Intensive Lecture – Day 1

Complex Analysis in Several Variables

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Starting point: complex analysis in one variable

We use the following as prerequisites, recalling formulas as needed.

- Holomorphic functions, Cauchy–Riemann equations, power series and radii of convergence.
- Cauchy's theorem and integral formula; Morera's theorem.
- Identity theorem, maximum principle, open mapping theorem, Liouville's theorem.
- Laurent series, isolated singularities, meromorphic functions, residue theorem.
- Uniform convergence on compact sets; Montel's theorem.
- Riemann sphere, rational functions, Möbius transformations.

A recurring tool: Cauchy's formula

For $a \in \mathbb{C}$ and $r > 0$, write $\Delta_r(a) := \{z \in \mathbb{C} \mid |z - a| < r\}$.

Theorem (Cauchy's integral formula and estimate)

If $f\colon \Delta_R(a) \to \mathbb{C}$ is holomorphic, then for $0 < r < R$ and $k \in \mathbb{Z}_{\geq 0}$,

$$f^{(k)}(a) = \frac{k!}{2\pi\sqrt{-1}} \int_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{k+1}}\, d\zeta, \qquad |f^{(k)}(a)| \leq \frac{k!}{r^k} \max_{|\zeta - a| = r} |f(\zeta)|.$$

□

- Holomorphic functions have local convergent power series: they are analytic, hence C^∞.
- Locally uniform limits are holomorphic. Locally bounded holomorphic families satisfy Montel's theorem.

We now review subharmonic functions before moving to several variables.

Subharmonic functions: center value and circle average

Let $U \subset \mathbb{C}$ be open.

Definition

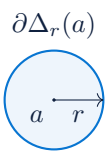
An upper semicontinuous function $u\colon U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic if for every $\overline{\Delta_r(a)} \subset U$,

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{\sqrt{-1}\theta})\, d\theta$$

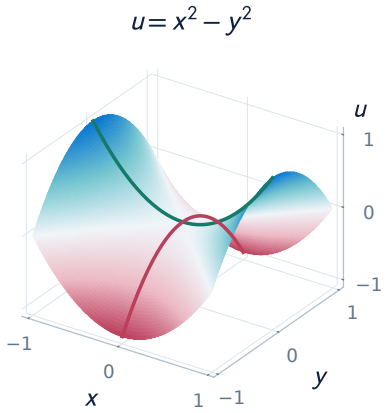
We allow $u \equiv -\infty$. □

Upper semicontinuity means $\limsup_{z \to a} u(z) \leq u(a)$.

Compare the center value with the average on a circle.



Can a saddle be subharmonic?



$$u(z) = \mathrm{Re}(z^2) = x^2 - y^2.$$

Real-line slices include both upward and downward parabolas.

On a circle centered at the origin,

$$u(re^{\sqrt{-1}\theta}) = r^2 \cos 2\theta.$$

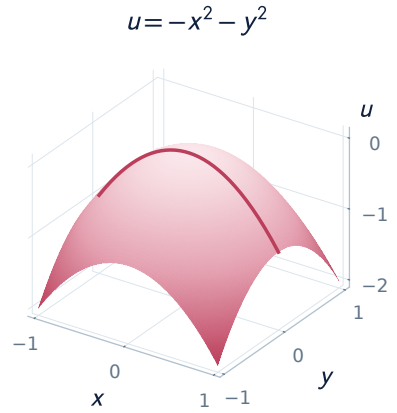
The circle average is $0 = u(0)$.

A saddle can be harmonic, hence subharmonic.

Fixed example: sh / std-b03. Rotate the graph; compare real-line slices and circle averages.

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A downward paraboloid and circle averages



$$u(z) = -|z|^2.$$

The value at the origin is 0. On the circle of radius r, every value is $-r^2$.

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{\sqrt{-1}\theta})\,d\theta = -r^2 < 0.$$

The submean inequality fails.

Reversing the sign gives the subharmonic function $|z|^2$.

Fixed example: sh / std-b02. Compare a circle centered at the origin with its center value.

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Theorem (Maximum principle)

If a subharmonic function u on a domain $U \subset \mathbb{C}$ attains its supremum at an interior point of U, then u is constant. □

Proposition

Finite maxima, sums, positive scalar multiples and decreasing limits preserve subharmonicity. □

Example

If h is harmonic, both h and $-h$ are subharmonic. For holomorphic f, the function $\log|f|^2$ is also subharmonic.

$$\max\{0, \operatorname{Re} z\}, \qquad \log|z|^2 \quad (\log 0 := -\infty).$$

□

Complex derivatives and ∂, $\overline{\partial}$

Write $z = x + \sqrt{-1}y$, $dz = dx + \sqrt{-1}\,dy$ and $d\bar{z} = dx - \sqrt{-1}\,dy$.

Definition

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right).$$

For a C^∞ function f, set

$$\partial f := \frac{\partial f}{\partial z}\,dz, \qquad \overline{\partial} f := \frac{\partial f}{\partial \bar{z}}\,d\bar{z}.$$

□

Extend to differential forms by differentiating coefficients and using the Leibniz rule. Then

$$d = \partial + \overline{\partial}, \qquad \partial^2 = \overline{\partial}^{\,2} = 0, \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

The C^2 case: testing second derivatives

Our sign convention for the Laplacian is

$$\Delta := -\left(\frac{\partial}{\partial x}\right)^2 - \left(\frac{\partial}{\partial y}\right)^2, \qquad \frac{\partial^2 u}{\partial z\, \partial \bar{z}} = -\frac{1}{4}\Delta u$$

Theorem

For a real-valued C^2 function u on U,

$$u \text{ is subharmonic} \iff \frac{\partial^2 u}{\partial z\, \partial \bar{z}} \geq 0 \iff \Delta u \leq 0.$$

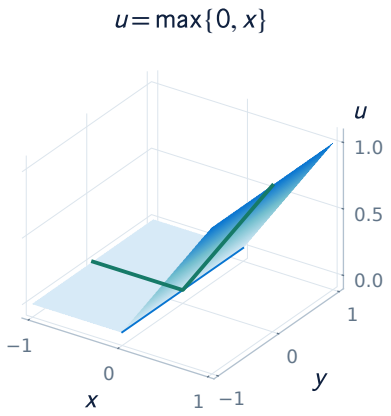
□

Taylor expansion of the circle average gives

$$\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{\sqrt{-1}\theta})\, d\theta - u(a) = -\frac{r^2}{4}\Delta u(a) + o(r^2).$$

The "Laplacian measure" in the interactive view is $-\Delta u$ in this notation.

Subharmonic even with a crease



$$u(z) = \max\{0, \operatorname{Re} z\}.$$

The maximum of two harmonic functions, 0 and $\operatorname{Re} z$.

There is a crease at $x = 0$; the function is not C^1.

$$\frac{1}{2\pi} \int_0^{2\pi} \max\{0, r\cos\theta\}\, d\theta = \frac{r}{\pi}.$$

Subharmonicity makes sense even when second derivatives are not ordinary functions.

Fixed example: sh / std-i02. This leads to our example of a distribution.

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Distributions: transferring derivatives to test functions

Definition

Let $C_c^\infty(U)$ be the space of compactly supported C^∞ functions on U. Its continuous linear functionals are distributions, forming $(C_c^\infty(U))^\vee$. The topology on the test-function space is explained next. □

Example

A locally integrable function $u \in L^1_{\mathrm{loc}}(U)$ and a point $a \in U$ define distributions by

$$T_u(\varphi) = \int_U u\varphi \, d\lambda, \qquad \delta_a(\varphi) = \varphi(a)$$

respectively. Here $d\lambda$ denotes Lebesgue measure. □

Motivated by integration by parts, define distributional derivatives by

$$\left(\frac{\partial T}{\partial x}\right)(\varphi) := -T\left(\frac{\partial \varphi}{\partial x}\right), \qquad (\Delta T)(\varphi) = T(\Delta\varphi).$$

A note on the topology of function spaces

Choose compact sets $K_\nu \subset \operatorname{Int} K_{\nu+1}$ with $U = \bigcup_{\nu \geq 0} K_\nu$. For $m \in \mathbb{Z}_{\geq 0}$, set

$$\rho_{\nu,m}(f) := \max_{p+q \leq m} \sup_{(x,y) \in K_\nu} \left| \frac{\partial^{p+q} f}{(\partial x)^p (\partial y)^q}(x,y) \right| \qquad (p, q \in \mathbb{Z}_{\geq 0})$$

Definition

A Fréchet space is a complete Hausdorff locally convex topological vector space whose topology is defined by countably many seminorms. □

Example

The seminorms above make $C^\infty(U)$ a Fréchet space. Writing $\operatorname{Supp} \varphi := \overline{\{\varphi \neq 0\}}$, the space $C^\infty_{K_\nu} := \{\varphi \in C^\infty(U) \mid \operatorname{Supp} \varphi \subset K_\nu\}$ is a closed subspace. Equip $C^\infty_c(U) = \varinjlim_\nu C^\infty_{K_\nu}$ with the finest **locally convex topology** making all inclusions continuous. Each inclusion is a closed embedding, so this is a strict countable LF-space. □

General subharmonic functions and $\sqrt{-1}\partial\overline{\partial}$

Theorem

A subharmonic function $u \not\equiv -\infty$ on a domain $U \subset \mathbb{C}$ is locally integrable. For a real-valued $L^1_{\mathrm{loc}}(U)$ equivalence class v,

$$\sqrt{-1}\partial\overline{\partial} v \geq 0 \quad \Longleftrightarrow \quad v \text{ has a unique subharmonic representative.}$$

□

Here ≥ 0 means that for every nonnegative test function $\varphi \in C_c^\infty(U)$, $(\sqrt{-1}\partial\overline{\partial} v)(\varphi) \geq 0$. For C^2 functions, $\sqrt{-1}\partial\overline{\partial} v = -\frac{1}{2}(\Delta v)\, dx \wedge dy$.

Example

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial} \log|z|^2 = \delta_0.$$

The singularity of $\log|z|^2$ at the origin appears as a Dirac measure. □

Moving to $\mathbb{C}^n$: notation and holomorphic functions

Let $\Omega \subset \mathbb{C}^n$ be open. A connected open set is a domain.

$$z = (z_1, \ldots, z_n), \qquad z_j = x_j + \sqrt{-1}y_j, \qquad |z|^2 = \sum_{j=1}^{n} |z_j|^2.$$

For $a = (a_1, \ldots, a_n)$ and $r_1, \ldots, r_n > 0$, $P(a; r_1, \ldots, r_n) := \prod_{j=1}^{n} \Delta_{r_j}(a_j)$ is a polydisc.

Definition

A function $f \colon \Omega \to \mathbb{C}$ is holomorphic if, for every $a \in \Omega$ and every j,

$$w \longmapsto f(a_1, \ldots, a_{j-1}, w, a_{j+1}, \ldots, a_n)$$

is holomorphic in one variable near $w = a_j$. □

Joint continuity has not been assumed. For real functions, separate continuity need not imply joint continuity. What happens here?

Hartogs' theorem on separate holomorphicity

For a multi-index $k = (k_1, \dots, k_n) \in (\mathbb{Z}_{\geq 0})^n$, set

$$|k| := k_1 + \cdots + k_n, \qquad (z-a)^k := (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n}$$

Theorem (Hartogs)

A separately holomorphic function $f \colon \Omega \to \mathbb{C}$ admits, near each point a, a uniformly convergent power series

$$f(z) = \lim_{N \to \infty} \sum_{\substack{k \in (\mathbb{Z}_{\geq 0})^n \\ |k| \leq N}} c_k (z-a)^k$$

In particular, f is C^∞ and $\overline{\partial} f = \sum_{j=1}^{n} \dfrac{\partial f}{\partial \bar{z}_j} \, d\bar{z}_j = 0.$ □

Separate holomorphicity implies holomorphicity in all variables together.
No continuity or local boundedness assumption is added. We omit the proof.

The identity theorem: agreement on an open set

Theorem

Holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ satisfy:

- Holomorphic functions agreeing on a nonempty open set agree on all of Ω.
- If $|f|$ has a local maximum at an interior point, then f is constant.
- Locally uniform limits are holomorphic. Locally bounded holomorphic families satisfy Montel's theorem.

□

Example

The function $f(z, w) = z$ is holomorphic on $\mathbb{C}^2$, with

$$f(0, 1/\nu) = 0 \quad (\nu = 1, 2, \ldots), \qquad f(1, 0) = 1.$$

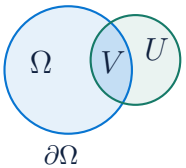
Its zeros $(0, 1/\nu)$ accumulate at $(0, 0)$, although f is not identically zero. □

What is a natural domain for holomorphic functions?

Write $\mathcal{O}_\Omega(\Omega)$ for the holomorphic functions on $\Omega \subset \mathbb{C}^n$.

Definition (Domain of holomorphy)

A domain Ω is a domain of holomorphy if, for every connected open set U meeting $\partial\Omega$ and every component V of $U \cap \Omega$, there is an $f \in \mathcal{O}_\Omega(\Omega)$ such that $f|_V$ has no holomorphic extension to all of U. □



In one variable, $(z-a)^{-1}$, with $a \in \partial\Omega$, prevents extension across that boundary point.

On a Hartogs figure, every holomorphic function extends

Write $\Delta_t := \Delta_t(0)$ and $\Delta := \Delta_1$. For $0 < r, s < 1$, set

$$H_{r,s} := (\Delta_r \times \Delta) \cup \big(\Delta \times (\Delta \setminus \overline{\Delta_s})\big) \subset \mathbb{C}^2.$$

Theorem (Hartogs extension)

Every holomorphic function $f\colon H_{r,s} \to \mathbb{C}$ has a unique holomorphic extension $\widetilde{f}$ to Δ^2. $\square$

The extension is given by a Cauchy integral:

$$\widetilde{f}(z,w) = \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta|=\rho} \frac{f(z,\zeta)}{\zeta - w}\, d\zeta, \qquad \max\{s, |w|\} < \rho < 1.$$

For $|z| < r$, Cauchy's formula gives agreement with f. Let us vary the fibers and visualize this extension.

How can we view a domain in $\mathbb{C}^2 = \mathbb{R}^4$?

Place two complex planes side by side. Move z and track the fiber

$$\Omega_z := \{w \in \mathbb{C} \mid (z, w) \in \Omega\}$$

Toshio Nishino writes that Kiyoshi Oka called such pictures *ryokuinzu* ("green-shade diagrams"). In Nishino's words:

> *"He drew and contemplated them day after day."*

(English translation of Nishino's account)

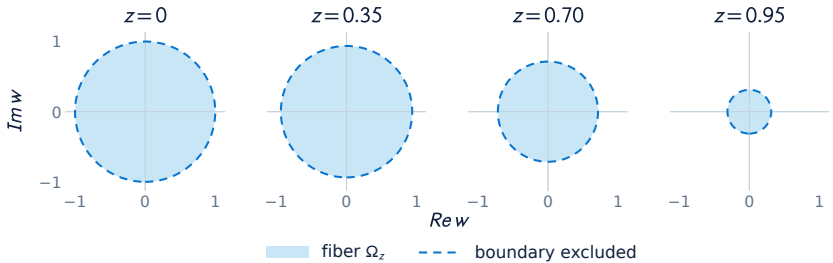
Open ball $\longrightarrow$ bidisc $\longrightarrow$ Hartogs figure

Move z on the left and compare the fibers in the w-plane on the right.

Toshio Nishino, "Oka Kiyoshi sensei no sūgaku" [The mathematics of Kiyoshi Oka], *Sugaku* 33(2), 1981, pp. 171–178, especially pp. 174–175 (in Japanese). doi:10.11429/sugaku1947.33.171

The open ball: shrinking disc fibers

$$B(0,1) = \{(z,w) : |z|^2 + |w|^2 < 1\}, \qquad B(0,1)_z = \{\, |w| < \sqrt{1-|z|^2} \,\} \quad (|z| < 1).$$

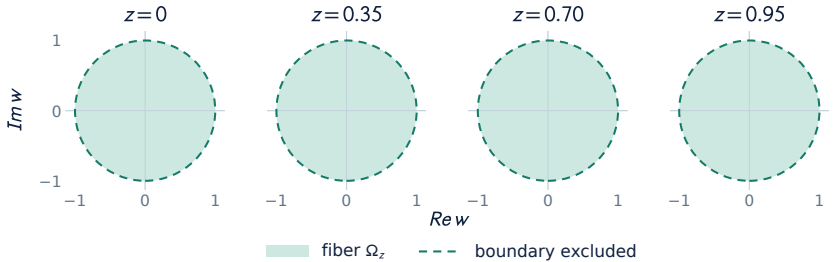


- Move z away from the origin on the left: the disc on the right shrinks.
- Changing the argument of z does not change the fiber when $|z|$ is fixed.

Fixed lecture example: ryokuinzu / lecture-ball. Dashed boundaries are excluded.
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The bidisc: constant fibers

$$\Delta^2 = \{(z,w) : |z| < 1,\ |w| < 1\}, \qquad (\Delta^2)_z = \Delta \quad (|z| < 1).$$

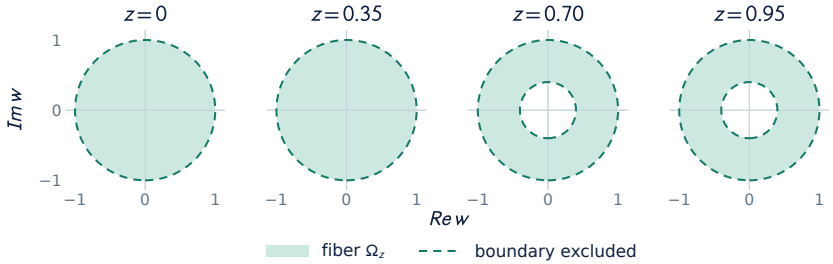


- As z moves, the fiber in the w-plane remains the same unit disc.
- This bidisc is where the next Hartogs figure will be filled in.

Fixed lecture example: ryokuinzu / lecture-polydisc. Dashed boundaries are excluded.
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The Hartogs figure: from discs to annuli

$$H_{.48,.40} = (\Delta_{.48} \times \Delta) \cup \big(\Delta \times \{.40 < |w| < 1\}\big).$$



A full disc for $|z| < .48$; an annulus for $.48 \le |z| < 1$.

Integrate on a circle $.40 < \rho < 1$. Agreement in the central part determines the extension.

Every holomorphic function extends to Δ^2: this is not a domain of holomorphy.

Fixed example: ryokuinzu / n18. Dashed boundaries are excluded.

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The tangent bundle and its complexification

Let $\Omega \subset \mathbb{C}^n$ be open, with $x_j = \operatorname{Re} z_j$ and $y_j = \operatorname{Im} z_j$.

Definition

Write $T\Omega$ for the tangent bundle of the real $2n$-dimensional manifold. Its fibers are

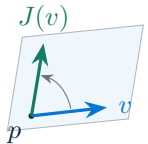
$$T_p\Omega = \bigoplus_{j=1}^{n} \left(\mathbb{R}\, \frac{\partial}{\partial x_j}\bigg|_p \oplus \mathbb{R}\, \frac{\partial}{\partial y_j}\bigg|_p \right).$$

Its complexification is $T^{\mathbb{C}}\Omega := T\Omega \otimes_{\mathbb{R}} \mathbb{C}$. □

Define the complex structure J on the real tangent bundle by

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

Then $J^2 = -\mathrm{id}$. Extend J complex-linearly to $T^{\mathbb{C}}\Omega$.



Inside $T_p\Omega$

Decomposing the tangent bundle by type

Definition

For each $p \in \Omega$, use these vector fields to define subspaces of $T_p^{\mathbb{C}}\Omega$:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right),$$

$$T_p^{1,0}\Omega := \bigoplus_{j=1}^{n} \mathbb{C}\left.\frac{\partial}{\partial z_j}\right|_p, \qquad T_p^{0,1}\Omega := \bigoplus_{j=1}^{n} \mathbb{C}\left.\frac{\partial}{\partial \overline{z}_j}\right|_p.$$

The corresponding subbundles are denoted by $T^{1,0}\Omega$ and $T^{0,1}\Omega$. □

$$T^{\mathbb{C}}\Omega = T^{1,0}\Omega \oplus T^{0,1}\Omega.$$

These rank-n complex subbundles are the $\sqrt{-1}$- and $-\sqrt{-1}$-eigenspaces of J, respectively. For example, a tangent vector $w \in T_p^{1,0}\Omega$ has the form $w = \sum_{j=1}^{n} a_j \left.\frac{\partial}{\partial z_j}\right|_p$.

Real hypersurfaces and complex tangent spaces

Let $M = \partial D \subset \Omega \subset \mathbb{C}^n$. Near $p \in M$, choose U with

$$D \cap U = \{\rho < 0\}, \qquad M \cap U = \{\rho = 0\}, \qquad d\rho \neq 0 \quad \text{on } M \cap U$$

Here ρ is real and C^2, and $T_pM = \ker(d\rho)_p$.

Definition

$$T_p^{1,0}M := \{w \in T_p^{1,0}\Omega \mid \langle(\partial\rho)_p, w\rangle = 0\} = (T_pM \otimes_{\mathbb{R}} \mathbb{C}) \cap T_p^{1,0}\Omega.$$

This is the complex tangent space of M; its complex dimension is $n-1$. □

$$\sum_{j=1}^{n} a_j \left.\frac{\partial}{\partial z_j}\right|_p \in T_p^{1,0}M \quad \Longleftrightarrow \quad \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(p) a_j = 0.$$

It is complex-linearly isomorphic to $H_pM := T_pM \cap J(T_pM)$ inside the real tangent space, via $v \mapsto v - \sqrt{-1}J(v)$.

The boundary Levi form

Definition

For $w = \sum_j a_j \frac{\partial}{\partial z_j}$ and $w' = \sum_k b_k \frac{\partial}{\partial z_k} \in T^{1,0}_p M$, define

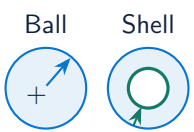
$$H_{M,p}(\rho)(w, w') := \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) a_j \overline{b_k}$$

This is the Levi form of M with respect to ρ. □

Example (Ball and spherical shell, $n \geq 2$)

For the ball, $\rho(z) = |z|^2 - 1$, so $H_{M,p}(\rho)(w, w) = \sum_j |a_j|^2 > 0$ $(w \neq 0)$.

On the inner boundary of the shell $\{a < |z| < 1\}$ $(0 < a < 1)$, take $\rho = a^2 - |z|^2$; the Levi form is $-\sum_j |a_j|^2$. □



Sign of the Levi form (schematic section)

The boundary in holomorphic coordinates

Let $n \geq 2$ and $p \in \partial D$. Fix the side $D = \{\rho < 0\}$.

Fact (Boundary normal form; proof omitted)

There are holomorphic coordinates $(z', w) = (z_1, \ldots, z_{n-1}, w)$ centered at p such that, in a sufficiently small neighborhood,

$$D = \left\{ \operatorname{Im} w > \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + R(z', \operatorname{Re} w) \right\}, \qquad R(z', u) = o(|z'|^2 + |u|^2).$$

The numbers of positive, negative and zero λ_j agree with those for the eigenvalues of the boundary Levi form. □

- z' gives complex tangential coordinates; w is a complex normal coordinate.
- The λ_j describe quadratic bending along $\{w = 0\}$.
- Keep the side of the domain fixed and compare positive and negative directions.

Positive Levi form: a local holomorphic obstruction

Proposition

If $\lambda_j > 0$ for every j, then in a small neighborhood U of the origin,

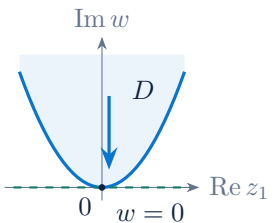
$$\{w = 0\} \cap \overline{D} \cap U = \{0\}.$$

Thus $F(z', w) = 1/w$ is holomorphic on $D \cap U$, and $|F| \to +\infty$ as we approach the origin within $D \cap U$. □

$F = 1/w$ cannot extend holomorphically across the origin.

The holomorphic hypersurface $w = 0$ locally supports the domain.

The function F constructed here is local. In the original coordinates, $\{w = 0\}$ need not be a linear hyperplane.



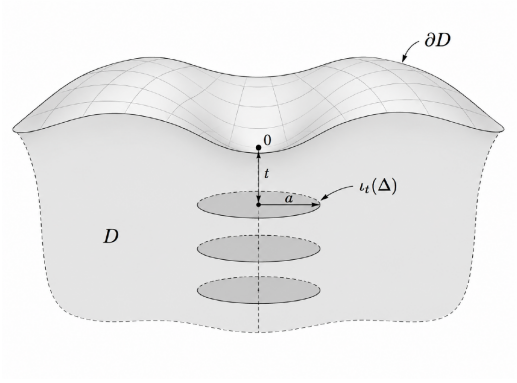
Real 2-dimensional section
of the quadratic approximation

A negative Levi direction: discs approach the boundary

Assume $\lambda_1 < 0$. Fix sufficiently small $a, t_0 > 0$ and set

$$\iota_t(\zeta) = (a\zeta, 0, \ldots, 0, \sqrt{-1}t), \qquad |\zeta| \le 1, \quad 0 < t \le t_0$$

The image $\iota_t(\Delta)$ is an analytic disc.



- For $t > 0$, the entire closed disc lies in D.
- As $t \downarrow 0$, its center approaches $0 \in \partial D$.
- Its boundary circle stays inside D, even at $t = 0$.

In this picture, the interior of the domain is drawn below the boundary.

Maximum principle: circle values control the center

Use the same formula for ι_t at $t = 0$. The union of the boundary circles,

$$\Gamma := \bigcup_{0 \le t \le t_0} \iota_t(\partial\Delta)$$

is a compact subset of D.

Proposition

For $F \in \mathcal{O}_D(D)$ and $0 < t \le t_0$, apply the one-variable maximum principle to $F \circ \iota_t$:

$$|F(\iota_t(0))| \le \max_{|\zeta|=1} |F(\iota_t(\zeta))| \le \sup_{\Gamma} |F| < +\infty.$$

□

The centers approach the boundary, but F stays uniformly bounded along them.
The bound depends on F, but not on t. We will relate this estimate to holomorphic convexity.

Cauchy's formula extends functions across the boundary

Theorem (Local extension in a negative Levi direction)

Suppose the boundary Levi form has a negative eigenvalue at a C^2 boundary point p of $D \subset \mathbb{C}^n$. There is a neighborhood U of p such that every $F \in \mathcal{O}_D(D)$ admits an extension $\widetilde{F} \in \mathcal{O}_U(U)$ with $\widetilde{F} = F$ on $D \cap U$. □

Set $z'' = (z_2, \ldots, z_{n-1})$. The integration circle remains in D when z'' and w vary slightly around 0.

$$\widetilde{F}(z_1, z'', w) = \frac{1}{2\pi\sqrt{-1}} \int_{|\eta|=a} \frac{F(\eta, z'', w)}{\eta - z_1}\, d\eta, \qquad |z_1| < a.$$

The right-hand side is holomorphic on a neighborhood crossing the boundary. Cauchy's formula gives agreement with F where the whole disc lies in D. Choose $D \cap U$ connected; the identity theorem gives agreement throughout $D \cap U$.

A domain of holomorphy cannot have a negative boundary Levi direction.

Levi pseudoconvexity: the signs of the eigenvalues

Definition

Let $D \subset \mathbb{C}^n$ be a domain with smooth boundary. Fix the side $D = \{\rho < 0\}$. At every $p \in \partial D$, require:

- all eigenvalues of $H_{\partial D,p}(\rho)$ to be ≥ 0 for Levi pseudoconvexity;
- all eigenvalues to be > 0 for strong Levi pseudoconvexity.

These are the $n-1$ eigenvalues on the complex tangent space. □

Proposition

Replacing ρ by $\widetilde{\rho} = a\rho$, with $a > 0$, gives

$$H_{M,p}(\widetilde{\rho}) = a(p) H_{M,p}(\rho).$$

The positive, negative and zero eigenvalue counts are therefore unchanged. □

Domains with corners, such as the bidisc and Hartogs figure, will be treated using exhaustion functions.

Plurisubharmonicity: test every complex line

Let $\Omega \subset \mathbb{C}^n$ be open.

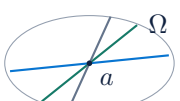
Definition

An upper semicontinuous function $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic (psh) if for every $a \in \mathbb{C}^n$ and $\xi \in \mathbb{C}^n \setminus \{0\}$,

$$w \longmapsto \varphi(a + w\xi)$$

is subharmonic on $\{w \in \mathbb{C} \mid a + w\xi \in \Omega\}$. We denote the space of such functions by $\mathrm{Psh}(\Omega)$. The value $-\infty$ is allowed identically on any connected component. □

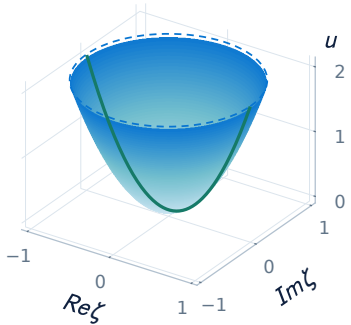
Require the submean inequality in every complex direction.
Next, change the direction of a complex line and observe.



Complex lines (schematic)

Positive curvature in every complex direction

$z = w = \zeta$: $u(\zeta, \zeta) = 2|\zeta|^2$



$|\zeta| < 1$; dashed boundary excluded

$$u(z, w) = |z|^2 + |w|^2.$$

On the complex line $(z, w) = a + \zeta\xi$,

$$u(a + \zeta\xi) = |\xi|^2|\zeta|^2 + 2\,\mathrm{Re}(c\zeta) + |a|^2$$

where $c = \xi_1\overline{a_1} + \xi_2\overline{a_2}$.

Every direction $\xi \neq 0$ gives a positive quadratic term.

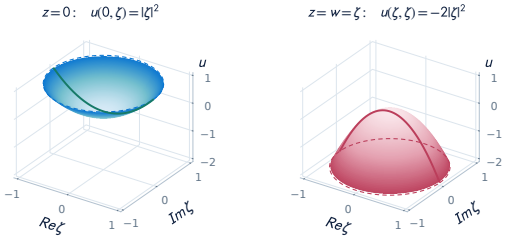
The graph shows the diagonal slice $z = w = \zeta$ through the origin.

Fixed example: psh / N01. Compare coordinate and diagonal complex discs.

Interactive view: open this slide in the HTML version

A non-example missed by coordinate slices

$$u(z,w) = |z|^2 + |w|^2 - 4\,\mathrm{Re}(z\overline{w}).$$



$|\zeta| < 1$; dashed boundary excluded

Every coordinate slice is strictly subharmonic. But on $z = w = \zeta$, we have $u = -2|\zeta|^2$.

$$H(u) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}, \qquad \text{eigenvalues } 3,\ -1.$$

Fixed example: psh / A03. Coordinate directions alone cannot detect plurisubharmonicity.
Interactive view: open this slide in the HTML version

C^2 plurisubharmonicity and the complex Hessian

Theorem

For a real C^2 function φ, the following are equivalent:

$$\varphi \in \mathrm{Psh}(\Omega) \quad \Longleftrightarrow \quad H(\varphi) := \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{j,k} \geq 0 \quad \Longleftrightarrow \quad \sqrt{-1} \partial \overline{\partial} \varphi \geq 0.$$

□

Indeed, the complex Hessian of $\varphi(a + w\xi)$ with respect to w is

$$\frac{\partial^2}{\partial w \partial \overline{w}} \varphi(a + w\xi) \Big|_{w=0} = \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}(a) \xi_j \overline{\xi_k}.$$

Definition

A function $\varphi \in \mathrm{Psh}(\Omega)$ is strictly psh if, near every point, $\varphi - \varepsilon |z|^2$ is psh for some $\varepsilon > 0$. For C^2 functions, this is equivalent to $H(\varphi) > 0$. □

More examples of psh functions

Example

Function	Property
$\lvert z\rvert^2 + \lvert w\rvert^2$	Strictly psh
$\mathrm{Re}\,(zw)$	Pluriharmonic: $H(\varphi) = 0$
$\lvert w - z^2\rvert^2$	Psh, but not strictly psh
$\max\{\lvert z\rvert^2, \lvert w\rvert^2\}$	Continuous psh; not generally C^2
$\log\left(\sum_{\nu=1}^{m} \lvert f_\nu\rvert^2\right)$	Psh when the f_ν are holomorphic

In the last example, set the value to $-\infty$ at common zeros. □

Proposition

Finite maxima, sums, positive scalar multiples and decreasing limits preserve plurisubharmonicity. □

General psh functions: positive $(1,1)$-currents

Theorem

On a domain Ω, every $\varphi \in \mathrm{Psh}(\Omega)$ with $\varphi \not\equiv -\infty$ belongs to $L^1_{\mathrm{loc}}(\Omega)$. For a real L^1_{loc} equivalence class u,

$$\sqrt{-1}\partial\overline{\partial} u \geq 0 \quad \Longleftrightarrow \quad u \text{ has a unique psh representative.}$$

□

Currents allow differential forms to have distributional coefficients. Positivity of this $(1,1)$-current means that, for every constant vector $\xi \in \mathbb{C}^n$ and every $0 \leq \chi \in C^\infty_c(\Omega)$,

$$\sum_{j,k=1}^{n} \xi_j \overline{\xi_k} \left\langle \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}, \chi \right\rangle \geq 0$$

Positivity of the complex Hessian also unifies the nonsmooth examples.

Holomorphic convexity and exhaustion functions

Definition (Holomorphic convexity)

The holomorphic hull of a compact set $\Gamma \subset \Omega$ is

$$\widehat{\Gamma}_\Omega := \{x \in \Omega \mid |f(x)| \leq \sup_\Gamma |f| \text{ for every } f \in \mathcal{O}_\Omega(\Omega)\}$$

If this hull is compact for every compact Γ, then Ω is holomorphically convex. □

With a negative Levi direction, $\iota_t(0) \in \widehat{\Gamma}_D$, yet $\iota_t(0) \to 0 \in \partial D$. Thus $\widehat{\Gamma}_D$ is not compact.

Definition (Exhaustion function)

A continuous function $\rho : \Omega \to \mathbb{R}$ is an exhaustion function if, for every $c \in \mathbb{R}$,

$$\Omega_c := \{x \in \Omega \mid \rho(x) < c\} \Subset \Omega$$



□

$A \Subset \Omega$ means that the closure of A is a compact subset of Ω.

Oka's theorem: characterizations of pseudoconvexity

Let $\Omega \Subset \mathbb{C}^n$ be a domain with C^∞ boundary.

Theorem (Solution of the Levi problem: Oka)

The following are equivalent. The heart of the Levi problem is (5)⇒(1).

1. Ω is a domain of holomorphy.
2. Ω is holomorphically convex.
3. Ω admits a C^∞ psh exhaustion function.
4. Ω admits a C^∞ strictly psh exhaustion function.
5. $\partial\Omega$ is Levi pseudoconvex on the side of Ω.

□

Definition

A domain Ω satisfying these equivalent conditions is pseudoconvex. The exhaustion condition also defines pseudoconvexity without boundary smoothness. □

Psh functions that diverge at the boundary

Example (Ball and polydisc)

$$B(0,1): \quad \rho_B(z) = -\log(1-|z|^2), \qquad \Delta^n: \quad \rho_{\Delta^n}(z) = -\sum_{j=1}^{n} \log(1-|z_j|^2).$$

Both are C^∞ strictly psh exhaustion functions. □

Theorem (Oka's lemma)

For a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$, let $\delta_\Omega(z) := \operatorname{dist}(z, \partial\Omega)$ be the distance to the boundary. Then $-\log \delta_\Omega$ is a continuous psh exhaustion function. □

Exhaustion functions are defined inside the domain; they tend to $+\infty$ at its boundary.

Complex manifolds: gluing holomorphic coordinates

Definition

Let X be a Hausdorff, second-countable space with an open cover $\{U_\alpha\}$ and homeomorphisms onto open subsets of $\mathbb{C}^n$,

$$z_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbb{C}^n$$

If each transition map $z_\beta \circ z_\alpha^{-1}$ on an overlap is biholomorphic, then X is a complex manifold of dimension n. □

Example

Domains in $\mathbb{C}^n$, Riemann surfaces, the Riemann sphere $\mathbb{P}^1$, and their products. □

Proposition

Holomorphic pullback preserves plurisubharmonicity. Thus $\mathrm{Psh}(X)$ is well-defined by requiring a function to be psh in every local coordinate chart. □

Strong and weak 1-convexity of functions

Definition

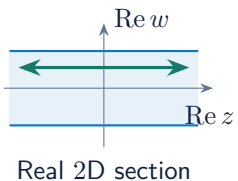
For a real C^2 function φ on a complex manifold, the case $q = 1$ reads

$$\begin{aligned} \text{strongly 1-convex at } x &\iff H(\varphi)(x) > 0, \\ \text{weakly 1-convex at } x &\iff H(\varphi)(x) \geq 0. \end{aligned}$$

Requiring this everywhere gives strict plurisubharmonicity and plurisubharmonicity, respectively. □

Example

On $\mathbb{C}^n$, the function $|z|^2$ is a strictly psh exhaustion. On $\mathbb{C}^2$, the function $\varphi(z, w) = |w|^2$ is psh, but $\{\varphi < c\} = \mathbb{C} \times \Delta_{\sqrt{c}}$, for $c > 0$, is not relatively compact. Thus φ is not an exhaustion. □



Strongly 1-convex and weakly 1-complete

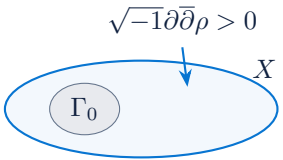
Consider C^∞ exhaustion functions $\rho : X \to \mathbb{R}$ on a complex manifold X.

Definition

- Strongly 1-convex: there exist such a ρ and a compact set $\Gamma_0 \subset X$ such that, on $X \setminus \Gamma_0$, $\sqrt{-1}\partial\overline{\partial}\rho > 0$.
- Weakly 1-complete: there exists such a ρ with $\sqrt{-1}\partial\overline{\partial}\rho \geq 0$ on all of X.
- Strongly 1-complete: there exists such a ρ with $\sqrt{-1}\partial\overline{\partial}\rho > 0$ on all of X.

□

Specify both the strength of positivity and where it holds. In every case, an exhaustion function is required.



Strongly 1-convex (schematic)

Example: $\mathbb{P}^1 \times \mathbb{C}$ is weakly 1-complete

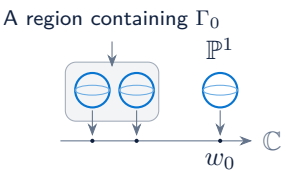
Example

On $X = \mathbb{P}^1 \times \mathbb{C}$, set $\rho(p, w) = |w|^2$. For $c > 0$, the set $\{\rho < c\} = \mathbb{P}^1 \times \Delta_{\sqrt{c}}$ is relatively compact; for $c \leq 0$, it is empty. In local coordinates, $H(\rho) = \mathrm{diag}(0, 1) \geq 0$ so ρ is a C^∞ psh exhaustion function. □

Proposition

X is not strongly 1-convex. □

Given any compact set Γ_0, choose $|w_0|$ so large that $\mathbb{P}^1 \times \{w_0\}$ is disjoint from Γ_0. Restricting a putative exhaustion, strictly psh there, to this compact curve contradicts the maximum principle.



What we have established

- Subharmonicity: circle averages, $\Delta u \leq 0$, distributional positivity.
- Plurisubharmonicity: every complex line, $H(\varphi) \geq 0$, $\sqrt{-1}\partial\overline{\partial}\varphi \geq 0$.
- Hartogs phenomenon and extension: at a boundary point with a negative Levi direction, every holomorphic function extends locally.
- Pseudoconvexity in $\mathbb{C}^n$: domains of holomorphy, exhaustions and boundary Levi forms.
- On manifolds: strongly 1-convex and weakly 1-complete.

Question

How does the geometry of a strongly 1-complete or strongly 1-convex space differ from that of a weakly 1-complete space? □

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