The University of Tokyo, A Semester 2026
Special Lectures in Mathematical Sciences V

Day 4: Supplementary Notes

Semi-positivity of effective divisors and analytic neighborhood structure

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This handout organizes the definitions, statements, and proofs presented in the Day 4 blackboard lecture, with details omitted from the lecture recorded in remarks. The underlying lecture manuscript was handwritten by the author. This handout was prepared by asking ChatGPT to transcribe that manuscript, compare it with the lecture notes [LN], and add supplementary explanations and references so that participants can consult the lecture notes for details not covered during the lecture. The author has checked the mathematics presented in the lecture and the contents of the lecture notes, but the transcription and the preparation of the supplementary explanations and references in this handout were largely entrusted to ChatGPT. The author has checked and corrected parts of the handout and, on the basis of that review, judged it suitable for publication, but has not verified the entire text line by line. Readers should therefore bear in mind that errors may remain in the details of the transcription, supplementary explanations, or references.

Only a proof sketch of the main theorem is given. Throughout, "the lecture notes" means *Complex Analytic Geometry and Dynamics on Analytic Neighborhood Spaces*, **Version 1.1 (October 28, 2026)** [LN]. Reference numbers beginning with Roman numerals, and equation numbers explicitly attributed to the lecture notes, refer to that version and are distinct from the numbering used in this handout.

1 Asking whether plurisubharmonic functions exist

Through Day 2, we mainly considered situations in which suitable plurisubharmonic functions exist on a given complex manifold. Building on the discussion of analytic neighborhood structure on Day 3, we now ask how geometric conditions can establish the existence or nonexistence of such functions.[1] Our aim is to relate plurisubharmonic functions to neighborhood structure through the curvature of the holomorphic line bundle associated with an effective divisor. We begin with the two settings that motivated the lecture.

1.1 Compact curves in surfaces and self-intersection numbers

Let X be a connected compact complex surface, and let $Z \subset X$ be a holomorphically embedded compact Riemann surface. Writing $N_{Z/X}$ for its normal bundle, its self-intersection number is

$$(Z^2) = \deg N_{Z/X}.$$

[1] As explained in connection with the Levi problem on Day 1, when sufficiently many holomorphic functions of suitable kinds are already known to exist, they can help us construct plurisubharmonic exhaustion functions and related objects. Here we ask about existence and nonexistence in situations that are more delicate from the viewpoint of pseudoconvexity.

Indeed, $N_{Z/X} \simeq [Z]|_Z$ (Example III.2.145 of the lecture notes; the line bundle $[Z]$ associated with the divisor is defined in the next section).

(i) The case $(Z^2) > 0$. Then X is projective algebraic, and there is a proper modification from $X \setminus Z$ onto an affine surface.[2] Moreover, $X \setminus Z$ is strongly 1-convex.[3] This is the case of a nonsingular irreducible curve in [KU, Corollary 5.1 and the preceding discussion]. See also Corollary V.5.7 of the lecture notes and its proof, and, for the general theory of contractions, Theorem III.2.183 of the lecture notes and Remark III.2.184 of the lecture notes.

(ii) The case $(Z^2) < 0$. The curve Z has a neighborhood basis consisting of strongly pseudoconvex neighborhoods, each of which is strongly 1-convex (Grauert's results and the Grauert–Mumford contraction criterion: [G], [R, Chapter 9, §9.8, Theorem 9.14]; see also §V.5.1 of the lecture notes). On the other hand, the complement $X \setminus Z$ of Z in the compact surface is not weakly 1-complete.

(iii) The case $(Z^2) = 0$. The self-intersection number alone does not determine the function theory of a neighborhood. The phenomena from Ueda theory discussed on Day 3 arise in this case.

1.2 Plurisubharmonic defining functions for domains

The second setting is a locally pseudoconvex domain Ω in a complex manifold X. Local pseudoconvexity is the local condition that each boundary point admit a coordinate neighborhood V for which $V \cap \Omega$ is pseudoconvex (§III.2.10 of the lecture notes). We ask whether a function defining Ω can itself be chosen to be plurisubharmonic.

A defining function for a domain with C^∞ boundary is a real-valued C^∞ function ρ on a neighborhood V of the boundary such that

$$\Omega \cap V = \{x \in V \mid \rho(x) < 0\}, \qquad \partial\Omega = \{x \in V \mid \rho(x) = 0\}, \qquad (d\rho)_p \neq 0 \quad (p \in \partial\Omega).$$

Even for a strongly 1-complete domain, a defining function need not be plurisubharmonic near the boundary. The following theorem makes this distinction explicit.

Theorem 1.1 (Nemirovski [Nem, Proposition 2 and the following corollary])**.** Let X be a complex surface and $\Omega \Subset X$ a relatively compact strongly 1-complete domain. Assume that $\partial\Omega$ is nonempty, C^∞, and Levi-flat. Let ρ be a C^∞ defining function for Ω, defined on a neighborhood of $\overline{\Omega}$. Then, for every compact subset $K \subset \Omega$, the restriction $\rho|_{\Omega\setminus K}$ is not plurisubharmonic.

Proof. Suppose that $\rho|_{\Omega\setminus K}$ is plurisubharmonic for some compact subset $K \subset \Omega$. By compactness of $\partial\Omega$ and the condition $d\rho \neq 0$, we can choose a sufficiently small tubular neighborhood V and $\varepsilon > 0$ such that

$$M_t := \{x \in V \mid \rho(x) = t\} \qquad (-\varepsilon < t \leq 0)$$

are compact C^∞ real hypersurfaces. Here $M_0 = \partial\Omega$, and we choose V so that $M_t \subset \Omega\setminus K$ whenever $t < 0$. Orient each M_t as the boundary of the side $\rho < t$.

[2] Being projective algebraic means that X can be realized as the common zero set of finitely many homogeneous polynomials in some complex projective space. For the relation to holomorphic embeddings into projective space, see Theorem III.2.172 of the lecture notes (Chow's theorem) and the surrounding discussion.

[3] The lecture used the model case $X = \mathbb{P}^2$ with Z the line at infinity. In this case $X \setminus Z \simeq \mathbb{C}^2$, and $|z_1|^2 + |z_2|^2$ is a C^∞ strictly plurisubharmonic exhaustion function, so the complement is strongly 1-complete.

We use the same normalization as the lecture notes:

$$d^c = \frac{1}{4\pi\sqrt{-1}}(\partial - \bar{\partial}), \qquad dd^c = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}.$$

With this normalization, define

$$F(t) := \int_{M_t} d^c\rho \wedge dd^c\rho \qquad (-\varepsilon < t \leq 0).$$

First, we check that $F(t) \geq 0$. Choose an auxiliary Hermitian metric on X, and let ν be the real unit normal to M_t pointing in the direction of increasing ρ. If J denotes the complex structure, then $J\nu$ is tangent to M_t, and

$$(d^c\rho)(J\nu) = -\frac{1}{4\pi}(d\rho)(J^2\nu) = \frac{1}{4\pi}(d\rho)(\nu) > 0.$$

Moreover, $d^c\rho$ vanishes on the complex tangent space $HM_t := TM_t \cap J(TM_t)$. The direction $J\nu$, followed by the complex orientation of HM_t, gives the boundary orientation of M_t. Since ρ is plurisubharmonic, the restriction of $dd^c\rho$ to HM_t is semi-positive. Thus $d^c\rho \wedge dd^c\rho$ is nonnegative with respect to this orientation. By continuity, the same assertion holds at the boundary level $t = 0$. These relations are also recorded in parts (i) and (iii) of Mini-Exercise III.1.16 of the lecture notes.

Next, Stokes' theorem gives, for $-\varepsilon < t < s \leq 0$,

$$\begin{aligned} F(s) - F(t) &= \int_{\{x\in V \mid t<\rho(x)<s\}} d(d^c\rho \wedge dd^c\rho) \\ &= \int_{\{x\in V \mid t<\rho(x)<s\}} (dd^c\rho)^2 \geq 0. \end{aligned}$$

The domain of integration has compact closure in V, and its oriented boundary is $M_s \sqcup (-M_t)$. Hence F is nondecreasing.

Since $M_0 = \partial\Omega$ is Levi-flat, $F(0) = 0$. It follows that

$$0 \leq F(t) \leq F(0) = 0 \qquad (-\varepsilon < t \leq 0).$$

On each M_t, the integrand is a nonnegative density given by a positive multiple of the tangential Levi form. Its integral being zero therefore forces the Levi form to vanish at every point. Consequently, every M_t is Levi-flat.

Fix $t < 0$. Take a C^∞ strictly plurisubharmonic exhaustion function on the strongly 1-complete domain Ω, and let p be a point where its restriction to M_t attains its maximum. Restricting this function to the Levi leaf through p gives a strictly subharmonic function with a local maximum at the interior point p, contrary to the maximum principle. □

Remark 1.2 (Relation to the theorem in the lecture notes)**.** Theorem III.2.191 of the lecture notes is stated as the nonexistence of a plurisubharmonic defining function on a two-sided neighborhood of a compact Levi-flat boundary. The proof above uses only regular levels inside the boundary and the regions between them, so it applies without change to Theorem 1.1 in the form used in the lecture. For the passage from Levi-flatness to the Levi foliation, see part (iv) of Mini-Exercise III.1.16 of the lecture notes and the discussion immediately following it.

2 Divisors and their associated holomorphic line bundles

We now return to the first setting and consider the relation between semi-positivity of the holomorphic line bundle associated with a divisor and function theory near its support. This section fixes the conventions for divisors, local defining functions, and associated line bundles. Throughout, X is a complex manifold.

2.1 Complex hypersurfaces and divisors

Definition 2.1 (Complex hypersurface)**.** A closed analytic subset $Z \subset X$ of pure codimension one is called a complex hypersurface. On a sufficiently small connected neighborhood U of each point, it can be written as

$$Z \cap U = \{x \in U \mid f(x) = 0\}$$

for a holomorphic function $f \in \mathcal{O}_X(U)$ that is not identically zero (Definition III.2.51 of the lecture notes, Proposition III.2.60 of the lecture notes).

A nonempty complex hypersurface is called irreducible if it cannot be written as the union of two proper closed analytic subsets. Intuitively, it cannot be decomposed further into several complex hypersurfaces. For the precise definition, see Definition III.2.39 of the lecture notes.

Remark 2.2 (Reduced local defining functions)**.** Write $\mathcal{I}_Z$ for the vanishing ideal sheaf of the hypersurface Z.[4] The function f above can moreover be chosen so that $\mathcal{I}_Z|_U = f\mathcal{O}_U$. The Weierstrass preparation theorem implies that the local ring $\mathcal{O}_{X,p}$ is a UFD, so multiplicities can be understood as exponents of prime factors, just as in one variable (Theorem III.1.29 of the lecture notes, Theorem III.1.33 of the lecture notes). The germ of f is a unit times the product of distinct irreducible function germs defining the local branches, each branch occurring with multiplicity one. In this sense, f is called a local defining function of Z. This does not assert that $df \neq 0$ at singular points: df may vanish there (Proposition III.2.60 of the lecture notes, Definition III.2.62 of the lecture notes). The passage from a generator in a single local ring to the sheaf equality $\mathcal{I}_Z|_U = f\mathcal{O}_U$ on a neighborhood is also nontrivial. The proof of Proposition III.2.60 of the lecture notes uses Oka's second coherence theorem (Theorem III.2.163 of the lecture notes) for this step.

Definition 2.3 (Divisor, effective divisor, and support)**.** A divisor on X is a locally finite formal sum

$$D = \sum_{\nu \in \Lambda} m_\nu Z_\nu, \qquad m_\nu \in \mathbb{Z},$$

where the Z_ν are distinct irreducible complex hypersurfaces. If all m_ν are nonnegative, D is called effective, and we write $D \geq 0$. Its support is defined by

$$\operatorname{Supp} D := \bigcup_{m_\nu \neq 0} Z_\nu$$

(Definition III.2.59 of the lecture notes).

Remark 2.4 (Local finiteness)**.** Local finiteness means that every point of X has a neighborhood meeting only finitely many Z_ν with $m_\nu \neq 0$. When X is noncompact, there need not be only finitely many components in total. If $\operatorname{Supp} D$ is compact, however, only finitely many components have nonzero coefficients.

[4] For every open set $W \subset X$, define $\mathcal{I}_Z(W) := \{g \in \mathcal{O}_X(W) \mid g|_{W \cap Z} = 0\}$, with restriction maps given by restriction of holomorphic functions. This is an ideal sheaf of $\mathcal{O}_X$ (equation (155) of the lecture notes).

Recording the zeros and poles of a function, together with their orders along hypersurfaces, produces a divisor. For a holomorphic function f that is not identically zero on any connected component, define

$$\operatorname{div}(f) = \sum_{\Gamma} \operatorname{ord}_{\Gamma}(f)\,\Gamma.$$

Here Γ ranges over the irreducible hypersurface components of the zero set of f, and $\operatorname{ord}_{\Gamma}(f)$ is the order of vanishing along Γ. This sum is locally finite. This follows from the local finite irreducible decomposition of analytic sets, based on the Noetherianity of the local rings $\mathcal{O}_{X,p}$ (Theorem III.1.32 of the lecture notes, Theorem III.2.45 of the lecture notes; for the connection with the definition of a divisor, see also Definition III.2.64 of the lecture notes and the following paragraph).

For a meromorphic unit $f \in \mathcal{M}_X^*(U)$, write $f = g/h$ locally and define

$$\operatorname{div}(f) := \operatorname{div}(g) - \operatorname{div}(h).$$

Here a meromorphic unit means a meromorphic function that is not identically zero on any connected component. Negative coefficients record the orders of poles. This definition is independent of the representation and satisfies $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$. For details, see Definition III.2.62 of the lecture notes and Definition III.2.64 of the lecture notes.

2.2 From local equations to line bundles

Fix a divisor D. On a sufficiently fine open cover $\{U_j\}_{j\in J}$, we can choose meromorphic units f_j on U_j such that

$$\operatorname{div}(f_j) = D|_{U_j}.$$

Locally, we raise an equation for each irreducible branch to the power specified by its coefficient and multiply the resulting factors. If D is effective, all f_j can be chosen holomorphic. On the overlaps $U_{jk} := U_j \cap U_k$, we have

$$g_{jk} := \frac{f_j}{f_k} \in \mathcal{O}_X^*(U_{jk}), \qquad g_{jk}g_{k\ell} = g_{j\ell}.$$

Thus $\{g_{jk}\}$ is a multiplicative Čech 1-cocycle.

Definition 2.5 (Line bundle associated with a divisor)**.** Write $[D] \to X$ for the holomorphic line bundle defined by this cocycle. In terms of local holomorphic frames e_j, our convention is

$$e_k = \frac{f_j}{f_k}\, e_j \qquad \text{on } U_{jk}. \tag{2.1}$$

Equivalently, $[D]$ corresponds to

$$\left[\left\{\left(U_{jk}, \frac{f_j}{f_k}\right)\right\}\right] \in \check{H}^1(X, \mathcal{O}_X^*) \simeq \operatorname{Pic}(X).$$

Remark 2.6 (Weil divisors and Cartier divisors)**.** A divisor viewed as a formal sum of irreducible hypersurfaces is called a Weil divisor. A Cartier divisor is represented by local meromorphic equations $\{f_j\}$ as above, with two systems identified after passing to a common refinement and multiplying local equations by holomorphic units. On a complex manifold, the existence of local equations identifies the two notions. Consequently, the construction of $[D]$ is independent, up to isomorphism, of the cover and the chosen equations. When the ambient space is singular, this identification cannot be used without further conditions (Definition III.2.67 of the lecture notes and the discussion immediately following it).

2.3 The canonical section of an effective divisor

Proposition 2.7 (Canonical section)**.** Let $D \geq 0$, and choose the functions f_j above to be holomorphic. The local sections $f_j e_j$ glue to a global holomorphic section of $[D]$ given by

$$\sigma_D|_{U_j} := f_j e_j.$$

This is called the canonical section of D. Its zero divisor, including multiplicities, is D.

Proof. Equation (2.1) gives $f_k e_k = f_j e_j$ on overlaps, so the local sections glue. Since each e_j is a nowhere-vanishing local frame, the zeros of the section and their orders agree with those of f_j (Proposition III.2.65 of the lecture notes). □

In particular, σ_D is nowhere zero on $X \setminus \operatorname{Supp} D$. In the next section, we use this section to construct functions on the complement from metrics on the line bundle.

3 Curvature and plurisubharmonic functions on the complement

3.1 Hermitian metrics and the canonical section of a divisor

Let h be a C^∞ Hermitian metric on a holomorphic line bundle $L \to X$. In a local holomorphic frame e_j, write

$$|e_j|_h^2 = e^{-\varphi_j}.$$

The real-valued C^∞ function φ_j is called a local weight. If $e_k = g_{jk} e_j$, then $\varphi_k = \varphi_j - \log|g_{jk}|^2$. Consequently, the local $(1,1)$-forms

$$\sqrt{-1}\Theta_h = \sqrt{-1}\partial\bar\partial\varphi_j, \qquad \frac{\sqrt{-1}}{2\pi}\Theta_h = dd^c\varphi_j$$

patch together on X. The latter represents the image of $c_1(L)$ in real cohomology (equation (149) of the lecture notes; Proposition III.2.22 of the lecture notes).

Definition 3.1. We say that L is *semi-positive* if it admits a C^∞ Hermitian metric h with $\sqrt{-1}\Theta_h \geq 0$. It is *positive* if it admits a metric with $\sqrt{-1}\Theta_h > 0$. It is *unitary flat* if its transition functions can be chosen to be locally constant functions of absolute value one. Unitary flatness is defined in the same way, using transition functions, for line bundles on analytic spaces.

Giving unitary flat frames norm one produces a metric of zero curvature. Conversely, if the curvature is zero, the local weights are pluriharmonic, and the holomorphic frames can be changed so that their norms are one. Thus unitary flatness is equivalent to the existence of a zero-curvature metric (Definition III.2.123 of the lecture notes and the explanation immediately following it).

From now on, let D be an effective divisor, put $Z := \operatorname{Supp} D$, and let σ_D be the canonical section of $[D]$. For a C^∞ metric h on $[D]$, set

$$\psi_h := -\log|\sigma_D|_h^2 \quad \text{on } X \setminus Z. \tag{3.1}$$

Writing $\sigma_D = s_j e_j$ locally gives

$$\psi_h = -\log|s_j|^2 + \varphi_j.$$

The Lelong–Poincaré formula gives, in the sense of currents,

$$dd^c\psi_h = -T_D + \frac{\sqrt{-1}}{2\pi}\Theta_h, \qquad \sqrt{-1}\partial\bar\partial\psi_h = -2\pi T_D + \sqrt{-1}\Theta_h. \tag{3.2}$$

For the Lelong–Poincaré formula itself, see Theorem III.2.99 of the lecture notes and equation (173) of the lecture notes. In particular, $\sqrt{-1}\partial\bar{\partial}\psi_h = \sqrt{-1}\Theta_h$ on $X \setminus Z$. It follows that h has semi-positive curvature if and only if ψ_h is plurisubharmonic on $X \setminus Z$. For the converse implication, we use the fact that if the Levi forms of the C^∞ weights φ_j are semi-positive on a dense open set, then they are also semi-positive on Z, by continuity.

Remark 3.2 (Currents and the correspondence between metrics and functions)**.** For $D = \sum_\nu m_\nu Z_\nu$, we have

$$T_D(\eta) = \sum_\nu m_\nu \int_{(Z_\nu)_{\mathrm{reg}}} \eta, \qquad \eta \in A_c^{n-1,n-1}(X).$$

These integrals converge even near singular points and define a closed current. See Theorem III.2.96 of the lecture notes, Remark III.2.98 of the lecture notes, and Lemma III.2.139 of the lecture notes. To recover a metric from a C^∞ function ψ on $X \setminus Z$, we require each $\psi + \log|s_j|^2$ to extend across Z as a C^∞ function. Under this condition, these extensions define the local weights of a metric (equations (351)–(352) of the lecture notes).

3.2 Nonsingular curves in surfaces and Ueda's classification

Let X be a complex surface and $Z \subset X$ a compact Riemann surface, and consider the case $D = Z$. Since $[Z]|_Z \simeq N_{Z/X}$, we have

$$(Z^2) = \deg N_{Z/X}.$$

The situation depends on the degree of the normal bundle as follows.

- If $\deg N_{Z/X} < 0$, then $[Z]$ is not semi-positive. Indeed, restricting semi-positive curvature to Z and integrating would give $\deg([Z]|_Z) \geq 0$, a contradiction.
- If $\deg N_{Z/X} > 0$, then $[Z]|_V$ is positive on some neighborhood V. The equivalence between positive degree and positivity on a curve is Theorem III.2.177 of the lecture notes; the extension of a positive-curvature metric to a neighborhood is the one-component case of Theorem V.5.1 of the lecture notes.
- If $\deg N_{Z/X} = 0$, then the normal bundle is unitary flat, and the analytic structure of the neighborhood becomes important.

In the last case, we use Ueda's classification, introduced on Day 3. The comparison here is between $[Z]$ and a unitary flat extension of the normal bundle $N_{Z/X}$ to a neighborhood. Class (α) is the case of finite Ueda type; class (β) is the case in which $[Z]$ is unitary flat on some neighborhood; and class (γ) is the case in which the formal comparison is trivial but the analytic comparison is not (Definition IV.8.4 of the lecture notes; Remark IV.8.5 of the lecture notes). In class (α), $[Z]$ is not semi-positive, whereas in class (β) it is semi-positive. The global semi-positivity in the latter case uses the locality of semi-positivity, which we prove in the next section.

Remark 3.3 (Non-semi-positivity from a growth estimate)**.** Let μ be the finite Ueda type, and let $r_Z = |\sigma_Z|_{h_0}$ be the function measured using a fixed C^∞ metric. By Ueda's theorem, for $0 < b < \mu$, a plurisubharmonic function $u = o(r_Z^{-b})$ on a punctured neighborhood is constant on a sufficiently small punctured neighborhood. If a C^∞ metric of semi-positive curvature existed, however, $\psi_h = -2\log r_Z + O(1)$ would satisfy this growth condition while tending to $+\infty$ as one approaches Z. This is a contradiction. See part (ii) of Theorem V.4.6 of the lecture notes and Lemma V.4.4 of the lecture notes.

For class (γ), the result of Koike–Ogawa shows that $[Z]$ is not semi-positive in the example constructed by Ueda. More precisely, for the suspension of an elliptic curve, linearizability of the transverse holomorphic map germ is equivalent to semi-positivity of $[Z]$. For Ueda's example and this equivalence, see Remark IV.7.4 of the lecture notes and the proof of Corollary V.3.6 of the lecture notes. The assertion of non-semi-positivity here concerns this particular example.

4 The main theorem and preliminaries

We state a criterion for semi-positivity based on [K, Theorem 1.1], under the hypotheses used in the lecture.

Theorem 4.1 (Semi-positivity of an effective divisor and a flat neighborhood)**.** Let X be a connected compact Kähler manifold with $n := \dim X \geq 2$. Let $D \neq 0$ be an effective divisor, and suppose that $Z := \operatorname{Supp} D$ is connected. Assume that $[D]|_Z$ is unitary flat. Then the following conditions are equivalent.

(i) $[D]$ is semi-positive.

(ii) $[D]|_V$ is unitary flat on some open neighborhood V of Z.

(iii) There is a family $\{V_j\}$ of open subsets of X covering Z, and, for each j, a holomorphic function $w_j \colon V_j \to \mathbb{C}$ such that
$$w_j^{-1}(0) = Z \cap V_j,$$
and each quotient w_j/w_k extends to a holomorphic unit of absolute value one on $V_j \cap V_k$.

In condition (iii), we do not assume in advance that the multiplicities of the zero divisor of w_j agree with those of D.

Remark 4.2 (Relation to the main theorem of the lecture notes)**.** Theorem V.2.5 of the lecture notes gives the same equivalence under the assumption that $[D]$ is nef of numerical dimension one. In our setting, unitary flatness of $[D]|_Z$ implies that, for a Kähler form ω,
$$\int_X c_1([D])^2 \wedge \omega^{n-2} = \sum_\nu m_\nu \int_{Z_\nu} c_1([D]) \wedge \omega^{n-2} = 0. \tag{4.1}$$
Moreover, since $D \neq 0$ is effective,
$$\int_X c_1([D]) \wedge \omega^{n-1} = \sum_\nu m_\nu \int_{Z_\nu} \omega^{n-1} > 0.$$
If semi-positivity is assumed, then $[D]$ is nef. If the integral of the square of a semi-positive curvature form is zero, its rank is at most one at every point, so the two identities above give numerical dimension one. Thus the main theorem of the lecture is the form of the theorem in the lecture notes appropriate to this setting. For the definitions of nefness and numerical dimension, see Definition V.1.9 of the lecture notes and Remark V.1.11 of the lecture notes.

4.1 The intersection matrix and comparison of multiplicities

We prepare a fact from linear algebra in order to recover the multiplicities of the divisor from condition (iii) of the main theorem. Let A be a real symmetric $N \times N$ matrix with nonnegative

off-diagonal entries, and suppose that the graph whose edges are given by $a_{\nu\mu} > 0$ is connected. Writing λ for its largest eigenvalue, we have

$$\begin{aligned} \lambda > 0 &\iff \exists m \in \mathbb{R}^N_{>0} \quad Am \in \mathbb{R}^N_{>0}, \\ \lambda = 0 &\iff \exists m \in \mathbb{R}^N_{>0} \quad Am = 0, \\ \lambda < 0 &\iff \exists m \in \mathbb{R}^N_{>0} \quad -Am \in \mathbb{R}^N_{>0}. \end{aligned}$$

The eigenspace for the largest eigenvalue is one-dimensional and is generated by a positive eigenvector. If A has integer entries, m can be chosen to be a positive integer vector in each of these conditions.

Remark 4.3. These facts are collected in Lemma II.2.1 of the lecture notes, Proposition II.2.2 of the lecture notes, and [KU, Proposition 3.2, Lemma 3.3]. One approach is to apply the Perron–Frobenius theorem to $A + cI$ for sufficiently large $c > 0$ (Theorem II.2.5 of the lecture notes). The one-dimensionality of the eigenspace can also be seen by observing that two independent eigenvectors would yield a nonzero eigenvector with a zero component.

In the setting of the main theorem, let $Z_1, \dots, Z_N$ be the irreducible components of Z, and set

$$A = (a_{\nu\mu}), \qquad a_{\nu\mu} := \int_X c_1([Z_\nu]) \wedge c_1([Z_\mu]) \wedge \omega^{n-2}.$$

This matrix satisfies the hypotheses above. Writing $D = \sum m_\nu Z_\nu$, unitary flatness of $[D]|_Z$ gives $Am = 0$. Thus, if an effective divisor $D' = \sum m'_\nu Z_\nu$ with the same support satisfies

$$\int_{Z_\nu} c_1([D']) \wedge \omega^{n-2} = 0 \quad (\nu = 1, \dots, N),$$

then $Am' = 0$. Since the kernel is one-dimensional, we obtain $D' = aD$ for some $a \in \mathbb{Q}_{>0}$. This is the content of Lemma V.2.6 of the lecture notes. In higher dimensions, the matrix A formed using a general ω has real entries, but rationality of this proportionality factor follows from the fact that m and m' have integer entries.

4.2 Locality of semi-positivity and the easy implications

Proposition 4.4 (Locality of semi-positivity)**.** Let X be a complex manifold and D an effective divisor with compact support Z. Then $[D]$ is semi-positive on X if and only if $[D]|_V$ is semi-positive on some open neighborhood $V \supset Z$. Moreover, a metric of semi-positive curvature on V can be extended to a metric of semi-positive curvature on X, while being kept unchanged on a smaller neighborhood of Z.

Proof. The restriction implication is immediate, so we prove the converse. If $Z = \varnothing$, take a zero-curvature metric for which σ_D is a global holomorphic unit frame. Henceforth suppose that $Z \neq \varnothing$. Starting with the metric on V, form $\psi = -\log|\sigma_D|^2_h$ and choose $Z \subset W \Subset V$. Choose C sufficiently large that $\psi < C - 2\varepsilon$ near ∂W, and put

$$\Psi = \begin{cases} \mathrm{RegMax}_\varepsilon(\psi, C) & \text{on } W \setminus Z, \\ C & \text{outside } W. \end{cases}$$

Here $\mathrm{RegMax}_\varepsilon$ is a C^∞ regularized maximum. The two expressions agree near ∂W, and Ψ is C^∞ and plurisubharmonic on $X \setminus Z$. Since $\psi \to +\infty$, we also have $\Psi = \psi$ sufficiently close to Z. The desired metric is therefore recovered using the correspondence between metrics and functions. $\square$

Remark 4.5. This proposition requires neither compactness nor the Kähler property of X; it uses compactness of the support of the divisor. The construction using $\chi \circ \psi$ in the lecture corresponds to taking $\chi(t) = \mathrm{RegMax}_\varepsilon(t, C)$. For details, see Proposition V.2.1 of the lecture notes; for the regularized maximum, see Example I.4.9 of the lecture notes.

The implication (ii) $\Rightarrow$ (i) in the main theorem follows from this proposition. Also, expressing the canonical section σ_D in flat holomorphic frames gives coefficient functions satisfying (iii). In this direction, the multiplicities of the zero divisor agree with those of D.

Conversely, patching together the zero divisors of the functions in (iii) gives $D' = \sum m'_\nu Z_\nu$. Its transition functions have absolute value one, so $[D']$ is unitary flat near Z. The numerical proportionality above gives $pD = qD'$ for positive integers p, q. Taking the positive p-th root of a flat metric on $[D']^{\otimes q} \simeq [D]^{\otimes p}$ yields a flat metric on $[D]$. This proves (iii) $\Rightarrow$ (ii). There is no need to choose roots of holomorphic functions: we take a root of a metric on the already given line bundle.

5 Outline of the proof of the main theorem

It remains to prove (i) $\Rightarrow$ (ii). Fix a Kähler form ω on X. Choose a C^∞ Hermitian metric h of semi-positive curvature on $[D]$, and put

$$\eta := |\sigma_D|_h^2, \qquad \psi := -\log \eta \quad \text{on } X \setminus Z.$$

Then ψ is plurisubharmonic and tends to $+\infty$ as one approaches Z. We first show that sufficiently high level sets are Levi-flat, and then split the argument into two cases according to the behavior of the resulting coefficient function. The outline below follows the proof of Theorem V.2.5 of the lecture notes.

5.1 Step 1: Levi-flatness of the level sets

First, $d\psi \neq 0$ sufficiently close to Z. Indeed, write $\eta = |s_j|^2 e^{-\varphi_j}$ locally and apply the Łojasiewicz gradient inequality to the real-analytic function $|s_j|^2$. For some $c > 0$ and $0 < \theta < 1$, we obtain

$$\left\| \frac{\partial s_j}{s_j} \right\| \geq c|s_j|^{2\theta - 2} \longrightarrow +\infty \quad (s_j \to 0).$$

Since $\partial \varphi_j$ is bounded, it follows that $\partial \psi = -\partial s_j / s_j + \partial \varphi_j \neq 0$. This argument applies even if Z is singular or D has multiplicities (Theorem III.2.57 of the lecture notes and the proof of Theorem V.2.5 of the lecture notes).

Consequently, for sufficiently large t,

$$M_t := \{\psi = t\}, \qquad \Omega_t := Z \cup \{\psi > t\} = \{\eta < e^{-t}\}$$

are respectively a compact C^∞ real hypersurface and a neighborhood of Z. We orient M_t as the boundary of the side $\{\psi < t\}$. This is the opposite of its orientation as the boundary of Ω_t.

Remark 5.1 (Connectedness of the level sets)**.** Each connected component of a sufficiently small sublevel set of η meets Z. Otherwise, η would attain a positive minimum in that component, contradicting $d\eta \neq 0$. Since Z is connected, the sublevel set is connected as well. Removing an analytic hypersurface preserves connectedness, and η is a proper submersion from the punctured neighborhood onto a small interval. Ehresmann's theorem then implies that the levels M_t are connected. See Remark III.2.95 of the lecture notes, Theorem I.8.7 of the lecture notes, and the proof of Theorem V.2.5 of the lecture notes (pp. 709–710).

Following Nemirovski's argument, consider

$$F(t) := \int_{M_t} d^c\psi \wedge dd^c\psi \wedge \omega^{n-2}. \tag{5.1}$$

The chosen orientation and plurisubharmonicity of ψ give $F(t) \geq 0$. For $t_1 < t_2$, Stokes' theorem also gives

$$F(t_2) - F(t_1) = \int_{\{t_1<\psi<t_2\}} (dd^c\psi)^2 \wedge \omega^{n-2} \geq 0. \tag{5.2}$$

Thus F is nondecreasing.

Remark 5.2 (Boundary orientation and nonnegativity of the integral)**.** If ν is the real unit normal pointing in the direction of increasing ψ, then $d^c\psi(J\nu) = d\psi(\nu)/(4\pi) > 0$. The direction $J\nu$, followed by the complex orientation of the complex tangent space $TM_t \cap JTM_t$, gives the boundary orientation chosen above. With this orientation, the integrand in (5.1) is a nonnegative density. See the proof of Theorem III.2.191 of the lecture notes and equations (353)–(354) of the lecture notes.

We next compute the limit. Choose a C^∞ metric $\widetilde{h}$ which agrees with h near Z and satisfies $|\sigma_D|_{\widetilde{h}} = 1$ outside a slightly larger neighborhood. Such a metric is obtained by the construction in Lemma III.2.146 of the lecture notes. Set

$$\alpha := \frac{\sqrt{-1}}{2\pi}\Theta_{\widetilde{h}}, \qquad \psi_{\text{aux}} := -\log|\sigma_D|^2_{\widetilde{h}}.$$

Then α is a C^∞ closed form representing $c_1([D])$, and $\alpha = dd^c\psi_{\text{aux}}$ on $X \setminus Z$. For large t, we have $\psi_{\text{aux}} = \psi$ near M_t, so Stokes' theorem gives

$$\begin{aligned} F(t) &= \int_{X\setminus\Omega_t} \alpha^2 \wedge \omega^{n-2} \\ &= \int_X \alpha^2 \wedge \omega^{n-2} - \int_{\Omega_t} \alpha^2 \wedge \omega^{n-2}. \end{aligned} \tag{5.3}$$

The neighborhoods Ω_t shrink to Z, and the analytic hypersurface Z has volume zero. Since α is C^∞, the last integral tends to zero. Together with (4.1), this yields

$$\lim_{t\to\infty} F(t) = \int_X c_1([D])^2 \wedge \omega^{n-2} = 0.$$

This is the computation in equations (355)–(356) of the lecture notes.

Since F is nonnegative and nondecreasing and tends to zero, $F(t) = 0$. Nonnegativity of the integrand implies that the trace of the Levi form on the complex tangent space of M_t is zero. The form is positive semidefinite, so it vanishes, and M_t is Levi-flat. Moreover, since the full complex Hessian is also positive semidefinite, its mixed components between complex tangential and normal directions vanish as well. Together with $d\psi \neq 0$, this shows that there is a C^∞ function $f \geq 0$ on the punctured neighborhood such that

$$\sqrt{-1}\partial\bar\partial\psi = f\sqrt{-1}\partial\psi \wedge \bar\partial\psi. \tag{5.4}$$

This is the conclusion of Step 1 (equation (357) of the lecture notes).

5.2 Step 2: The case of nonconstant restrictions to level sets

First, differentiate the identity obtained above,

$$\partial\bar{\partial}\psi = f\,\partial\psi \wedge \bar{\partial}\psi.$$

Using this identity again to replace the occurrences of $\partial\bar{\partial}\psi$ produced by differentiating the right-hand side, we obtain

$$df \wedge \partial\psi \wedge \bar{\partial}\psi = 0.$$

Thus f is constant along each Levi leaf of every M_t.

Suppose that, for every sufficiently large t_0, there exists $t \in (t_0, +\infty)$ such that $f|_{M_t}$ is nonconstant, and fix one such t. By Sard's theorem, choose an interval of regular values and three distinct points r_1, r_2, r_3 in that interval. Choose connected components of the level sets

$$\{x \in M_t \mid f(x) = r_j\}$$

consistently along this interval, and denote them by Γ_j. Since f is constant along the Levi leaves, each Γ_j is a connected nonsingular compact complex hypersurface in X. Moreover, they are pairwise disjoint and represent the same homology class.

Now apply the construction of a fibration based on Neeman's argument ([Nee, pp. 109–110]; for the formulation on compact Kähler manifolds, see [KJDG, proof of Lemma 3.6, p. 1220] and Lemma V.2.7 of the lecture notes). This yields a surjective holomorphic map with connected fibers

$$\pi : X \longrightarrow R$$

to a compact Riemann surface R, such that at least one of the Γ_j is the support of a fiber. By Remmert's theorem, $\pi(Z)$ is an analytic subset of R. Since Z is connected and disjoint from this fiber, $\pi(Z)$ consists of a single point. Comparing the support and multiplicities of the fiber by means of intersection numbers gives

$$\pi^*\{p\} = aD$$

for some $p \in R$ and $a \in \mathbb{Q}_{>0}$. Pulling back a local coordinate on the base around p shows that the line bundle associated with the divisor on the left-hand side is trivial near Z. Taking a positive tensor power and then a positive root of the resulting metric, we conclude that $[D]$ is unitary flat near Z.

Remark 5.3 (Three hypersurfaces and the comparison of multiplicities)**.** This step was simplified in the lecture. Ehresmann's theorem allows one to track a connected component over the interval of regular values. The resulting three hypersurfaces are isotopic, so the associated line bundles have the same integral first Chern class. The construction of the fibration is given in Lemma V.2.7 of the lecture notes, and the numerical proportionality to the original divisor D in Lemma V.2.6 of the lecture notes. For details, including the fact that Z is the entire support of the fiber, see the proof of Theorem V.2.5 of the lecture notes (pp. 708–711).

5.3 Step 3: The case of constant restrictions to level sets

It remains to consider the case in which $f|_{M_t}$ is constant for every sufficiently large t. Since the level sets are connected and $d\psi \neq 0$, we can write

$$f = g \circ \psi$$

for a real-valued C^∞ function g. Solve the ordinary differential equation in one variable

$$\chi'' + g\chi' = 0 \tag{5.5}$$

and put $\widetilde{\psi} := \chi \circ \psi$. Then

$$\begin{aligned}\partial\bar\partial\widetilde{\psi} &= \chi'(\psi)\,\partial\bar\partial\psi + \chi''(\psi)\,\partial\psi\wedge\bar\partial\psi \\ &= \big(\chi'(\psi)g(\psi) + \chi''(\psi)\big)\partial\psi\wedge\bar\partial\psi = 0.\end{aligned}$$

Thus $\widetilde{\psi}$ is pluriharmonic.

With a suitable normalization of the solution χ, we can arrange that $\chi(s) - s \to 0$ as $s \to +\infty$. Let s_j be local equations for D, and write

$$|\sigma_D|_h^2 = |s_j|^2 e^{-\varphi_j}, \qquad \psi = -\log|s_j|^2 + \varphi_j.$$

Then $\widetilde{\psi} + \log|s_j|^2$ is locally bounded near Z and pluriharmonic outside Z. Denote its pluriharmonic extension across Z by $\widetilde{\varphi}_j$. We have

$$\widetilde{\psi} = -\log|s_j|^2 + \widetilde{\varphi}_j.$$

The local metrics $e^{-\widetilde{\varphi}_j}$ glue to give a C^∞ Hermitian metric of zero curvature on $[D]$. This proves the implication from semi-positivity to unitary flatness in Theorem 4.1.

Remark 5.4 (Normalization of the ordinary differential equation and growth estimates)**.** The normalization of the solution and the growth estimates were omitted in the lecture. The following supplement is based on the proof of Theorem V.2.5 of the lecture notes (pp. 711–712). Put $\eta := |\sigma_D|_h^2 = e^{-\psi}$; then $f = B(\eta)$. Choose a nonsingular point of an irreducible component of Z, away from the other components, and let m be its multiplicity in D. Writing $\eta = |w|^{2m}e^{-\varphi}$ locally, we have

$$f = \frac{|w|^2\,\dfrac{\partial^2\varphi}{\partial w\,\partial\bar w}}{\left|-m + w\dfrac{\partial\varphi}{\partial w}\right|^2}.$$

Using $\eta^{1/(2m)}$ as a local coordinate along the real arc $w = r \geq 0$, we obtain

$$B(t) = t^{1/m}A(t),$$

where A is continuous up to $t = 0$. Since f depends only on η, this single estimate applies to all sufficiently small levels.

Changing variables to η gives

$$\partial\bar\partial\eta = G_\eta(\eta)\,\partial\eta\wedge\bar\partial\eta, \qquad G_\eta(t) = \frac{1}{t} - t^{1/m-1}A(t). \tag{5.6}$$

This is equation (358) of the lecture notes. The lecture notes denote this coefficient by G. Here we write G_η to distinguish it from the function g in $f = g(\psi)$.

More explicitly, put

$$\begin{aligned}Q(t) &:= \int_0^t s^{1/m-1}A(s)\,ds, \\ H(t) &:= \int_0^t \frac{e^{Q(s)} - 1}{s}\,ds.\end{aligned}$$

Since $Q(t) = O(t^{1/m})$, we have $H(t) = O(t^{1/m})$, and H extends continuously with $H(0) = 0$. The function $u(t) := \log t + H(t)$ satisfies $u'' + G_\eta u' = 0$, and

$$\widetilde{\psi} = -u(\eta) = \psi - H(\eta)$$

is pluriharmonic. Returning to the original variable ψ, we obtain

$$g(s) = e^{-s/m} A(e^{-s}), \qquad \chi(s) = s - H(e^{-s}).$$

This gives (5.5) and $\chi(s) - s = O(e^{-s/m}) \to 0$. The general form of this ODE appears in Mini-Exercise I.4.15 of the lecture notes; the solution to that exercise (pp. 808–809) contains a detailed proof of the existence and uniqueness of a solution with a logarithmic leading term.

Remark 5.5 (Extension across singular Z)**.** The preceding argument also applies when Z is singular or D has multiplicities. Indeed,

$$\widetilde{\psi} + \log |s_j|^2 = \varphi_j - H(\eta)$$

is locally bounded near every point of Z. An analytic hypersurface in a complex manifold is locally pluripolar, so the removable singularity theorem for pluriharmonic functions, Lemma III.2.94 of the lecture notes, applies. The transition rule required for the extended functions to define a metric follows by continuity from the corresponding identity on the dense complement. For details, see the end of the proof of Theorem V.2.5 of the lecture notes (p. 712).

6 Complements of semi-positive effective divisors and Hartogs-type extension

Finally, without assuming that $[D]|_Z$ is unitary flat, we consider holomorphic functions on the complement of a semi-positive effective divisor.

Corollary 6.1 (The dichotomy between a fibration and Hartogs-type extension)**.** Let X be a connected compact Kähler manifold and $D \neq 0$ an effective divisor, and suppose that $Z := \operatorname{Supp} D$ is connected. If $[D]$ is semi-positive, exactly one of the following holds.

(i) There exist a compact Riemann surface R, a surjective holomorphic map $\pi : X \to R$ with connected fibers, a point $p \in R$, and a positive rational number a such that $\pi^*\{p\} = aD$.

(ii) For every connected open neighborhood $V \subset X$ of Z, each holomorphic function on $V \setminus Z$ extends uniquely to all of $X \setminus Z$.

This is the assertion of Theorem V.2.9 of the lecture notes. In (ii), a holomorphic function given on the side close to the divisor extends to the entire complement. In case (i), a local coordinate on the base can be used to construct a function near Z with a pole along another fiber, so (ii) does not hold.

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