My lecture slides

Definitions, theorems and interactive observation

Your name

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Questions for today

Connect mathematical definitions with the intuition gained from pictures.

- How does the value of a function compare with its circular average?
- Can derivatives detect this property?
- Can a function with a saddle-shaped graph be subharmonic?

Subharmonic functions

Let $\Omega \subset \mathbb{C}$ be a domain.

Definition

An upper semicontinuous function $u\colon \Omega \to [-\infty, \infty)$, with $u \not\equiv -\infty$, is subharmonic if

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta$$

for every closed disc $\overline{D(a,r)} \subset \Omega$.

The pictures mainly show continuous examples, but continuity is not assumed in the definition.

A criterion for smooth functions

Theorem

A function $u \in C^2(\Omega, \mathbb{R})$ is subharmonic if and only if $\Delta u = u_{xx} + u_{yy} \geq 0$.

Example

For $u(x+iy) = \mathrm{Re}(z^2) = x^2 - y^2$,

$$u_{xx} = 2, \qquad u_{yy} = -2, \qquad \Delta u = 0.$$

This saddle-shaped graph belongs to a harmonic, hence subharmonic, function.

Explore a saddle-shaped function

Example

$u(z) = \mathrm{Re}(z^2) = x^2 - y^2.$

The real-axis slice bends upward; the imaginary-axis slice bends downward.

Yet on every circle, the center value equals the circular average.

In the HTML version, move the slices and rotate the view to explore this example.

The interactive view is available in the HTML presentation

The next question

- We studied subharmonicity through circular averages and the Laplacian.
- Subharmonicity does not require convexity along every real line.

Question

Which slices should we consider in $\mathbb{C}^n$?