z_0) = a_m \neq 0$. Therefore g is invertible in $\mathcal{O}_{\mathbb{C},z_0}$, and we obtain

$$([f]) = \big((z - z_0)^m\big) \qquad \text{as ideals of } \mathcal{O}_{\mathbb{C},z_0}.$$

Now let $I \subset \mathcal{O}_{\mathbb{C},z_0}$ be a non-zero ideal. Choose $[f] \in I \setminus \{0\}$ such that $\operatorname{ord}_{z_0}(f)$ is minimal among the vanishing orders of non-zero elements of I, and set $m := \operatorname{ord}_{z_0}(f) \in \mathbb{Z}_{\geq 0}$. We claim that $I = \big((z - z_0)^m\big)$. Since $[f] \in I$ and $([f]) = \big((z - z_0)^m\big)$ as shown above, we have $\big((z - z_0)^m\big) \subset I$. Conversely, take any $[h] \in I$. If $[h] = 0$, the desired inclusion is immediate. Otherwise, write $h = (z - z_0)^\ell u$ with $\ell := \operatorname{ord}_{z_0}(h) \in \mathbb{Z}_{\geq 0}$ and u invertible in $\mathcal{O}_{\mathbb{C},z_0}$. By the minimality of m, we have $\ell \geq m$. Hence $[h] \in \big((z - z_0)^m\big)$. Therefore $I \subset \big((z - z_0)^m\big)$, and the claim follows.

In particular, every ideal of $\mathcal{O}_{\mathbb{C},z_0}$ is principal: it is either the zero ideal or of the form $\big((z - z_0)^m\big)$ for some $m \in \mathbb{Z}_{\geq 0}$.

As a supplementary remark, the map $\operatorname{ord}_{z_0} \colon \mathcal{O}_{\mathbb{C},z_0} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ defines a discrete valuation on the fraction field of $\mathcal{O}_{\mathbb{C},z_0}$. In particular, $\mathcal{O}_{\mathbb{C},z_0}$ is a discrete valuation ring.

Mini-Exercise I.9.3

There are several possible proofs.

One proof uses Harnack's inequality. It suffices to show that $h(z) = h(0)$ for every $z \in \mathbb{C}$. For each $R > |z|$, Harnack's inequality gives

$$\frac{R - |z|}{R + |z|}\, h(0) \leq h(z) \leq \frac{R + |z|}{R - |z|}\, h(0).$$

Keeping z fixed and letting $R \to \infty$, we obtain the claim.

Another proof is available if the reader knows the existence of harmonic conjugates on simply connected domains. Since $\mathbb{C}$ is simply connected, one can choose a harmonic function $g \colon \mathbb{C} \to \mathbb{R}$ such that $f := g + \sqrt{-1}h$ is an entire holomorphic function on $\mathbb{C}$. Since $h > 0$, we have Image $f \subset \mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Let $\Phi \colon \mathbb{H} \to \Delta$ be the Cayley transform. Then $\Phi \circ f$ is a bounded entire holomorphic function on $\mathbb{C}$, and therefore is constant by Liouville's theorem. It follows that f is constant, and hence $h = \operatorname{Im} f$ is also constant.

Mini-Exercise I.9.5

We first show that the function $z \mapsto \log|f(z)|^2$ is subharmonic on U. As this function is clearly upper semicontinuous, by our definition of subharmonicity, it suffices to check the mean-value inequality at each point. At a zero of f, this function takes the value $-\infty$, so the mean-value inequality clearly holds. Now let $z_0 \in U$ be a point such that $f(z_0) \neq 0$. Then on a sufficiently small disc centered at z_0, one can choose a holomorphic branch F of $\log f$. As $\log|f(z)|^2 = 2\operatorname{Re} F$ holds on this disc, it follows that $z \mapsto \log|f(z)|^2$ is harmonic near z_0, and therefore satisfies the mean-value equality there. This proves the subharmonicity of $\log|f|^2$.

Next we prove the Lelong–Poincaré formula. First note that the sum on the right-hand side is locally finite. Indeed, if $\varphi \in C_c^\infty(U)$, then $\operatorname{Supp}\varphi$ is compact, and since f is holomorphic and not identically zero, its zero set is discrete. Hence $\{z \in \operatorname{Supp}\varphi \mid f(z) = 0\}$ is a finite set, say $\{a_1, a_2, \dots, a_N\}$, and therefore

$$\sum_{a\in U} \operatorname{ord}_a(f)\,\delta_a(\varphi) = \sum_{j=1}^{N} \operatorname{ord}_{a_j}(f)\,\delta_{a_j}(\varphi) = \sum_{j=1}^{N} \operatorname{ord}_{a_j}(f)\,\varphi(a_j)$$

is a finite sum.

As already observed, the function $\log|f|^2$ is harmonic on the complement of the zero set of f. Therefore the left-hand side of (58) vanishes there. The right-hand side is also supported on the zero set of f. Thus it suffices to prove the equality in a neighborhood of each zero of f.

Let $z_0 \in U$ be a zero of f. By translating the coordinate, we may assume that $z_0 = 0$. Let $m := \operatorname{ord}_0(f)$. Then there exists a holomorphic function g defined near 0, with $g(0) \neq 0$, such that $f(z) = z^m g(z)$ near 0 (refer to the discussion in the solution to Mini-Exercise I.9.2). Hence

$$\log|f(z)|^2 = m\log|z|^2 + \log|g(z)|^2.$$

Since $g(0) \neq 0$, the function $\log|g|^2$ is harmonic near 0, and therefore

$$\partial\overline{\partial}\log|f(z)|^2 = m\,\partial\overline{\partial}\log|z|^2 + \partial\overline{\partial}\log|g(z)|^2 = m\,\partial\overline{\partial}\log|z|^2.$$

Thus the proof of the desired formula is reduced to the case $U = \mathbb{C}$ and $f(z) = z$, namely to the identity

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|z|^2 = \delta_0.$$

We now prove this identity. Let $\varphi \in C_c^\infty(\mathbb{C})$. Since

$$d\big(\overline{\partial}\log|z|^2\big) = \partial\overline{\partial}\log|z|^2 + \overline{\partial}\,\overline{\partial}\log|z|^2 = \partial\overline{\partial}\log|z|^2,$$

we obtain

$$\begin{aligned}\int_{\mathbb{C}} \varphi\cdot\partial\overline{\partial}\log|z|^2 &= \int_{\mathbb{C}} \varphi\cdot d\big(\overline{\partial}\log|z|^2\big) = -\int_{\mathbb{C}} d\varphi\wedge\overline{\partial}\log|z|^2 \\ &= -\int_{\mathbb{C}} d\varphi\wedge\frac{d\overline{z}}{\overline{z}} = -\int_{\mathbb{C}} \frac{\varphi_z}{\overline{z}}\,dz\wedge d\overline{z},\end{aligned}$$

where φ_z denotes $\frac{\partial \varphi}{\partial z}$.

Since the integrand has compact support, the domain of integration may be replaced by $\Delta_M := \{z \in \mathbb{C} \mid |z| < M\}$ for sufficiently large $M > 0$. For $0 < \varepsilon < M$, define

$$\chi_\varepsilon(z) := \begin{cases} 1 & |z| \geq \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\left| \frac{\varphi_z}{\overline{z}} (1 - \chi_\varepsilon) \right| \leq \left| \frac{\varphi_z}{\overline{z}} \right|,$$

and the right-hand side is Lebesgue integrable on $\mathbb{C}$. Hence, by the dominated convergence theorem,

$$\int_{\mathbb{C}} \varphi \cdot \partial\overline{\partial} \log |z|^2 = -\lim_{\varepsilon \searrow 0} \int_{\Delta_M \setminus \Delta_\varepsilon} \frac{\varphi_z}{\overline{z}}\, dz \wedge d\overline{z}.$$

Now, by Stokes' theorem,

$$\int_{\Delta_M \setminus \Delta_\varepsilon} \frac{\varphi_z}{\overline{z}}\, dz \wedge d\overline{z} = \int_{\Delta_M \setminus \Delta_\varepsilon} d\left(\frac{\varphi}{\overline{z}}\, d\overline{z}\right) = \int_{\partial \Delta_M} \frac{\varphi}{\overline{z}}\, d\overline{z} - \int_{\partial \Delta_\varepsilon} \frac{\varphi}{\overline{z}}\, d\overline{z} = -\int_{\partial \Delta_\varepsilon} \frac{\varphi}{\overline{z}}\, d\overline{z},$$

because $\varphi \equiv 0$ on $\partial \Delta_M$ for M sufficiently large. Therefore

$$\begin{aligned} \int_{\mathbb{C}} \varphi \cdot \partial\overline{\partial} \log |z|^2 &= -\lim_{\varepsilon \searrow 0} \int_{\Delta_M \setminus \Delta_\varepsilon} \frac{\varphi_z}{\overline{z}}\, dz \wedge d\overline{z} = \lim_{\varepsilon \searrow 0} \int_{\partial \Delta_\varepsilon} \frac{\varphi}{\overline{z}}\, d\overline{z} \\ &= \lim_{\varepsilon \searrow 0} \int_0^{2\pi} \frac{\varphi(\varepsilon e^{\sqrt{-1}\theta})}{\varepsilon e^{-\sqrt{-1}\theta}}\, d(\varepsilon e^{-\sqrt{-1}\theta}) = -\sqrt{-1} \lim_{\varepsilon \searrow 0} \int_0^{2\pi} \varphi(\varepsilon e^{\sqrt{-1}\theta})\, d\theta \\ &= -2\pi\sqrt{-1}\, \varphi(0). \end{aligned}$$

Thus we have

$$\int_{\mathbb{C}} \varphi \cdot \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log |z|^2 = \varphi(0) = \delta_0(\varphi),$$

which completes the proof.

Mini-Exercise I.9.6

We first recall the C^2-case. If $u \in C^2(U)$, then

$$\partial\overline{\partial} u = \frac{\partial^2 u}{\partial z\, \partial \overline{z}}\, dz \wedge d\overline{z}.$$

Hence, in the C^2-case, condition (ii) is equivalent to the usual differential criterion for subharmonicity, which can be easily shown by applying the standard formula

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{\sqrt{-1}\theta})\, d\theta - u(z_0) = \frac{2}{\pi} \int_0^r \left(\int_{\Delta_t(z_0)} \frac{\partial^2 u}{\partial z \partial \overline{z}}(\zeta)\, d\lambda(\zeta) \right) \frac{dt}{t}$$

for $z_0 \in U$ and every $r > 0$ such that $\overline{\Delta_r(z_0)} \subset U$ (one can also deduce this formula by computing the derivative $(t \cdot M'(t))'$, where $M(t) := \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + te^{\sqrt{-1}\theta})\, d\theta$).

We now treat the general case.

$(i) \Longrightarrow (ii)$. Assume that the L^1_{loc}-equivalence class of u admits a subharmonic representative. In this case, we may assume that u is subharmonic. Let $\varphi \in \mathcal{A}^0_c(U)$ be a non-negative test function. Since $\operatorname{Supp}\varphi$ is compact and contained in U, we may choose an open set $V \subset U$ with compact closure contained in U such that $\operatorname{Supp}\varphi \subset V$. By the standard regularization of subharmonic functions by convolution, there exists a family

$$u_\varepsilon \in C^\infty(V) \qquad (\varepsilon > 0 \text{ sufficiently small})$$

such that each u_ε is subharmonic on V, and $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(V)$.

Since each u_ε is smooth and subharmonic, the C^2-case shows that $\sqrt{-1}\partial\overline{\partial}u_\varepsilon$ is semi-positive on V. Hence, for every $\varepsilon > 0$ sufficiently small,

$$(\sqrt{-1}\partial\overline{\partial}u_\varepsilon)(\varphi) \geq 0.$$

On the other hand, because $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(V)$, we obtain $(\sqrt{-1}\partial\overline{\partial}u)(\varphi) \geq 0$. Since $\varphi \geq 0$ was arbitrary, condition (ii) follows.

$(ii) \Longrightarrow (i)$. Assume that $\sqrt{-1}\partial\overline{\partial}u$ is semi-positive. Let $V \subset U$ be an open subset with compact closure contained in U. Choose a non-negative radial standard mollifier ρ on $\mathbb{C} \cong \mathbb{R}^2$, and set

$$\rho_\varepsilon(z) := \frac{1}{\varepsilon^2}\rho\left(\frac{z}{\varepsilon}\right).$$

For $\varepsilon > 0$ sufficiently small, define

$$u_\varepsilon := u * \rho_\varepsilon$$

on V. Then $u_\varepsilon \in C^\infty(V)$, and $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(V)$. For every non-negative test function $\varphi \in C^\infty_c(V)$, we have

$$(\sqrt{-1}\partial\overline{\partial}u_\varepsilon)(\varphi) = \int_V u_\varepsilon\, \sqrt{-1}\partial\overline{\partial}\varphi = \int_V (u * \rho_\varepsilon)\, \sqrt{-1}\partial\overline{\partial}\varphi.$$

For sufficiently small ε, one has $\rho_\varepsilon * \varphi \in C^\infty_c(U)$, and the preceding expression equals

$$\int_U u\, \sqrt{-1}\partial\overline{\partial}(\rho_\varepsilon * \varphi) = (\sqrt{-1}\partial\overline{\partial}u)(\rho_\varepsilon * \varphi).$$

Since $\rho_\varepsilon * \varphi \geq 0$, the semi-positivity of $\sqrt{-1}\partial\overline{\partial}u$ implies

$$(\sqrt{-1}\partial\overline{\partial}u_\varepsilon)(\varphi) \geq 0.$$

Hence $\sqrt{-1}\partial\overline{\partial}u_\varepsilon$ is semi-positive on V. By the C^2-case, u_ε is subharmonic on V.

We next observe that, if $0 < \varepsilon_1 < \varepsilon_2$ are sufficiently small, then

$$u_{\varepsilon_1} \leq u_{\varepsilon_2}$$

on every relatively compact open subset of V on which both sides are defined. Indeed, for $\eta > 0$ sufficiently small, u_η is smooth and subharmonic. Since ρ is radial, the mean-value inequality for u_η shows that

$$u_\eta * \rho_{\varepsilon_1} \leq u_\eta * \rho_{\varepsilon_2}.$$

Letting $\eta \to 0$ gives

$$u_{\varepsilon_1} \leq u_{\varepsilon_2}.$$

Thus, as $\varepsilon \downarrow 0$, the functions u_ε decrease pointwise. Define

$$v(z) := \lim_{\varepsilon \downarrow 0} u_\varepsilon(z)$$

locally on V.

A decreasing limit of subharmonic functions is subharmonic. On the other hand, since $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(V)$, there exists a sequence $\varepsilon_\nu \downarrow 0$ such that

$$u_{\varepsilon_\nu} \to u$$

almost everywhere on V. Therefore it follows that $v = u$ almost everywhere on V (In particular, v is not identically $-\infty$). Hence v is subharmonic on V and is a representative of the L^1_{loc}-equivalence class of u.

Since the open set $V \subset U$ with compact closure contained in U was arbitrary, and the locally constructed representatives agree on overlaps, the L^1_{loc}-equivalence class of u admits a subharmonic representative on U. Thus (i) follows.

Finally, we prove uniqueness. Suppose that v_1 and v_2 are two subharmonic representatives of the L^1_{loc}-equivalence class of u. Then $v_1 = v_2$ almost everywhere. Using the same non-negative radial mollifier as above, we have $v_1 * \rho_\varepsilon = v_2 * \rho_\varepsilon$ wherever the convolutions are defined. For a subharmonic function v, the standard regularization by a radial mollifier satisfies $v * \rho_\varepsilon \to v$ pointwise as $\varepsilon \downarrow 0$, from which the assertion follows.

Mini-Exercise I.9.7

(i) When $z_0 = 0$, the assertion is precisely the mean-value inequality. Thus, assume that $z_0 \neq 0$. If $u(z_0) = -\infty$, the desired inequality is immediate, so we may further assume that $u(z_0) > -\infty$.

Consider the automorphism

$$\psi(z) := \lambda \frac{\frac{z}{\lambda} - \frac{z_0}{\lambda}}{1 - \frac{\overline{z_0}}{\lambda}\frac{z}{\lambda}} = \lambda^2 \frac{z - z_0}{\lambda^2 - \overline{z_0} z}$$

of Δ_λ. Then $\psi(z_0) = 0$, and its inverse is given by

$$\psi^{-1}(w) = \lambda^2 \frac{w + z_0}{\lambda^2 + \overline{z_0} w}.$$

The only pole of ψ^{-1} is $-\lambda^2/\overline{z_0}$, whose absolute value is $\lambda^2/|z_0| > \lambda$. Thus ψ^{-1} extends holomorphically to a neighborhood of $\overline{\Delta_\lambda}$. Moreover,

$$\psi^{-1}(\overline{\Delta_\lambda}) = \overline{\Delta_\lambda} \subset \Delta_r.$$

It follows that there exists a neighborhood W of $\overline{\Delta_\lambda}$ such that $\psi^{-1}(W) \subset \Delta_r$. Therefore

$$v := u \circ \psi^{-1}$$

is subharmonic on W.

The function v is upper semicontinuous and hence bounded above on $\partial\Delta_\lambda$. Since $v(0) = u(z_0) > -\infty$, the mean-value inequality shows that the restriction of v to $\partial\Delta_\lambda$ is integrable. Consequently, the following changes of variables are legitimate, where the line integrals are understood as Lebesgue integrals along the parametrized circle.

Let γ denote the positively oriented circle $\partial\Delta_\lambda$. Since $\psi|_\gamma$ is an orientation-preserving diffeomorphism of γ, the mean-value inequality gives

$$\begin{aligned} u(z_0) &= v(0) \\ &\le \frac{1}{2\pi} \int_0^{2\pi} v(\lambda e^{\sqrt{-1}\theta})\, d\theta \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\gamma v(w)\, \frac{dw}{w} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\gamma u(z)\, \frac{\psi'(z)}{\psi(z)}\, dz. \end{aligned}$$

A direct computation gives

$$\frac{\psi'(z)}{\psi(z)} = \frac{\lambda^2 - |z_0|^2}{(z - z_0)(\lambda^2 - \overline{z_0} z)}.$$

For $z \in \partial\Delta_\lambda$, we have $\lambda^2 = z\overline{z}$, and hence

$$(z - z_0)(\lambda^2 - \overline{z_0} z) = z(z - z_0)(\overline{z} - \overline{z_0}) = z|z - z_0|^2.$$

Therefore

$$\begin{aligned} u(z_0) &\le \frac{1}{2\pi\sqrt{-1}} \int_\gamma u(z)\, \frac{\lambda^2 - |z_0|^2}{|z - z_0|^2}\, \frac{dz}{z} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\lambda^2 - |z_0|^2}{|\lambda e^{\sqrt{-1}\theta} - z_0|^2}\, u(\lambda e^{\sqrt{-1}\theta})\, d\theta. \end{aligned}$$

This proves (i).

(ii) Set

$$\mu := \frac{\rho + \lambda}{2}.$$

Then

$$\rho < \mu < \lambda.$$

By condition (a), there exists $B \in \mathbb{R}$ such that

$$\sup_{\Delta_\lambda} u_k \leq B$$

for every $k \in \mathbb{Z}_{\geq 1}$. Replacing u_k, A, and α by $u_k - B$, $A - B$, and $\alpha - B$, respectively, and retaining the same notation, we may assume that

$$u_k \leq 0 \quad \text{on } \Delta_\lambda$$

for every $k \in \mathbb{Z}_{\geq 1}$.

If $\alpha \geq 0$, then

$$\sup_{\Delta_\rho} u_k \leq 0 \leq \alpha$$

for every k, and there is nothing to prove. Thus we may assume that

$$A < \alpha < 0.$$

Set

$$\varepsilon := \frac{\alpha - A}{2} \qquad \text{and} \qquad c := A + \varepsilon = \frac{A + \alpha}{2}.$$

Then

$$A < c < \alpha < 0.$$

For each positive integer ν, set

$$E_\nu := \left\{ \theta \in [0, 2\pi] \;\middle|\; u_k(\mu e^{\sqrt{-1}\theta}) \leq c \text{ for every } k \in \mathbb{Z}_{>\nu} \right\}.$$

For every k, the function

$$\theta \longmapsto u_k(\mu e^{\sqrt{-1}\theta})$$

is upper semicontinuous and hence Borel measurable. Therefore

$$E_\nu = \bigcap_{k \in \mathbb{Z}_{>\nu}} \left\{ \theta \in [0, 2\pi] \mid u_k(\mu e^{\sqrt{-1}\theta}) \leq c \right\}$$

is a Borel subset of $[0, 2\pi]$.

The sequence $\{E_\nu\}_{\nu \geq 1}$ is increasing. Moreover, condition (b) gives, for every $\theta \in [0, 2\pi]$,

$$\limsup_{k \to \infty} u_k(\mu e^{\sqrt{-1}\theta}) \leq A < c.$$

Hence, for every $\theta \in [0, 2\pi]$, there exists a positive integer ν such that

$$u_k(\mu e^{\sqrt{-1}\theta}) \leq c$$

for every $k \in \mathbb{Z}_{>\nu}$. It follows that

$$\bigcup_{\nu=1}^{\infty} E_\nu = [0, 2\pi].$$

Consequently, by the continuity from below of Lebesgue measure,

$$m(E_\nu) \longrightarrow m([0, 2\pi]) = 2\pi.$$

Set

$$X_\nu := \frac{2\pi - m(E_\nu)}{2\pi}.$$

Then

$$X_\nu \longrightarrow 0$$

as $\nu \to \infty$.

For $z_0 \in \Delta_\rho$ and $\theta \in [0, 2\pi]$, set

$$P_{z_0}(\theta) := \frac{\mu^2 - |z_0|^2}{|\mu e^{\sqrt{-1}\theta} - z_0|^2}.$$

Since

$$|\mu e^{\sqrt{-1}\theta} - z_0| \geq \mu - |z_0| > \mu - \rho,$$

we have

$$0 < P_{z_0}(\theta) \leq \frac{\mu^2}{(\mu - \rho)^2} =: M.$$

Since

$$c(1 - MX_\nu) \longrightarrow c < \alpha,$$

there exists a positive integer N such that

$$c(1 - MX_N) < \alpha.$$

Take $k \in \mathbb{Z}_{>N}$ and $z_0 \in \Delta_\rho$. Applying (i) with μ in place of λ, we obtain

$$\begin{aligned} u_k(z_0) \leq \frac{1}{2\pi} \int_{E_N} P_{z_0}(\theta) u_k(\mu e^{\sqrt{-1}\theta})\, d\theta \\ + \frac{1}{2\pi} \int_{[0,2\pi]\setminus E_N} P_{z_0}(\theta) u_k(\mu e^{\sqrt{-1}\theta})\, d\theta. \end{aligned}$$

Here the integrals are understood as extended Lebesgue integrals. Since $u_k \leq 0$ on Δ_λ, all the integrands appearing above are bounded above by 0, so no indeterminate expression occurs.

On E_N, we have

$$u_k(\mu e^{\sqrt{-1}\theta}) \leq c.$$

On the other hand, since $\mu < \lambda$,

$$u_k(\mu e^{\sqrt{-1}\theta}) \leq 0$$

for every $\theta \in [0, 2\pi]$. Thus

$$u_k(z_0) \leq c \cdot \frac{1}{2\pi} \int_{E_N} P_{z_0}(\theta)\, d\theta.$$

By Poisson's formula applied to the constant function 1,

$$\frac{1}{2\pi}\int_0^{2\pi} P_{z_0}(\theta)\,d\theta = 1.$$

Hence

$$\begin{aligned}\frac{1}{2\pi}\int_{E_N} P_{z_0}(\theta)\,d\theta &= 1 - \frac{1}{2\pi}\int_{[0,2\pi]\setminus E_N} P_{z_0}(\theta)\,d\theta \\ &\geq 1 - \frac{M}{2\pi} m([0,2\pi]\setminus E_N) \\ &= 1 - MX_N.\end{aligned}$$

Since $c < 0$, multiplication by c reverses the inequality, and therefore

$$u_k(z_0) \leq c\cdot\frac{1}{2\pi}\int_{E_N} P_{z_0}(\theta)\,d\theta \leq c(1-MX_N) < \alpha.$$

As $z_0 \in \Delta_\rho$ was arbitrary, we conclude that

$$\sup_{\Delta_\rho} u_k \leq \alpha$$

for every $k \in \mathbb{Z}_{>N}$.

Exercise I.1

(i) Consider the map

$$F : \mathbb{C}^n \longrightarrow \mathbb{C}^n, \qquad (\lambda_1, \lambda_2, \ldots, \lambda_n) \longmapsto (a_{n-1}, a_{n-2}, \ldots, a_0),$$

where $(a_{n-1}, a_{n-2}, \ldots, a_0)$ are the elementary symmetric polynomials with the appropriate alternating signs in $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, so that

$$z^n + a_{n-1}z^{n-1} + \cdots + a_0 = \prod_{j=1}^{n}(z-\lambda_j).$$

The map F is polynomial, hence continuous, and it is invariant under the natural action of the symmetric group S_n on $\mathbb{C}^n$. Therefore it descends to a continuous map

$$\bar{F} : \mathrm{Sym}^n(\mathbb{C}) \longrightarrow \mathbb{C}^n.$$

We claim that $\bar{F}$ is a homeomorphism. First, $\bar{F}$ is bijective, since every monic polynomial of degree n has exactly n complex roots counted with multiplicities. Next, we verify that $\bar{F}$ is proper. By Cauchy's estimate for the roots of a polynomial (Lemma I.1.2), any root λ of a monic polynomial

$$z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

satisfies

$$|\lambda| \leq 1 + \max_{0\leq j\leq n-1} |a_j|.$$

Hence the preimage under $\bar{F}$ of a compact subset of $\mathbb{C}^n$ is contained in a compact subset of $\mathrm{Sym}^n(\mathbb{C})$. Therefore $\bar{F}$ is proper. Thus $\bar{F}$ is a homeomorphism, since it is a continuous, proper bijection between Hausdorff locally compact spaces (see Mini-Exercise I.3.16).

Now consider the map

$$\Omega \longrightarrow \mathbb{C}^n, \qquad t \longmapsto (a_{n-1}(t), a_{n-2}(t), \ldots, a_0(t)),$$

which is continuous by assumption. Composing this with $\bar{F}^{-1}$ yields the desired continuous map

$$t \longmapsto R(t) = \bar{F}^{-1}\big(a_{n-1}(t), a_{n-2}(t), \ldots, a_0(t)\big) \in \mathrm{Sym}^n(\mathbb{C}).$$

(ii) Fix $t_0 \in \Omega$. Let $r_1, \ldots, r_m \in \mathbb{C}$ be the distinct roots of P_{t_0}, with multiplicities $k_1, \ldots, k_m$, so that $\sum_{j=1}^m k_j = n$.

Choose $\varepsilon > 0$ so small that the open disks

$$D_j := \{z \in \mathbb{C} \mid |z - r_j| < \varepsilon\}, \qquad j = 1, \ldots, m,$$

are pairwise disjoint and satisfy $P_{t_0}(z) \neq 0$ for all $z \in \partial D_j$. Set

$$\delta_j := \min_{z\in\partial D_j} |P_{t_0}(z)| > 0.$$

Since each coefficient $a_i(t)$ is continuous at t_0, for any non-empty compact $K \subset \mathbb{C}$ we have

$$\sup_{z\in K} |P_t(z) - P_{t_0}(z)| \leq \sum_{i=0}^{n-1} |a_i(t) - a_i(t_0)| \sup_{z\in K} |z|^i \xrightarrow[t\to t_0]{} 0.$$

In particular, for each j there exists a neighborhood U_j of t_0 such that for all $t \in U_j$,

$$|P_t(z) - P_{t_0}(z)| < \delta_j \quad \text{for all } z \in \partial D_j.$$

Hence $|P_t(z) - P_{t_0}(z)| < |P_{t_0}(z)|$ on ∂D_j, and by Rouché's theorem P_t and P_{t_0} have the same number of zeros (counted with multiplicities) in D_j. Therefore P_t has exactly k_j roots in D_j for each j.

Endow $\mathrm{Sym}^n(\mathbb{C})$ with the distance

$$d([z_1, \ldots, z_n], [w_1, \ldots, w_n]) := \min_{\sigma\in S_n} \max_{1\leq i\leq n} |z_i - w_{\sigma(i)}|,$$

which induces the quotient topology of $\mathbb{C}^n/S_n$. Then the previous paragraph implies $d(R(t), R(t_0)) \leq \varepsilon$ for all t sufficiently close to t_0. This proves continuity at t_0, and since t_0 was arbitrary, the lemma follows.

Exercise I.2

For each $t \in U$, set

$$A(t) := \omega(t)^{-1/2}\, H(t)\, \omega(t)^{-1/2}.$$

Then $A(t)$ is Hermitian, and by definition the numbers

$$\gamma_1(t) \geq \gamma_2(t) \geq \cdots \geq \gamma_n(t)$$

are precisely the eigenvalues of $A(t)$ counted with algebraic multiplicity.

First let us show the assertion by assuming that $t \mapsto A(t)$ is continuous. Note that it is clearly the case when $\omega(t)$ is the identity matrix for any $t \in U$: i.e., when $A = H$. Consider the characteristic polynomial of $A(t)$:

$$P_t(z) := \det(zI - A(t)).$$

This is a monic polynomial of degree n. Each coefficient of P_t is a polynomial expression in the entries of $A(t)$, hence depends continuously on t when $t \mapsto A(t)$ is continuous. By Lemma I.1.3, the multiset of roots of P_t (counted with multiplicities) depends continuously on t as an element of $\mathrm{Sym}^n(\mathbb{C})$.

Let $t_0 \in U$ and let $\varepsilon > 0$. By the continuity of the multiset of the eigenvalues of A in $\mathrm{Sym}^n(\mathbb{C})$, there exists a neighborhood V of t_0 such that for every $t \in V$ the multiset of eigenvalues of $A(t)$ can be matched with that of $A(t_0)$ so that each matched pair differs by at most ε. We claim that for all $t \in V$,

$$|\gamma_j(t) - \gamma_j(t_0)| \leq \varepsilon, \qquad j = 1, 2, \ldots, n.$$

Indeed, suppose first that

$$\gamma_j(t) > \gamma_j(t_0) + \varepsilon$$

for some j and $t \in V$. Then the j largest eigenvalues $\gamma_1(t), \gamma_2(t), \ldots, \gamma_j(t)$ of $A(t)$ are all $> \gamma_j(t_0) + \varepsilon$. Since the multisets of eigenvalues of $A(t)$ and $A(t_0)$ can be matched so that each matched pair differs by at most ε, each of these j eigenvalues of $A(t)$ must be matched with an eigenvalue of $A(t_0)$ which is $> \gamma_j(t_0)$. However, by the ordering $\gamma_1(t_0) \geq \gamma_2(t_0) \geq \cdots \geq \gamma_n(t_0)$, there are at most $j - 1$ eigenvalues of $A(t_0)$ which are $> \gamma_j(t_0)$. This is a contradiction. Similarly, if

$$\gamma_j(t) < \gamma_j(t_0) - \varepsilon,$$

then the $n - j + 1$ smallest eigenvalues $\gamma_j(t), \gamma_{j+1}(t), \ldots, \gamma_n(t)$ of $A(t)$ are all $< \gamma_j(t_0) - \varepsilon$. Hence each of them must be matched with an eigenvalue of $A(t_0)$ which is $< \gamma_j(t_0)$, but there are at most $n - j$ such eigenvalues. This is again a contradiction, which proves the claim.

Thus we have $\gamma_j(t) \to \gamma_j(t_0)$ as $t \to t_0$ for each j. As t_0 was arbitrary, γ_j is continuous on U for all $j = 1, 2, \ldots, n$ when $\omega(t)$ is the identity matrix for any $t \in U$.

In the general case, denote by

$$\lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_n(t) > 0$$

the eigenvalues of $\omega(t)$ counted with their algebraic multiplicity. By the special case already proved (applied to the Hermitian matrix $\omega(t)$ itself), each λ_j is continuous on U.

Fix $t_0 \in U$. Since $\lambda_n(t_0) > 0$, by continuity of λ_n there exist a neighborhood W of t_0 and constants c, C with $0 < c < C$ such that every eigenvalue of $\omega(t)$ belongs to $[c, C]$ for all $t \in W$. Set

$$x_0 := \frac{c + C}{2}.$$

Since the function $x \mapsto x^{-1/2}$ is real analytic on $(0, \infty)$, its Taylor series at x_0,

$$x^{-1/2} = \sum_{\nu=0}^{\infty} a_\nu (x - x_0)^\nu,$$

converges uniformly on $[c, C]$. For each $k \in \mathbb{Z}_{\geq 0}$ and $t \in W$, define

$$B_k(t) := \sum_{\nu=0}^{k} a_\nu \big(\omega(t) - x_0 I\big)^\nu.$$

Then $t \mapsto B_k(t)$ is continuous on W, since it is obtained from $\omega(t)$ by finitely many matrix additions and multiplications.

We claim that $B_k(t) \to \omega(t)^{-1/2}$ uniformly on W. Indeed, since $\omega(t)$ is Hermitian, there exists a unitary matrix $U(t)$ such that

$$\omega(t) = U(t)^\dagger \operatorname{diag}(\lambda_1(t), \ldots, \lambda_n(t))\, U(t).$$

Hence

$$B_k(t) = U(t)^\dagger \operatorname{diag}\big(p_k(\lambda_1(t)), \ldots, p_k(\lambda_n(t))\big) U(t),$$

where

$$p_k(x) := \sum_{\nu=0}^{k} a_\nu (x - x_0)^\nu.$$

Similarly,

$$\omega(t)^{-1/2} = U(t)^\dagger \operatorname{diag}\big(\lambda_1(t)^{-1/2}, \ldots, \lambda_n(t)^{-1/2}\big) U(t).$$

Therefore

$$\|B_k(t) - \omega(t)^{-1/2}\|_{\mathrm{op}} = \max_{1 \leq j \leq n} |p_k(\lambda_j(t)) - \lambda_j(t)^{-1/2}| \leq \sup_{x \in [c,C]} |p_k(x) - x^{-1/2}|,$$

where $\|\bullet\|_{\mathrm{op}}$ denotes the operator norm of a matrix, viewed as a linear map $\mathbb{C}^n \to \mathbb{C}^n$ endowed with the standard Euclidean norm. Since $p_k \to x^{-1/2}$ uniformly on $[c, C]$, the right-hand side tends to 0 as $k \to \infty$, uniformly in $t \in W$. Thus $B_k \to \omega(\,\cdot\,)^{-1/2}$ uniformly on W.

Hence $t \mapsto \omega(t)^{-1/2}$ is continuous on W, being the uniform limit of continuous matrix-valued functions. Since t_0 was arbitrary, $t \mapsto \omega(t)^{-1/2}$ is continuous on U. Consequently,

$$A(t) = \omega(t)^{-1/2} H(t) \omega(t)^{-1/2}$$

is continuous on U. The proof is therefore reduced to the special case already treated.

Exercise I.3

For $U_j \neq X$, the function $x \mapsto d(x, X \setminus U_j)$ is continuous, so $U_j^{(r)}$ is open in Y by the strict inequality in its definition. The same assertion is immediate when $U_j = X$. Hence it suffices to show that there exists a positive number $r_0 > 0$ such that, for every $r \in (0, r_0)$,

$$Y = \bigcup_{j \in J} U_j^{(r)}.$$

Since Y is compact and $\{U_j\}_{j \in J}$ is an open covering of Y by open subsets of X, we may apply Lebesgue's covering theorem [Ke, Chapter 5]. Thus there exists a positive number $r_0 > 0$ such that for every point $y \in Y$ there exists an index $j \in J$ with the property that the open r_0-sphere about y,

$$B_X(y, r_0) := \{x \in X \mid d(x, y) < r_0\},$$

is contained in U_j.

Fix $y \in Y$ and $r \in (0, r_0)$. Choose $j \in J$ with $B_X(y, r_0) \subset U_j$. If $U_j \neq X$, this inclusion implies

$$d(y, X \setminus U_j) \geq r_0 > r.$$

Thus $y \in U_j^{(r)}$; the same conclusion is immediate if $U_j = X$. Since y was arbitrary, the family $\{U_j^{(r)}\}_{j \in J}$ is an open covering of Y for every $r \in (0, r_0)$.

If the compactness assumption on Y is dropped, the statement is false in general. For example, take $X = \mathbb{R}$ with its Euclidean distance, $Y = (0, 1)$, and the covering consisting of $U_0 = (0, 1)$. For $0 < r < 1/2$, one has $U_0^{(r)} = (r, 1 - r)$, which does not cover Y.

Exercise I.4

(*i*) Fix $\tau \geq 1$. For each $n \in \mathbb{Z}_{>0}$, set

$$E_{\tau,n} := \Big\{\alpha \in \mathbb{R} \;\Big|\; d(q\alpha, \mathbb{Z}) \geq \tfrac{1}{n}\, q^{-\tau} \text{ for all } q \in \mathbb{Z}_{>0}\Big\}.$$

Then

$$\mathrm{Dio}(\tau) = \bigcup_{n=1}^{\infty} E_{\tau,n}.$$

We claim that each $E_{\tau,n}$ is nowhere dense in $\mathbb{R}$. First, $E_{\tau,n}$ is closed: for each fixed q, the map $\alpha \mapsto d(q\alpha, \mathbb{Z})$ is continuous on $\mathbb{R}$, hence the set $\{\alpha \mid d(q\alpha, \mathbb{Z}) \geq \frac{1}{n} q^{-\tau}\}$ is closed, and $E_{\tau,n}$ is the intersection of these closed sets over $q \geq 1$.

Next we show that $E_{\tau,n}$ has empty interior. Let $I \subset \mathbb{R}$ be a non-empty open interval. Choose a rational number $p/q \in I$. Since I is open, there exists $\alpha \in I$ such that

$$|\alpha - p/q| < \frac{1}{n} q^{-\tau-1}.$$

Then

$$d(q\alpha, \mathbb{Z}) \le |q\alpha - p| = q\,|\alpha - p/q| < \frac{1}{n} q^{-\tau},$$

and hence $\alpha \notin E_{\tau,n}$. Therefore $E_{\tau,n}$ is nowhere dense for each n, and $\mathrm{Dio}(\tau)$ is a countable union of nowhere dense sets.

(ii) Fixing $\tau > 1$, for each $q \in \mathbb{Z}_{>0}$, define

$$A_q := \Big\{ \alpha \in [0,1] \;\Big|\; d(q\alpha, \mathbb{Z}) < q^{-\tau} \Big\}.$$

We estimate $\lambda(A_q)$, where λ denotes the Lebesgue measure. For an integer m with $0 \le m \le q$, the inequality $|q\alpha - m| < q^{-\tau}$ is equivalent to

$$\alpha \in \Big(\frac{m}{q} - q^{-\tau-1},\ \frac{m}{q} + q^{-\tau-1} \Big).$$

Intersecting with $[0,1]$, we see that A_q is contained in a union of at most $q+1$ intervals, each of length $2q^{-\tau-1}$. Hence

$$\lambda(A_q) \le (q+1) \cdot 2q^{-\tau-1} \le 4q^{-\tau}.$$

For $Q \in \mathbb{Z}_{>0}$ put $B_Q := \bigcup_{q>Q} A_q$. By subadditivity of λ we have

$$\lambda(B_Q) \le \sum_{q>Q} \lambda(A_q) \le 4 \sum_{q>Q} q^{-\tau}.$$

Since $\tau > 1$, the tail tends to 0, and in fact

$$\sum_{q>Q} q^{-\tau} \le \int_Q^\infty x^{-\tau}\,dx = \frac{1}{\tau - 1} Q^{1-\tau},$$

so

$$\lambda(B_Q) \le \frac{4}{\tau - 1} Q^{1-\tau} \xrightarrow[Q\to\infty]{} 0.$$

Now set

$$E := \bigcap_{Q=1}^{\infty} B_Q = \Big\{ \alpha \in [0,1] \;\Big|\; d(q\alpha, \mathbb{Z}) < q^{-\tau} \text{ for infinitely many } q \Big\}.$$

Since $E \subset B_Q$ for every Q, we obtain $\lambda(E) \le \lambda(B_Q)$ for all Q, hence $\lambda(E) = 0$.

Let $\alpha \in [0,1] \setminus (E \cup \mathbb{Q})$. Then $\alpha \notin \mathbb{Q}$ and there exists Q_0 such that $\alpha \notin A_q$ for all $q > Q_0$, i.e.

$$d(q\alpha, \mathbb{Z}) \ge q^{-\tau} \qquad \text{for all } q > Q_0.$$

Moreover, $d(q\alpha, \mathbb{Z}) > 0$ for every $1 \le q \le Q_0$ because $\alpha \notin \mathbb{Q}$. Define

$$C := \min\Big\{ 1,\ \min_{1 \le q \le Q_0} q^{\tau}\, d(q\alpha, \mathbb{Z}) \Big\} > 0.$$

Then $d(q\alpha, \mathbb{Z}) \geq Cq^{-\tau}$ holds for all $q \in \mathbb{Z}_{>0}$. Thus $\alpha \in \mathrm{Dio}(\tau)$. Consequently, $[0,1]\setminus(E\cup\mathbb{Q}) \subset \mathrm{Dio}(\tau)$, and hence $\lambda\big([0,1] \setminus \mathrm{Dio}(\tau)\big) \leq \lambda(E \cup \mathbb{Q}) \leq \lambda(E) + \lambda(\mathbb{Q}) = 0$.

(iii) By *(ii)*, for each fixed $\tau > 1$ the set $\mathrm{Dio}(\tau)$ has full Lebesgue measure in $\mathbb{R}$. Hence $\mathrm{Dio}_{>1}$ also has full Lebesgue measure.

On the other hand, by *(i)*, $\mathrm{Dio}(\tau)$ is meagre for every $\tau \geq 1$. Moreover,

$$\mathrm{Dio}_{>1} = \bigcup_{\tau \in \mathbb{Q}_{>1}} \mathrm{Dio}(\tau),$$

since if $\alpha \in \mathrm{Dio}(\tau_0)$ for some $\tau_0 > 1$, then $\alpha \in \mathrm{Dio}(\tau)$ for every $\tau \geq \tau_0$, and we can choose a rational $\tau > \tau_0$. Therefore $\mathrm{Dio}_{>1}$ is a countable union of meagre sets, hence meagre in $\mathbb{R}$.

Exercise I.5

First note that, as X is compact, any analytic object on X has compact support. Hence $C_c^\infty(X) = C^\infty(X)$, $\mathcal{K}_c^\bullet(X) = \mathcal{K}^\bullet(X)$, and so on, hold on X. Since X is a compact connected smooth manifold, the pairing

$$H^2(\mathcal{K}^\bullet(X), d) \times H^0_{\mathrm{DR}}(X) \longrightarrow \mathbb{C}$$

is nondegenerate by Lemma I.8.50. Moreover, X is connected, so $H^0_{\mathrm{DR}}(X) \cong \mathbb{C}$, represented by constant functions. For a constant function $a \in H^0_{\mathrm{DR}}(X)$, one has

$$\left(\sum_{j=1}^N n_j\, \delta_{p_j}\right)(a) = a \sum_{j=1}^N n_j.$$

Therefore, a necessary and sufficient condition for the existence of a current $T \in \mathcal{K}^1(X)$ such that

$$dT = \sum_{j=1}^N n_j\, \delta_{p_j}$$

holds, or equivalently for the triviality of the class

$$\sum_{j=1}^N n_j\, \delta_{p_j} \mod \mathrm{Image}(\mathrm{d}\colon \mathcal{K}^1(\mathrm{X}) \to \mathcal{K}^2(\mathrm{X})) \in \mathrm{H}^2(\mathcal{K}^\bullet(\mathrm{X}), \mathrm{d}),$$

can be expressed as

$$\sum_{j=1}^N n_j = 0.$$

Now assume that this condition is satisfied. If $N = 1$, then necessarily $n_1 = 0$, and one may simply take $T = 0$. Assume therefore that $N > 1$. In what follows we regard X as the Riemann sphere $\mathbb{P}^1$ by choosing an inhomogeneous coordinate z in such a way that p_1

corresponds to the infinity point $z = \infty$. For each $j = 2, 3, \ldots, N$, let $\zeta_j \in \mathbb{C}$ denote the z-coordinate of p_j. Set

$$f(z) := \prod_{j=2}^{N} (z - \zeta_j)^{n_j},$$

and regard it as a rational function on $\mathbb{P}^1$. Then one has

$$\operatorname{ord}_{p_j}(f) = n_j \qquad (j = 1, 2, \ldots, N).$$

Moreover, f has no zeros or poles outside the set $\{p_1, \ldots, p_N\}$. Hence, by the Lelong–Poincaré formula (58),

$$\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log |f|^2 = \sum_{j=1}^{N} n_j \, \delta_{p_j}.$$

Therefore, if we set

$$T := \frac{\sqrt{-1}}{2\pi} \overline{\partial} \log |f|^2,$$

then

$$dT = \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log |f|^2 = \sum_{j=1}^{N} n_j \, \delta_{p_j},$$

as desired, since $d\overline{\partial} = \partial\overline{\partial}$.

Exercise I.6

By considering the Laurent expansion of a holomorphic function on the annulus $U_0 \cap U_1 = \{z \in \mathbb{C} \mid r < |z| < \ell\}$, or equivalently by the discussion in Remark I.9.1, one sees that δ is surjective. Hence we have $\operatorname{Coker} \delta = 0$.

Next let us determine the kernel. An element $(f_0, f_1) \in W_0 \oplus W_1$ belongs to $\operatorname{Ker} \delta$ if and only if $f_0 = f_1$ on the intersection $U_0 \cap U_1$. Thus such a pair determines a continuous function f on $\mathbb{P}^1$, obtained by gluing f_0 and f_1, which is holomorphic on U_0 and on U_1 (Readers who are already familiar with the notion of holomorphic functions on Riemann surfaces may simply say that f is a holomorphic function on $\mathbb{P}^1$). In particular, the restriction $f|_{\mathbb{C}}$ is a bounded entire holomorphic function. By Liouville's theorem, $f|_{\mathbb{C}}$ must be constant. It follows that both f_0 and f_1 are constant, with the same constant value. Therefore $\operatorname{Ker} \delta = \{(c, c) \mid c \in \mathbb{C}\}$.

Here let us also note that one may alternatively prove the constancy of f by using the maximum modulus principle. Indeed, since f is continuous on the compact space $\mathbb{P}^1$, the function $|f|$ attains its maximum at some point of $\mathbb{P}^1$. Such a point lies in either U_0 or U_1, and hence is an interior point of the corresponding domain. The maximum modulus principle (and the identity theorem on $U_0 \cap U_1$) then implies that f is constant.

If one replaces holomorphic functions by smooth functions and considers

$$\delta \colon C^\infty(U_0) \oplus C^\infty(U_1) \to C^\infty(U_0 \cap U_1), \qquad \delta(f_0, f_1) := -f_0 + f_1,$$

then δ is still surjective, by a standard cut-off argument. Hence $\operatorname{Coker}\delta = 0$ holds also in this case. On the other hand, the kernel now consists of pairs of smooth functions on U_0 and U_1 which agree on the overlap, and therefore glue together to a smooth function on $\mathbb{P}^1$. Thus $\operatorname{Ker}\delta \cong C^\infty(\mathbb{P}^1)$, which is an infinite-dimensional Fréchet space.

If one instead considers locally constant functions, then U_0, U_1, and $U_0 \cap U_1$ are all connected, so the corresponding spaces are all naturally identified with $\mathbb{C}$. Thus δ becomes

$$\delta\colon \mathbb{C}^2 \to \mathbb{C}, \qquad \delta(c_0, c_1) := -c_0 + c_1.$$

Hence

$$\operatorname{Ker}\delta = \{(c,c) \mid c \in \mathbb{C}\}, \qquad \operatorname{Coker}\delta = 0,$$

exactly as in the original holomorphic setting.

The reader is encouraged to reinterpret the above computations later as computations of Čech cohomology, after Chapter II, and to compare them with the general theory.

Exercise I.7

By the Riemann mapping theorem, U is biholomorphic either to $\mathbb{C}$ or to the strip $\{z \in \mathbb{C} \mid -1 < \operatorname{Im} z < 1\}$. Therefore it suffices to treat the cases $U = \mathbb{C}$ or $U = \{z \in \mathbb{C} \mid -1 < \operatorname{Im} z < 1\}$.

For each positive integer N, define an ideal $I_N \subset \mathcal{O}_{\mathbb{C}}(U)$ by

$$I_N := \{f \in \mathcal{O}_{\mathbb{C}}(U) \mid f(n) = 0 \text{ for every } n \in \mathbb{Z} \text{ with } |n| \geq N\}.$$

It is clear from the definition that

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

is an increasing sequence of ideals.

To show that $\mathcal{O}_{\mathbb{C}}(U)$ is not Noetherian, it suffices to prove that this sequence does not stabilize. For each N, consider the function

$$f_N(z) := \frac{\sin(\pi z)}{\prod_{n=-N}^{N}(z-n)}.$$

Since the zeros of $\sin(\pi z)$ are precisely the integers, the apparent singularities at $z = -N, -N+1, \ldots, N$ are removable. Therefore f_N is holomorphic on $\mathbb{C}$, hence also on U. Moreover, f_N vanishes at every integer m with $|m| \geq N+1$, while it does not vanish at $m = \pm N$. Thus $f_N \in I_{N+1} \setminus I_N$. Hence $I_N \neq I_{N+1}$ for every N, which implies that $\mathcal{O}_{\mathbb{C}}(U)$ is not Noetherian.

Now define

$$I_\infty := \{f \in \mathcal{O}_{\mathbb{C}}(U) \mid \#\{n \in \mathbb{Z} \mid f(n) \neq 0\} < \infty\}.$$

Then I_∞ is an ideal of $\mathcal{O}_{\mathbb{C}}(U)$, and

$$I_\infty = \bigcup_{N=1}^{\infty} I_N.$$

We claim that I_∞ is not finitely generated. Indeed, suppose that I_∞ were generated by finitely many elements $g_1, g_2, \dots, g_r \in I_\infty$. Since each g_j belongs to I_∞, there exists N_j such that $g_j \in I_{N_j}$. If $N := \max\{N_1, N_2, \dots, N_r\}$, then $g_1, g_2, \dots, g_r \in I_N$, and therefore we have $I_\infty \subset I_N$, which contradicts $I_{N+1} \not\subset I_N$.

3.2 Hints and solutions to the exercises in Chapter II

Mini-Exercise II.2.3

Refer to [KUed2].

Mini-Exercise II.2.4

Let A be a square matrix of order N satisfying $\mathbf{(P_1)}$, $\mathbf{(P_2)}$, and $\mathbf{(P_3)}$. Choose a sufficiently large real number $c > 0$ and set

$$B := A + cI_N,$$

where I_N denotes the identity matrix of order N. By construction, all diagonal entries of B are non-negative, and by $\mathbf{(P_2)}$ all off-diagonal entries of B are also non-negative. Moreover, since adding a scalar multiple of the identity matrix does not affect off-diagonal entries, condition $\mathbf{(P_3)}$ remains valid for B.

Since $B = A + cI_N$, the eigenvectors of A and B coincide, and all eigenvalues of B are obtained by shifting those of A by c. In particular,

$$\lambda(B) = \lambda(A) + c.$$

By Theorem II.2.5, there exists a unique element $\mathfrak{p} \in \mathbb{R}^N_+$ such that

$$B\mathfrak{p} = \lambda(B)\mathfrak{p} \qquad \text{and} \qquad \|\mathfrak{p}\|_{L^\infty} = 1.$$

Moreover, $\lambda(B)$ is algebraically simple, hence (by symmetry of B) the eigenspace of $\lambda(B)$ is one-dimensional.

Proof of Lemma II.2.1. Let $\mathfrak{m}$ be an eigenvector of A associated with $\lambda(A)$. Then $\mathfrak{m}$ is an eigenvector of B associated with $\lambda(B)$. Since the eigenspace of $\lambda(B)$ is one-dimensional, there exists $t \in \mathbb{R} \setminus \{0\}$ such that $\mathfrak{m} = t\mathfrak{p}$. Hence either $\mathfrak{m}$ or $-\mathfrak{m}$ belongs to $\mathbb{R}^N_+$.

Proof of Proposition II.2.2. We first prove the "only if" parts. If $\lambda(A) > 0$, then we have

$$A\mathfrak{p} = (\lambda(B) - c)\mathfrak{p} = \lambda(A)\mathfrak{p} \in \mathbb{R}^N_+,$$

which gives (i). If $\lambda(A) = 0$, then $A\mathfrak{p} = 0$, proving (ii). If $\lambda(A) < 0$, then

$$-A\mathfrak{p} = (-\lambda(A))\mathfrak{p} \in \mathbb{R}^N_+,$$

which gives (iii).

Conversely, assume that there exists a suitable $\mathfrak{m} \in \mathbb{R}_+^N$. When $A\mathfrak{m} \in \mathbb{R}_+^N$, as $B\mathfrak{m} = A\mathfrak{m} + c\mathfrak{m} \in \mathbb{R}_+^N$, we have $B\mathfrak{m} > c\mathfrak{m}$ componentwise. Taking the Euclidean inner product with $\mathfrak{p}$ and using $B\mathfrak{p} = \lambda(B)\mathfrak{p}$ and $\mathfrak{p} > 0$, we obtain

$$\lambda(B)\langle \mathfrak{p}, \mathfrak{m} \rangle = \langle B\mathfrak{p}, \mathfrak{m} \rangle = \langle \mathfrak{p}, B\mathfrak{m} \rangle > c\langle \mathfrak{p}, \mathfrak{m} \rangle.$$

Since $\langle \mathfrak{p}, \mathfrak{m} \rangle > 0$, we conclude $\lambda(B) > c$, hence $\lambda(A) = \lambda(B) - c > 0$. When $A\mathfrak{m} = 0$, $B\mathfrak{m} = c\mathfrak{m}$. By Theorem II.2.5 (iv), we have $\lambda(B) = c$, hence $\lambda(A) = 0$. When $-A\mathfrak{m} \in \mathbb{R}_+^N$, as $B\mathfrak{m} = A\mathfrak{m} + c\mathfrak{m} \in \mathbb{R}_+^N$, we have $B\mathfrak{m} < c\mathfrak{m}$ componentwise. Taking the inner product with $\mathfrak{p}$ as above, we obtain

$$\lambda(B)\langle \mathfrak{p}, \mathfrak{m} \rangle = \langle \mathfrak{p}, B\mathfrak{m} \rangle < c\langle \mathfrak{p}, \mathfrak{m} \rangle,$$

hence $\lambda(B) < c$, and therefore $\lambda(A) = \lambda(B) - c < 0$.

Finally, let us consider the case where all entries of A are integers. When $\lambda(A)$ is positive (resp. negative), $A\mathfrak{m} \in \mathbb{Q}_+^N$ (resp. $-A\mathfrak{m} \in \mathbb{Q}_+^N$) holds for an element $\mathfrak{m} \in \mathbb{Q}_+^N$ which is sufficiently close to $\mathfrak{p}$, since the linear map defined by the matrix A is continuous. One can modify such $\mathfrak{m}$ to construct an element of $\mathbb{Z}_+^N$ as in the assertion by multiplying by a sufficiently divisible integer. When $\lambda(A) = 0$, choose j such that the j-th component of $\mathfrak{p}$ is 1, using the normalization above. Since $\operatorname{Ker} A$ is one-dimensional, $\mathfrak{p}$ is the unique solution of the rational-coefficient system $A\mathfrak{r} = 0$, $r_j = 1$, where r_j denotes the j-th component of $\mathfrak{r}$. It follows that $\mathfrak{p} \in \mathbb{Q}_+^N$. Therefore $M\mathfrak{p}$ is an element of $\mathbb{Z}_+^N$ for a sufficiently divisible integer M.

Mini-Exercise II.3.2

For every object $X \in \mathrm{ob}(\mathfrak{C}^{\mathrm{op}})$, we define its identity morphism in $\mathfrak{C}^{\mathrm{op}}$ to be the identity morphism

$$\mathrm{id}_X \colon X \to X$$

in $\mathfrak{C}$.

Let

$$f \colon X \to Y, \qquad g \colon Y \to Z$$

be morphisms in $\mathfrak{C}^{\mathrm{op}}$. Regarded as morphisms in $\mathfrak{C}$, they have the directions

$$f \colon Y \to X, \qquad g \colon Z \to Y.$$

Therefore, they can be composed in $\mathfrak{C}$ in the order

$$f \circ_{\mathfrak{C}} g \colon Z \to X.$$

This is a morphism from X to Z in $\mathfrak{C}^{\mathrm{op}}$. We thus define

$$g \circ_{\mathfrak{C}^{\mathrm{op}}} f := f \circ_{\mathfrak{C}} g.$$

When no confusion can arise, the subscripts are omitted and both composition laws are denoted simply by $\circ$.

We first verify associativity. Let

$$f\colon X \to Y, \qquad g\colon Y \to Z, \qquad h\colon Z \to W$$

be morphisms in $\mathfrak{C}^{\mathrm{op}}$. As morphisms in $\mathfrak{C}$, they have the directions

$$f\colon Y \to X, \qquad g\colon Z \to Y, \qquad h\colon W \to Z.$$

By the definition of composition in $\mathfrak{C}^{\mathrm{op}}$, we have

$$\begin{aligned} h \circ_{\mathfrak{C}^{\mathrm{op}}} \big(g \circ_{\mathfrak{C}^{\mathrm{op}}} f\big) &= \big(f \circ_{\mathfrak{C}} g\big) \circ_{\mathfrak{C}} h \\ &= f \circ_{\mathfrak{C}} \big(g \circ_{\mathfrak{C}} h\big) \\ &= \big(h \circ_{\mathfrak{C}^{\mathrm{op}}} g\big) \circ_{\mathfrak{C}^{\mathrm{op}}} f, \end{aligned}$$

where the middle equality follows from the associativity of composition in $\mathfrak{C}$. Hence composition in $\mathfrak{C}^{\mathrm{op}}$ is associative.

Next, let

$$f\colon X \to Y$$

be a morphism in $\mathfrak{C}^{\mathrm{op}}$. Regarded as a morphism in $\mathfrak{C}$, it is a morphism

$$f\colon Y \to X.$$

We then have

$$f \circ_{\mathfrak{C}^{\mathrm{op}}} \mathrm{id}_X = \mathrm{id}_X \circ_{\mathfrak{C}} f = f$$

and

$$\mathrm{id}_Y \circ_{\mathfrak{C}^{\mathrm{op}}} f = f \circ_{\mathfrak{C}} \mathrm{id}_Y = f.$$

Thus id_X is the identity morphism of X in $\mathfrak{C}^{\mathrm{op}}$. It follows that $\mathfrak{C}^{\mathrm{op}}$ is a category.

Finally, the objects of $(\mathfrak{C}^{\mathrm{op}})^{\mathrm{op}}$ are the objects of $\mathfrak{C}$, and, for objects $X, Y \in \mathrm{ob}(\mathfrak{C})$,

$$\begin{aligned} \mathrm{Hom}_{(\mathfrak{C}^{\mathrm{op}})^{\mathrm{op}}}(X, Y) &= \mathrm{Hom}_{\mathfrak{C}^{\mathrm{op}}}(Y, X) \\ &= \mathrm{Hom}_{\mathfrak{C}}(X, Y). \end{aligned}$$

Moreover, reversing the order of composition twice recovers the original composition law of $\mathfrak{C}$, and the identity morphisms remain unchanged. Therefore,

$$(\mathfrak{C}^{\mathrm{op}})^{\mathrm{op}} = \mathfrak{C}.$$

Mini-Exercise II.3.26

(*i*) The morphisms $F(u_i)\colon F(i_0) \longrightarrow F(i)$ form a cone over F: Indeed, for every morphism $a\colon i \longrightarrow j$ in $\mathfrak{I}$, the uniqueness of the morphism from i_0 to j gives

$$a \circ u_i = u_j,$$

and hence

$$F(a) \circ F(u_i) = F(u_j).$$

Let

$$\big(T, \{\tau_i \colon T \longrightarrow F(i)\}_{i \in \mathrm{ob}(\mathfrak{I})}\big)$$

be any cone over F. Set

$$h := \tau_{i_0} \colon T \longrightarrow F(i_0).$$

Applying the compatibility condition for the cone to

$$u_i \colon i_0 \longrightarrow i,$$

we obtain

$$F(u_i) \circ h = F(u_i) \circ \tau_{i_0} = \tau_i$$

for every $i \in \mathrm{ob}(\mathfrak{I})$. Thus h is a morphism from the cone T to the cone with vertex $F(i_0)$.

This morphism is unique. Indeed, if

$$h' \colon T \longrightarrow F(i_0)$$

also satisfies

$$F(u_i) \circ h' = \tau_i$$

for every i, then, on taking $i = i_0$ and using

$$u_{i_0} = \mathrm{id}_{i_0},$$

we obtain

$$h' = \tau_{i_0} = h.$$

Therefore $F(i_0)$, together with the morphisms $F(u_i)$, is a limit of F.

(ii) The morphisms $F(v_i) \colon F(i) \longrightarrow F(i_1)$ form a cocone under F, by the same reason as (i). Let

$$\big(T, \{\sigma_i \colon F(i) \longrightarrow T\}_{i \in \mathrm{ob}(\mathfrak{I})}\big)$$

be any cocone under F. Set

$$h := \sigma_{i_1} \colon F(i_1) \longrightarrow T.$$

The compatibility condition for the cocone gives

$$h \circ F(v_i) = \sigma_{i_1} \circ F(v_i) = \sigma_i$$

for every $i \in \mathrm{ob}(\mathfrak{I})$. Thus h is a morphism from the cocone with vertex $F(i_1)$ to the cocone with vertex T.

If another morphism

$$h' \colon F(i_1) \longrightarrow T$$

has the same property, then, on taking $i = i_1$ and using

$$v_{i_1} = \mathrm{id}_{i_1},$$

we obtain

$$h' = \sigma_{i_1} = h.$$

Therefore $F(i_1)$, together with the morphisms $F(v_i)$, is a colimit of F.

(*iii*) Regarded as a category, the directed set I has m as a terminal object. Hence (*ii*) gives

$$\varinjlim_{i \in I} F(i) \cong F(m).$$

The object m is an initial object of the opposite category I^{op}. Therefore (*i*) applied to

$$G\colon I^{\mathrm{op}} \xrightarrow{\text{functor}} \mathfrak{C}$$

gives

$$\varprojlim_{i \in I} G(i) \cong G(m).$$

Mini-Exercise II.3.29

Suppose that a smooth map

$$e\colon M \to \mathbb{R}^2$$

from a smooth manifold M were an equalizer of h and 0. In particular,

$$h \circ e = 0.$$

Applying the universal property with the one-point manifold $\{*\}$ as the source, composition with e gives a bijection

$$\mathrm{Hom}_{(C^\infty\mathsf{Mfd})}(\{*\}, M) \longrightarrow \left\{ b \in \mathrm{Hom}_{(C^\infty\mathsf{Mfd})}(\{*\}, \mathbb{R}^2) \;\middle|\; h \circ b = 0 \circ b \right\}.$$

Since smooth maps from $\{*\}$ to a smooth manifold are naturally identified with its points, this means that e induces a bijection from the underlying set of M onto

$$\left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; xy = 0 \right\}.$$

Let $p \in M$ be the unique point such that

$$e(p) = (0, 0).$$

The two smooth curves

$$c_1\colon \mathbb{R} \to \mathbb{R}^2, \qquad c_1(t) := (t, 0),$$

and

$$c_2\colon \mathbb{R} \to \mathbb{R}^2, \qquad c_2(t) := (0, t),$$

satisfy

$$h \circ c_1 = 0 \qquad \text{and} \qquad h \circ c_2 = 0.$$

Hence, by the universal property of the equalizer, there exist unique smooth maps

$$\widetilde{c}_1, \widetilde{c}_2 \colon \mathbb{R} \to M$$

such that

$$e \circ \widetilde{c}_1 = c_1 \qquad \text{and} \qquad e \circ \widetilde{c}_2 = c_2.$$

Since

$$e\big(\widetilde{c}_1(0)\big) = e\big(\widetilde{c}_2(0)\big) = (0,0),$$

the uniqueness of p gives

$$\widetilde{c}_1(0) = \widetilde{c}_2(0) = p.$$

Therefore,

$$de_p\left(\widetilde{c}_1'(0)\right) = (1,0)$$

and

$$de_p\left(\widetilde{c}_2'(0)\right) = (0,1).$$

Thus

$$de_p \colon T_pM \to T_{(0,0)}\mathbb{R}^2$$

is surjective.

By the submersion theorem, the image under e of a neighborhood of p contains a neighborhood of $(0,0)$ in $\mathbb{R}^2$. On the other hand, the identity

$$h \circ e = 0$$

implies that

$$e(M) \subset \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; xy = 0\right\},$$

and this set contains no neighborhood of $(0,0)$, which is a contradiction.

Mini-Exercise II.3.37

The two subsets coincide in the first, third, and fourth cases. They need not coincide in the second case.

Let $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ denote the power sets of X and Y, regarded as categories by inclusion. Thus, for $A, A' \subset X$, there is a unique morphism

$$A \to A'$$

if and only if $A \subset A'$, and similarly for subsets of Y.

Define functors

$$\exists_f \colon \mathcal{P}(X) \xrightarrow{\text{functor}} \mathcal{P}(Y), \qquad \exists_f(A) := f(A),$$

and

$$f^{-1} \colon \mathcal{P}(Y) \xrightarrow{\text{functor}} \mathcal{P}(X), \qquad B \longmapsto f^{-1}(B).$$

For $A \subset X$ and $B \subset Y$, one has

$$f(A) \subset B \quad \Longleftrightarrow \quad A \subset f^{-1}(B).$$

Therefore,

$$\exists_f \dashv f^{-1}.$$

There is also a functor

$$\forall_f \colon \mathcal{P}(X) \xrightarrow{\text{functor}} \mathcal{P}(Y)$$

defined by

$$\forall_f(A) := \left\{ y \in Y \;\middle|\; f^{-1}(\{y\}) \subset A \right\}.$$

For $A \subset X$ and $B \subset Y$, one has

$$f^{-1}(B) \subset A \quad \Longleftrightarrow \quad B \subset \forall_f(A).$$

Hence

$$f^{-1} \dashv \forall_f.$$

Combining the two adjunctions, we obtain

$$\exists_f \dashv f^{-1} \dashv \forall_f.$$

In the categories $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, unions are colimits and intersections are limits. Proposition II.3.36 therefore implies that $\exists_f$ preserves unions, while f^{-1} preserves both unions and intersections. Consequently,

$$f\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda} f(A_\lambda),$$

$$f^{-1}\left(\bigcup_{\lambda\in\Lambda} B_\lambda\right) = \bigcup_{\lambda\in\Lambda} f^{-1}(B_\lambda),$$

and

$$f^{-1}\left(\bigcap_{\lambda\in\Lambda} B_\lambda\right) = \bigcap_{\lambda\in\Lambda} f^{-1}(B_\lambda).$$

On the other hand, a left adjoint is not required to preserve limits. The direct-image functor $\exists_f$ need not preserve intersections.

For a concrete counterexample, let

$$X := \{0, 1\}, \qquad Y := \{*\},$$

and let

$$f \colon X \to Y$$

be the unique map. Put

$$A_0 := \{0\}, \qquad A_1 := \{1\}.$$

Then

$$A_0 \cap A_1 = \emptyset,$$

and hence

$$f(A_0 \cap A_1) = \emptyset.$$

However,

$$f(A_0) = f(A_1) = \{*\},$$

so that

$$f(A_0) \cap f(A_1) = \{*\}.$$

Thus the two subsets in the second case need not coincide.

Mini-Exercise II.3.88

(*i*) By Proposition II.3.81, there are natural isomorphisms $H^2(X, \underline{\mathbb{Z}}_X) \cong H^2(X, \mathbb{Z})$ and $H^2(X, \underline{\mathbb{C}}_X) \cong H^2(X, \mathbb{C})$. Under these isomorphisms, the morphism $H^2(X, \underline{\mathbb{Z}}_X) \to H^2(X, \underline{\mathbb{C}}_X)$ is identified with the morphism $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$ induced by the inclusion $\mathbb{Z} \to \mathbb{C}$.

By Mini-Exercise I.7.23, the torsion subgroup of $H^2(X, \mathbb{Z})$ is naturally isomorphic to the torsion subgroup of $H_1(X)$. Since $H_1(X)$ is torsion-free, it follows that $H^2(X, \mathbb{Z})$ is torsion-free. Moreover, by Mini-Exercise I.7.22, there is a natural isomorphism $H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^2(X, \mathbb{C})$. Under this isomorphism, the homomorphism induced by $\mathbb{Z} \longrightarrow \mathbb{C}$ is identified with

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}, \qquad \alpha \longmapsto \alpha \otimes 1.$$

Since $H^2(X, \mathbb{Z})$ is torsion-free, this homomorphism is injective, from which the assertion follows.

(*ii*) Consider the commutative diagram of exact sequences of sheaves of abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathbb{Z}}_X & \longrightarrow & \underline{\mathbb{C}}_X & \longrightarrow & \underline{\mathbb{C}^*}_X & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \underline{\mathbb{Z}}_X & \longrightarrow & \mathcal{C}_X & \longrightarrow & \mathcal{C}_X^* & \longrightarrow & 1. \end{array}$$

Here the vertical morphisms are the natural inclusions of locally constant functions into continuous functions.

Let $L \to X$ be a $\mathbb{C}^*$-flat line bundle. By Lemma I.7.36 and Remark II.3.85, the flat transition functions of L determine a class

$$\alpha_L \in H^1(X, \underline{\mathbb{C}^*}_X).$$

Its image under the natural morphism

$$H^1(X, \underline{\mathbb{C}^*}_X) \longrightarrow H^1(X, \mathcal{C}_X^*)$$

is the class

$$[L]_{\text{top}} \in H^1(X, \mathcal{C}_X^*)$$

of the underlying topological line bundle.

The naturality of the long exact cohomology sequence gives the commutative diagram

$$\begin{array}{ccccc} H^1\left(X, \underline{\mathbb{C}^*}_X\right) & \xrightarrow{\delta_{\mathrm{lc}}} & H^2\left(X, \underline{\mathbb{Z}}_X\right) & \longrightarrow & H^2\left(X, \underline{\mathbb{C}}_X\right) \\ \downarrow & & \| & & \downarrow \\ H^1\left(X, \mathcal{C}_X^*\right) & \xrightarrow{\delta_{\exp}} & H^2\left(X, \underline{\mathbb{Z}}_X\right) & \longrightarrow & H^2\left(X, \mathcal{C}_X\right). \end{array}$$

By the exactness of the upper row, $\delta_{\mathrm{lc}}(\alpha_L)$ belongs to the kernel of $H^2\left(X, \underline{\mathbb{Z}}_X\right) \to H^2\left(X, \underline{\mathbb{C}}_X\right)$, which is injective by (i). Therefore

$$\delta_{\mathrm{lc}}(\alpha_L) = 0.$$

The commutativity of the preceding diagram now gives

$$\delta_{\exp}\left([L]_{\mathrm{top}}\right) = 0.$$

By Example II.3.86, the connecting morphism

$$\delta_{\exp} = c_1 \colon H^1\left(X, \mathcal{C}_X^*\right) \longrightarrow H^2\left(X, \underline{\mathbb{Z}}_X\right)$$

is an isomorphism, from which the assertion follows.

Mini-Exercise II.3.94

We first recall the elementary ring-theoretic fact that local homomorphisms are closed under composition. Let

$$(A, \mathfrak{m}_A), \qquad (B, \mathfrak{m}_B), \qquad (C, \mathfrak{m}_C)$$

be local rings, and let

$$\varphi \colon A \longrightarrow B, \qquad \psi \colon B \longrightarrow C$$

be local homomorphisms. By the definition of a local homomorphism,

$$\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A, \qquad \psi^{-1}(\mathfrak{m}_C) = \mathfrak{m}_B.$$

It follows that

$$(\psi \circ \varphi)^{-1}(\mathfrak{m}_C) = \varphi^{-1}\big(\psi^{-1}(\mathfrak{m}_C)\big) = \varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A.$$

Hence $\psi \circ \varphi$ is again a local homomorphism. The identity homomorphism of a local ring is also local.

We now apply this observation to the objects and morphisms described in the definition of $(\mathsf{LocRingSp}_{\mathbb{C}})$. Let $(X, \mathcal{R}_X)$ be one of its objects. The identity morphism

$$\big(\mathrm{id}_X, \mathrm{id}_{\mathcal{R}_X}\big) \colon (X, \mathcal{R}_X) \longrightarrow (X, \mathcal{R}_X)$$

is a morphism of locally ringed spaces, and $\mathrm{id}_{\mathcal{R}_X}$ is a morphism of sheaves of $\mathbb{C}$-algebras. Thus every object has an identity morphism of the required kind.

Next, let

$$(f, f^\sharp)\colon (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$$

and

$$(g, g^\sharp)\colon (Y, \mathcal{R}_Y) \longrightarrow (Z, \mathcal{R}_Z)$$

be morphisms of the prescribed kind. Their composite as morphisms of ringed spaces has underlying continuous map

$$g \circ f\colon X \longrightarrow Z$$

and structure sheaf morphism

$$(g \circ f)^\sharp\colon \mathcal{R}_Z \xrightarrow{g^\sharp} g_*\mathcal{R}_Y \xrightarrow{g_* f^\sharp} g_* f_* \mathcal{R}_X = (g \circ f)_* \mathcal{R}_X.$$

Since both $f^\sharp$ and $g^\sharp$ are morphisms of sheaves of $\mathbb{C}$-algebras, so is $(g \circ f)^\sharp$.

For every $x \in X$, the induced homomorphism on stalks is

$$(g \circ f)^\sharp_x = f^\sharp_x \circ g^\sharp_{f(x)}\colon \mathcal{R}_{Z,g(f(x))} \longrightarrow \mathcal{R}_{X,x}.$$

Both

$$g^\sharp_{f(x)}\colon \mathcal{R}_{Z,g(f(x))} \longrightarrow \mathcal{R}_{Y,f(x)}$$

and

$$f^\sharp_x\colon \mathcal{R}_{Y,f(x)} \longrightarrow \mathcal{R}_{X,x}$$

are local homomorphisms. By the ring-theoretic observation above, their composite is local. Therefore

$$\big(g \circ f, (g \circ f)^\sharp\big)$$

is again a morphism of the prescribed kind.

Associativity and the identity laws are inherited from the category (LocRingSp), since the composition just described is its usual composition. Hence the objects and morphisms described above form a category.

Exercise II.1

(i) Define

$$f(z) := -\frac{1}{\pi} \int_{\mathbb{C}} \varphi(\zeta)\, \frac{1}{\zeta - z}\, d\lambda(\zeta)$$

and

$$A := -\frac{1}{\pi} \int_{\mathbb{C}} \varphi(\zeta)\, \frac{1}{\zeta}\, d\lambda(\zeta).$$

These integrals are well defined, since the functions

$$\zeta \longmapsto \frac{1}{|\zeta - z|} \qquad \text{and} \qquad \zeta \longmapsto \frac{1}{|\zeta|}$$

are locally integrable on $\mathbb{C}$ and φ has compact support.

By definition,

$$P\varphi = f - A.$$

Moreover,

$$f(0) = -\frac{1}{\pi}\int_{\mathbb{C}} \varphi(\zeta)\,\frac{1}{\zeta}\,d\lambda(\zeta) = A,$$

and hence

$$(P\varphi)(0) = 0.$$

We first verify the regularity of f. Making the change of variables $\xi := \zeta - z$, we obtain

$$f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \varphi(\xi + z)\,\frac{1}{\xi}\,d\lambda(\xi).$$

The function $\xi \mapsto 1/\xi$ is locally integrable. Moreover, when z varies in a fixed compact subset of $\mathbb{C}$, the functions

$$\xi \longmapsto \varphi(\xi + z)$$

and all their derivatives have supports contained in a common compact subset and are uniformly bounded there. Therefore differentiation under the integral sign is valid to every order. It follows that f, and hence $P\varphi$, is of class C^∞.

We now compute its $\overline{\partial}$-derivative. Since

$$d\lambda(\xi) = \frac{\sqrt{-1}}{2}\,d\xi \wedge d\overline{\xi}$$

and

$$\partial \log|\xi|^2 = \frac{d\xi}{\xi}$$

away from the origin, we have

$$\begin{aligned} f(z) &= -\frac{\sqrt{-1}}{2\pi}\int_{\mathbb{C}} \varphi(\xi + z)\,\frac{1}{\xi}\,d\xi \wedge d\overline{\xi} \\ &= \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{C}} \varphi(\xi + z)\,d\overline{\xi} \wedge \partial \log|\xi|^2. \end{aligned}$$

For a fixed $z \in \mathbb{C}$, set

$$\psi^z(\xi) := \varphi(\xi + z).$$

Since

$$\frac{\partial \psi^z}{\partial \overline{\xi}}(\xi) = \frac{\partial \varphi}{\partial \overline{z}}(\xi + z),$$

differentiation under the integral sign gives

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}}(z) &= \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{C}} \frac{\partial \varphi}{\partial \overline{\xi}}(\xi + z)\,d\overline{\xi} \wedge \partial \log|\xi|^2 \\ &= \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{C}} \overline{\partial}\psi^z \wedge \partial \log|\xi|^2. \end{aligned}$$

Since ψ^z has compact support, integration by parts in the sense of currents yields

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}}(z) &= -\frac{\sqrt{-1}}{2\pi}\int_{\mathbb{C}} \psi^z\, \overline{\partial}\partial \log|\xi|^2 \\ &= \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{C}} \psi^z\, \partial\overline{\partial} \log|\xi|^2.\end{aligned}$$

By the Lelong–Poincaré formula $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|\xi|^2 = \delta_0$, we conclude that

$$\frac{\partial f}{\partial \overline{z}}(z) = \psi^z(0) = \varphi(z).$$

Since A is constant, the assertion follows.

(*ii*) By Proposition II.3.58, it is enough to check exactness on the stalks.

The first morphism

$$\mathcal{O}_\Omega \longrightarrow \mathcal{A}^0_\Omega$$

is the natural inclusion and is injective. Moreover, the Cauchy–Riemann equation gives

$$\operatorname{Ker}\bigl(\overline{\partial}\colon \mathcal{A}^0_{\Omega,x} \longrightarrow \mathcal{A}^{0,1}_{\Omega,x}\bigr) = \mathcal{O}_{\Omega,x}$$

for every $x \in \Omega$. Thus the sequence is exact at $\mathcal{A}^0_\Omega$.

It remains to prove that

$$\overline{\partial}\colon \mathcal{A}^0_{\Omega,x} \longrightarrow \mathcal{A}^{0,1}_{\Omega,x}$$

is surjective for every $x \in \Omega$. Let a germ in $\mathcal{A}^{0,1}_{\Omega,x}$ be represented on an open neighborhood $W \subset \Omega$ of x by

$$g\, d\overline{z}, \qquad g \in C^\infty(W).$$

Choose a C^∞ function

$$\chi \in C^\infty_c(W)$$

which is identically equal to 1 on an open neighborhood W' of x. Define a function

$$\varphi\colon \mathbb{C} \longrightarrow \mathbb{C}$$

by

$$\varphi := \begin{cases} \chi g & \text{on } W, \\ 0 & \text{on } \mathbb{C}\setminus W. \end{cases}$$

Since χ has compact support in W, the function φ is of class C^∞ and has compact support. By (*i*), the function $u := P\varphi$ belongs to $C^\infty(\mathbb{C})$ and satisfies

$$\frac{\partial u}{\partial \overline{z}} = \varphi.$$

On W', one has $\varphi = g$, and therefore $\overline{\partial} u = g\, d\overline{z}$. Thus the germ of $g\, d\overline{z}$ at x belongs to the image of the stalk morphism induced by $\overline{\partial}$, which proves the required surjectivity.

(*iii*) The domain Ω is a (paracompact Hausdorff) C^∞ manifold. Moreover, $\mathcal{A}^0_\Omega = \mathcal{C}^\infty_\Omega$, and both $\mathcal{A}^0_\Omega$ and $\mathcal{A}^{0,1}_\Omega$ are sheaves of $\mathcal{C}^\infty_\Omega$-modules. Smooth partitions of unity therefore show, by Example II.3.64, that both sheaves are fine. Hence they are $\Gamma(\Omega, -)$-acyclic by Proposition II.3.63 (*iii*). It follows from (*ii*) that

$$0 \longrightarrow \mathcal{O}_\Omega \longrightarrow \mathcal{A}^0_\Omega \xrightarrow{\overline{\partial}} \mathcal{A}^{0,1}_\Omega \longrightarrow 0$$

is a resolution of $\mathcal{O}_\Omega$ by $\Gamma(\Omega, -)$-acyclic sheaves. Therefore the assertion directly follows from the acyclicity theorem (Proposition II.3.49).

Exercise II.2

Let

$$\gamma\colon [0,1] \longrightarrow \Omega, \qquad \gamma(t) := e^{2\pi\sqrt{-1}\,t}.$$

This loop represents the positively oriented generator used in the identification

$$H^1(\Omega, \underline{\mathbb{Z}}_\Omega) \cong \mathbb{Z}.$$

(*i*) Let

$$f \in H^0(\Omega, \mathcal{O}^*_\Omega).$$

Choose a subdivision

$$0 = t_0 < t_1 < \cdots < t_N = 1$$

and open subsets $U_\nu \subset \Omega$ such that

$$\gamma([t_{\nu-1}, t_\nu]) \subset U_\nu$$

and $f|_{U_\nu}$ admits a holomorphic logarithm with respect to the exponential map appearing in the exact sequence. Thus there exist

$$g_\nu \in \mathcal{O}_\Omega(U_\nu)$$

such that

$$\exp(2\pi\sqrt{-1}\,g_\nu) = f|_{U_\nu}.$$

The local existence of such logarithms is precisely the local surjectivity of

$$\mathcal{O}_\Omega \xrightarrow{\exp(2\pi\sqrt{-1}\,\bullet)} \mathcal{O}^*_\Omega.$$

On an overlap, two such local logarithms differ by a locally constant integer-valued function. By adding suitable integers to the g_ν, we may arrange that their restrictions along γ fit together to define a continuous function

$$G\colon [0,1] \longrightarrow \mathbb{C}$$

such that

$$\exp(2\pi\sqrt{-1}\,G(t)) = f(\gamma(t)).$$

Since $\gamma(1) = \gamma(0)$, one has

$$G(1) - G(0) \in \mathbb{Z}.$$

The differences of the local logarithms form the Čech cocycle representing

$$\delta_{\exp}(f) \in H^1(\Omega, \mathbb{Z}_\Omega).$$

At a crossing from U_ν to $U_{\nu+1}$ the cocycle has value $g_{\nu+1} - g_\nu$. To continue a single logarithm, its integer correction changes by the opposite amount. Including the final crossing back to the initial chart, the sum of the cocycle values is therefore $G(0) - G(1)$. Consequently, $\delta_{\exp}(f)$ evaluates to $G(0) - G(1)$ on the positively oriented generator.

Consider the map

$$h\colon \mathrm{U}(1) \longrightarrow \mathrm{U}(1), \qquad h(z) := \frac{f(z)}{|f(z)|}.$$

Since the winding number in Chapter I §7.1 was defined for loops based at 1, consider the based loop

$$\widehat{h}(z) := \frac{|f(1)|}{f(1)}\, h(z).$$

Multiplication by this constant does not change the winding number. Moreover,

$$\widetilde{h}(t) := \operatorname{Re} G(t) - \operatorname{Re} G(0)$$

is the lift of $\widehat{h} \circ \gamma$ to $\mathbb{R}$ under

$$\mathbb{R} \longrightarrow \mathrm{U}(1), \qquad s \longmapsto e^{2\pi\sqrt{-1}\,s}.$$

Indeed,

$$e^{2\pi\sqrt{-1}\,\widetilde{h}(t)} = \frac{|f(1)|}{f(1)}\,\frac{f(\gamma(t))}{|f(\gamma(t))|}.$$

Since $G(1) - G(0)$ is an integer, it is real, and hence

$$\widetilde{h}(1) = \operatorname{Re}\big(G(1) - G(0)\big) = G(1) - G(0).$$

By the definition of the winding number in Chapter I §7.1,

$$\operatorname{wind}(h) = G(1) - G(0),$$

and hence $\delta_{\exp}(f) = -\operatorname{wind}(h)$ under the chosen identification. This proves (i).

Alternative proof. Choose an open covering

$$\Omega = \bigcup_{i \in I} U_i$$

such that each U_i is a contractible open subset of $\mathbb{C}^*$, and holomorphic functions

$$g_i \in \mathcal{O}_\Omega(U_i)$$

such that

$$\exp(2\pi\sqrt{-1}\, g_i) = f|_{U_i}.$$

On every overlap, set

$$n_{ij} := g_j - g_i.$$

Then

$$n_{ij} \in \underline{\mathbb{Z}}_\Omega(U_i \cap U_j),$$

and the Čech cocycle (n_{ij}) represents $\delta_{\exp}(f)$.

Consider the morphism induced by the inclusion of coefficients

$$H^1(\Omega, \underline{\mathbb{Z}}_\Omega) \longrightarrow H^1(\Omega, \underline{\mathbb{C}}_\Omega) \cong H^1_{\mathrm{DR}}(\Omega).$$

Under the degree-one Čech–de Rham correspondence of Example II.3.84, the image of the cocycle (n_{ij}) is represented by the global closed 1-form obtained by gluing the local forms $-dg_i$. Differentiating

$$\exp(2\pi\sqrt{-1}\, g_i) = f$$

gives

$$dg_i = \frac{1}{2\pi\sqrt{-1}} \frac{df}{f}.$$

Hence the image of $\delta_{\exp}(f)$ in $H^1_{\mathrm{DR}}(\Omega)$ is

$$-\left[\frac{1}{2\pi\sqrt{-1}} \frac{df}{f}\right].$$

The integer corresponding to $\delta_{\exp}(f)$ is therefore the period of this class along the positively oriented generator:

$$-\frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{df}{f}.$$

As reviewed in Chapter I § I.9, this argument integral without the preceding minus sign is the winding number of the loop

$$f \circ \gamma \colon [0,1] \longrightarrow \mathbb{C}^*$$

around the origin. The radial deformation retraction

$$\mathbb{C}^* \longrightarrow \mathrm{U}(1), \qquad w \longmapsto \frac{w}{|w|}$$

does not change the winding number, from which the assertion follows.

(*ii*) Apply part (*i*) to the nowhere vanishing holomorphic function

$$f(z) := z.$$

Since

$$\frac{f(z)}{|f(z)|} = z \qquad \text{for } z \in \mathrm{U}(1),$$

the corresponding map

$$\mathrm{U}(1) \longrightarrow \mathrm{U}(1)$$

is the identity map and has winding number 1. Hence

$$\delta_{\exp}(z) = -1$$

under the identification

$$H^1(\Omega, \underline{\mathbb{Z}}_\Omega) \cong \mathbb{Z}.$$

The exponential exact sequence induces an exact sequence

$$H^0(\Omega, \mathcal{O}_\Omega) \xrightarrow{\exp(2\pi\sqrt{-1}\bullet)} H^0(\Omega, \mathcal{O}_\Omega^*) \xrightarrow{\delta_{\exp}} H^1(\Omega, \underline{\mathbb{Z}}_\Omega).$$

Since

$$\delta_{\exp}(z) \neq 0,$$

exactness shows that

$$z \notin \mathrm{Image}\left(H^0(\Omega, \mathcal{O}_\Omega) \xrightarrow{\exp(2\pi\sqrt{-1}\bullet)} H^0(\Omega, \mathcal{O}_\Omega^*) \right).$$

Exercise II.3

(i) Let

$$[c] \in \check{H}^1(\mathcal{U}, \mathcal{F})$$

and suppose that

$$\rho^*[c] = 0$$

in $\check{H}^1(\mathcal{V}, \mathcal{F})$. Choose a representative

$$c = \{(U_{jk}, c_{jk})\} \in \check{Z}^1(\mathcal{U}, \mathcal{F}).$$

By the assumption, there exists

$$b = \{(V_a, b_a)\} \in \check{C}^0(\mathcal{V}, \mathcal{F})$$

such that

$$\rho^* c = \check{\delta}^0(b).$$

Thus, on $V_a \cap V_b$, one has

$$c_{\rho(a)\rho(b)} = b_b - b_a.$$

For every $j \in J$ and every $a \in A$ such that $U_j \cap V_a \neq \emptyset$, define

$$s_{j,a} := b_a|_{U_j \cap V_a} - c_{j,\rho(a)}|_{U_j \cap V_a} \in \mathcal{F}(U_j \cap V_a).$$

This is well-defined. Indeed,

$$b_a \in \mathcal{F}(V_a), \qquad c_{j,\rho(a)} \in \mathcal{F}(U_j \cap U_{\rho(a)}),$$

and, since ρ is a refinement map, one has $V_a \subset U_{\rho(a)}$. Hence

$$U_j \cap V_a \subset U_j \cap U_{\rho(a)},$$

so both sections restrict to $U_j \cap V_a$.

For each fixed $j \in J$, the nonempty open subsets

$$\big\{U_j \cap V_a \;\big|\; a \in A,\ U_j \cap V_a \neq \emptyset\big\}$$

form an open covering of U_j. On each $U_j \cap V_a \cap V_b$, one has

$$\begin{aligned} s_{j,b} - s_{j,a} &= b_b - b_a - c_{j,\rho(b)} + c_{j,\rho(a)} \\ &= c_{\rho(a),\rho(b)} - c_{j,\rho(b)} + c_{j,\rho(a)} = 0, \end{aligned}$$

where the last equality follows from the cocycle identity $c_{j,\rho(a)} + c_{\rho(a),\rho(b)} + c_{\rho(b),j} = 0$. Since

$$U_j = \bigcup_{a \in A} (U_j \cap V_a),$$

the sheaf condition gives a unique section

$$s_j \in \mathcal{F}(U_j)$$

such that

$$s_j|_{U_j \cap V_a} = s_{j,a}$$

for every $a \in A$.

We claim that

$$c_{jk} = s_k|_{U_j \cap U_k} - s_j|_{U_j \cap U_k}$$

for every $j, k \in J$. Indeed, on each $U_j \cap U_k \cap V_a$, one has

$$\begin{aligned} s_k - s_j &= s_{k,a} - s_{j,a} \\ &= c_{j,\rho(a)} - c_{k,\rho(a)} = c_{jk} \end{aligned}$$

(The last equality follows from $c_{jk} + c_{k,\rho(a)} + c_{\rho(a),j} = 0$). Since the open subsets $\{U_j \cap U_k \cap V_a\}_{a \in A}$ cover $U_j \cap U_k$, the claim follows from the uniqueness part of the sheaf condition.

Therefore

$$c = \check{\delta}^0(s) \qquad (s := \{(U_j, s_j)\} \in \check{C}^0(\mathcal{U}, \mathcal{F})),$$

from which the assertion follows.

(ii) Let $\iota_{\mathcal{U}} \colon \check{H}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^1(X, \mathcal{F})$ be the canonical morphism, and suppose that $\iota_{\mathcal{U}}(\xi) = 0$ for some $\xi \in \check{H}^1(\mathcal{U}, \mathcal{F})$. By the description of the inductive limit over the filtered category $(\mathsf{OpCov}(X))$, there exists an open covering $\mathcal{W}$ refining $\mathcal{U}$ such that the image of ξ in $\check{H}^1(\mathcal{W}, \mathcal{F})$

is zero. The refinement morphism $\check{H}^1(\mathcal{U},\mathcal{F}) \to \check{H}^1(\mathcal{W},\mathcal{F})$ is injective by (i). It follows that $\xi = 0$. Therefore $\iota_{\mathcal{U}}$ is injective.

(iii) Let $X := \mathrm{U}(1)$ and $\mathcal{F} := \underline{\mathbb{Z}}_X$. Consider the open covering

$$\mathcal{U} := \{X\}$$

and its refinement

$$\mathcal{V} := \{V_0, V_1, V_2\},$$

where

$$\begin{aligned} V_0 &:= \{e^{2\pi\sqrt{-1}\theta} \mid -0.1 < \theta < 0.26\},\\ V_1 &:= \{e^{2\pi\sqrt{-1}\theta} \mid 0.24 < \theta < 0.51\},\\ V_2 &:= \{e^{2\pi\sqrt{-1}\theta} \mid 0.49 < \theta < 1.1\}. \end{aligned}$$

The unique map from the index set of $\mathcal{V}$ to that of $\mathcal{U}$ is a refinement map.

For the covering $\mathcal{U}$, the group of Čech 1-cocycles is zero. Indeed, if $\{(X\cap X, c_{00})\}$ $(c_{00} \in \mathbb{Z})$ is a 1-cocycle, then its cocycle condition on the ordered triple $(0,0,0)$ gives

$$c_{00} - c_{00} + c_{00} = c_{00} = 0.$$

Consequently, $\check{H}^1(\mathcal{U}, \underline{\mathbb{Z}}_X) = 0$.

On the other hand, the three pairwise intersections are

$$\begin{aligned} V_0 \cap V_1 &= \left\{ e^{2\pi\sqrt{-1}\theta} \;\middle|\; 0.24 < \theta < 0.26 \right\},\\ V_1 \cap V_2 &= \left\{ e^{2\pi\sqrt{-1}\theta} \;\middle|\; 0.49 < \theta < 0.51 \right\},\\ V_2 \cap V_0 &= \left\{ e^{2\pi\sqrt{-1}\theta} \;\middle|\; -0.1 < \theta < 0.1 \right\}. \end{aligned}$$

They are all nonempty and connected, while $V_0\cap V_1\cap V_2 = \emptyset$. Since each V_j is also connected, one has

$$\underline{\mathbb{Z}}_X(V_j) \cong \mathbb{Z}$$

and

$$\underline{\mathbb{Z}}_X(V_j \cap V_k) \cong \mathbb{Z}$$

whenever $j \neq k$.

Let

$$c = \{(V_{jk}, c_{jk})\} \in \check{Z}^1\big(\mathcal{V}, \underline{\mathbb{Z}}_X\big).$$

The cocycle condition on the ordered triples (j,j,k) and (j,k,j) gives

$$c_{jj} = 0 \qquad \text{and} \qquad c_{kj} = -c_{jk},$$

respectively. Since there is no nonempty intersection involving three distinct members of $\mathcal{V}$, there are no further conditions. Thus a 1-cocycle is uniquely determined by

$$(c_{01}, c_{12}, c_{20}) \in \mathbb{Z}^3,$$

and hence

$$\check{Z}^1\big(\mathcal{V}, \underline{\mathbb{Z}}_X\big) \cong \mathbb{Z}^3.$$

Under the identifications

$$\check{C}^0\big(\mathcal{V}, \underline{\mathbb{Z}}_X\big) \cong \mathbb{Z}^3, \qquad \{(V_j, a_j)\}_{j=0}^2 \mapsto (a_0, a_1, a_2)$$

and

$$\check{Z}^1\big(\mathcal{V}, \underline{\mathbb{Z}}_X\big) \cong \mathbb{Z}^3, \qquad \{(V_{jk}, c_{jk})\} \mapsto (c_{01}, c_{12}, c_{20}),$$

the Čech coboundary morphism is represented by

$$\begin{pmatrix} c_{01} \\ c_{12} \\ c_{20} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}.$$

The image of this homomorphism is

$$\big\{(b_{01}, b_{12}, b_{20}) \in \mathbb{Z}^3 \;\big|\; b_{01} + b_{12} + b_{20} = 0\big\}.$$

Therefore

$$\begin{aligned} \check{H}^1\big(\mathcal{V}, \underline{\mathbb{Z}}_X\big) &\cong \frac{\mathbb{Z}^3}{\{(b_{01}, b_{12}, b_{20}) \in \mathbb{Z}^3 \mid b_{01} + b_{12} + b_{20} = 0\}} \\ &\cong \mathbb{Z}, \end{aligned}$$

where the last isomorphism is induced by

$$\mathbb{Z}^3 \longrightarrow \mathbb{Z}, \qquad (b_{01}, b_{12}, b_{20}) \longmapsto b_{01} + b_{12} + b_{20}.$$

Consequently, the refinement morphism

$$\rho^* \colon \check{H}^1\big(\mathcal{U}, \underline{\mathbb{Z}}_X\big) \longrightarrow \check{H}^1\big(\mathcal{V}, \underline{\mathbb{Z}}_X\big)$$

is a homomorphism

$$0 \longrightarrow \mathbb{Z},$$

and hence is not surjective.

(*iv*) Let $\xi \in \check{H}^q(\mathcal{U}, \mathcal{F})$ and suppose that $\rho^*(\xi) = 0$. By the compatibility of the comparison morphisms with refinements established in Theorem II.3.79 (*i*), one has

$$\kappa^q_{\mathcal{U}} = \kappa^q_{\mathcal{V}} \circ \rho^*.$$

Consequently,

$$\kappa^q_{\mathcal{U}}(\xi) = \kappa^q_{\mathcal{V}}\big(\rho^*(\xi)\big) = 0.$$

Since

$$\kappa^q_{\mathcal{U}} \colon \check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F})$$

is injective by assumption, it follows that $\xi = 0$. Thus ρ^* is injective.

(*v*) Suppose that $\mathcal{U}$ is a Leray covering for $\mathcal{F}$. By Theorem II.3.79 (*ii*), the comparison morphism

$$\kappa^q_{\mathcal{U}} \colon \check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F})$$

is an isomorphism, and in particular is injective. The assertion therefore follows immediately from (*iv*).

3.3 Hints and solutions to the exercises in Chapter III

Mini-Exercise III.1.2

Let η be a compactly supported smooth form of bidegree $(n-q, n-p)$. By the definition of the complex conjugate of a current, we have

$$\overline{\alpha}(\eta) = \overline{\alpha(\overline{\eta})} = \overline{\int_\Omega \alpha \wedge \overline{\eta}} = \int_\Omega \overline{\alpha \wedge \overline{\eta}} = \int_\Omega \overline{\alpha} \wedge \eta,$$

from which the assertion follows, since the last expression is precisely the value at η of the (q,p)-current induced by the smooth form $\overline{\alpha}$.

Mini-Exercise III.1.4

By the definitions of the Wirtinger derivatives, we have

$$\frac{\partial}{\partial x^j} = \frac{\partial}{\partial z^j} + \frac{\partial}{\partial \overline{z^j}}, \qquad \frac{\partial}{\partial y^j} = \sqrt{-1}\frac{\partial}{\partial z^j} - \sqrt{-1}\frac{\partial}{\partial \overline{z^j}}.$$

Since $T^{1,0}\Omega$ and $T^{0,1}\Omega$ are orthogonal for $h^{\mathbb{C}}$ (by definition of $h^{\mathbb{C}}$), we get

$$\begin{aligned} g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) &= h\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right) + \overline{h}\left(\frac{\partial}{\partial \overline{z^j}}, \frac{\partial}{\partial \overline{z^k}}\right) \\ &= h_{j\overline{k}} + \overline{h_{j\overline{k}}} = 2\,\mathrm{Re}\, h_{j\overline{k}}. \end{aligned}$$

Similarly,

$$\begin{aligned} g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\right) &= h\left(\frac{\partial}{\partial z^j}, \sqrt{-1}\frac{\partial}{\partial z^k}\right) + \overline{h}\left(\frac{\partial}{\partial \overline{z^j}}, -\sqrt{-1}\frac{\partial}{\partial \overline{z^k}}\right) \\ &= -\sqrt{-1}\, h_{j\overline{k}} + \sqrt{-1}\,\overline{h_{j\overline{k}}} \\ &= 2\,\mathrm{Im}\, h_{j\overline{k}}. \end{aligned}$$

Finally,

$$\begin{aligned} g\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) &= h\left(\sqrt{-1}\frac{\partial}{\partial z^j}, \sqrt{-1}\frac{\partial}{\partial z^k}\right) + \overline{h}\left(-\sqrt{-1}\frac{\partial}{\partial \overline{z^j}}, -\sqrt{-1}\frac{\partial}{\partial \overline{z^k}}\right) \\ &= h_{j\overline{k}} + \overline{h_{j\overline{k}}} = 2\,\mathrm{Re}\, h_{j\overline{k}}. \end{aligned}$$

This proves the three displayed formulae.

It remains to show that $g(u,v) = \omega(u, Jv)$ for $u, v \in T\Omega$. Since both sides are real bilinear forms on $T\Omega$, it is enough to check this equality on the frame

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}.$$

First, using

$$\frac{\partial}{\partial x^j} = \frac{\partial}{\partial z^j} + \frac{\partial}{\partial \overline{z^j}}, \qquad \frac{\partial}{\partial y^k} = \sqrt{-1}\frac{\partial}{\partial z^k} - \sqrt{-1}\frac{\partial}{\partial \overline{z^k}},$$

we obtain

$$\begin{aligned}
\omega\left(\frac{\partial}{\partial x^j}, J\frac{\partial}{\partial x^k}\right) &= \omega\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\right) \\
&= -\sqrt{-1}\,\omega\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z^k}}\right) + \sqrt{-1}\,\omega\left(\frac{\partial}{\partial \overline{z^j}}, \frac{\partial}{\partial z^k}\right) \\
&= -\sqrt{-1}\cdot\sqrt{-1}h_{j\bar{k}} + \sqrt{-1}\cdot\left(-\sqrt{-1}h_{k\bar{j}}\right) \\
&= h_{j\bar{k}} + h_{k\bar{j}} = 2\mathrm{Re}\, h_{j\bar{k}}.
\end{aligned}$$

Next,

$$\begin{aligned}
\omega\left(\frac{\partial}{\partial x^j}, J\frac{\partial}{\partial y^k}\right) &= -\omega\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) \\
&= -\omega\left(\frac{\partial}{\partial z^j} + \frac{\partial}{\partial \overline{z^j}}, \frac{\partial}{\partial z^k} + \frac{\partial}{\partial \overline{z^k}}\right) \\
&= -\omega\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z^k}}\right) - \omega\left(\frac{\partial}{\partial \overline{z^j}}, \frac{\partial}{\partial z^k}\right) \\
&= -\sqrt{-1}h_{j\bar{k}} + \sqrt{-1}h_{k\bar{j}} = 2\mathrm{Im}\, h_{j\bar{k}}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\omega\left(\frac{\partial}{\partial y^j}, J\frac{\partial}{\partial y^k}\right) &= -\omega\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^k}\right) \\
&= -\omega\left(\sqrt{-1}\frac{\partial}{\partial z^j} - \sqrt{-1}\frac{\partial}{\partial \overline{z^j}}, \frac{\partial}{\partial z^k} + \frac{\partial}{\partial \overline{z^k}}\right) \\
&= -\sqrt{-1}\,\omega\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z^k}}\right) + \sqrt{-1}\,\omega\left(\frac{\partial}{\partial \overline{z^j}}, \frac{\partial}{\partial z^k}\right) \\
&= -\sqrt{-1}\cdot\sqrt{-1}h_{j\bar{k}} + \sqrt{-1}\cdot\left(-\sqrt{-1}h_{k\bar{j}}\right) \\
&= h_{j\bar{k}} + h_{k\bar{j}} = 2\mathrm{Re}\, h_{j\bar{k}}.
\end{aligned}$$

Comparing these formulae with the already proved identities

$$g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = 2\,\mathrm{Re}\, h_{j\bar{k}}, \quad g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^k}\right) = 2\,\mathrm{Im}\, h_{j\bar{k}}, \quad g\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = 2\,\mathrm{Re}\, h_{j\bar{k}},$$

we obtain $g(u,v) = \omega(u, Jv)$ for all $u, v \in T\Omega$.

Mini-Exercise III.1.10

We first prove the assertion for smooth forms. Let $\alpha \in \mathcal{A}^{p,q}(\Omega)$, and write

$$\alpha = \sum_{\#J=p,\ \#K=q} f_{J,\overline{K}}\, dz^J \wedge d\overline{z^K}.$$

Then

$$\partial\alpha = \sum_{\#J=p,\ \#K=q} \sum_{\nu=1}^{n} \frac{\partial f_{J,\overline{K}}}{\partial z^\nu}\, dz^\nu \wedge dz^J \wedge d\overline{z^K}.$$

Applying ∂ once more, we obtain

$$\partial\partial\alpha = \sum_{\#J=p,\ \#K=q} \sum_{\nu,\mu=1}^{n} \frac{\partial^2 f_{J,\overline{K}}}{\partial z^\mu \partial z^\nu}\, dz^\mu \wedge dz^\nu \wedge dz^J \wedge d\overline{z^K}.$$

Since $f_{J,\overline{K}}$ is C^∞, we have

$$\frac{\partial^2 f_{J,\overline{K}}}{\partial z^\mu \partial z^\nu} = \frac{\partial^2 f_{J,\overline{K}}}{\partial z^\nu \partial z^\mu}.$$

On the other hand,

$$dz^\mu \wedge dz^\nu = -dz^\nu \wedge dz^\mu.$$

Therefore the terms with (μ,ν) and (ν,μ) cancel, and the terms with $\mu = \nu$ vanish because $dz^\mu \wedge dz^\mu = 0$. Hence

$$\partial\partial\alpha = 0.$$

The assertion for $\overline{\partial\partial}$ follows by complex conjugation. Indeed, complex conjugation commutes with d and exchanges the components of type $(1,0)$ and $(0,1)$; hence

$$\overline{\partial}\beta = \overline{\partial\,\overline{\beta}}$$

for every smooth form β. Therefore

$$\overline{\partial\partial}\alpha = \overline{\partial\,\overline{\partial}\alpha} = \overline{\partial\partial\,\overline{\alpha}} = 0.$$

Equivalently, one may prove the same identity directly by repeating the preceding computation with the variables $\overline{z^1}, \ldots, \overline{z^n}$ instead of $z^1, \ldots, z^n$.

We next prove the assertion for currents. Let $T \in \mathcal{K}^{p,q}(\Omega)$ be a (p,q)-current. By Proposition I.6.49, the operators ∂ and $\overline{\partial}$ on currents are defined as the natural extensions of the corresponding operators on test forms. Thus, up to the signs appearing in the definition of these extensions, the value of $\partial\partial T$ on a test form φ is given by applying T to $\partial\partial\varphi$. More precisely, for every test form φ of the appropriate type, one has

$$(\partial\partial T)(\varphi) = \pm T(\partial\partial\varphi).$$

Since $\partial\partial\varphi = 0$ by the smooth case already proved, it follows that

$$(\partial\partial T)(\varphi) = 0.$$

As this holds for every test form φ, we obtain

$$\partial\partial T = 0.$$

The same argument gives

$$(\overline{\partial\partial}T)(\varphi) = \pm T(\overline{\partial\partial}\varphi) = 0$$

for every test form φ of the appropriate type. Hence

$$\overline{\partial\partial}T = 0.$$

This proves

$$\partial\partial \equiv 0, \qquad \overline{\partial\partial} \equiv 0$$

on both smooth forms and currents.

Mini-Exercise III.1.12

(i) Since ρ is a local defining function of M, after shrinking U and using ρ as the first real coordinate, we may choose C^2 real coordinates

$$(t, \xi^2, \ldots, \xi^{2n})$$

on U such that

$$t = \rho \qquad \text{and} \qquad M \cap U = \{t = 0\}.$$

Since $\widetilde{\rho}$ vanishes when $t = 0$, the fundamental theorem of calculus gives

$$\begin{aligned}\widetilde{\rho}(t, \xi^2, \ldots, \xi^{2n}) &= \widetilde{\rho}(t, \xi^2, \ldots, \xi^{2n}) - \widetilde{\rho}(0, \xi^2, \ldots, \xi^{2n}) \\ &= t \int_0^1 \frac{\partial\widetilde{\rho}}{\partial t}(st, \xi^2, \ldots, \xi^{2n})\, ds.\end{aligned}$$

Thus, setting

$$a(t, \xi^2, \ldots, \xi^{2n}) := \int_0^1 \frac{\partial\widetilde{\rho}}{\partial t}(st, \xi^2, \ldots, \xi^{2n})\, ds,$$

we obtain

$$\widetilde{\rho} = a\rho.$$

Since $\widetilde{\rho}$ is of class C^2, the function a is of class C^1. Moreover, on $M \cap U$,

$$a = \frac{\partial\widetilde{\rho}}{\partial t},$$

and this function does not vanish there because $d\widetilde{\rho} \neq 0$ on $M \cap U$. After shrinking U, we may therefore assume that a is nowhere vanishing.

If M, ρ, and $\widetilde{\rho}$ are of class C^∞, the same integral expression shows that a is of class C^∞.

(ii) Since $\widetilde{\rho} = a\rho$, we have $d\widetilde{\rho} = a\, d\rho + \rho\, da$. Hence, on $M \cap U$,

$$d\widetilde{\rho} = a\, d\rho.$$

Taking the component of type $(1,0)$ gives

$$\partial\widetilde{\rho} = a\,\partial\rho \quad \text{on } M \cap U.$$

Since a is nowhere zero, the kernels of $(\partial\widetilde{\rho})_q$ and $(\partial\rho)_q$ in $T^{1,0}_q\Omega$ agree for every $q \in M \cap U$. Therefore $T^{1,0}_q M$ is independent of the choice of a local defining function.

(iii) Fix $q \in M \cap U$, and use the standard complex coordinates $z = (z^1, z^2, \ldots, z^n)$ of $\mathbb{C}^n$. Since $\widetilde{\rho} = a\rho$, we have

$$\frac{\partial\widetilde{\rho}}{\partial\overline{z^k}} = a\frac{\partial\rho}{\partial\overline{z^k}} + \rho\frac{\partial a}{\partial\overline{z^k}} \qquad (k \in \{1, 2, \ldots, n\}).$$

Taking the derivative in the z^j-direction at q, and using $\rho(q) = 0$, we obtain

$$\frac{\partial^2\widetilde{\rho}}{\partial z^j\partial\overline{z^k}}(q) = a(q)\frac{\partial^2\rho}{\partial z^j\partial\overline{z^k}}(q) + \frac{\partial a}{\partial z^j}(q)\frac{\partial\rho}{\partial\overline{z^k}}(q) + \frac{\partial\rho}{\partial z^j}(q)\frac{\partial a}{\partial\overline{z^k}}(q). \tag{393}$$

Note that, although a is only assumed to be of class C^1, the last calculation is valid at q. Indeed, since $\rho(q) = 0$ and $\partial a/\partial\overline{z^k}$ is continuous, the derivative at q in the z^j-direction of the function

$$\rho\frac{\partial a}{\partial\overline{z^k}}$$

is

$$\frac{\partial\rho}{\partial z^j}(q)\frac{\partial a}{\partial\overline{z^k}}(q).$$

Thus no second derivative of a is required.

Let

$$w = \sum_{j=1}^{n} c^j\frac{\partial}{\partial z^j}, \qquad w' = \sum_{k=1}^{n} d^k\frac{\partial}{\partial z^k}$$

be elements of $T^{1,0}_q M$. By the definition of $T^{1,0}_q M$, we have

$$\sum_{j=1}^{n}\frac{\partial\rho}{\partial z^j}(q)c^j = 0$$

and

$$\sum_{k=1}^{n}\frac{\partial\rho}{\partial z^k}(q)d^k = 0.$$

Since ρ is real-valued, the latter equality also gives

$$\sum_{k=1}^{n}\frac{\partial\rho}{\partial\overline{z^k}}(q)\overline{d^k} = 0.$$

Therefore, by (393), we obtain

$$\begin{aligned} H_{M,q}(\widetilde{\rho})(w,w') &= \sum_{j,k=1}^{n} \frac{\partial^2 \widetilde{\rho}}{\partial z^j \partial \overline{z^k}}(q) c^j \overline{d^k} \\ &= a(q) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z^j \partial \overline{z^k}}(q) c^j \overline{d^k} + \left(\sum_{j=1}^{n} \frac{\partial a}{\partial z^j}(q) c^j \right) \left(\sum_{k=1}^{n} \frac{\partial \rho}{\partial \overline{z^k}}(q) \overline{d^k} \right) \\ &\quad + \left(\sum_{j=1}^{n} \frac{\partial \rho}{\partial z^j}(q) c^j \right) \left(\sum_{k=1}^{n} \frac{\partial a}{\partial \overline{z^k}}(q) \overline{d^k} \right) \\ &= a(q) H_{M,q}(\rho)(w,w'). \end{aligned}$$

Hence we have $H_{M,q}(\widetilde{\rho}) = a(q) H_{M,q}(\rho)$.

(iv) The two functions ρ and $\widetilde{\rho}$ have the same sign on $D \cap U$ and also the same sign on the other side of M. Thus

$$a > 0 \quad \text{on } U \setminus M.$$

Since a is continuous and nowhere vanishing, it follows, after shrinking U if necessary, that

$$a > 0 \quad \text{on } U.$$

Consequently, $H_{M,q}(\widetilde{\rho}) = a(q) H_{M,q}(\rho)$ is obtained from $H_{M,q}(\rho)$ by multiplication by a positive real number. Hence the numbers of positive, negative, and zero eigenvalues are unchanged.

Mini-Exercise III.1.16

(i) Let v be a real tangent vector, and write its decomposition in $T^{\mathbb{C}}\Omega$ as

$$v = v^{1,0} + v^{0,1}.$$

Since

$$J(v^{1,0}) = \sqrt{-1}\, v^{1,0}, \qquad J(v^{0,1}) = -\sqrt{-1}\, v^{0,1},$$

we obtain

$$\begin{aligned} \langle df, J(v) \rangle &= \langle df,\ \sqrt{-1}\, v^{1,0} - \sqrt{-1}\, v^{0,1} \rangle \\ &= \sqrt{-1}\, \langle \partial f, v^{1,0} \rangle - \sqrt{-1}\, \langle \overline{\partial} f, v^{0,1} \rangle \\ &= \sqrt{-1}\, \langle \partial f - \overline{\partial} f, v \rangle. \end{aligned}$$

Therefore,

$$\langle \partial f - \overline{\partial} f, v \rangle = -\sqrt{-1}\, \langle df, J(v) \rangle.$$

By the definition of d^c,

$$\langle d^c f, v \rangle = \frac{1}{4\pi\sqrt{-1}} \langle \partial f - \overline{\partial} f, v \rangle = -\frac{1}{4\pi} \langle df, J(v) \rangle.$$

(ii) By the formula for the exterior derivative of a 1-form,

$$(dd^c\rho)(u,v) = u\big(\langle d^c\rho, v\rangle\big) - v\big(\langle d^c\rho, u\rangle\big) - \langle d^c\rho, [u,v]\rangle.$$

Using (i), we obtain

$$(dd^c\rho)(u,v) = -\frac{1}{4\pi}u\big(\langle d\rho, J(v)\rangle\big) + \frac{1}{4\pi}v\big(\langle d\rho, J(u)\rangle\big) + \frac{1}{4\pi}\langle d\rho, J([u,v])\rangle.$$

Multiplying by 4π gives the assertion.

(iii) As $w = u - \sqrt{-1}\,J(u)$, one has $\overline{w} = u + \sqrt{-1}\,J(u)$. As the real $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}\rho$ corresponds to the Hermitian form defined by the complex Hessian of ρ, one has

$$H_{M,p}(\rho)(w,w) = -\sqrt{-1}\left(\sqrt{-1}\partial\overline{\partial}\rho\right)_p(w,\overline{w}).$$

As $\sqrt{-1}\partial\overline{\partial}\rho = 2\pi dd^c\rho$, since $dd^c\rho$ is a real 2-form,

$$\begin{aligned}(dd^c\rho)_p(w,\overline{w}) &= (dd^c\rho)_p\left(u - \sqrt{-1}\,J(u),\ u + \sqrt{-1}\,J(u)\right)\\ &= 2\sqrt{-1}\,(dd^c\rho)_p(u, J(u)).\end{aligned}$$

Therefore,

$$H_{M,p}(\rho)(w,w) = -2\pi\sqrt{-1}\,(dd^c\rho)_p(w,\overline{w}) = 4\pi(dd^c\rho)_p(u, J(u)).$$

It follows that $H_{M,p}(\rho)(w,w) = 0$ for every $w \in T^{1,0}_pM$ if and only if $(dd^c\rho)_p(u, J(u)) = 0$ for every $u \in H_pM$.

Set

$$\omega := (dd^c\rho)_p|_{H_pM\times H_pM}.$$

Since ω is of type $(1,1)$, the real bilinear form

$$B(u,v) := \omega(u, J(v)) \qquad (u,v \in H_pM)$$

is symmetric. Its associated quadratic form is

$$Q(u) := B(u,u) = \omega(u, J(u)).$$

By the polarization identity,

$$B(u,v) = \frac{1}{2}\left(Q(u+v) - Q(u) - Q(v)\right).$$

Therefore, if $\omega(u, J(u)) = 0$ for every $u \in H_pM$, then $B \equiv 0$. Since $\omega(u,v) = B(u, -J(v))$, it follows that $\omega \equiv 0$. Consequently,

$$H_{M,p}(\rho) \equiv 0 \quad \Longleftrightarrow \quad (dd^c\rho)_p|_{H_pM\times H_pM} \equiv 0.$$

(iv) Let u and v be smooth sections of HM. After extending them locally to smooth real vector fields in a neighborhood of M, we may apply (107). Since

$$u,\ v,\ J(u),\ J(v) \in TM,$$

we have, on M,

$$\langle d\rho, u\rangle = \langle d\rho, v\rangle = \langle d\rho, J(u)\rangle = \langle d\rho, J(v)\rangle = 0.$$

Therefore,

$$4\pi(dd^c\rho)(u,v) = \langle d\rho, J([u,v])\rangle \quad \text{on } M.$$

Moreover, since u and v are tangent to M, their Lie bracket is also tangent to M. Equivalently,

$$\langle d\rho, [u,v]\rangle = [u,v](\rho) = u\big(v(\rho)\big) - v\big(u(\rho)\big) = 0 \quad \text{on } M.$$

Suppose first that M is Levi-flat. By (iii),

$$(dd^c\rho)|_{HM\times HM} \equiv 0.$$

Hence

$$\langle d\rho, J([u,v])\rangle = 0.$$

Together with $\langle d\rho, [u,v]\rangle = 0$, this shows that $[u,v] \in HM$. Thus HM is involutive.

Conversely, suppose that HM is involutive. Then $[u,v] \in HM$, and hence both $[u,v]$ and $J([u,v])$ are tangent to M. It follows that $\langle d\rho, J([u,v])\rangle = 0$. Therefore,

$$(dd^c\rho)(u,v) = 0$$

for all smooth sections u, v of HM. By (iii), the Levi form of M vanishes identically. Thus M is Levi-flat.

Mini-Exercise III.1.21

When $k = 0$, we have $a_0(z) = f(z,0)$, which is holomorphic by the condition (i) of Theorem III.1.18.

Let N be a positive integer, and assume that $a_0, a_1, \ldots, a_{N-1}$ are holomorphic on Δ_r. For every $w \in \Delta_\ell \setminus \{0\}$, define

$$F_{N,w}(z) := \frac{f(z,w) - \sum_{k=0}^{N-1} a_k(z)w^k}{w^N}.$$

By the induction hypothesis and the holomorphicity of f, the function $F_{N,w}\colon \Delta_r \to \mathbb{C}$ is holomorphic.

For every $z \in \Delta_r$, we have

$$|F_{N,w}(z) - a_N(z)| = \frac{\left|f(z,w) - \sum_{k=0}^{N} a_k(z)w^k\right|}{|w^N|} = \left|\sum_{k=1}^{\infty} a_{N+k}(z)w^k\right|.$$

By the Cauchy estimate (109),

$$|a_{N+k}(z)| \le \frac{M}{\ell^{N+k}}.$$

Therefore,

$$|F_{N,w}(z) - a_N(z)| \le \sum_{k=1}^{\infty} \frac{M}{\ell^{N+k}}|w|^k = \frac{M}{\ell^N}\sum_{k=1}^{\infty}\Big|\frac{w}{\ell}\Big|^k = \frac{M}{\ell^N}\frac{|w|/\ell}{1-|w|/\ell}.$$

The right-hand side is independent of z and tends to zero as $w \to 0$. Hence $F_{N,w}$ converges uniformly to a_N on Δ_r as $w \to 0$.

Since a uniform limit of holomorphic functions is holomorphic, it follows that a_N is holomorphic on Δ_r, which completes the induction.

Mini-Exercise III.2.8

(i) In a sufficiently small adapted coordinate neighborhood U from (141), Taylor expansion gives $\mathcal{I}_Y|_U = \sum_{k=m+1}^{n}(\mathcal{O}_X|_U)z^k$. Outside Y, this ideal sheaf equals $\mathcal{O}_X$. Thus Proposition III.1.43, applied in charts, proves coherence.

(ii) In the same coordinates, every holomorphic germ on Y extends by being independent of the remaining coordinates, and the kernel of restriction at $y \in Y$ is $(\mathcal{I}_Y)_y$. These restriction maps induce the asserted isomorphism of sheaves.

Mini-Exercise III.2.10

Let $p \in \mathcal{L}$ and choose $z\colon U \to \Omega$ as in the exercise. Put $M_z := z(M \cap U)$. The differential of z takes HM onto HM_z, since it is complex-linear; this is the complex tangent invariance in Chapter III, §1.3. The map $\iota_z = z \circ \iota$ is defined on $\iota^{-1}(U)$. Since $\mathcal{F}$ is the Levi foliation of M, we have $d(\iota_z)_p(T_p\mathcal{L}) = H_{\iota_z(p)}M_z$. The latter space is invariant under J, so it is a complex subspace of $(T_{\iota_z(p)}\Omega, J)$ of dimension $n-1$.

We first prove (i). Put $a := \iota_z(p)$ and choose a real-valued C^∞ local defining function ρ of M_z near a. Since ρ is real-valued and $d\rho(a) \neq 0$, there is an index $j \in \{1, 2, \dots, n\}$ such that $(\partial\rho/\partial z^j)(a) \neq 0$. Take the coordinate projection which omits this component:

$$P\colon \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}, \qquad P(\zeta^1, \zeta^2, \dots, \zeta^n) := (\zeta^k)_{k\neq j},$$

where the remaining components are listed in their original order. By (100), a vector $\sum_{k=1}^{n} c^k \partial/\partial z^k$ belongs to $T^{1,0}_a M_z$ if and only if $\sum_{k=1}^{n}(\partial\rho/\partial z^k)(a)c^k = 0$. The coefficient of c^j is nonzero, so c^j is uniquely determined by the other components. Thus the restriction of dP_a to $T^{1,0}_a M_z$ is a complex-linear isomorphism onto $T^{1,0}_{P(a)}\mathbb{C}^{n-1}$. The isomorphisms of Remark III.1.1, including their restrictions to the complex tangent spaces of real hypersurfaces, commute with the differential of the complex-linear map P. Hence $P|_{H_aM_z}$ is a complex-linear isomorphism onto $\mathbb{C}^{n-1}$. Since $d(\iota_z)_p$ identifies $T_p\mathcal{L}$ with H_aM_z, the map $d(P\circ\iota_z)_p = P\circ d(\iota_z)_p$ is an isomorphism of real vector spaces.

We next prove (ii). By the real inverse function theorem, after shrinking an open neighborhood V of p in $\mathcal{L}$, the map $P \circ \iota_z|_V\colon V \longrightarrow W$ is a C^∞ diffeomorphism onto a domain $W \subset \mathbb{C}^{n-1}$. Define

$$\Phi := \iota_z \circ (P \circ \iota_z|_V)^{-1}\colon W \longrightarrow \Omega.$$

Then

$$P \circ \Phi = \mathrm{id}_W. \tag{394}$$

After shrinking V further if necessary, we may assume that $P|_{H_{\iota z(q)}M_z}$ is an isomorphism onto $\mathbb{C}^{n-1}$ for every $q \in V$: this holds at p and persists nearby because $H_{\iota z(q)}M_z$ depends continuously on q.

For $\zeta \in W$, the image of $d\Phi_\zeta$ is $H_{\Phi(\zeta)}M_z$. Differentiating (394) shows that $d\Phi_\zeta$ is the inverse of the complex-linear isomorphism $P|_{H_{\Phi(\zeta)}M_z}$. Hence $d\Phi_\zeta \circ J = J \circ d\Phi_\zeta$, so the components of Φ satisfy the Cauchy–Riemann equations on W. The characterization of holomorphic functions (Theorem III.1.18, $(iii) \Rightarrow (i)$) therefore shows that Φ is holomorphic.

Finally, we prove (iii), allowing the ambient chart to vary. Let $z_\nu \colon U_\nu \to \Omega_\nu$ and P_ν give leaf charts

$$\varphi_\nu = P_\nu \circ z_\nu \circ \iota \colon V_\nu \longrightarrow W_\nu \subset \mathbb{C}^{n-1} \qquad (\nu \in \{1,2\}).$$

On the open set $\varphi_1(V_1 \cap V_2)$, their transition map is

$$\varphi_2 \circ \varphi_1^{-1} = P_2 \circ (z_2 \circ z_1^{-1}) \circ \big(z_1 \circ \iota \circ \varphi_1^{-1}\big).$$

The last factor is holomorphic by (ii) and takes values in $z_1(U_1 \cap U_2)$ on this open set. The middle factor is a holomorphic transition map of X, and P_2 is complex-linear. Thus the transition map is holomorphic. Interchanging 1 and 2 shows that its inverse is holomorphic as well. This also proves compatibility for different choices of the projections. Hence all these charts define a complex atlas on the existing C^∞ leaf $\mathcal{L}$, of dimension $n-1$.

In a leaf chart $\varphi = P \circ z \circ \iota$, the ambient coordinate expression of the inclusion is $z \circ \iota \circ \varphi^{-1} = \Phi$, which is holomorphic by (ii). Its differential has real rank $2n-2$, and, being complex-linear, has complex rank $n-1$. Therefore $\iota \colon \mathcal{L} \to X$ is a holomorphic immersion. The atlas is independent of the choices just made. More precisely, any complex structure on the given smooth leaf for which ι is a holomorphic immersion makes $P \circ z \circ \iota$ holomorphic; its differential is invertible, so the holomorphic inverse function theorem makes it a chart for that structure as well. Thus the structure constructed above is natural in this sense.

Mini-Exercise III.2.17

(i) The essential check is the change of frame. In the trivializations of (142), the columns of scalar forms satisfy $\eta_j = g_{jk}\eta_k$. Since $\overline{\partial} g_{jk} = 0$, one has $\overline{\partial}\eta_j = g_{jk}\overline{\partial}\eta_k$, which is exactly the transition rule for an E-valued form. Coordinate independence follows from (125), and $\overline{\partial}_E^{\,2} = 0$ follows from the scalar identity in each frame; see also [De4, Chapter V, §11, pp. 267–268].

(ii) There is the extra term $(dg_{jk}) \wedge \eta_k$. For example, in the trivial line bundle on $\mathbb{C}$, let e be the constant frame and put $e' = e^z e$. The constant section e has coefficient 1 in the first frame and e^{-z} in the second. Applying d to these coefficients gives 0 in the first frame but $-dz \otimes e$ in the second. Thus this prescription for d is not intrinsic.

In the special case where a holomorphic flat structure on E has been fixed, choose local holomorphic frames with locally constant transition matrices in $\mathrm{GL}_r(\mathbb{C})$. In these flat frames, coefficientwise application of d does define an operator D, since $dg_{jk} = 0$. This is the flat connection of Chapter I, defined in (41), for the chosen flat structure.

Mini-Exercise III.2.21

The positive ratio $|e_j|^2_{h'}/|e_j|^2_h$ is independent of the local frame; its negative logarithm defines ψ. The weights satisfy $\varphi'_j = \varphi_j + \psi$, so (150) follows from (149). Its right-hand side is d-exact, proving the last assertion.

Mini-Exercise III.2.26

On the chart where a defining covector of H has nonzero first coordinate, write it as $(1, z^1, z^2, \dots, z^n)$. The class e of the first standard basis vector is a holomorphic frame of V/H. The vector of smallest norm on which this covector has value one has squared norm

$$|e|^2_h = \frac{1}{1 + \sum_{j=1}^n |z^j|^2}.$$

Thus the local weight is $\varphi = \log(1 + |z|^2)$, where $|z|^2 := \sum_{j=1}^n |z^j|^2$, and

$$\sqrt{-1}\Theta_h = \sqrt{-1}\partial\overline{\partial}\log(1 + |z|^2) = \sqrt{-1}\sum_{j,k=1}^n \frac{(1 + |z|^2)\delta_{jk} - \overline{z^j}z^k}{(1 + |z|^2)^2}\, dz^j \wedge d\overline{z^k}. \tag{395}$$

For $\xi \neq 0$, its Hermitian quadratic form is

$$\frac{(1 + |z|^2)|\xi|^2 - |\sum_{j=1}^n \overline{z^j}\xi^j|^2}{(1 + |z|^2)^2} \geq \frac{|\xi|^2}{(1 + |z|^2)^2} > 0.$$

The other charts give the same conclusion. With the normalization of this note, the Fubini–Study form is $\omega_{\mathrm{FS}} = \sqrt{-1}\Theta_h/(2\pi)$.

Mini-Exercise III.2.34

Fix $x \in S$, put $b = f(x)$, and let $R = \mathcal{O}_{S,x} = \mathcal{O}_{U,x}/\mathcal{I}_x$ with maximal ideal $\mathfrak{m} = \mathfrak{m}_{S,x}$. Consider the two local $\mathbb{C}$-algebra homomorphisms

$$\varphi, \psi \colon \mathcal{O}_{V,b} \longrightarrow R, \qquad \varphi(h) = f_x^\sharp([h]), \quad \psi(h) = [h \circ F].$$

By the choice of the lifts F^j, these homomorphisms agree on the coordinate germs, hence on every polynomial in them. For $n \geq 1$, let P_{n-1} be the Taylor polynomial of h at b containing all terms of total degree less than n. Then $h - P_{n-1} \in \mathfrak{m}^n_{V,b}$. Both homomorphisms send $\mathfrak{m}_{V,b}$ into $\mathfrak{m}$, so $\varphi(h) - \psi(h) \in \mathfrak{m}^n$ for every n. The ring R is Noetherian, being a quotient of a holomorphic local ring (Theorem III.1.32). The Krull intersection theorem (Theorem I.5.15), applied to the R-module R and the ideal $\mathfrak{m}$, gives $\bigcap_{n\geq 1} \mathfrak{m}^n = 0$. Consequently $\varphi(h) = \psi(h)$. This argument uses finite Taylor polynomials; it does not assume that an abstract ring homomorphism commutes with an infinite sum.

If $h \in \mathcal{J}_b$, its class is zero, so $h \circ F \in \mathcal{I}_x$. Apply this to local generators of $\mathcal{J}$ after shrinking the ambient neighborhoods. At points of S their compositions belong to the stalks of $\mathcal{I}$, and outside S one has $\mathcal{I}_x = \mathcal{O}_{U,x}$. Thus F pulls $\mathcal{J}$ into $\mathcal{I}$ as sheaves. The established identity on all germs identifies the induced morphism of structure sheaves with $f^\sharp$; together with $F|_S = f$, this proves the assertion.

Mini-Exercise III.2.63

The assertion is immediate if the germ of f is zero. Otherwise, boundedness of the difference gives a constant $C > 1$ with

$$C^{-1} \prod_{j=1}^{n} |z^j|^{-2\gamma_j} \le e^{-\varphi(z)} \le C \prod_{j=1}^{n} |z^j|^{-2\gamma_j}$$

off the coordinate hyperplanes in a small neighborhood. It suffices to test the product weight. Write $f(z) = \sum_k c_k z^k$ on a polydisc slightly larger than Δ_ε^n. In polar coordinates $z^j = r_j e^{\sqrt{-1}\theta_j}$, integration in the angular variables and then Tonelli's theorem give

$$\begin{aligned} &\int_{\Delta_\varepsilon^n} |f(z)|^2 \prod_{j=1}^{n} |z^j|^{-2\gamma_j} \, d\lambda(z) \\ &\qquad = (2\pi)^n \sum_{k:\, c_k \neq 0} |c_k|^2 \prod_{j=1}^{n} \left(\int_0^\varepsilon r_j^{2k_j - 2\gamma_j + 1} \, dr_j \right), \end{aligned}$$

with ∞ allowed on both sides. By Lemma I.4.2, a radial factor is finite exactly when $k_j > \gamma_j - 1$, or equivalently $k_j \ge \lfloor \gamma_j \rfloor$. Thus finiteness gives the required orders along the coordinate hyperplanes. Conversely, these orders give a factorization $f(z) = \prod_{j=1}^n (z^j)^{\lfloor \gamma_j \rfloor} g(z)$ with g holomorphic near zero. Boundedness of g on a smaller polydisc reduces the integral to the product of the radial integrals with exponents $2\lfloor \gamma_j \rfloor - 2\gamma_j + 1 > -1$. They are finite by the same lemma.

The bounded difference of the weights reduces the question to $\int_{B_\varepsilon} |f(z)|^2 |z|^{-2\alpha} \, d\lambda(z)$. If f vanishes identically near zero, the assertion is immediate. Otherwise put $k := \operatorname{ord}_0(f)$ and let P_k be its first nonzero homogeneous Taylor term. Uniformly on the unit sphere, $f(r\xi) = r^k (P_k(\xi) + O(r))$, so

$$\int_{|\xi|=1} |f(r\xi)|^2 \, d\sigma(\xi) = r^{2k}(c + O(r)), \qquad c := \int_{|\xi|=1} |P_k(\xi)|^2 \, d\sigma(\xi) > 0.$$

Polar integration therefore gives the condition $\int_0^\varepsilon r^{2k - 2\alpha + 2n - 1} \, dr < \infty$, equivalently $k > \alpha - n$. Since k is a nonnegative integer, this is exactly the stated power of the maximal ideal.

Mini-Exercise III.2.78

For (i), if $h = \operatorname{Re} F$, then $\partial h = dF/2$. Conversely, a smooth f satisfying $df = \partial h$ is holomorphic by comparison of types, and the proof of Proposition III.2.77 gives $h = \operatorname{Re}(2f + c)$ for a real constant c. The singular–de Rham correspondence (Theorem I.8.54) shows that $H^1(X, \mathbb{C}) = 0$ implies $H^1_{\mathrm{DR}}(X) = 0$, so every closed complex-valued 1-form is exact. For contractible X, one can also use Corollary I.8.43 directly.

For (ii), locally $\log |z|$ is the real part of a holomorphic branch of $\log z$. Also,

$$2\partial h = \frac{dz}{z}, \qquad \int_{|z|=1} 2\partial h = 2\pi\sqrt{-1} \neq 0.$$

An exact form has zero integral over a closed curve. Thus (i) excludes a global F.

Mini-Exercise III.2.79

Set $X = \mathbb{C}^*$. The long exact cohomology sequence of (170) gives an injection

$$\frac{\mathcal{H}_X(X)}{\mathcal{O}_X(X) + \overline{\mathcal{O}}_X(X)} \longrightarrow H^1(X, \underline{\mathbb{C}}_X) \cong \mathbb{C}.$$

Here the last isomorphism follows from the deformation retraction of X onto the unit circle and the singular–sheaf comparison in Example II.3.65. Thus the quotient has dimension at most one. The class of $\log|z|$ is not zero: if $\log|z| = f+g$ globally, then $\partial \log|z| = df$, whereas Mini-Exercise III.2.78 gives $\int_{|z|=1} \partial \log|z| = \pi\sqrt{-1} \neq 0$. Consequently this class spans the quotient, which gives the required decomposition. Finally, applying ∂ to $h = f + g + \lambda \log|z|$ and integrating over the positively oriented unit circle yields

$$\lambda = \frac{1}{\pi\sqrt{-1}} \int_{|z|=1} \partial h,$$

since $\partial f = df$ and $\partial g = 0$. This also proves the uniqueness of λ.

Mini-Exercise III.2.89

Locally lift the reductions of the f_j to holomorphic functions F_j in an ambient domain Ω. By Mini-Exercise I.9.5 on each complex line, $\log|F_j|^2$ is plurisubharmonic; an identically zero restriction gives the allowed value $-\infty$. For each integer $m \geq 1$, set $v_{j,m} := \max\{\log|F_j|^2, -m\}$. These are finite-valued plurisubharmonic functions. By Mini-Exercise I.4.10, the function

$$\chi(t^1, t^2, \ldots, t^r) := \log \sum_{j=1}^{r} e^{t^j}$$

is convex; it is also increasing in each variable. The convex-composition property in §2.7 therefore shows that $\chi(v_{1,m}, v_{2,m}, \ldots, v_{r,m})$ is plurisubharmonic. As $m \to \infty$, these functions decrease to $\log \sum_{j=1}^{r} |F_j|^2$, including the value $-\infty$ at the common zeros. The decreasing-limit property and restriction to X in the chosen embedding prove the assertion.

Mini-Exercise III.2.105

The metric compatibility of D_h and Stokes' theorem give, for u of total degree k and v of degree $k-1$, with one compactly supported,

$$\langle\!\langle u, D_h v \rangle\!\rangle_{h,\omega} = (-1)^{(2n-k+1)k} \langle\!\langle *D_h * u, v \rangle\!\rangle_{h,\omega}.$$

The exponent is even. Thus $D^\dagger_{h,\omega,\text{formal}} = *D_h*$, and comparison of types gives the first two identities, since $*$ sends type (p,q) to $(n-q, n-p)$. For the last identity, fix a point a and take the normal coordinates of (129). At a, $\omega = \sqrt{-1} \sum_{j=1}^{n} dz^j \wedge d\overline{z^j}$, and the dz^j and $d\overline{z^j}$ form

an orthonormal coframe. Let ι_j and $\bar{\iota}_j$ denote contraction by $\partial/\partial z^j$ and $\partial/\partial\overline{z^j}$, respectively. Taking the adjoint of exterior multiplication on this monomial basis gives

$$\Lambda_\omega = -\sqrt{-1}\sum_{j=1}^{n} \bar{\iota}_j \iota_j.$$

More explicitly, for increasing multi-indices $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$, its value on $dz^I \wedge d\overline{z^J}$ is

$$\sqrt{-1}\sum_{i_r = j_s} (-1)^{p+r+s} dz^{I\setminus\{i_r\}} \wedge d\overline{z^{J\setminus\{j_s\}}}.$$

In particular, $\Lambda_\omega \omega = n$ at a. The defining pairing for $*$ shows on the same basis that $*^{-1} L_\omega *$ has this contraction formula. All computations act coefficientwise on E-valued forms. The absence of a minus sign in the first two formulas reflects our convention of placing $*u$ to the left of $\overline{v}$.

Mini-Exercise III.2.109

Fix a point $x \in X$. Since the operator in (183) acts pointwise, it suffices to compute at x. Choose holomorphic coordinates centered at x in which ω has the form (97) at x. By the Hermitian spectral theorem (Lemma III.1.8), a further unitary change of coordinates diagonalizes $\sqrt{-1}\Theta_h$ at x while preserving this form of ω. Choose a local holomorphic frame e of L with $|e(x)|_h = 1$.

Let a_j and b_j denote exterior multiplication by dz^j and $d\overline{z^j}$, respectively, and let $a_j^\dagger$ and $b_j^\dagger$ be their pointwise adjoints. The exterior-algebra relations give

$$[a_j b_j, b_j^\dagger a_j^\dagger] = a_j a_j^\dagger + b_j b_j^\dagger - \mathbf{I}.$$

Here $a_j a_j^\dagger$ and $b_j b_j^\dagger$ are the projections onto forms containing dz^j and $d\overline{z^j}$, respectively, and $\mathbf{I}$ is the identity on the exterior algebra at the point. Since $\Lambda_\omega = -\sqrt{-1}\sum_j b_j^\dagger a_j^\dagger$ and the terms with different indices commute, the contribution of the j-th curvature term on u is γ_j times the number of these two differentials present, minus γ_j. Summing over j proves the formula. This calculation uses only the Hermitian inner products at the point.

Mini-Exercise III.2.149

Let U be a connected open neighborhood of $\overline{\Omega}$ and $F \in \mathcal{O}_U(U)$. Since $\overline{\Omega}$ is compact, $|F|$ attains its maximum on $\overline{\Omega}$ at some point p. If $p \in \Omega$, the maximum modulus principle makes F constant on Ω, and the identity theorem then makes it constant on U. We may therefore assume $p \in \partial\Omega$.

Choose local holomorphic coordinates $(z^1, z^2, \dots, z^n)$ centered at p as in Lemma III.1.14, with $\lambda_1 < 0$. For sufficiently small $a > 0$, the holomorphic disc given in these coordinates by

$$\iota(\zeta) = (a\zeta, 0, \dots, 0), \qquad \zeta \in \Delta,$$

where Δ is the unit disc, satisfies $\iota(0) = p$ and $\iota(\Delta \setminus \{0\}) \subset \Omega$. Indeed, on this disc the right-hand side of the boundary normal form is $\lambda_1 a^2|\zeta|^2 + o(a^2|\zeta|^2) < 0$ for $\zeta \neq 0$, whereas $\operatorname{Im} z^n = 0$. The function $F \circ \iota$ is holomorphic also at 0, since F is defined on U. Its absolute value has a maximum at 0, so the one-variable maximum modulus principle makes it constant. Thus $|F|$ attains its maximum on $\overline{\Omega}$ also at an interior point of Ω. The maximum modulus principle on Ω, followed by the identity theorem on U, proves the assertion.

Mini-Exercise III.2.168

For (i), group the even and odd powers of y in the convergent Taylor series of $F \in \mathbb{C}\{x, y\}$ to write $F(x, y) = A(x, y^2) + yB(x, y^2)$, with A, B holomorphic near $(0, 0)$. Modulo $y^2 - x^3$, this is $a(x) + yb(x)$, where $a(x) := A(x, x^3)$ and $b(x) := B(x, x^3)$ are convergent near 0. Thus every element of $\mathcal{O}_{C,o} = \mathbb{C}\{x, y\}/(y^2 - x^3)$ has such a representative. Its pullback is $a(t^2) + t^3 b(t^2)$. The even and odd powers show both injectivity and

$$\nu^* \mathcal{O}_{C,o} = \mathbb{C}\{t^2, t^3\} = \mathbb{C} + t^2 \mathbb{C}\{t\} = \{h \in \mathbb{C}\{t\} \mid h'(0) = 0\}.$$

The extension to $\mathbb{C}\{t\}$ is finite, generated as a module by $1, t$, and has the same fraction field, since $t = t^3/t^2$. Since $\mathbb{C}\{t\}$ is integrally closed, it is the integral closure of this image. Globally, ν is a proper bijection: $|t|^2 = |x|$ controls inverse images of compact subsets. It is biholomorphic off o, with inverse y/x, and its source is nonsingular. Together with the local-ring computation this identifies ν as the normalization, including at o.

For (ii), set $g(o) = 0$ and $g(x, y) = y/x$ off o. Then $g \circ \nu = t$ and $|g(x, y)|^2 = |x|$. Thus g is continuous and locally bounded, and is holomorphic on $C \setminus \{o\}$, so it is weakly holomorphic. It is not holomorphic at o, since t does not belong to the image computed in (i).

Mini-Exercise III.2.171

Locally, choose a finite presentation $\mathcal{O}_X^{\oplus s} \xrightarrow{M} \mathcal{O}_X^{\oplus r} \to \mathcal{F} \to 0$, with M a matrix of holomorphic sections. At a point x, the stalk $\mathcal{F}_x$ vanishes exactly when M_x is surjective. Surjectivity implies that its reduction modulo the maximal ideal of $\mathcal{O}_{X,x}$ is surjective, so an r by r minor is nonzero at x. Conversely, such a minor is a unit in $\mathcal{O}_{X,x}$; its square submatrix is invertible, so M_x is surjective. Thus, if $s \geq r > 0$, $\operatorname{Supp} \mathcal{F}$ is locally the common zero set of these minors, which are holomorphic sections. If $s < r$, the support is the whole local model; if $r = 0$, it is empty. In all cases it is analytic.

Mini-Exercise III.2.193

For (i), the cited proof gives a smooth exhaustion ψ with $\operatorname{tr}_{\omega_A}(\sqrt{-1}\partial\overline{\partial}\psi) > 0$ on Ω. Choose $b_1 < b_2 < 0$ so that $\operatorname{tr}_{\omega_A}(\sqrt{-1}\partial\overline{\partial}\Phi) \geq c_0 > 0$ on $\{\Phi \geq b_1\}$. The set $\{\Phi \leq b_2\}$ is compact. Choose $t > 0$ small and $B \in \mathbb{R}$ so that $b_1 < t\psi + B < b_2$ on this compact set, with a fixed positive margin at both ends. For a regularization scale smaller than this margin, define

$$\varphi = \begin{cases} \operatorname{RegMax}(\Phi, t\psi + B) & \text{on } \{\Phi < b_2\}, \\ \Phi & \text{on } \{\Phi \geq b_2\}. \end{cases}$$

These formulas agree near $\{\Phi = b_2\}$. On $\{\Phi \le b_1\}$ the regularized maximum equals $t\psi + B$. On its transition set both arguments have positive Levi trace, so the chain rule and convexity preserve positive trace. Compactness gives a positive minimum on $\{\Phi \le b_2\}$; outside it the lower bound is c_0. The function φ is bounded.

For (ii), give $\mathbb{I}_\Omega$ the metric $h = h_0 e^{-\varphi}$. Its Chern curvature is $\sqrt{-1}\partial\overline{\partial}\varphi$, and (183) gives $A^{n,n}_{h,\omega_A} = \mathrm{tr}_{\omega_A}(\sqrt{-1}\partial\overline{\partial}\varphi)\mathbf{I} \ge c\mathbf{I}$. Every (n,n)-form is $\overline{\partial}$-closed. Apply Theorem III.2.107, then use the uniform equivalence of h and h_0, which follows from boundedness of φ.

Mini-Exercise III.3.9

Take $h(w) = \sin w$. The triple-angle identity gives $f(h(w)) = 3\sin w - 4(\sin w)^3 = \sin(3w) = h(3w)$. Since $h(0) = 0$ and $h'(0) = 1$, h is locally biholomorphic and is a linearizing function.

Exercise III.1

The smooth proof glues local lifts of a base coordinate vector field using a partition of unity. This gluing does not preserve holomorphicity. Local holomorphic lifts exist by submersion coordinates, but they need not glue holomorphically over the whole inverse image of a base neighborhood.

Conversely, over a coordinate disc U centered at 0 with coordinate t, suppose that a holomorphic vector field v on $f^{-1}(U)$ satisfies $f_* v = \partial/\partial t$. Put $A = v + \overline{v}$ and $B = \sqrt{-1}(v - \overline{v}) = JA$. The flow calculation following (123) makes their real flows holomorphic in the initial point. Since $[v, \overline{v}] = 0$, these flows commute. The identity $B = JA$ then gives the Cauchy–Riemann equations in the time variable for $\Phi(s + \sqrt{-1}u, x) := \phi^u_B \circ \phi^s_A(x)$. Thus Φ is a local holomorphic complex-time flow, with $f(\Phi(\tau, x)) = f(x) + \tau$. Compactness of $f^{-1}(0)$ and properness give a common domain after shrinking U; properness also ensures that the inverse image of a smaller disc lies in this neighborhood of the central fiber. Consequently

$$(\tau, p) \longmapsto \Phi(\tau, p), \qquad x \longmapsto \big(f(x), \Phi(-f(x), x)\big)$$

are inverse holomorphic maps between $U \times f^{-1}(0)$ and $f^{-1}(U)$. In a higher-dimensional coordinate neighborhood, if holomorphic lifts of all coordinate vector fields are available over its whole inverse image, use their flows in a fixed order and invert them in the reverse order. Different coordinate lifts need not commute.

Exercise III.2

For (i), holomorphic functions on X are constant by the maximum principle. The Hodge calculation in Example III.2.127 and the Dolbeault isomorphism (Theorem III.2.116) give $\dim H^1(X, \mathcal{O}_X) = g$ and $\dim H^0(X, \mathcal{O}_X(K_X)) = g$. Together with $\deg \mathbb{I}_X = 0$, this proves both assertions.

For (ii), fix $p \in X$ and a positive integer m. The quotient in the exact sequence

$$0 \longrightarrow \mathcal{O}_X(L) \longrightarrow \mathcal{O}_X(L \otimes [mp]) \longrightarrow \mathcal{Q}_m \longrightarrow 0$$

is supported at p. Here the first inclusion is interpreted as allowing a pole of order at most m at p in a meromorphic section of L, equivalently as multiplication by σ_{mp}. In a local coordinate z vanishing at p and a local frame of L, its sections are represented by the principal parts $a_1 z^{-1} + \cdots + a_m z^{-m}$. Thus $\dim H^0(X, \mathcal{Q}_m) = m$. The group $H^1(X, \mathcal{O}_X(L))$ is finite-dimensional by Lemma III.2.121. Take m larger than its dimension. The connecting map $H^0(X, \mathcal{Q}_m) \to H^1(X, \mathcal{O}_X(L))$ then has a nonzero kernel. Exactness lifts a nonzero principal part to a global section of $\mathcal{O}_X(L \otimes [mp])$, hence to a nonzero meromorphic section s of L with no poles outside p. Its divisor $D := \operatorname{div}(s)$ has finite support. If $s = f_j e_j$ in local frames, then the transition functions of L are f_j/f_k. Proposition III.2.65 therefore identifies L with $[D]$. Splitting the positive and negative coefficients of D gives effective divisors A, B with $D = A - B$.

For (iii), near p write $D = kp$ with $k \in \mathbb{Z}$. In a coordinate z as above, the quotient is $z^{-k-1}\mathcal{O}_{X,p}/z^{-k}\mathcal{O}_{X,p}$. It is a one-dimensional vector space annihilated by the maximal ideal, so the quotient sheaf is a skyscraper at p, with one-dimensional space of global sections. Such a sheaf is flabby: its restriction maps are either the identity or a map to zero. Its higher cohomology vanishes by Proposition II.3.63 (i). The long exact cohomology sequence of

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D+p) \longrightarrow \mathcal{O}_X(D+p)/\mathcal{O}_X(D) \longrightarrow 0$$

therefore gives the claimed difference of Euler characteristics. All dimensions are finite. There are no terms in degree two for the line-bundle sheaves, by the Dolbeault resolution on the curve. The argument permits negative k; D was not assumed effective.

For (iv), write D as a finite integral sum of points and use (iii) successively, in either direction for negative coefficients. Starting with (i) gives $\chi(\mathcal{O}_X(D)) = 1 - g + \deg D$. The degree equals the sum of the coefficients by (197), and (ii) gives the same identity for L. Compact Serre duality (Lemma III.2.121 (ii)) identifies the dual of $H^1(X, \mathcal{O}_X(L))$ with $H^0(X, \mathcal{O}_X(K_X \otimes L^{-1}))$. Taking dimensions yields the required formula.

Exercise III.3

For (i), choose coordinates (z_j, w_j) with $Z = \{w_j = 0\}$. On an overlap write $z_k = \varphi_{kj}(z_j, w_j)$ and $w_k = a_{kj}(z_j, w_j) w_j$, where a_{kj} is a holomorphic unit. Put $c_{kj} := (\partial \varphi_{kj}/\partial z_j)|_Z$ and $n_j := [\partial/\partial w_j] \in N_{Z/X}$. The change of tangent vectors modulo T_Z and the change of top-degree forms give

$$n_k = (a_{kj}|_Z)^{-1} n_j, \qquad (dz_k \wedge dw_k)|_Z = c_{kj}(a_{kj}|_Z)(dz_j \wedge dw_j)|_Z.$$

The tensor frames consequently have transition factor c_{kj}, the same as $dz_k|_{T_Z} = c_{kj} dz_j|_{T_Z}$. Thus the assignments $((dz_j \wedge dw_j)|_Z) \otimes n_j \mapsto dz_j|_{T_Z}$ glue to the required isomorphism. A local choice of a transverse vector is sufficient; no holomorphic splitting of $0 \to T_Z \to T_X|_Z \to N_{Z/X} \to 0$ has been assumed.

For (ii), on the charts $Z_0 \neq 0$ and $Z_1 \neq 0$ put $(x, y) = (Z_1/Z_0, Z_2/Z_0)$ and $(u, v) = (Z_0/Z_1, Z_2/Z_1)$, respectively. Then

$$u = 1/x, \qquad v = y/x, \qquad du \wedge dv = -x^{-3} dx \wedge dy.$$

The analogous formulas on the third chart identify the canonical bundle with L_{taut}^3. More explicitly, using the chart frame $(-1)^j \bigwedge_{k\neq j} d(Z_k/Z_j)$, with the remaining indices in increasing order, absorbs the permutation signs, and the transition factor from chart j to chart k is $(Z_j/Z_k)^3$, the cube of the tautological transition. The local defining functions P/Z_j^d have ratios $(Z_k/Z_j)^d$, so (165) gives $[Z] \cong L_{\text{taut}}^{-d}$. The local normal-bundle calculation in Example III.2.145 also gives $N_{Z/\mathbb{P}^2} \cong [Z]|_Z$, on each component.

Write $H := L_{\text{taut}}^{-1}$ and $h := c_1(H)$. Two distinct projective lines meet transversely in one point, so Proposition III.2.143 and the intersection formula give $\int_{\mathbb{P}^2} h^2 = 1$. The divisor-current interpretation of the first Chern class then yields

$$\deg(H|_Z) = \int_Z h = \int_{\mathbb{P}^2} h\, c_1([Z]) = d.$$

Here both degree and integration sum over the components; the formula has not used connectedness.

For (iii), the CW computation (18) and the universal coefficient theorem give $H^2(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}h$; $\int_{\mathbb{P}^2} h^2 = 1$ shows that h is a generator. If Z were disconnected, split it into two nonempty disjoint unions Z_1, Z_2 of connected components. They are compact effective divisors. Write $c_1([Z_i]) = m_i h$. Since the Fubini–Study form representing h is positive,

$$m_i = \int_{Z_i} h > 0 \qquad (i = 1, 2).$$

Their intersection would then be $m_1 m_2 > 0$. But the canonical section of $[Z_2]$ never vanishes on Z_1, so $[Z_2]|_{Z_1}$ is trivial and their intersection is zero, a contradiction. Thus Z is a compact Riemann surface. Part (i) gives $K_Z \cong H^{d-3}|_Z$, and hence $\deg K_Z = d(d-3)$ by (ii). Combining this with (219) gives $2g - 2 = d(d-3)$, or $g = (d-1)(d-2)/2$. For $d = 1, 2, 3$ the genera are $0, 0, 1$, respectively.

3.4 Hints and solutions to the exercises in Chapter IV

Mini-Exercise IV.2.13

Every open subset of $\mathbb{C}$ is Stein (See the statement about open Riemann surfaces following Definition III.2.157), so the neighborhood assertion holds. Put $A := \Gamma(K, \mathcal{O}_{\mathbb{C}})$. For each positive integer n, the isolated point $1/n$ and the compact set $K \setminus \{1/n\}$ have disjoint open neighborhoods. The function equal to 1 on the first and 0 on the second is holomorphic on their union and determines an element $e_n \in A$. These elements satisfy $e_n^2 = e_n$ and $e_n e_m = 0$ for $n \neq m$. The chain

$$(e_1) \subsetneq (e_1, e_2) \subsetneq (e_1, e_2, e_3) \subsetneq \cdots$$

is strictly increasing: every element of $(e_1, \ldots, e_n)$ vanishes at $1/(n+1)$, whereas e_{n+1} has value 1 there. Thus A is not Noetherian.

Revision History

Version 1.2 (November 4, 2026)
Minor corrections following the intensive lecture series.

Version 1.1 (October 28, 2026)
Minor revisions to improve the exposition, mainly in Chapters IV and V.

Version 1.0 (October 25, 2026)
First public release, prepared for the intensive lecture series scheduled to be held at the University of Tokyo during 26–30 October 2026. Chapters IV and V added, with minor revisions to Chapters I–III.

v 0.4 (September 26, 2026)
Expanded and revised Chapter III, with related minor revisions to Chapters I and II.

v 0.31 (August 3, 2026)
Chapter I, Chapter II, Chapter III §1.1, and Appendix §1–§2 are (almost) complete. Mini-Exercise I.4.1 added to Chapter I.

v 0.3 (July 27, 2026)
Chapter I, Chapter II, and Appendix §1–§2 are (almost) complete. Mini-Exercise I.3.10 added to Chapter I.

v 0.23 (April 1, 2026)
Chapter I, Chapter II §1–2, and Appendix §1 – §2 are (almost) complete. Corrected some minor mistakes in Chapter I.

v 0.22 (March 3, 2026)
Corrected a few minor mistakes in Chapter I §1, and cleaned up the rough notes in Chapter II, §2.

v 0.2, 0.21 (February 18, 2026)
Experimental release. Chapter I §1 – §8 and Appendix §1 – §2 are (almost) complete, while the subsequent sections and chapters are included in preliminary and partially drafted form. This version was made available for limited distribution to students in the author's research group and some collaborators.

v 0.1 (December 18, 2025)
Initial experimental release. Chapter I §1 – §5.1 and Appendix §1 are (almost) complete, while the subsequent sections and chapters are included in preliminary and partially drafted form. This version was made available for limited distribution to students in the author's research group.